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Some Distributions for a 2-Identical-Unit System

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Summary: A 2-identical-unit system, with either 1 or 2 repair facilities, is considered. Initially, only the existence of the unit failure and repair time densities is assumed. Equations are derived describing the distributions of the system up, down and total repair times, number of repairs, times of the nth unit failure and nth repair completion, and the probability that i units are operating at time t. These equations are then solved under the assumption that the unit times to failure have a density which is a linear combination of ERLANGian densities. For some arrangements of the system, the first moments are exhibited for arbitrary failure/repair distributions.

1. Introduction

The 2-unit system has been analysed under a variety of restrictions on the distributions of the unit failure times ([3], [4], [5]). In those models in which both units may operate simultaneously, it has been assumed that at least one of the failure times is distributed as the sum of exponential random variables. Techniques unique to MARKOVian distributions are then used both in the derivations of the equations and in their analysis. In Section 3 of this paper, equations describing a 2-unit system are derived, assuming only the existence of the relevant densities. Solutions to these equations give the joint distribution of the up, down and total repair time in a cycle, the probability that i units are operating at time t, and the distributions of the number of repairs, the time of the nth unit failure, and the time of the nth repair completion. In Section 4 the equations are solved, assuming that the common operating-time distribution is a linear combination of Erlangian distributions. The final results are complicated, but are given explicitly in a form suitable for numerical evaluation.

The model assumptions are:

a) There are two identical units, each of which operates whenever so capable. There is either one repair facility ("Model 1") or two such facilities ("Model 2"). No distinction is made between an "active" and a "standby" unit.

b) The system is up whenever one or more units are functioning, down otherwise.

c) Switching and sensing are perfect. A repaired unit is like-new.

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2. Notation

The time point "0" is defined to be the beginning of a period of up time, prior to which one unit is under repair, and the other is either under repair (Model 2) or awaiting repair (Model 1). A "*cycle*" is an up time followed by a down time. The following random variables are considered:

$$
\begin{aligned}
T &\equiv \text{Cycle time}\\
U &\equiv \text{Up time in } T\\
Z &\equiv \text{Total repair time in } U\\
R &\equiv \text{Number of completed repairs in } U\\
U_n &\equiv \text{Time of the } n\text{th unit failure in } T (U_0 \equiv 0)\\
V_n &\equiv \text{Time of the } n\text{th repair completion in } T (V_0 \equiv 0)\\
Z_n &\equiv \text{Total repair time in } [0, V_n]\\
D &\equiv \text{Down time in } T (D = T - U)\,.
\end{aligned}
$$

The distribution function (d.f.) of the unit times to failure is $F(t)$, with density $f(t)$ and mean α. The d.f. of all unit repair times except possibly the first and last is $G(t)$, with density $g(t)$, mean β, and $\bar{G}(t) \equiv 1 - G(t)$. The density of the first repair is $g_0(t)$. In Model 1, $g_0 \equiv g$. In Model 2, $g_0(t)$ is given exactly by $g(a+t)/\bar{G}(a)$ if it is known that the first repair has been underway for "a" units of time prior to the commencement of the up time. Otherwise $g_0(t)$ may be approximated by $\bar{G}(t)/\beta$, the stationary forward recurrence time density. The density of a repair which commences while another is underway is $g_1(t)$. In Model 2, $g_1 \equiv g$ while in Model 1, $g_1 \equiv 0$, $\bar{G} \equiv 1$. Define

$$\delta_c(x) \equiv \text{Dirac's impulse function, with mass at } x = c$$

$$\gamma_0(t, u) \equiv g_1(t)\, \bar{G}_0\, (t+u) + \bar{G}_1(t)\, g_0\, (t+u) \quad (0 \leqq u,\ t < \infty)$$

$$\gamma(t, u) \equiv g_1(t)\, \bar{G}\, (t+u) + \bar{G}_1(t)\, g\, (t+u) \quad (0 \leqq u,\ t < \infty)$$

$$H_n(t, u, z) \equiv \mathsf{P}\, (T \leqq t,\ U \leqq u,\ Z \leqq z,\ R = n) \quad (0 \leqq z \leqq u \leqq t < \infty)$$

$$H(t, u, z) \equiv \mathsf{P}\, (T \leqq t,\ U \leqq u,\ Z \leqq z) = \sum_{n=0}^{\infty} H_n(t, u, z)$$

$$L_n(u, v, w, z) = \mathsf{P}\, (U_{n+1} \leqq u,\ U_n \leqq v,\ V_n \leqq w,\ Z_n \leqq z,\ R \geqq n) \quad (n \geqq 1,\ 0 \leqq z \leqq w \leqq v \leqq u < \infty)$$

$$L(u, v, w, z) = \sum_{n=1}^{\infty} L_n(u, v, w, z)$$

$$q_n(t) \equiv \mathsf{P}\, (U_n \leqq t \mid t \leqq T), \quad q(t) \equiv \sum_{n=1}^{\infty} q_n(t)$$

$$r_n(t) \equiv \mathsf{P}\, (V_n \leqq t \mid t \leqq T), \quad r(t) \equiv \sum_{n=1}^{\infty} r_n(t)$$

$$z_n(t) \equiv \mathsf{P}\, (Z_n \leqq t \mid t \leqq T)$$

$$\pi(x) \equiv \sum_{n=1}^{\infty} x^n\, \mathsf{P}\, (R \geqq n)$$

$$p_i(t) \equiv \mathsf{P}\, (i \text{ units are operating at } t \text{ and } t \leqq T)\,.$$

The density and survivor function of H are denoted by h and $\bar{H}$, with analogous notation for H_n, L and L_n. LAPLACE transforms are denoted by $\hat{f}(s)$ or $L_s(f)$. It follows from the above definitions that

$$q(t) = \mathsf{E}\,[\text{Number of unit failures in } [0, t] \mid t \leqq T]$$

$$r(t) = \mathsf{E}\,[\text{Number of completed repairs in } [0, t] \mid t \leqq T]$$

$$\hat{z}_n(0) = \mathsf{E}[Z_n]$$

$$\pi(1) = \bar{L}(0, 0, 0, 0) = \mathsf{E}[R] \tag{2.1}$$

$$\hat{p}_1(0) = \mathsf{E}[Z] \tag{2.2}$$

$$\hat{p}_1(0) + \hat{p}_2(0) = \mathsf{E}[U] \tag{2.3}$$

$$\hat{p}_0(0) = \mathsf{E}[D] \tag{2.4}$$

3. Derivation of the Equations

The changes of state of the system are represented schematically below. In the figures, a solid line represents a unit operating period, a broken line a unit repair period. The possible sequences of events described by $l_1(u, v, w, z)$ are given in Fig. 1.

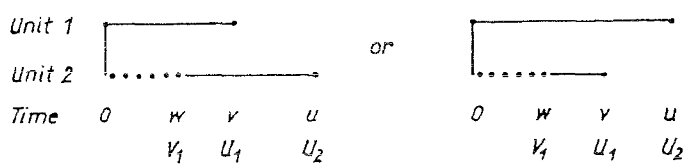

Fig. 1

The sequence $\{l_n\}$ is then defined inductively, using $l_n(u, x, \cdot, \cdot)$ in Fig. 2a, and $l_n(v, x, \cdot, \cdot)$ in Fig. 2b, to define l_{n+1}.

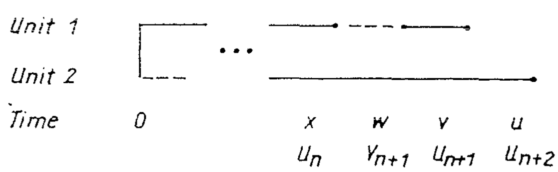

Fig. 2a

or

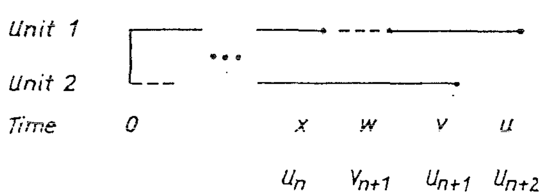

Fig. 2b

From the figures, it is evident that

$$l_1(u, v, w, z) = g_0(w)\,[f(v)\,f\,(u-w) + f(u)\,f\,(v-w)]\,\delta_w(z) \tag{3.1}$$

and

$$l_{n+1}(u, v, w, z) = \int_{w-z}^{w} \int_{z-(w-x)}^{x} g(w-x)\,[f(v-w)\,l_n(u, x, y, z-(w-x)) + f(u-w)\,l_n(v, x, y, z-(w-x))]\,\mathrm{d}y\mathrm{d}x \tag{3.2}$$

from which l is expressible by iterating 3.2), or by summing over n to yield an integral equation. Figure 3 gives the possible events described by $h_n(t, u, z)$, given $l_n(u, v, \cdot, \cdot)$.

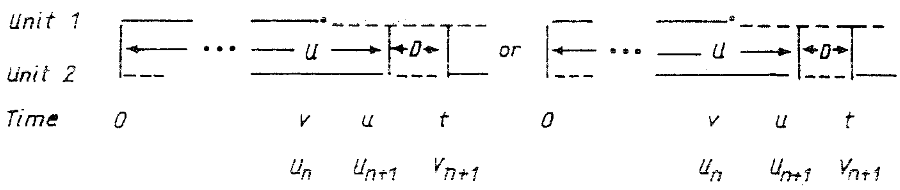

Fig. 3

Thus

$$h_n(t, u, z) = \int_{u-z}^{u} \int_{z-(u-v)}^{v} l_n(u, v, w, z-(u-v)) \times \gamma(t-u, u-v)\,\mathrm{d}w\mathrm{d}v \quad (n \geqq 1) \tag{3.3}$$

and similarly

$$h_0(t, u, z) = f(u)\,\gamma_0(t-u, u)\,\delta_u(z)\,. \tag{3.4}$$

The distributions $r(t)$, $z_n(t)$, $q(t)$ and $p_i(t)$ are obtained from h and l. Clearly, $r_0(t) = z_0(t) = q_0(t) = 1$. For $n \geqq 1$,

$$\begin{aligned} r_n(t) &= \mathsf{P}\,(V_n \leqq t \text{ and } R \geqq n) + \mathsf{P}\,(V_n \leqq t \text{ and } R = n-1) \\ &= \mathsf{P}\,(V_n \leqq t \text{ and } R \geqq n) + \mathsf{P}\,(T \leqq t \text{ and } R = n-1) \\ &= L_n(\infty, \infty, t, \infty) + H_{n-1}(t, \infty, \infty) \end{aligned}$$

so that

$$r(t) = L(\infty, \infty, t, \infty) + H(t, \infty, \infty)\,. \tag{3.5}$$

Similarly,

$$z_n(t) = L_n(\infty, \infty, \infty, t) + H_{n-1}(\infty, \infty, t) \tag{3.6}$$

$$q(t) = L(\infty, t, \infty, \infty) + H(\infty, t, \infty)\,. \tag{3.7}$$

Also,

$$p_0(t) = \mathsf{P}\,(U < t \leqq T) = \bar{H}(t, 0, 0) - \bar{H}(0, t, 0) \tag{3.8}$$

$$\begin{aligned} p_2(t) &= \sum_{n=1}^{\infty} \mathsf{P}\,(V_n \leqq t \leqq U_n \text{ and } R \geqq n) \\ &= \sum_{n=1}^{\infty} (\bar{L}_n(0, t, 0, 0) - \bar{L}_n(0, 0, t, 0)) \\ &= \bar{L}(0, t, 0, 0) - \bar{L}(0, 0, t, 0) \end{aligned} \tag{3.9}$$

$$p_1(t) = \mathsf{P}\,(T \geqq t) - p_0(t) - p_2(t)$$
$$= \overline{H}(0, t, 0) - p_2(t) \tag{3.10}$$

$$\pi(x) = \sum_{n=1}^{\infty} x^n \overline{L}_n(0, 0, 0, 0)\,. \tag{3.11}$$

4. Solutions for a Class of Failure Time Densities

Assume now that $f(t)$ is of the form

$$f(t) = \sum_{a=0}^{A} \sum_{b=0}^{B} r_{ab}\lambda_a \frac{(\lambda_a t)^b}{b!}\, \mathrm{e}^{-\lambda_a t} \equiv \sum_{a=0}^{A} \sum_{b=0}^{B} r_{ab}\lambda_a f_b^a(t)\,, \tag{4.1}$$

where the r_{ab} and λ_a are real, $\sum_{a,b} r_{ab} = 1$, $\lambda_a > 0$. Any density, whose LAPLACE transform is a rational function with real poles, is of the form 4.1). As well, any failure time distribution may be approximated arbitrarily closely by d.f.s with densities of this form. Proper choices of the parameters with $A = 1$ will yield any positive mean, and any coefficient of variation in $[(B+1)^{-1/2}, \infty)$. See [1], [2] for modeling considerations.

The POISSON probabilities $f_b^a(t)$ are characterized by the property

$$f_b^a(t) = \sum_{i=0}^{b} f_i^a\,(t-u)\, f_{b-i}^a(u) \quad (0 \leqq u \leqq t)\,.$$

This property may be used to put the integrals in (3.1)–(3.3) into convolution forms, and thus show that l_n is of the form

$$l_n(u, v, w, z) = \sum_{a=0}^{A} \sum_{b=0}^{B} r_{ab}\lambda_a \sum_{i=0}^{b} f_i^a\,(u-v)\, \varphi_i^n\,(v-w, w-z, z;, a, b)$$

where

$$\varphi_i^1\,(v-w, w-z, z; a, b) = \sum_{c=0}^{A} \sum_{b=0}^{B} r_{cd}\lambda_c \Bigg[\sum_{j=0}^{d} (f_j^c f_{b-i}^a)\,(v-w)\,(f_{d-j}^c g_0)\,(w)\,\delta_w(z)$$
$$+ \sum_{j=0}^{b-i} (f_d^c f_{b-i-j}^a)\,(u-v)\,(f_j^a g_0)\,(w)\,\delta_w(z) \Bigg]\,,$$

$$\varphi_i^{n+1}\,(v-w, w-z, z; a, b) = \sum_{c=0}^{A} \sum_{d=0}^{B} \Bigg[\sum_{j=i}^{b} \sum_{k=i}^{j} (r_{cd}\lambda_c f_d^c f_{k-i}^a)\,(v-w)$$
$$\times \int_0^z (f_{j-k}^a g)\,(z-x) \int_0^{w-z} \varphi_j^n\,(w-z-y, y, x; a, b)\,\mathrm{d}y\mathrm{d}x$$
$$+ \sum_{j=0}^{d} \sum_{k=0}^{j} (r_{cd}\lambda_c f_{b-i}^a f_{j-k}^c)\,(v-w)$$
$$\times \int_0^z (f_k^c g)\,(z-x) \int_0^{w-z} \varphi_j^n\,(w-z-y, y, x; c, d)\,\mathrm{d}y\mathrm{d}x \Bigg]\,.$$

Taking LAPLACE transforms,

$$\hat{l}_n(s_1, s_2, s_3, s_4) = \sum_{a=0}^{A} \sum_{b=0}^{B} \sum_{i=0}^{b} r_{a_b}\lambda_a \hat{f}_i^a(s_1)\, \hat{\varphi}_i^n\, (s_1+s_2,\, s_1+s_2+s_3,\, s_1+s_2+s_3+s_4;\, a, b) \tag{4.2}$$

$$\hat{\varphi}_i^1(s_1, s_2, s_3;\, a, b) = \sum_{c=0}^{A} \sum_{d=0}^{B} r_{cd}\lambda_c \left[\sum_{j=0}^{d} L_{s_1}(f_j^c f_{b-i}^a)\, L_{s_3}(f_{d-j}^c g_0) + \sum_{j=0}^{b-i} L_{s_1}(f_d^c f_{b-i-j}^a)\, L_{s_3}(f_j^a g_0)\right] \tag{4.3}$$

$$\hat{\varphi}_i^{n+1}(s_1, s_2, s_3;\, a, b) = \sum_{c=0}^{A} \sum_{d=0}^{B} r_{cd} \left[\sum_{j=i}^{b} \sum_{k=i}^{j} L_{s_1}(\lambda_c f_d^c f_{k-i}^a)\, L_{s_3}(f_{j-k}^a g) \times \hat{\varphi}_j^n(s_2, s_2, s_3;\, a, b) + \sum_{j=0}^{d} \sum_{k=0}^{j} L_{s_1}(\lambda_c f_{b-i}^a f_k^c)\, L_{s_3}\,(f_{j-k}^c g)\, \hat{\varphi}_j^n(s_2, s_2, s_3;\, c, d)\right] \tag{4.4}$$

These equations may be expressed in matrix terms. Below, vectors $\boldsymbol{x}$ are defined by their representative elements x_i^{ab} $(0 \leqq i \leqq b)$. These are to be arranged lexicographically, with the precedence relation $\prec$ defined by $x_i^{ab} \prec x_{i'}^{a'b'}$ if $b < b'$, or $b = b'$ and $a < a'$, or $b = b'$, $a = a'$ and $i < i'$. Thus $\boldsymbol{x}_{ab} = (x_0^{ab}, \ldots, x_b^{ab})'$, $\boldsymbol{x}_b = (\boldsymbol{x}_{0b}', \ldots, \boldsymbol{x}_{Ab}')'$, $\boldsymbol{x} = (\boldsymbol{x}_0', \ldots, \boldsymbol{x}_B')'$. Define

$$\tau_i^{ab}(s_1) \equiv r_{ab}\lambda_a \hat{f}_i^a(s_1) = r_{ab}\, (\lambda_a/(s_1+\lambda_a))^{i+1}$$

$$\psi_i^{ab}(s_3) \equiv L_{s_3}(f_{b-i}^a g_0) = (-\lambda_a)^{b-i}\, \hat{g}_{0(s_3+\lambda_a)}^{(b-i)}/(b-i)!$$

$$\varphi_i^{n;a,b}(s_1, s_2, s_3) \equiv \hat{\varphi}_i^n(s_1, s_2, s_3;\, a, b)$$

$$\sigma_i^{ab}(s_1, s) \equiv r_{ab} L_s(f_i^a(\cdot)\, \hat{\gamma}(s_1, \cdot)) = r_{ab}\lambda_a \frac{(-\lambda_a)^i}{i!} \frac{d^i}{ds^i}\, \hat{\gamma}\, (s_1, s+\lambda_a)$$

$$\dot{\psi}_i^{ab} \equiv \frac{\mathrm{d}}{\mathrm{d}s_1}\, \psi_i^{ab}(s_1)\Big|_{s_1=0}, \quad \dot{\sigma}_i^{ab} \equiv \frac{-\mathrm{d}}{\mathrm{d}s_1}\, \sigma_s^{ab}(s_1, 0)\Big|_{s_1=0}, \quad \ddot{\sigma}_i^{ab} \equiv \frac{-\mathrm{d}}{\mathrm{d}s}\, \sigma_i^{ab}(0, s)\Big|_{s=0}.$$

The necessary block matrices are built up in an analogous fashion. Define

$$E_{ij}^{ab,cd}(s_1) \equiv r_{cd}\lambda_c L_{s_1}(f_{b-i}^a f_j^c) = r_{cd} \binom{b-i+j}{j} \frac{\lambda_a^{b-i}\lambda_c^{j+1}}{(s_1+\lambda_a+\lambda_c)^{b-i+j+1}} \qquad (0 \leqq i \leqq b,\ 0 \leqq j \leqq d)$$

$$E_{(s_1)}^{bd} \equiv (E_{(s_1)}^{ab,cd})_{a,c=0,\ldots,A} \qquad E(s_1) \equiv (E_{(s_1)}^{bd})_{b,d=0,\ldots,B}$$

$$D_{ij}^{ab}(s_1) \equiv \begin{cases} L_{s_1}(ff_{j-i}^a) = (-\lambda_a)^{j-i}\, \hat{f}_{(s_1+\lambda_a)}^{(j-i)}/(j-i)! & (0 \leqq i \leqq j \leqq b) \\ 0 & (i > j) \end{cases}$$

$$D^b(s_1) \equiv \operatorname{diag}\,(D_{(s_1)}^{0b}, \ldots, D_{(s_1)}^{Ab}) \quad D(s_1) \equiv \operatorname{diag}\,(D_{(s_1)}^{0}, \ldots, D_{(s_1)}^{B})$$

$$M_{ij}^{ab}(s_3) \equiv \begin{cases} L_{s_3}(gf_{j-i}^a) = (-\lambda_a)^{j-i}\, \hat{g}_{(s_3+\lambda_a)}^{(j-i)}/(j-i)! & (0 \leqq i \leqq j \leqq b) \\ 0 & (i > j) \end{cases}$$

$$M^b(s_3) \equiv \operatorname{diag}\,(M_{(s_3)}^{0b}, \ldots, M_{(s_3)}^{Ab}) \quad M(s_3) \equiv \operatorname{diag}\,(M_{(s_3)}^{0}, \ldots, M_{(s_3)}^{B})$$

$$\Sigma(s_1) \equiv D(s_1) + E(s_1), \quad \dot{\Sigma} \equiv \frac{-\mathrm{d}}{\mathrm{d}s_1}\, \Sigma(s_1)\Big|_{s_1=0}, \quad \dot{M} = \frac{-\mathrm{d}}{\mathrm{d}s_3}\, M(s_3)\Big|_{s_3=0}.$$

Equations (4.3) and (4.4) now become

$$\boldsymbol{\varphi}^1(s_1, s_2, s_3) = \Sigma(s_1)\ \boldsymbol{\psi}(s_3)$$

$$\boldsymbol{\varphi}^{n+1}(s_1, s_2, s_3) = \Sigma(s_1)\ M(s_3)\ \boldsymbol{\varphi}^n(s_2, s_2, s_3)\ .$$

By induction,

$$\boldsymbol{\varphi}^n(s_1, s_2, s_3) = \Sigma(s_1)\ [M(s_3)\ \Sigma(s_2)]^{n-1}\ \boldsymbol{\psi}(s_3) \quad n = 1, 2, \ldots \tag{4.5}$$

From (4.2) and (4.5),

$$\begin{aligned}\hat{l}_n(s_1, s_2, s_3, s_4) &= \boldsymbol{\tau}'(s_1)\ \boldsymbol{\varphi}^n\ (s_1+s_2, s_1+s_2+s_3, s_1+s_2+s_3+s_4)\\ &= \boldsymbol{\tau}'(s_1)\ \Sigma\ (s_1+s_2)\ [M\ (s_1+s_2+s_3+s_4)\ \Sigma\ (s_1+s_2+s_3)]^{n-1}\\ &\quad \cdot \boldsymbol{\psi}\ (s_1+s_2+s_3+s_4)\ .\end{aligned}$$

The condition

$$0 \leqq \hat{l}_n(s_1, s_2, s_3, s_4) \leqq \hat{l}_n(0, 0, 0, 0) = \mathsf{P}\ (R \geqq n) \to 0 \quad \text{as} \quad n \to \infty$$

implies that $[M\Sigma]^{n-1}$ tends to the zero matrix as $n \to \infty$, which in turn implies that

$$\sum_{n=1}^{\infty} [M\Sigma]^{n-1} = [I - M\Sigma]^{-1}\ ,$$

so that

$$\begin{aligned}\hat{l}(s_1, s_2, s_3, s_4) = \boldsymbol{\tau}'(s_1)\ \Sigma\ (s_1+s_2)\ &[I - M\ (s_1+s_2+s_3+s_4)\ \Sigma\ (s_1+s_2+s_3)]^{-1}\\ &\cdot \boldsymbol{\psi}\ (s_1+s_2+s_3+s_4)\ .\end{aligned}$$

Expressions (3.3)–(3.11) may now be evaluated. Straightforward calculations yield

$$\hat{h}_0(s_1, s_2, s_3) = L_{s_1+s_2+s_3}\ (f(\cdot)\ \hat{\gamma}_0(s_1, \cdot))$$

$$\hat{h}(s_1, s_2, s_3) = \hat{h}_0(s_1, s_2, s_3) + \boldsymbol{\sigma}'\ (s_1, s_1+s_2+s_3)\ \Sigma\ (s_1+s_2)$$

$$[I - M\ (s_1+s_2+s_3)\ \Sigma\ (s_1+s_2)]^{-1}\ \boldsymbol{\psi}\ (s_1+s_2+s_3)\ .$$

Writing Σ for $\Sigma(0)$, $\boldsymbol{\tau}$ for $\boldsymbol{\tau}(0)$, etc;

$$\pi(x) = x\boldsymbol{\tau}'\Sigma\ [I - xM\Sigma]^{-1}\ \boldsymbol{\psi}$$

$$\hat{z}_1(s) = s^{-1}\ \{\boldsymbol{\tau}'\Sigma\boldsymbol{\psi}(s) + L_s(f\bar{G}_0)\}$$

$$\hat{z}_n(s) = s^{-1}\ \{(\boldsymbol{\tau}'\Sigma M(s) + \boldsymbol{\sigma}'(0, s))\ \Sigma[M(s)\ \Sigma]^{n-2}\boldsymbol{\psi}(s)\} \quad (n > 1)$$

$$\hat{q}(s) = s^{-1}\ \{(\boldsymbol{\tau}' + \boldsymbol{\sigma}'(0, s))\ \Sigma(s)\ [I - M(s)\ \Sigma(s)]^{-1}\ \boldsymbol{\psi}(s) + L_s(f\bar{G}_0)\}$$

$$\hat{r}(s) = s^{-1}\ \{(\boldsymbol{\tau}'\Sigma + \boldsymbol{\sigma}'(s, s)\ \Sigma(s))\ [I - M(s)\ \Sigma(s)]^{-1}\ \boldsymbol{\psi}(s) + L_s(f(\cdot)\ \hat{\gamma}_0(s, \cdot))\}$$

$$\begin{aligned}p_0(s) = s^{-1}\ \{&L_s(f(\cdot)\ (\hat{\gamma}_0(0, \cdot) - \hat{\gamma}_0(s, \cdot)))\\ &+ (\boldsymbol{\sigma}'(0, s) - \boldsymbol{\sigma}'(s, s))\ \Sigma(s)\ [I - M\Sigma(s)]^{-1}\ \boldsymbol{\psi}(s)\}\end{aligned}$$

$$\hat{p}_2(s) = s^{-1}\ \{\boldsymbol{\tau}'[\Sigma(0) - \Sigma(s)]\ [I - M(s)\ \Sigma(s)]^{-1}\ \boldsymbol{\psi}(s)\}$$

$$\hat{p}_1(s) = s^{-1}\ \{1 - \hat{h}(0, s, 0)\} - \hat{p}_2(s)\ .$$

The moments of R, Z, D and U may be determined numerically, since the matrices involved are matrices of constants. The first moments are, from (2.1)–(2.4),

$$\mathsf{E}[R] = \boldsymbol{\tau}'\Sigma\ [I - M\Sigma]^{-1}\ \boldsymbol{\psi}$$

$$\mathsf{E}[D] = \int_0^\infty \int_0^\infty \bar{G}_0\ (t+u)\ \bar{G}_1(t)\ \mathrm{d}F(u)\ \mathrm{d}t + \dot{\boldsymbol{\sigma}}'\Sigma\ [I - M\Sigma]^{-1}\ \boldsymbol{\psi}$$

$$\mathsf{E}[U] = \int_0^\infty t\bar{G}_0(t)\,\mathrm{d}F(t) + \sigma'\Sigma\,[I - M\Sigma]^{-1}\,\dot{\psi}$$
$$+ \{\sigma'\Sigma + \sigma'\dot{\Sigma} + \sigma'\Sigma\,[I - M\Sigma]^{-1}\,[\dot{M}\Sigma + M\dot{\Sigma}]\}\,[I - M\Sigma]^{-1}\,\psi$$
$$\mathsf{E}[Z] = \mathsf{E}[U] - \tau'\dot{\Sigma}\,[I - M\Sigma]^{-1}\psi\,.$$

Remarks: a) In practice, many of the coefficients in (4.1) will be zero. If $r_{ab} = 0$ for $b \in S = \{b_1, \ldots, b_k\}$, then omit the k corresponding blocks from all of the vectors and matrices defined above. Their order is then $(A+1) \sum_{b \notin S} (b+1)$.

b) Suppose $f(t)$ is better approximated by a function of the form (4.1) in which the failure rates λ_a depend upon the number of units in operation, i.e.

$$\lambda_a = \begin{cases} \mu_a & \text{when both units are operating} \\ \nu_a & \text{when the unit is operating alone.} \end{cases}$$

An examination of the roles played by Σ, M, ψ and σ in the preceding derivation shows that this change in the model may be accommodated by changing λ_a to μ_a in Σ, and to ν_a in M, ψ and σ. The vector τ does not lend itself to such a change, but $\tau(s)$ is used only in $\hat{q}(s)$, and elsewhere as $\tau(0)$, which is independent of the λ_a. Besides, $0 \leqq r(t) - q(t) \leqq 1$ for all t.

c) In Model 2, the two one-unit sub-processes are asymptotically independent, and the stationary forward recurrence density may be used in the determination of the mean times. From Fig. 3, with $x = t - u$, it is seen that D has density

$$\bar{G}(x)\,\frac{\bar{G}(x)}{\beta} + g(x) \int_x^\infty \frac{\bar{G}(u)}{\beta}\,\mathrm{d}u\,,$$

with $\mathsf{E}[D] = \beta/2$. Elementary arguments then give $\mathsf{E}[Z] = \alpha$, $\mathsf{E}[U] = \alpha + \alpha^2/2\beta$, $\mathsf{E}[R] = \alpha/\beta$. Note that the mean total repair time is the same as the mean unit time to failure.

In Model 1 approximate mean times may be obtained, by approximating the density of the residual service time of a unit at the end of a repair phase of the other unit by $F(t)/\alpha$. Standard renewal-theoretic methods then give

$$\mathsf{E}[R] \approx \alpha \int_0^\infty G_0(t)\,\mathrm{d}F(t) \Big/ \int_0^\infty \bar{F}(t)\,\bar{G}(t)\,\mathrm{d}t$$
$$\mathsf{E}[D] \approx \int_0^\infty\!\!\int_0^\infty \bar{G}_0\,(t+u)\,\mathrm{d}F(u)\,\mathrm{d}t + \mathsf{E}[R] \int_0^\infty\!\!\int_0^\infty \bar{G}\,(t+u)\,\frac{\bar{F}(u)}{\alpha}\,\mathrm{d}u\,\mathrm{d}t$$
$$\mathsf{E}[U] \approx \int_0^\infty \bar{G}_0(t)\,\bar{F}(t)\,\mathrm{d}t + \mathsf{E}[R] \left\{\frac{\alpha}{2} + \int_0^\infty\!\!\int_0^\infty \bar{G}(u)\,\frac{\bar{F}\,(t+u)}{\alpha}\,\mathrm{d}u\,\mathrm{d}t\right\}$$
$$\mathsf{E}[Z] \approx \mathsf{E}[U] - \frac{\alpha}{2}\,\mathsf{E}[R]\,.$$

If $g_0 = \bar{G}/\beta$ (i.e. if two repair facilities are available until the commencement of the up-time), or if $f(t)$ is exponential, then these means are exact.

(d) If

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{when both units are operating} \\ \nu e^{-\nu t} & \text{when the unit is operating alone} \end{cases}$$

then in both Models 1 and 2, the marginal density of Z is $\nu e^{-\nu z}$, regardless of the form of $g(t)$. If, as well, $g(t) = \sigma e^{-\sigma t}$, then the LAPLACE transform of h is easily inverted. With r denoting the number of repair facilities, it is found that the joint density of (U, Z, D) is

$$(r\sigma e^{-r\sigma d})\, \nu e^{-(\sigma+\nu)z} \left[\delta_u(z) + \left(\frac{2\lambda\sigma z}{u-z}\right)^{1/2} e^{-2\lambda(u-z)} I_1\left(\left[8\lambda\sigma z\,(u-z)\right]^{1/2}\right) \right],$$

where $I_1(\cdot)$ is the first hyperbolic BESSEL function.

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Zusammenfassung

Betrachtet werden Systeme mit zwei identischen Einheiten, in denen entweder eine oder zwei Reparaturmöglichkeiten auftreten. Zunächst werden nur die Ausfallzeit- der Einheiten und die Reparaturzeitverteilungsdichte vorausgesetzt. Es werden Gleichungen hergeleitet, die die Verteilungen der Arbeits- und Stillstandszeiten der Systeme, der totalen Reparaturzeiten, der Anzahl der Reparaturen, der Ausfallzeiten der n-ten Einheit und der Beendigung der n-ten Reparatur und die Wahrscheinlichkeit, daß i Einheiten zur Zeit t arbeiten, beschreiben. Diese Gleichungen werden dann unter der Annahme gelöst, daß die Ausfallzeiten der Einheiten eine Dichte besitzen, die sich als Linearkombination von ERLANGdichten darstellen läßt. Für einige Anordnungen der Systeme werden die ersten Momente für beliebige Ausfall- und Reparaturzeitverteilung angegeben.

Резюме

Рассмотрим системы с двумя идентичными единицами в которой или одна или два возможности ремонта существуют. Сначало предпологаются известными только плотности распределения времени выхода из строя и ремонта единицы. Выписаны уравнения, описывающие распределения периодов работы и остановка системы, полного времени ремонта, числа ремонтов, моментов времени выхода из строя n-ой единицы и конца n-го

ремонта и вероятность того что в момент t работают i единиц. Эти уравнения потом решаются в предположении что времена выхода из строя единиц обладают плотностью, представимой в виде линейной комбинации плотностей распределения Эрланга. Для некоторых размещении системы выписываются первые моменты для произвольных времен выхода из строя и ремонта.

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