MAXIMIN EFFICIENT BOUNDED BIAS DESIGNS

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In the notation of Wiens (2018, "I-Robust and D-Robust Designs on a Finite Design Space," *Statistics and Computing*, 28, 241-258), suppose that loss in Ψ_τ (defined by $\sum_{\mathbf{x}\in\chi}\psi^2(\mathbf{x})\leq\tau^2/n$ and $\sum_{\mathbf{x}\in\chi}\mathbf{f}(\mathbf{x})\psi(\mathbf{x})=\mathbf{0}$) is

$$\mathcal{I}(\psi,\boldsymbol{\xi})=\frac{1}{n}\left\{\sigma_\varepsilon^2 tr\mathbf{R}^{-1}(\boldsymbol{\xi})+\ \tau^2\boldsymbol{c}_\psi'\mathbf{B}(\boldsymbol{\xi})\boldsymbol{c}_\psi\right\},$$

where none of $\boldsymbol{c}_\psi$ ($\|\boldsymbol{c}_\psi\|\leq 1$), $\mathbf{R}(\boldsymbol{\xi})$, $\mathbf{B}(\boldsymbol{\xi})$ depend on τ, and

$$\max_{\Psi_\tau}\mathcal{I}(\psi,\boldsymbol{\xi})=\frac{1}{n}\left\{\sigma_\varepsilon^2 tr\mathbf{R}^{-1}(\boldsymbol{\xi})+\ \tau^2 ch_{\max}\mathbf{B}(\boldsymbol{\xi})\right\}.$$

Fix σ_ε and define $V(\boldsymbol{\xi})=\sigma_\varepsilon^2 tr\mathbf{R}^{-1}(\boldsymbol{\xi})$. For given τ define $B_\tau(\boldsymbol{\xi})=\max_{\Psi_\tau}\left\{\tau^2\boldsymbol{c}_\psi'\mathbf{B}(\boldsymbol{\xi})\boldsymbol{c}_\psi\right\}$ $=\tau^2 ch_{\max}\mathbf{B}(\boldsymbol{\xi})$. The minimax (over Ψ_τ) design $\boldsymbol{\xi}_\tau$ minimizes $V(\boldsymbol{\xi})+B_\tau(\boldsymbol{\xi})$:

$$\boldsymbol{\xi}_\tau=\arg\min_{\boldsymbol{\xi}}\max_{\Psi_\tau}\mathcal{I}(\psi,\boldsymbol{\xi}). \tag{1}$$

Consider the problem

(B): Minimize $V(\boldsymbol{\xi})$ in the class Ξ_b of designs for which $B_\tau(\boldsymbol{\xi})\leq b^2$.

Theorem 1 *If there are (unique?) τ and $\boldsymbol{\xi}_\tau$ satisfying (1) such that $b^2=B_\tau(\boldsymbol{\xi}_\tau)$, then $\boldsymbol{\xi}_\tau$ is the solution to (**B**).*

Proof: We have that $\boldsymbol{\xi}_\tau\in\Xi_b$. For any $\boldsymbol{\xi}\in\Xi_b=\Xi_{b_\tau}$ we must have that $\sigma_\varepsilon^2 tr\mathbf{R}^{-1}(\boldsymbol{\xi})\geq\sigma_\varepsilon^2 tr\mathbf{R}^{-1}(\boldsymbol{\xi}_\tau)$, since otherwise

$$\begin{aligned}\max_{\Psi_\tau}\mathcal{I}(\psi,\boldsymbol{\xi}) &= \left\{\sigma_\varepsilon^2 tr\mathbf{R}^{-1}(\boldsymbol{\xi})+\ \tau^2 ch_{\max}\mathbf{B}(\boldsymbol{\xi})\right\}/n\\ &< \left\{\sigma_\varepsilon^2 tr\mathbf{R}^{-1}(\boldsymbol{\xi}_\tau)+b^2\right\}/n\\ &= \left\{V(\boldsymbol{\xi}_\tau)+B_\tau(\boldsymbol{\xi}_\tau)\right\}/n\\ &= \min_{\boldsymbol{\xi}}\max_{\Psi_\tau}\mathcal{I}(\psi,\boldsymbol{\xi}),\end{aligned}$$

a contradiction. □

To establish the existence of τ and $\boldsymbol{\xi}_\tau$ it suffices to show that $B_\tau(\boldsymbol{\xi}_\tau)$ ranges over $[0,\infty)$ as τ does. That $B_0(\boldsymbol{\xi}_0)=0$ and $B_\infty(\boldsymbol{\xi}_\infty)=\infty$ seem pretty obvious, so that if $B_\tau(\boldsymbol{\xi}_\tau)$ is continuous then existence will follow; if it is strictly increasing then uniqueness will follow as well.

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