inscribed in a circle. (Cf. G. Pólya, *Mathematics and Plausible Reasoning*, Volume 1, Princeton, 1954, pp. 174–177.) Let $B(n)$ be the area of a regular polygon with n sides and perimeter $1 + 2 + \cdots + n$. Prove that

$$1 - \frac{A(n)}{B(n)} \sim \frac{\pi^2}{3n^2}. \qquad (n \to \infty).$$

(b) For $1/2 < q < 1$ let $A(q, n)$ be the maximum area of a polygon with n sides of lengths $1, q, q^2, \ldots, q^{n-1}$, respectively, where n is large enough so that $q + q^2 + \cdots + q^{n-1} > 1$. Let $B(q, n)$ be the area of a regular polygon with n sides and perimeter $1 + q + q^2 + \cdots + q^{n-1}$. Prove that $c(q) = \lim_{n \to \infty} A(q, n)/B(q, n)$ exists and find

$$\lim_{q \to 1^-} \frac{1 - c(q)}{(1 - q)^2}.$$

SOLUTIONS OF ADVANCED PROBLEMS

Persistence of a Distribution Function

6522 [1986, 485]. *Proposed by Gunnar Blom, University of Lund and Lund Institute of Technology, Sweden.*

Let $X_1, X_2, \ldots$ be an infinite sequence of independent random variables with the common continuous distribution function F. Let X_N be the first variable that is less than exactly one of all its predecessors $X_1, \ldots, X_{N-1}$. Determine the distribution function of X_N.

Solution by Robert B. Israel, University of British Columbia, Vancouver, BC, Canada. The distribution function of X_N is F. In fact, for any positive integer m this statement is true if "exactly one" is replaced by "exactly m".

For any $k > m$, the probability that X_k is less than exactly m of its predecessors is $1/k$ (since X_k is equally likely to be the first, second, $\ldots$, kth order statistic). Note that this is independent of the ordering of $X_1, \ldots, X_{k-1}$ among themselves. Thus the probability that X_k is the first one less than exactly m of its predecessors is

$$P(X_k = X_N) = \left(1 - \frac{1}{m+1}\right)\left(1 - \frac{1}{m+2}\right) \cdots \left(1 - \frac{1}{k-1}\right)\frac{1}{k} = \frac{m}{k(k-1)}.$$

Since the set of values of a given set of i.i.d. random variables is independent of the order in which they occur, the conditional distribution of X_N given $X_N = X_k$ is the distribution of the $(k - m)$th order statistic for $X_1, \ldots, X_k$. Namely, for any x, if $P(X_i \le x) = p$, and $q = 1 - p$, then

$$P(X_N \le x | X_N = X_k) = \sum_{j=0}^{m} \binom{k}{j} p^{k-j} q^j$$

so that

$$P(X_N \le x) = \sum_{k=m+1}^{\infty} \frac{m}{k(k-1)} \sum_{j=0}^{m} \binom{k}{j} p^{k-j} q^j.$$

Write this as R_m in order to make the dependence on m explicit. We shall prove by induction on m that $R_m = p$. Note that the series converges absolutely for $0 \le p \le 1$.

First consider the case $m = 1$ (which is the problem as stated). We have

$$\begin{aligned} R_1 &= \sum_{k=2}^{\infty} \frac{1}{k(k-1)} (p^k + kp^{k-1}q) \\ &= \sum_{k=2}^{\infty} \frac{1}{k(k-1)} (kp^{k-1} - (k-1)p^k) \\ &= \sum_{k=2}^{\infty} \left(\frac{p^{k-1}}{k-1} - \frac{p^k}{k} \right) = p. \end{aligned}$$

Now suppose that $R_{m-1} = p$. We have

$$R_m = \frac{m}{m-1} R_{m-1} - \frac{1}{m-1} \sum_{j=0}^{m-1} \binom{m}{j} p^{m-j} q^j + \sum_{k=m+1}^{\infty} \frac{m}{k(k-1)} \binom{k}{m} p^{k-m} q^m,$$

where the sum over j comprises the terms for $k = m$ that are present in R_{m-1} but not in R_m, and the sum over k comprises the terms for $j = m$ that are present in R_m but not R_{m-1}. Now

$$\sum_{j=0}^{m-1} \binom{m}{j} p^{m-j} q^j = (p+q)^m - q^m = 1 - q^m$$

while

$$\begin{aligned} \sum_{k=m+1}^{\infty} \frac{m}{k(k-1)} \binom{k}{m} p^{k-m} q^m &= q^m \sum_{i=1}^{\infty} \frac{(i+m-2)!}{i!(m-1)!} p^i \\ &= \frac{q^m}{m-1} (q^{1-m} - 1) = \frac{1}{m-1} (q - q^m). \end{aligned}$$

(Here we have used the binomial series

$$\sum_{i=0}^{\infty} \frac{(i+n)!}{i!n!} p^i = (1-p)^{-1-n},$$

which converges for $|p| < 1$; for $p = 1$ the formula is trivial.) Thus

$$R_m = \frac{m}{m-1}p - \frac{1}{m-1}(1 - q^m) + \frac{1}{m-1}(q - q^m) = p$$

as required.

The above generalization of 6522 was also proved by Barthel W. Huff, Eugene Salamin, and Glenn A. Stoops. Both Marcel F. Neuts and the proposer remark that it is sufficient to establish the result for the case in which F is the uniform distribution on $(0, 1)$. The proposer (who also provided a solution for $m = 1$ based on order statistics) used this to provide a noncalculational argument, based on the notion of "records," for the truth of the result. He adds that the result exists in the literature on "records" and is implicit in a paper of Charles M. Goldie and L. C. G. Rogers, The k-record Processes are i.i.d., *Z. für Wahrscheinlichkeitstheorie*, 67 (1984) 197–211.

Neuts added the following remark to his solution. "This result is quite remarkable. It shows that the distribution of the first near-record X_N is the same as that of the underlying random variables. This would be very difficult to infer, for example, from simulation runs. Because of the heavy tail of the distribution of N, the empirical distribution of X_N over many replicated runs may be expected to converge only very slowly to F."

Also solved by Thomas N. Delmer, Ellen Hertz, James M. Meehan, G. S. Rogers, Kenneth Schilling, David G. Weinman, Western Maryland College Problems Group, and Douglas P. Wiens (Canada).