Solution by Jean-Marie Monier, Lyon, France. A direct calculation yields

$$\frac{a_{n+1}}{a_n} = \frac{2n+1}{2\sqrt{(n+1)n}} = \left(1 + \frac{1}{4(n^2+n)}\right)^{1/2} > 1.$$

Hence, the sequence $\{a_n\}$ is strictly increasing. Since

$$\log a_{n+1} - \log a_n = \frac{1}{2}\log\left(1 + \frac{1}{4(n^2+n)}\right) < \frac{1}{2} \cdot \frac{1}{4(n^2+n)}$$
$$= \frac{1}{8}\left(\frac{1}{n} - \frac{1}{n+1}\right), \tag{$*$}$$

we find that

$$\log a_n - \log a_1 = \sum_{j=1}^{n-1} (\log a_{j+1} - \log a_j) < \frac{1}{8}\sum_{j=1}^{n-1}\left(\frac{1}{j} - \frac{1}{j+1}\right)$$
$$= \frac{1}{8}\left(1 - \frac{1}{n}\right) < \frac{1}{8}.$$

Thus, $a_n < a_1 e^{1/8} = (1/2)\,e^{1/8}$ for each positive integer n. Since $\{a_n\}$ is a bounded strictly increasing sequence, (i) and the upper bound on a_n in (ii) follow.

To obtain the lower bound on a_n in (ii), we observe that ($*$) implies that

$$\log a_{n+1} + \frac{1}{8(n+1)} < \log a_n + \frac{1}{8n},$$

so that the sequence $\{\log a_n + 1/(8n)\}$ is strictly decreasing. Furthermore, $\{\log a_n + 1/(8n)\}$ converges to $\log L$ so that $\log a_n + 1/(8n) > \log L$ for all positive integers n, which implies the lower bound on a_n in (ii).

Editorial comment. Most of the solutions received were similar to the one given above. Some solvers observed that the upper bound $(1/2)\,e^{1/8}$ obtained for a_n above is a remarkably close elementary estimate. More specifically, $(1/2)\,e^{1/8} = 0.56657\ldots$, while the least upper bound is $L = 1/\sqrt{\pi} = 0.56418\ldots$.

Solved also by the proposer and 31 other readers. One partial solution was received.

The Longest Expected World Series

E 3386 [1990, 427]. *Proposed by Eugene F. Schuster, University of Texas, El Paso, TX.*

Let L be the length of a $(2N-1)$-game World Series, modeled as a sequence of independent identically distributed Bernoulli trials which terminates as soon as one team wins N games. (The length is the number of games actually played.) Prove the seemingly obvious observation that the expected length $E(L)$ of the series is maximized when the two teams are evenly matched.

Composite solution I by C. Georghiou, University of Patras, Greece, and Kumar Joag-Dev, University of Illinois at Urbana-Champaign. Let $L = N + k$, for $k \geq 0$. The probability distribution for the random variable L is given by

$$P(L = N + k) = \binom{N-1+k}{k}\left[p^N q^k + q^N p^k\right], \qquad k \geq 0,$$

where p, q are the win probabilities for the two teams in a single game. We have

$$E(L) = \sum_{k=0}^{N-1} (N+k)\binom{N-1+k}{k}\left[p^N q^k + q^N p^k\right]$$
$$= N\sum_{k=0}^{N-1}\binom{N+k}{N}\left[p^N q^k + q^N p^k\right].$$

Let $\varepsilon_N = E(L)/N$; note that $\varepsilon_1 = 1$. We claim that $\varepsilon_N - \varepsilon_{N-1} = (1/N)\binom{2N-2}{N-1}(pq)^{N-1}$, from which it follows that

$$E(L) = N\left\{1 + \sum_{k=2}^{N}(\varepsilon_k - \varepsilon_{k-1})\right\} = N\sum_{k=0}^{N-1}\frac{(pq)^k}{k+1}\binom{2k}{k}.$$

Hence $E(L)$ is maximized when pq is maximized, i.e., when $p = q = \frac{1}{2}$.

To prove the claim, we write $\varepsilon_N = \sum_{k=0}^{N-1}\binom{N+k}{N}w^k g(N-k)$, where $w = pq$ and $g(j) = p^j + q^j$. Note that $g(0) = 2$, $g(1) = 1$, and $g(j) = g(j-1) - wg(j-2)$ for $j \geq 2$. In the summation for ε_N, we separate out the last term, apply the recurrence for g to the other terms, and separate out the last term of the second resulting sum to obtain

$$\varepsilon_N = \binom{2N-1}{N}w^{N-1} + \sum_{k=0}^{N-2}\binom{N+k}{N}w^k g(N-1-k)$$
$$- \sum_{k=0}^{N-3}\binom{N+k}{N}w^{k+1}g(N-2-k) - 2\binom{2N-2}{N}w^{N-1}.$$

By collecting the terms involving w^{N-1}, shifting the index of the final summation, and applying the recurrence for the binomial coefficients, this becomes

$$\varepsilon_N = \left(\frac{2N-1}{N} - 2\frac{N-1}{N}\right)\binom{2N-2}{N-1}w^{N-1}$$
$$+ \sum_{k=0}^{N-2}\left[\binom{N+k}{N} - \binom{N-1+k}{N}\right]w^k g(N-1-k)$$
$$= \frac{1}{N}\binom{2N-2}{N-1}(pq)^{N-1} + \varepsilon_{N-1}.$$

Solution II by Fred Richman, TCI Software Research, Las Cruces, NM. We prove the stronger result that for every n, the probability that the nth game is played is maximized when $p = \frac{1}{2}$. This implies the desired result, because $E(L) = N + \sum_{n=N+1}^{2N-1} E(X_n)$, where X_n is 1 if the nth game is played and 0 otherwise. The value of $E(X_{n+1})$ is the probability that the $(n+1)$th game is played, which is the probability that the first team wins between $n - N + 1$ and $N - 1$ of the first n games. Letting $B(x, p)$ denote the cumulative probability in the binomial distribution with parameters n and p, we want to maximize $E(X_{n+1}) = B(N-1, p) - B(n-N, p)$, the middle part of the distribution.

We prove that this is maximized at $p = \frac{1}{2}$ by considering the derivative of $B(x, p)$ with respect to p. If we increase p by an infinitesimal amount, the probability that the number of successes is at most x decreases by the probability of having exactly x successes before the increase times the probability that one of the failures becomes a success when we increase p, which is $(n-x)\,dp/q$. Hence

$B(x, p + dp) = B(x, p) - \binom{n}{x} p^x q^{n-x}(n - x)\, dp/q$, or $dB(x, p)/dp = -(n - x)\binom{n}{x} p^x q^{n-x-1}$. (This differentiation formula can also be proved algebraically.) Noting that $(n - x)\binom{n}{x} = (x + 1)\binom{n}{x+1}$, we have $dE(X_{n+1})/dp = \binom{n}{N}(pq)^{n-N}(q^{2N-n-1} - p^{2N-n-1})$, which is positive if $p < \frac{1}{2}$ and negative if $p > \frac{1}{2}$

Editorial comment. It is interesting to note the appearance of the Catalan numbers $\binom{2k}{k}/(k + 1)$ in the formula for $E(L)$. K. Hinderer and M. Steiglitz refer to a discussion of this and related problems in their paper in *Didaktik der Mathematik* 15(2)(1987), 81–114 (see p. 102). The second solution above is equivalent to showing $P(L > n)$ is maximized at $p = \frac{1}{2}$ for every n, as shown directly by several solvers. John H. Lindsey II took the approach of proving the stronger result that $P(L = n + 1)/P(L = n)$ is maximized at $p = \frac{1}{2}$ for every n. Since $P(L = N + j)$ is proportional to $p^j q^N + p^N q^j$, it suffices to verify that, for every $j, (p^{j+1} q^N + p^N q^{j+1})/(p^j q^N + p^N q^j)$ has its maximum at $p = \frac{1}{2}$. This is easily proved by induction. There were a variety of other approaches.

Michael Perlman noted that any nondecreasing function of $L(N)$ has maximum expectation at $p = \frac{1}{2}$ and that similar conclusions hold for k-contestant series involving k-person games in which the series concludes when any contestant wins N of them. The fact that the expected series length is maximized when each player has probability $1/k$ of winning each game is implied by the Schur-concavity of the appropriate cumulative density function and a theorem of Y. Rinott (see *Israel J. Math.*, 15(1973) 60–77, and Marshal and Olkins' *Inequalities, Theory of Majorization and Its Applications*, Academic Press, 1979). Perlman also noted that if the series is prolonged until each contestant has won N games, then the expected length is minimized in the symmetric $1/k$ case, by Schur-convexity of the corresponding cumulative density function.

Solved also by A. Adler, R. A. Agnew, D. Callan, N. J. Fine, P. Griffin, E. Hertz, K. Hinderer & M. Steiglitz (Germany), R. D. Hurwitz, B. R. Johnson, B. G. Klein, A. Kozek (Poland), O. Krafft & M. Schaefer (Germany), K.-W. Lau (Hong Kong), J. H. Lindsey II, H. Lipman, M. D. Perlman, D. S. Romano, O. Saleh & S. Byrd, R. Stong, M. Vowe (Switzerland), D. P. Wiens, and the proposer. Three incorrect solutions were received.

Infinite Almost Everywhere

6632 [1990, 433]. *Proposed by Gilbert Muraz, Institut Fourier, Université de Grenoble I, St. Martin d'Héres, France, and Pawel Szeptycki and Fred Galvin, University of Kansas, Lawrence.*

Let E be a measurable subset of $\mathbb{R}$ modulo 1 having positive measure. For real t let N_t be the set of positive integers n such that nt modulo 1 is in E. Suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive real numbers such that $\sum a_n = \infty$. Prove that

$$\sum_{n \in N_t} a_n = \infty$$

for almost all t in $[0, 1]$.

Solution by Nathan J. Fine, Deerfield Beach, Florida. By an abuse of notation we may consider E to be a subset of $[0, 1)$. Then let $E_0 = \bigcup_{j=0}^{\infty}(E + j)$, and let $\chi(t)$