An Asymptotic Study of Generalized Moment Estimators of the Nakagami Fading Parameters

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Abstract — **Moment estimation of the Nakagami-m fading parameters is further investigated. The asymptotic large-sample properties of the generalized moment estimators are studied. The limiting case of this family is particularly of interest, because this new estimator is simple to implement and is almost fully efficient.**

I. INTRODUCTION

The Nakagami-m distribution is flexible in modeling a variety of fading conditions. This model is of practical importance because the Nakagami-m distribution was originally deduced from experimental data [1]. Extensive empirical measurements have confirmed the usefulness of the Nakagami-m distribution for modeling radio links over a wide range of frequency bands (see [2] and references therein). The Nakagami-m model is also of theoretical interest because the distribution function contains only elementary functions. Thus, analytical expressions can often be obtained for system performance metrics such as bit error rate and outage probability.

The probability density function (PDF) of the Nakagami-m distribution is given by [3]

$$f_R(r) = \frac{2}{\Gamma(m)} \left(\frac{m}{\Omega}\right)^m r^{2m-1} e^{-mr^2/\Omega}, \ \ r \geq 0$$

where Ω is the second moment, i.e., $\Omega = \mathbb{E}[R^2]$, and the m parameter, also known as inverse of the fading figure (FF), is defined as

$$m = \frac{\Omega^2}{\mathbb{E}[(R^2 - \Omega)^2]}, \ \ m \geq 1/2.$$

The Nakagami-m distribution covers a wide range of fading conditions; when $m = 1/2$, it is a one-sided Gaussian distribution and when $m = 1$, it is a Rayleigh distribution. In the limit as m approaches infinity, the channel becomes static and its corresponding PDF becomes an impulsive function located at $\sqrt{\Omega}$, its root-mean-square value. The kth moment expression for the Nakagami-m distribution is given by

$$\mu_k = \mathbb{E}[R^k] = \frac{\Gamma(m+k/2)}{\Gamma(m)} \left(\frac{\Omega}{m}\right)^{k/2}.$$

The motivations of this work are as follows. In order to use the Nakagami-m distribution to model a given set of empirical fading data, one must determine or estimate the value of m from these measured data. Knowledge of the m parameter is also required by the receiver for optimal reception of signals in Nakagami fading [4]. The later application calls for a good m estimator with a form which is suitable for fast and efficient on-line implementation. The knowledge of the m parameter can also be fed back to the transmitter side so that the transmission can be adapted by taking account of the channel information.

Throughout this work we assume a high signal-to-noise power ratio environment. Therefore, the background noise can be neglected in our analysis. The estimation problem can be formally stated as follows. Given $r_1, r_2, \ldots, r_N$ as N independent realizations of the Nakagami-m random variable (RV), we seek good estimators $\hat{m}$, which are functions of $r_1, r_2, \ldots, r_N$, for m.

Maximum-likelihood (ML) estimation of the Nakagami fading parameter was considered in [5], where two approximate ML-based estimators were proposed. In a recent note [6], Zhang pointed out that estimation of the m parameter can be put in the framework of Gamma density parameter estimation. This is because the square of a Nakagami RV follows the Gamma distribution. The mapping from Nakagami to Gamma is one-to-one; therefore, by the data processing theorem, we can estimate the Nakagami m parameter based on Gamma samples without loss of information. It can be shown that the log-likelihood function (LLF) of Nakagami-m and Gamma samples will both lead to non-linear equations as

$$g(\hat{m}) = \ln(\hat{m}) - \psi(\hat{m}) = \Delta; \ \ \ \hat{m} = g^{-1}(\Delta)$$

where $\psi(\cdot)$ is the psi function, also called the digamma function, defined in [7], and $\Delta = \ln\left[\frac{1}{N}\sum_{i=1}^{N} r_i^2\right] - \frac{1}{N}\sum_{i=1}^{N} \ln r_i^2$, a function of N observed Nakagami-m samples. Unfortunately, there exists no direct way to compute $g^{-1}(\cdot)$ other than iterative numerical techniques. This has, in part, motivated some researchers to approach this estimation problem using the method of moments.

Moment-based estimation of the Nakagami-m parameter was first considered by Abdi and Kaveh in [8] where a moment estimator was proposed based on the second and the fourth Nakagami sample moments. A better moment estimator which uses the first and the third sample moments was proposed by Cheng and Beaulieu [9]. Using both integer and non-integer sample moments, Cheng and Beaulieu [10] proposed a family of new moment estimators for m and conducted a simulation study. It was shown that this family of moment estimators also includes known integer moment estimators as special cases, therefore, the name generalized moment estimators. More recently, Tepedelenlioglu [11] proposed moment estimators of the form

$$f_{l,k}(\hat{m}) = \frac{\hat{\mu}_l^k}{\hat{\mu}_k^l}; \ \ \ \hat{m} = f_{l,k}^{-1}\left(\frac{\hat{\mu}_l^k}{\hat{\mu}_k^l}\right)$$

where $\hat{\mu}_k$ is the kth Nakagami sample moment. It has been shown that $f_{1,2}^{-1}(\hat{\mu}_1^2/\hat{\mu}_2)$ outperforms all published integer-based moment estimators. However the inverse function $f_{l,k}^{-1}(\cdot)$ in general does not have closed-form expression and it needs to be numerically computed and stored in an implementation.

In this paper, we study the asymptotic large-sample properties of the generalized moment estimators proposed in [10]. As will be seen, the limiting case of this family is particularly of interest, because this estimator is simple to implement and is almost fully efficient.

II. GENERALIZED MOMENT ESTIMATORS

In [10], Cheng and Beaulieu have shown

$$m_{\frac{1}{p}} = \frac{\mu_{\frac{1}{p}}\mu_2}{2p(\mu_{2+\frac{1}{p}} - \mu_{\frac{1}{p}}\mu_2)} \quad (1)$$

where p is real and non-negative. We now investigate the limiting case when p approaches $+\infty$. Putting $k = 1/p$, we rewrite (1) as

$$m_k = \frac{k\mu_k\mu_2}{2(\mu_{2+k} - \mu_k\mu_2)} = \frac{\mu_2}{\rho_k} \quad (2)$$

where

$$\rho_k = \frac{2(\mu_{k+2} - \mu_k\mu_2)}{k\mu_k}. \quad (3)$$

Note that when the real number k approaches zero, ρ_k approaches the form of $0/0$! Therefore, by invoking l'Hôpital's rule, we can obtain the limiting value of ρ_k when $k \to 0$ as

$$\begin{aligned}
\lim_{k \to 0} \rho_k &= \lim_{k \to 0} \frac{2(\mu_{k+2} - \mu_k\mu_2)}{k\mu_k} \\
&= \lim_{k \to 0} \frac{2(\mathbb{E}[R^{k+2}] - \mathbb{E}[R^k]\mathbb{E}[R^2])}{k\mathbb{E}[R^k]} \\
&= \lim_{k \to 0} \frac{2\left(\mathbb{E}[\frac{d}{dk}R^{k+2}] - \mathbb{E}[\frac{d}{dk}R^k]\mathbb{E}[R^2]\right)}{\frac{d}{dk}(k\mathbb{E}[R^k])} \\
&= \mathbb{E}[R^2 \ln R^2] - \mathbb{E}[R^2]\mathbb{E}[\ln R^2] \\
&= \text{cov}[R^2, \ln R^2]. \quad (4)
\end{aligned}$$

Combining (2) and (4), we obtain, in essence, a new compact definition for the Nakagami-m fading parameter as

$$m = \frac{\mathbb{E}[R^2]}{\text{cov}[R^2, \ln R^2]} . \quad (5)$$

This result is intuitively correct. As a sanity check, when m approaches infinity (no fading), both R^2 and $\ln R^2$ approach their respective means, and thus, $\lim_{m \to +\infty} \text{cov}[R^2, \ln R^2] = 0$, which in turn suggests that m is infinity according to (5).

The fading figure for Nakagami-m fading is given by FF = $1/m$ [1]. According to (5), we can obtain an alternative expression for the Nakagami fading figure as

$$\begin{aligned}
\text{FF} &= \frac{\text{cov}[R^2, \ln R^2]}{\mathbb{E}[R^2]} \\
&= \frac{\text{cov}[R^2, 10\log_{10}R^2]}{c\mathbb{E}[R^2]} \quad (6)
\end{aligned}$$

where $c = 10\log_{10}e$. In words, eqn. (6) says that the fading figure for the Nakagami-m fading model can be interpreted as the covariance of the instantaneous fading power and its value in dB [1] normalized by its average fading power and a constant. The covariance term $\text{cov}[R^2, \ln R^2]$ has its own interesting physical interpretation. To see this, since FF = $\text{var}[R^2]/(\mathbb{E}[R^2])^2$, from (6), we have

$$\begin{aligned}
\text{cov}[R^2, \ln R^2] &= \frac{\text{var}[R^2]}{\mathbb{E}[R^2]} \\
&= \frac{\text{Variance of the instantaneous fading power}}{\text{Mean of the instantaneous fading power}} . \quad (7)
\end{aligned}$$

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[1]We comment that, coincidentally, the original Nakagami distribution was deduced from the distribution of the intensity in dB [1].

The new compact definition of the m fading parameter in (5) suggests that we can have a new estimator for m based on sample moments and sample covariance. The new estimator becomes

$$\hat{m}_0 = \frac{a}{\frac{1}{N-1}\sum_{i=1}^{N}(r_i^2 - a)(\ln r_i^2 - b)} \quad (8)$$

where

$$a = \frac{1}{N}\sum_{i=1}^{N}r_i^2, \quad b = \frac{1}{N}\sum_{i=1}^{N}\ln r_i^2, \quad (9)$$

and where the factor $1/(N-1)$ in the denominator of $\hat{m}_0$ is required to obtain an unbiased estimator of the covariance for a finite sample size. This new m parameter estimator is compact and computationally simpler than those estimators presented in [10]. We remark that $\hat{m}_0$ is also a function of the complete sufficient statistic derived in [9] for m and Ω.

III. ASYMPTOTIC PROPERTIES OF THE GENERALIZED MOMENT ESTIMATORS

It is difficult to obtain the analytical performances of our generalized moment estimators for m with finite sample size. We can, however, obtain their asymptotic large-sample performances analytically. To do so, we first rewrite our generalized moment estimators in forms which are suitable for our ensuing analysis, as

$$\hat{m}_k = \begin{cases} \frac{k}{2\left(\frac{\mu_{k+2}}{\mu_k\mu_2} - 1\right)} & k > 0 \\ \left(\frac{\frac{1}{N}\sum_{i=1}^{N}r_i^2 \ln r_i^2}{\hat{\mu}_2} - \frac{1}{N}\sum_{i=1}^{N}\ln r_i^2\right)^{-1} & k = 0 \end{cases} \quad (10)$$

where, for mathematical convenience, we have replaced $1/(N-1)$ by $1/N$ in $\hat{m}_0$ since these two factors are equivalent when N approaches infinity. We now summarize our main result on the large-sample properties of our new moment-based estimators in the following theorem.

Theorem *The estimators in (10), indexed by $k \geq 0$, are consistent, i.e.*

$$\hat{m}_k \xrightarrow{Pr} m \quad as \quad N \longrightarrow +\infty \quad (11)$$

and asymptotically normal-distributed

$$\sqrt{N}(\hat{m}_k - m) \xrightarrow{L} \mathcal{N}(0, \sigma_k^2) \quad as \quad N \longrightarrow +\infty \quad (12)$$

where for $k > 0$, the asymptotic variance is

$$\sigma_k^2 = m^2\left[\frac{v_{2k}}{v_k^2} + \frac{v_{2k+2} - \frac{v_{k+2}^2}{v_2}}{(k/2)^2 v_k^2}\right] \quad (13)$$

where $v_k = \Gamma(m+k/2)/\Gamma(m)$; and where for $k = 0$, the asymptotic variance is

$$\sigma_0^2 = m^2[1 + m\psi'(m+1)]. \quad (14)$$

Proof: Recall that if R is a Nakagami-m distributed RV, then $G = (m/\Omega)R^2$ has a one-parameter Gamma density function with shape parameter m, which is restricted to values $m > 1/2$, i.e.,

$$f_G(g) = \frac{g^{m-1}e^{-g}}{\Gamma(m)}, \quad g > 0. \quad (15)$$

The mean and variance of G are $\mathbb{E}[G] = m$ and $\text{var}[G] = m$, respectively. The kth moment of G is given by

$$\mathbb{E}[G^k] = \frac{\Gamma(m+k)}{\Gamma(m)} \quad (16)$$

and therefore,

$$v_k = \frac{\Gamma(m+k/2)}{\Gamma(m)} = \mathbb{E}[G^{k/2}]. \qquad (17)$$

We first observe that the estimators in (10) share a property of *scale invariance* such that if $\hat{m}_k = T_k(r_1^2, r_2^2, \ldots, r_N^2)$, then

$$T_k(cr_1^2, cr_2^2, \ldots, cr_N^2) = T_k(r_1^2, r_2^2, \ldots, r_N^2) \qquad (18)$$

for any constant $c > 0$. To see this, when $k > 0$,

$$
\begin{aligned}
T_k(cr_1^2, cr_2^2, \ldots, cr_N^2) &= \cfrac{k}{2\left\{ \cfrac{\frac{1}{N}\sum_{i=1}^N (cr_i^2)^{\frac{k+2}{2}}}{\left[\frac{1}{N}\sum_{i=1}^N (cr_i^2)^{\frac{k}{2}}\right]\left[\frac{1}{N}\sum_{i=1}^N (cr_i^2)\right]} - 1 \right\}} \\
&= \cfrac{k}{2\left[\cfrac{c^{\frac{k+2}{2}}\hat{\mu}_{k+2}}{c^{\frac{k}{2}+1}\hat{\mu}_k\hat{\mu}_2} - 1 \right]} \\
&= \cfrac{k}{2\left[\cfrac{\hat{\mu}_{k+2}}{\hat{\mu}_k\hat{\mu}_2} - 1 \right]} \\
&= T_k(r_1^2, r_2^2, \ldots, r_N^2).
\end{aligned}
$$

Similarly, one can show that $\hat{m}_0$ is also scale invariant. As a consequence, if we choose $c = m/\Omega$, this invariance property allows us to write

$$
\begin{aligned}
\hat{m}_k &= T_k(r_1^2, r_2^2, \ldots, r_N^2) \\
&= T_k((m/\Omega)r_1^2, (m/\Omega)r_2^2, \ldots, (m/\Omega)r_N^2) \\
&= T_k(g_1, g_2, \ldots, g_N) \qquad (19)
\end{aligned}
$$

where $g_1, g_2, \ldots, g_N$ are samples from the one-parameter Gamma density given in (15). Furthermore, we have the identity $v_{k/2} = (\frac{m}{\Omega})^{k/2}\mu_k$. We emphasize that the above substitution is carried out for mathematical convenience only, as in general $c = m/\Omega$ is unknown, m and Ω are the parameters we need to estimate.

For $\hat{m}_k$ $(k > 0)$, to establish consistency, we note that

$$
\begin{aligned}
\hat{\mu}_k &= \frac{1}{N}\sum_{i=1}^N (r_i^2)^{k/2} \\
&= \frac{1}{N}\left[(r_1^2)^{k/2} + (r_2^2)^{k/2} + \ldots + (r_N^2)^{k/2} \right] \\
&= \frac{(\Omega/m)^{k/2}}{N}\left[g_1^{k/2} + g_2^{k/2} + \ldots + g_N^{k/2} \right].
\end{aligned}
\qquad (20)
$$

Therefore, by the Weak Law of Large Number (WLLN), we have

$$\hat{\mu}_k \xrightarrow{Pr} (\Omega/m)^{k/2}\mathbb{E}[G^{k/2}] = (\Omega/m)^{k/2}v_{k/2} = \mu_k. \qquad (21)$$

Since $\hat{m}_k$ is a continuous function of $\hat{\mu}_k$, we have

$$\hat{m}_k \xrightarrow{Pr} \frac{k}{2\left(\frac{v_{k+2}}{v_k v_2} - 1\right)} = m. \qquad (22)$$

To find the asymptotic variance σ_k^2, we note that, by the multivariate Central Limit Theorem (CLT),

$$\sqrt{N}\begin{bmatrix} \hat{\mu}_2 - v_2 \\ \hat{\mu}_k - v_k \\ \hat{\mu}_{k+2} - v_{k+2} \end{bmatrix} \xrightarrow{L} \mathcal{N}(\mathbf{0}, \Sigma_k) \qquad (23)$$

where $\mathcal{N}(\mathbf{0}, \Sigma_k)$ is a trivariate normal distribution with mean vector $\mathbf{0} = [0, 0, 0]^T$ and covariance matrix Σ_k given by

$$\Sigma_k =$$
$$\begin{bmatrix} \mathrm{var}[G] & \mathrm{cov}[G, G^{k/2}] & \mathrm{cov}[G, G^{(k+2)/2}] \\ \mathrm{cov}[G, G^{k/2}] & \mathrm{var}[G^{k/2}] & \mathrm{cov}[G^{k/2}, G^{(k+2)/2}] \\ \mathrm{cov}[G, G^{(k+2)/2}] & \mathrm{cov}[G^{k/2}, G^{(k+2)/2}] & \mathrm{var}[G^{(k+2)/2}] \end{bmatrix}.$$

Using (16) and (17), we can express entries of Σ_k in terms of v_k and obtain

$$\Sigma_k = \begin{bmatrix} v_4 - v_2^2 & v_{k+2} - v_2 v_k & v_{k+4} - v_2 v_{k+2} \\ v_{k+2} - v_2 v_k & v_{2k} - v_k^2 & v_{2k+2} - v_k v_{k+2} \\ v_{k+4} - v_2 v_{k+2} & v_{2k+2} - v_k v_{k+2} & v_{2k+4} - v_{k+2}^2 \end{bmatrix}. \qquad (24)$$

If we define

$$f_k(x, y, z) = \frac{k}{2\left(\frac{z}{xy}\right) - 1}, \qquad (25)$$

the gradient of $f_k(x, y, z)$ can be found to be

$$\nabla f_k(x, y, z) = \frac{2f_k^2(x, y, z)}{kxy}\left[\frac{z}{x}, \frac{z}{y}, -1\right]^T. \qquad (26)$$

Evaluating $\nabla f_k(x, y, z)$ at $[v_2, v_k, v_{k+2}]^T$ and using the fact that $f_k(v_2, v_k, v_{k+2}) = m$ and $v_2 = m$, we have

$$\nabla f_k(v_2, v_k, v_{k+2}) = \frac{m}{(k/2)v_k}\left[\frac{v_{k+2}}{v_2}, \frac{v_{k+2}}{v_k}, -1\right]^T. \qquad (27)$$

By the multivariate *delta method* described in [12], we obtain the asymptotic variance σ_k^2 as

$$\sigma_k^2 = \left(\nabla f_k(v_2, v_k, v_{k+2})\right)^T \Sigma_k \left(\nabla f_k(v_2, v_k, v_{k+2})\right) \qquad (28)$$

where the above expression can be evaluated using a quadratic form expansion. After some tedious algebra and simplifications, we obtain (13).

For $\hat{m}_0$, we first note that the following identity is true

$$\mathbb{E}[G^i(\ln G)^j] = \frac{\frac{d^j}{dm^j}\Gamma(m+i)}{\Gamma(m)}, \quad i > 0, \; j = 0, 1, 2, \ldots \quad. \qquad (29)$$

Let $\gamma_1 = \mathbb{E}[G\ln G]$, $\gamma_2 = \mathbb{E}[G]$, $\gamma_3 = \mathbb{E}[\ln G]$, and let $\hat{\gamma}_1$, $\hat{\gamma}_2$, $\hat{\gamma}_3$ be the sample means for $G\ln G$, G, and $\ln G$, respectively. By WLLN, we have $\hat{\gamma}_k \xrightarrow{Pr} \gamma_k$. Since $\hat{m}_0$ is also a continuous function of $\hat{\gamma}_k$, we establish the consistency for $\hat{m}_0$ as

$$\hat{m}_0 \xrightarrow{Pr} \left(\frac{\gamma_1}{\gamma_2} - \gamma_3\right)^{-1} = m. \qquad (30)$$

By the multivariate CLT, we have

$$\sqrt{N}\begin{bmatrix} \hat{\gamma}_1 - \gamma_1 \\ \hat{\gamma}_2 - \gamma_2 \\ \hat{\gamma}_3 - \gamma_3 \end{bmatrix} \xrightarrow{L} \mathcal{N}(\mathbf{0}, \Sigma_0) \qquad (31)$$

where $\mathcal{N}(\mathbf{0}, \Sigma_0)$ is a trivariate normal distribution with mean vector $\mathbf{0} = [0, 0, 0]^T$ and covariance matrix Σ_0 given by

$$\Sigma_0 = \begin{bmatrix} \mathrm{var}[G\ln G] & \mathrm{cov}[G, G\ln G] & \mathrm{cov}[G\ln G, \ln G] \\ \mathrm{cov}[G, G\ln G] & \mathrm{var}[G] & \mathrm{cov}[G, \ln G] \\ \mathrm{cov}[G\ln G, \ln G] & \mathrm{var}[G, \ln G] & \mathrm{var}[\ln G] \end{bmatrix} \qquad (32)$$

and where the entries of Σ_0 can be calculated using (29). If we now define

$$f_0(x,y,z) = \frac{1}{\frac{x}{y} - z},\qquad(33)$$

the gradient of $f_0(x,y,z)$ can be found as

$$\nabla f_0 = \frac{f_0^2(x,y,z)}{y}\left[-1, \frac{x}{y}, y\right]^T.\qquad(34)$$

Evaluating $\nabla f_0(x,y,z)$ at $[\gamma_1, \gamma_2, \gamma_3]^T$ and using the fact that

$$f_0(\gamma_1,\gamma_2,\gamma_3) = f_0(\mathbb{E}[G\ln G], \mathbb{E}[G], \mathbb{E}[\ln G]) = m,$$

we have

$$\nabla f_0(\gamma_1,\gamma_2,\gamma_3) = m\left[-1, \frac{\Gamma'(m+1)}{m\Gamma(m)}, m\right]^T.\qquad(35)$$

Again, the asymptotic variance σ_0^2 can be found by the multivariate delta method as

$$\sigma_0^2 = \left(\nabla f_0(\gamma_1,\gamma_2,\gamma_3)\right)^T \Sigma_0 \left(\nabla f_0(\gamma_1,\gamma_2,\gamma_3)\right).\qquad(36)$$

Substituting (35) and (32) into (36), and after some tedious algebra, we obtain (14) as required.

IV. DISCUSSION

A confidence interval on m, with asymptotic coverage $1-\alpha$, is given by $\hat{m}_k \pm z_{\alpha/2}\hat{\sigma}_k/\sqrt{N}$ where the consistent estimate $\hat{\sigma}_k$ can be obtained by replacing m by $\hat{m}$ in (13). With the aid of Maple, it can be shown that the variance σ_k^2 is $2m^2(1+O(m^{-1}))$. It suggests that the width of the confidence interval increases, approximately, linear with m. This undesirable feature can be improved by using the delta method to obtain

$$\sqrt{N}(\ln\hat{m}_k - \ln m) \xrightarrow{L} \mathcal{N}(0, V_k^2)\qquad(37)$$

where $V_k^2 = \sigma_k^2/m^2$. The log transformation is chosen since $\left(\frac{d}{dm}\ln m\right)^2 = 1/m^2$, which is used to compensate the m^2 factor in σ_k^2. We comment that the delta method is used here to stabilize the variance, while the delta method is used to obtain the asymptotic variance in the proof of the Theorem in Section III. After the log transformation on $\hat{m}_k$, the confidence interval on $\ln m$ becomes $\ln\hat{m}_k \pm z_{\alpha/2}\hat{V}_k/\sqrt{N}$.

The asymptotic variance V_k^2 of $\ln\hat{m}_k$ is plotted against m in Fig. 1 for $k=2$, $k=0$ and for the maximum likelihood estimator $\ln\hat{m}_{ML}$. Using the Cramér-Rao lower bound (CRLB) derived in [5], the asymptotic attainable variance for $\ln\hat{m}_{ML}$ is

$$V_{ML}^2 = \left[m^2\left(\psi'(m) - \frac{1}{m}\right)\right]^{-1}.\qquad(38)$$

Therefore the asymptotic relative efficiency (ARE) of $\ln\hat{m}_0$ with respect to $\ln\hat{m}_{ML}$ (or equivalently, ARE of $\hat{m}_0$ with respect to $\hat{m}_{ML}$) is

$$\begin{aligned}
\text{ARE} &= \frac{V_{ML}^2}{V_0^2}\\
&= \frac{1}{m^2(\psi'(m) - \frac{1}{m})(1 + m\psi'(m+1))}\\
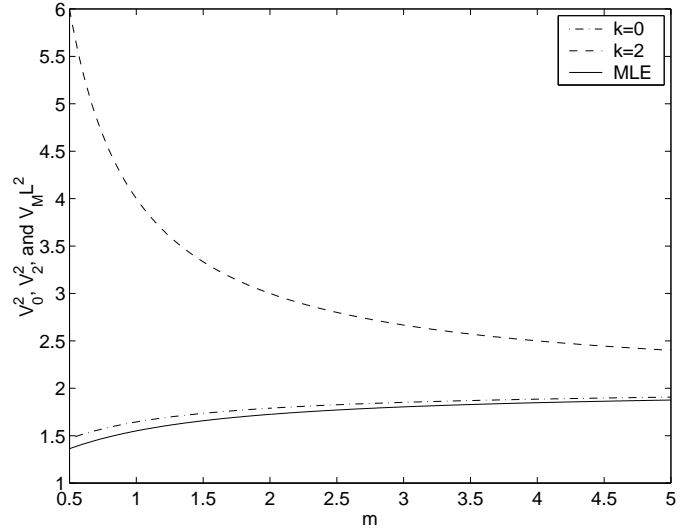&= 1 - \frac{1}{12m} + O(m^{-2})
\end{aligned}\qquad(39)$$

Figure 1: Asymptotic variances of $\ln\hat{m}_{k=2}$, $\ln\hat{m}_{k=0}$, and $\ln\hat{m}_{ML}$ vs. m.

where the last equality is obtained using Maple and the asymptotic expansion for the digamma function. Therefore, we conclude that $\hat{m}_0$ is almost fully efficient, in particular for large values of m, as shown in Fig. 1.

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