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DISTRIBUTIONS FOR A TWO-UNIT REPAIRABLE SYSTEM

by

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ABSTRACT

In this thesis, we look at the superposition of two alternating renewal processes, thought of as a two-unit system in which each unit may be operating, under repair, or waiting for repair. We review those results previously obtained under the assumption that only one unit operates at a time. We then replace this by the assumption that each unit operates whenever it is so capable. For both the series and parallel arrangements of such a system, we obtain expressions for the distributions of the up and down times with either one or two repairmen. In the six non-trivial cases, these expressions are functions of the solution to an integral equation. This equation is solved for two classes of special cases, and the application of numerical methods is discussed.

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CHAPTER 1. INTRODUCTION

Consider two units -- machines, computers, power lines, telecommunications systems, etc. -- each of which is subject to failure when it operates, and is repaired when it fails. The operating and repair times are random variables, and a repaired unit is considered to be as good as new. The sequence of operating, repair, and, if there is only one repair facility, "waiting for repair" states, forms, for each unit, an alternating renewal process.

Now consider the superposition of these processes, henceforth referred to merely as the "system". We say that the two units function "in parallel" if the system is "up", i.e. operative, whenever one or more units are functioning, and "down" otherwise. Alternatively, the units are said to function "in series" if the system is "up" when, and only when, both units are functioning.

For a given parallel or series system, some questions which arise are:

i) If the system is up (down) at time $t = 0$, what is the distribution of time to the next passage into the down (up) state? These distributions are referred to as those of the up (down) times for the given system.

ii) What is the "availability" of the system; i.e. what is the probability, as $t \to \infty$, that the system will be up at time t?

The above considerations form the basis of the "two unit repairman problem". This problem has been extensively studied, with a myriad of variations. Some of them are described below. In each of them, the bulk of the analysis is concerned with the determination of the parallel up time, given certain initial conditions. It will later be seen that the three remaining cases -- parallel down, series up, series down -- render themselves amenable to analysis which is either trivial, or analagous to the parallel up case.

The previously studied models of the problem may, with some overlapping, be divided into two categories, termed "Active Standby" models and "Passive Standby" models. We note that the existence of the densities of the relevant distributions is always assumed, and, unless otherwise specified, the passage between operating and repair states is assumed to be instantaneous.

1. The Active Standby Models

In these models, each unit functions whenever it is capable of so doing, and the operating times for each unit are independent random variables. Gaver (1963) appears to have been the first to derive results pertaining to the parallel up time. In the model he first considered, the two units had identically, independently, exponentially distributed operating times, and were serviced by one, arbitrarily distributed repair facility (i.e. the repair *times* were arbitrarily distributed -- this grammatical abuse shall be maintained for convenience). It was assumed that both units were functioning at time $t = 0$, and the Laplace transform of the density of time spent in the up state, the mean time spent in this state, and the availability were obtained. The analysis was performed using the method of supplementary variables. The availability of the two repairman system was also obtained, by an argument given for exponentially distributed operating times but valid in general. This argument will be given in a later chapter.

In a subsequent paper Gaver (1964) generalized the above results to allow for distinct, exponentially distributed operating times. He considered also a model in which, when one unit fails, the other immediately fails due to overloading, with probability α or, if not immediately overloaded experiences an overloading failure after time t with probability $(1-S_i(t))e^{-\lambda_i t}$ for some arbitrary distribution function $S_i(t)$.

Osaki (1970) obtained similar results to those of Gaver (1963), using the integral equation of renewal theory. If the operating times are exponentially distributed, renewal points may be identified at the end of each repair phase. We shall return to this later.

In the Branson and Shah (1971) model, the state of the system depends entirely on the state of one crucial, "on line" unit. The operating times are exponentially distributed, with failure rate λ_1 for the "on line" unit, λ_2 for the "off line" unit. When an "on line" unit fails, it is repaired in a time governed by an arbitrary p.d.f. $g_1(t)$, after which time it is considered to be "off line" until the failure of the now crucial, but previously "off line" unit. A unit may fail while "off line", in which case the repair time has arbitrary p.d.f. $g_2(t)$.

The transition probabilities between the various states of the system are determined, and the Laplace transform of the parallel up time p.d.f. is obtained using the method of semi-Markov processes.

2. The Passive Standby Models

In most of these models, only one unit functions at a time. When it fails, it is replaced by the other, "standby" unit, assuming that this unit is not being repaired. In the latter eventuality, the system goes down. Gaver (1963) studied such a model in which the operating times were identically and exponentially distributed, and generalized his results, in the direction mentioned in the discussion of his work on the Active Standby model, in his 1964 paper. Muth (1966) obtained results for the case in which the single, gamma-distributed repair facility is capable of only n repairs to each exponentially distributed unit, after which time the system goes down if it has not already done so. A later generalization by Enns (1966) allowed for general operating and repair distributions.

Osaki (1970) studied three variations of the Passive Standby model. In the first, identical to the basic model described above, the two operating and two repair p.d.f.'s are arbitrary, and the Laplace transform of the parallel up time p.d.f. is obtained. In the second, only one unit is repaired, and it is used whenever both are available. The operating times are exponentially distributed. This "priority" model is also investigated by Buzacott (1971), but here both units are repairable and the four p.d.f.'s are arbitrary. In Osaki's third Passive Standby model, the time required to switch a unit from the active to the inactive state is itself a random variable, arbitrarily distributed. A later generalization (Osaki, 1972) allows for failure and repair of the switching mechanism. Here, the repair times of the two units are arbitrarily distributed, all other random variables are exponentially distributed. Osaki and Nakagawa (1971) also looked at a model in which the units may fail while in the standby state. The operating times in standby are exponentially distributed; the operating times while active, and repair times are arbitrarily distributed.

Other variations of all of the aforementioned models have been investigated by Srinivisan and Gopalan (1972), Osaki (1972), Arora (1975), Gopalan (1975), Kodamo, Nakamichi and Takamatsu (1976), Nakagawa (1977) and Srinivisan (1977).

The reader familiar with renewal theory may have noticed one feature shared in common by the models thus far discussed. In each of them, renewal points may be easily identified and exploited. A renewal point of a process is a point at which the past history of the process ceases to furnish any worthwhile information. For example, when a unit passes from the operating to the repair state there is a renewal of that one-unit sub-process at the point of passage, if each repair for that unit is independently distributed. Knowledge of times previously spent in the various states is of no benefit, and the process may be considered to have "renewed" itself. In general, however, the superposition of continuous renewal processes is not itself a renewal process, since the two component processes will never undergo changes of state simultaneously. Unless special conditions are imposed renewal points will not exist, and in the analysis the history of the system must be recorded for the entire duration of the time being considered.

Two such special conditions have been illustrated in the preceding discussion of Active Standby and Passive Standby models. We note first that for the duration of the parallel up time, it is never the case that two units are simultaneously under repair. Only the operating states are superimposed, either on each other or on repair states. In the Active Standby models, the operating times have been assumed to be exponentially distributed. Because of the "memoryless" property of the exponential distribution, at no point of a process so distributed is a knowledge of its past history beneficial. Thus, at any point at which one unit comes out of repair, the other is being governed by an exponential law and a renewal may be identified.

The other condition employed to circumvent this problem is common to the Passive Standby models. Here, it is specified that the two units do not operate simultaneously, or, if they do, the distribution law governing the operation of one changes at the point at which the other fails. Renewal points are thus imposed at each unit-failure point and the analysis becomes quite tractable, even if most of the states are arbitrarily distributed.

Both of these types of models were also examined by Kodama and Deguchi (1974), assuming Erlangian operating distributions. Some of their results will be independently obtained here. Some of the other aforementioned results, pertaining to what we term the "Renewal Models" will be obtained and extended in the next chapter.

In the bulk of this thesis, an attempt is made to treat the Active Standby model in more generality than has previously been done. The series up, series down and parallel down cases are briefly discussed, but the majority of our work is concerned with the determination of the distribution of the parallel up time. The system is characterized by the number of times it passes through that state in which both units are simultaneously operative. Inductive methods are employed to determine the time spent in $n+1$ such passages as a function of the time spent in n passages. In this way the history of the system may be preserved, and the exponential restriction becomes unnecessary.

In the most general case here considered, the operating times are identically but arbitrarily distributed, as are the repair times. The inductive methods described will lead to the derivation of a double integral equation containing all of the desired information. It is loath to relinquish this information, however, and at this point much of the generality must be abandoned. The equation is solved in some special cases, and numerical methods for approximating a solution are discussed.

CHAPTER 2. THE RENEWAL MODELS

Throughout this chapter, the operating times of the two units will have distribution functions F_i $(i = 1,2)$, with densities f_i and complementary distribution functions $\overline{F}_i$, all with support in $[0,\infty)$. The corresponding functions for the repair times will be denoted by G_i, g_i, $\overline{G}_i$ respectively. We define

$$\alpha_i = \int_0^\infty t dF_i(t), \qquad \beta_i = \int_0^\infty t dG_i(t).$$

The convolution of two functions f and g will be written as $f*g$, with $(f*g)^n$ indicating n-fold convolution and $(f*g)^0$ being Dirac's impulse function $\delta(t)$, which has the property that $(h*\delta)(t) = h(t)$ for any function h. The concatenation of functions will indicate pointwise multiplication; thus for example

$$f(g*h)(t) = f(t) \int_0^t g(t-u)h(u)du.$$

The Laplace transform

$$\int_0^\infty e^{-st} f(t)dt$$

of f will be denoted by $\hat{f}(s)$, or by $L_s(f)$ in the case of complicated expressions. We list some manipulative properties of the Laplace transform, all of which are easily verified:

i) $L_s(f*g) = \hat{f}(s)\hat{g}(s)$

ii) $L_s(\int_0^t f(u)du) = \hat{F}(s) = \frac{\hat{f}(s)}{s}$

iii) $\hat{\bar{F}}(s) = \frac{1-\hat{f}(s)}{s}$

iv) $L_s(t^n f(t)) = (-1)^n \frac{d^n}{ds^n} \hat{f}(s).$

From iii) and iv), if T is a random variable with distribution function F and density f, and if E is the expectation operator:

v) $E(T^n) = (-1)^n \hat{f}^{(n)}(0) = (-1)^{n-1} \hat{\bar{F}}^{(n-1)}(0)$.

In particular,

vi) $E(T) = -\hat{f}'(0) = \hat{\bar{F}}(0).$

We treat several of the parallel systems with one repair facility in this chapter. A discussion of series systems is deferred until the next chapter. There, we will develop the machinery to treat such systems, without requiring the assumption of exponentially distributed operating times.

1. The Parallel Down Time, with Passive Standby

Suppose that at time $t = 0$ the parallel system fails, due to the failure of unit i during a repair of unit j. The system will then remain down until the completion of this repair. Since only one unit operates at a time, the operating phase of unit i must have begun at the same time as the repair phase of unit j, hence the p.d.f. of the time remaining in this repair is

$$\frac{\int_0^\infty f_i(u)g_j(t+u)du}{\int_0^\infty \int_0^\infty f_i(u)g_j(t+u)dudt} = \frac{\int_0^\infty f_i(u)g_j(t+u)du}{\int_0^\infty F_i(t)dG_j(t)} \; .$$

The system will fail due to this cause with probability

$$\frac{\int_0^\infty F_i(t)dG_j(t)}{\int_0^\infty F_i(t)dG_j(t) + \int_0^\infty F_j(t)dG_i(t)} \; .$$

The parallel down time p.d.f. is therefore

$$h(t) = \frac{\int_0^\infty f_1(u)g_2(t+u) + f_2(u)g_1(t+u)du}{\int_0^\infty F_1(t)dG_2(t) + \int_0^\infty F_2(t)dG_1(t)} \; . \tag{2.1}$$

For future reference we evaluate the Laplace transform of (2.1) for $f_i(t) = \lambda_i e^{-\lambda_i t}$. A simple calculation yields

$$\hat{h}(s) = \frac{\frac{\lambda_1}{\lambda_1 - s}(\hat{g}_2(s) - \hat{g}_2(\lambda_1)) + \frac{\lambda_2}{\lambda_2 - s}(\hat{g}_1(s) - \hat{g}_1(\lambda_2))}{1-\hat{g}_1(\lambda_2) + 1-\hat{g}_2(\lambda_1)} \tag{2.2}$$

and so the expected down time, upon simplification, is

$$E_D = \frac{\beta_1 - \hat{\bar{G}}_1(\lambda_2) + \beta_2 - \hat{\bar{G}}_2(\lambda_1)}{\lambda_1 \hat{\bar{G}}_2(\lambda_1) + \lambda_2 \hat{\bar{G}}_1(\lambda_2)} \; . \tag{2.3}$$

In the active standby model, unit j may have been operative at the time of commencement of operation of unit i, so that (2.1) is invalid in this case. This time is still a renewal point, however, if the operating times are exponentially distributed, so that (2.2) and (2.3) remain true for this model as well.

2. <u>The Parallel Up Time</u>

In this model, we assume that only one unit operates at a time, but allow for unit failure while in standby. Define

$$W_i(t) = P\{\text{Unit } i\text{, newly repaired, fails to survive a standby interval of length } t\} \qquad (i = 1,2).$$

The system comes up upon the completion of the repair of one of the two failed units (a renewal point) and remains up as long as it is the case that the standby unit is repaired and ready for service when the active unit fails. Define $h_1(t)$ to be the p.d.f. of the up time, given the initial condition that unit 2 begins to function at time 0. We represent the possible sequences of events pictorially.

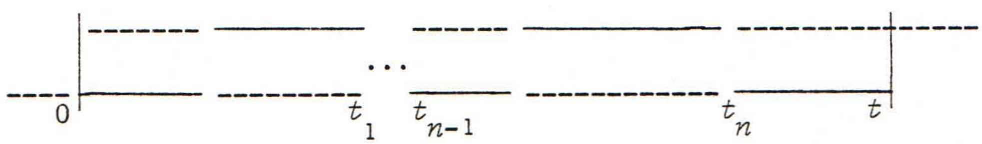


Figure 2.1.

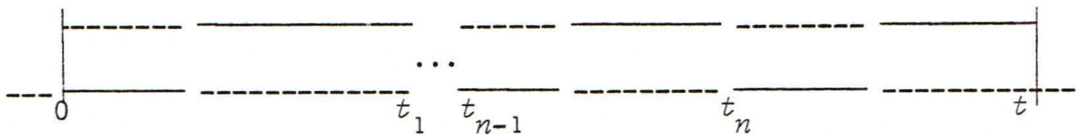


Figure 2.2.

Here, a solid line represents an operating interval, a broken line a repair/standby interval. In Fig. 2.1, system breakdown occurs due to the failure of unit 2 when unit 1 is either not repaired or has experienced a standby failure. Fig. 2.2 illustrates the dual cause of system breakdown. The three dots indicate that the events in $[0,t_1]$ are repeated n times in $[0,t_n]$. We note that each point t_i $(i = 1,\ldots,n)$ is a renewal point, so that the density of the time spent in any interval $[t_i,t_{i+1}]$ $(i = 0,\ldots,n-1; t_0 \equiv 0)$ is

$$[(g_1*\overline{W}_1)f_2]*[(g_2*\overline{W}_2)f_1](t_{i+1}-t_i).$$

From Fig. 2.1, the density of the time spent in (t_n,t) is

$$(\overline{G}_1f_2+(g_1*W_1)f_2)(t-t_n),$$

allowing for an incomplete repair, or standby failure. From Fig. 2.2, the density of this time is

$$[(g_1*\overline{W}_1)f_2]*[\overline{G}_2f_1+(g_2*W_2)f_1](t-t_n).$$

Collecting terms:

$$h_1(t) = \left\{\sum_{n=0}^{\infty} ([(g_1*\overline{W}_1)f_2]*[(g_2*\overline{W}_2)f_1])^n\right\} * \left\{[\overline{G}_1f_2+(g_1*W_1)f_2] + [(g_1*\overline{W}_1)f_2]*[\overline{G}_2f_1+(g_2*W_2)f_1]\right\}(t) \quad .$$

Taking Laplace transforms and simplifying gives

$$\hat{h}_1(s) = 1 - \frac{s[\hat{\bar{F}}_2(s) + \hat{\bar{F}}_1(s)L_s(f_2(g_1*\bar{W}_1))]}{1 - L_s(f_1(g_2*\bar{W}_2))L_s(f_2(g_1*\bar{W}_1))} \qquad (2.4)$$

so that

$$\hat{\bar{H}}_1(s) = \frac{\hat{\bar{F}}_2(s) + \hat{\bar{F}}_1(s)L_s(f_2(g_1*\bar{W}_1))}{1 - L_s(f_1(g_2*\bar{W}_2))L_s(f_2(g_1*\bar{W}_1))} \quad . \qquad (2.5)$$

We can now obtain as special cases some of the results mentioned in Chapter 1. If we put $W_i(t) \equiv 0$, we have the basic Passive Standby model in which the standby unit does not operate. The above expression then becomes

$$\hat{\bar{H}}_1(s) = \frac{\hat{\bar{F}}_2(s) + \hat{\bar{F}}_1(s)L_s(f_2G_1)}{1 - L_s(f_1G_2)L_s(f_2G_1)} \quad , \qquad (2.6)$$

with mean

$$\mu_1 = \frac{\alpha_2 + \alpha_1 \int_0^\infty G_1(t)dF_2(t)}{1 - \int_0^\infty G_2(t)dF_1(t) \int_0^\infty G_1(t)dF_2(t)} \quad . \qquad (2.7)$$

This was first obtained by Osaki (1970).

Suppose now that a unit may operate, fail, and be repaired while in standby. If we wish to retain the assumption that the operating p.d.f. of unit i when not in standby is $f_i(t)$, regardless of how much time it has spent in standby, we must utilize the memoryless characteristic of the exponential distribution.

Thus, let $k_i(t) = \lambda_i e^{-\lambda_i t}$ and $l_i(t)$ be the operating and repair p.d.f.'s of the units while in standby. Then

$$\overline{W}_i(t) = \overline{K}_i * \sum_{n=0}^{\infty} (k_i * l_i)^n(t) \quad . \tag{2.8}$$

Substituting this expression into (2.5) gives the "Warm Standby" results of Osaki and Nakagawa (1971) in the case $g_i = l_i$, and of Branson and Shah (1971) in the case $f_1 = f_2 = \lambda_1 e^{-\lambda_1 t}$, $k_1 = k_2 = \lambda_2 e^{-\lambda_2 t}$.

Now put $g_i = l_i$, $f_i = k_i = \lambda_i e^{-\lambda_i t}$; i.e. for each unit i, all repairs are identically but arbitrarily distributed, and all operating times are identically and exponentially distributed.

Returning to Figures 2.1 and 2.2, we see that the last phase of each process governed by $\overline{W}_i(t)$ is a standby operating interval which must last at least until time t_j in order for the system to remain up, at which point a new, identically distributed operating interval begins. If this last standby operating interval begins at time $t_j - t_j'$, say, the probability that the unit will still be operating at time $t > t_j$ is

$$\overline{K}_i(t_j')\overline{F}_i(t-t_j) = e^{-\lambda_i(t-(t_j-t_j'))} ,$$

which is just the probability of uninterrupted operation, either in or out of standby, from time $t_j - t_j'$ to at least time t.

In other words, by utilizing the memoryless property of the exponential distribution, we may assume that the units operate, with equal status, whenever they are so capable. This is exactly the defining assumption of the Active Standby model.

Substituting into (2.8) and (2.5), then, and simplifying, we get that the distribution of the parallel up time in the Active Standby model with exponential operating times has

$$\hat{\hat{H}}_1(s) = \frac{[1+\lambda_2\hat{\hat{G}}_2(s+\lambda_1)][1+(\lambda_1-\lambda_2)\hat{\hat{G}}_1(s+\lambda_2)]}{s+\lambda_1(s+\lambda_2)\hat{\hat{G}}_1(s+\lambda_2)+\lambda_2(s+\lambda_1)\hat{\hat{G}}_2(s+\lambda_1)} \quad . \tag{2.9}$$

This is Gaver's (1964) result, after adjusting for the differing initial conditions. To determine the probability of our initial condition, we note that this condition could only result if, at the beginning of the repair phase which is underway at time 0, unit 1 was operating and has since ceased to do so. This has probability

$$p_1 = \frac{\lambda_1\hat{\hat{G}}_2(\lambda_1)}{\lambda_1\hat{\hat{G}}_2(\lambda_1) + \lambda_2\hat{\hat{G}}_1(\lambda_2)} \quad .$$

The mean up time E_U for this system is then $p_1\hat{\hat{H}}_1(0) + (1-p_1)\hat{\hat{H}}_2(0)$, where $\hat{\hat{H}}_2$ is obtained from $\hat{\hat{H}}_1$ by permuting the indices. In the case $\lambda_1 = \lambda_2$, $G_1 = G_2$ this is

$$E_U = \frac{1 + \lambda\hat{\hat{G}}(\lambda)}{2\lambda^2\hat{\hat{G}}(\lambda)} \tag{2.10}$$

so that the availability is

$$A = \frac{E_U}{E_U + E_D} = \frac{1 + \lambda\hat{\hat{G}}(\lambda)}{1 - \lambda\hat{\hat{G}}(\lambda) + 2\lambda\beta} \quad , \tag{2.11}$$

in agreement with Gaver (1963).

In the simple case $f_1(t) = f_2(t) = \lambda e^{-\lambda t}$, $g_1(t) = g_2(t) = \nu e^{-\nu t}$, the up time p.d.f. is easily and explicitly obtainable by inverting the Laplace transform. From (2.9) we have, in this case, that

$$\hat{h}(s) = \frac{\lambda(s+2\lambda)}{s^2 + (\nu+3\lambda)s + 2\lambda^2} \quad . \tag{2.12}$$

Define constants A and B by

$$A = \frac{\nu + 3\lambda}{2} , \qquad B = \frac{(\nu^2+6\lambda\nu+\lambda^2)^{\frac{1}{2}}}{2} ,$$

then (2.12) becomes

$$\hat{h}(s) = \frac{\lambda(s+2\lambda)}{(s+A)^2 - B^2}$$

which may be decomposed as

$$\hat{h}(s) = \frac{\lambda}{2B}\left[\frac{A+B}{s+(A+B)} - \frac{A-B}{s+(A-B)}\right] + \frac{2\lambda^2}{2B}\left[\frac{1}{s+(A-B)} - \frac{1}{s+(A+B)}\right]$$

$$= L_s\left\{\frac{\lambda}{2B}\left[(A+B)e^{-(A+B)t} - (A-B)e^{-(A-B)t}\right] + \frac{2\lambda^2}{2B}\left[e^{-(A-B)t} - e^{-(A+B)t}\right]\right\} \quad .$$

After some algebraic manipulations, and upon invoking the uniqueness property of the Laplace transform, we get

$$h(t) = \lambda e^{-(\frac{\nu+3\lambda}{2})t}\left\{\cosh Bt + \frac{(\lambda-\nu)}{2B}\sinh Bt\right\} \quad . \tag{2.13}$$

CHAPTER 3. THE GENERAL ACTIVE STANDBY MODEL

In this chapter, we assume that the operating times for the two units are independently and identically distributed, with an arbitrary distribution function $F(t)$, density $f(t)$, mean α. When there are two repair facilities they are also i.i.d., with an arbitrary distribution function $G(t)$, density $g(t)$, mean β.

We first derive an integral equation, the solution to which yields the p.d.f. of the parallel up time with two repair facilities. The function satisfying this equation will then be modified and adapted to the cases of the parallel and series up and down times with one repair facility, and to the series down time with two repair facilities. The distributions of the parallel down and series up times, with two repair facilities, are also obtained, as are several of the mean times relating to these models.

1. Derivation of the Integral Equation

Define state i ($i = 0,1,2$) to be that state of the system in which exactly i units are inoperative. The parallel system is up as long as it is in states 0 or 1, and if we define

$$h_n(t) = \text{pdf}\{\text{up time} \mid \text{state 0 is entered exactly } n \text{ times}\}$$
$$\times\ P\{\text{state 0 is entered exactly } n \text{ times}\},$$

then the parallel up time p.d.f. is

$$h(t) = \sum_{n=0}^{\infty} h_n(t). \tag{3.1}$$

The first p.d.f./probability $h_0(t)$ is that of sequence of events represented pictorially as

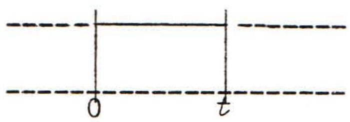


Figure 3.1.

and hence

$$h_0(t) = f(t) \int_t^{\infty} \frac{\overline{G}(u)}{\beta}\, du \ . \tag{3.2}$$

Our use of the equilibrium pdf $\overline{G}/\beta$, in the cases in which there are two repair facilities, implies that the process is assumed to have been underway for an infinite period of time prior to time 0. See Cox (1962) for a derivation.

State 0 may be entered exactly once in two ways:

Figure 3.2.

and so

$$h_1(t) = \int_0^t \left\{ \int_0^u \frac{\overline{G}(v)}{\beta} [f(u)f(t-v)+f(t)f(u-v)]dv \right\} \overline{G}(t-u)du. \tag{3.3}$$

Now define $l_{n+1}(t,u)$ $(n \geq 0)$ to be the joint p.d.f. of the time spent in n+1 passages through state 0, given that these passages end with unit failures at times u and t $(t > u)$, multiplied by the probability that this event occurs after n+1 passages through state 0. Then in the interval $[u,t]$ a repair is underway but not completed, and so

$$h_{n+1}(t) = \int_0^t l_{n+1}(t,u)\, \bar{G}(t-u)\, du \quad . \tag{3.4}$$

In (3.3), for instance, $l_1(t,u)$ is the term in braces. We use this term to begin the construction of the sequence $\{l_n\}_{n=1}^{\infty}$. If we are given l_n, we may construct l_{n+1} by considering the ways in which n passages through state 0 generate an (n+1)th such passage:

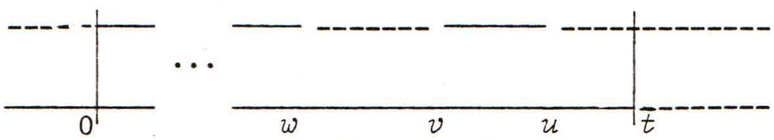


Figure 3.3.

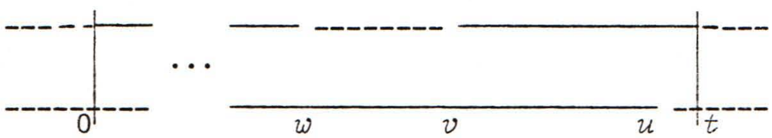


Figure 3.4.

In Fig. 3.3, we assume as the induction hypothesis that $l_n(t,w)$ is the p.d.f. of the time spent by the upper unit in the interval $[0,w]$, and by the lower unit in $[0,t]$, multiplied by the corresponding probability. The three dots indicate that n passages through state 0 occur in $[0,w]$. Similarly, in Fig. 3.4, it is $l_n(u,w)$ which we are given. The figures show two distinct ways in which any sequence of n passages through state 0 may generate a sequence of n+1 such passages. At the (n+1)th stage, we thus have 2^{n+1} sequences generated. It is easy to see that there can be no more than 2^{n+1} such sequences, and so we completely characterize the system by setting

$$l_{n+1}(t,u) = \int_0^u \int_0^v g(v-w)f(u-v)l_n(t,w) + g(v-w)f(t-v)l_n(u,w)dwdv. \quad (3.5)$$

We introduce some further notation. Define

$$l(t,u) = \sum_{n=1}^{\infty} l_n(t,u),$$

$$\rho(t,u) = l_1(t,u) = \int_0^u \frac{\overline{G}(v)}{\beta} [f(u)f(t-v)+f(t)f(u-v)]dv , \quad (3.6)$$

and for any function $\sigma(t,u)$ put

$$*\sigma(t,u) = \int_0^u \int_0^v g(v-w)f(u-v)\sigma(t,w)dwdv$$

$$\otimes\sigma(t,u) = \int_0^u \int_0^v g(v-w)f(t-v)\sigma(u,w)dwdv.$$

In this notation, the above equations become

$$h(t) = h_0(t) + \int_0^t l(t,u)\overline{G}(t-u)du \quad (3.7)$$

$$l_{n+1}(t,u) = *l_n(t,u) + \otimes l_n(t,u). \quad (3.8)$$

We can express $l(t,u)$ either as the solution to an integral equation, by summing over n in (3.8), or as the sum of an integral series, by repeatedly applying (3.8).

Thus

$$l(t,u) = \rho(t,u) + *l(t,u) + \otimes l(t,u) \quad (3.9)$$

$$= (\rho+(*\rho+\otimes\rho)+(**\rho+*\otimes\rho+\otimes*\rho+\otimes\otimes\rho)+\ldots)(t,u). \quad (3.10)$$

In general, of course, $*$ and $\otimes$ do not commute. If, however, $f(t) = \lambda e^{-\lambda t}$, then the two summands of $\rho(t,u)$ are identical, $*\rho(t,u) = \otimes\rho(t,u)$, and $*$ and $\otimes$ commute, so that in this case

$$l(t,u) = \sum_{n=0}^{\infty} (*2\rho)^n(t,u)$$

where exponentiation indicates n-fold iteration.

Of the two operations, $*$ and $\otimes$, the latter is by far the more troublesome. The former represents merely a double convolution, with respect to the second argument of the function being operated upon, of that function with f and g. It will be seen in Chapter 5 that if $f(t)$ is a linear combination of exponential functions, then $\otimes$ is in a certain pathological sense the "mirror image" of $*$. For $f(t)$ a member of the class of Erlangian densities, we show in Chapter 6 that $*$ and $\otimes$ are related by a certain transformation which is its own inverse.

We now show how the solutions to the above equations may be adapted to some of the other arrangements of the system.

2. Analogous Models

Because of the symmetry of the two unit, two repair system, one need only read "down" for "up", "operating" for "repair", etc., in order to derive results for the series arrangements from those for the parallel arrangements. Thus, if one exchanges $\overline{F}$ for $\overline{G}$, f for g, α for β in all of the equations of Section 1, then $h(t)$ becomes the series down time p.d.f. with two repair facilities. The parallel up time p.d.f. with one repair facility is also easily obtainable. Merely replace $\frac{\overline{G}}{\beta}$ by g in equations (3.2), (3.3) and (3.6), since in this case the initial repair in $[0,t]$ could not begin until time 0. We will write $\tilde{h}_n$ for h_n, $\tilde{\rho}$ for ρ, etc., when this replacement is necessary.

Obtaining the series up time p.d.f. with one repair facility requires more extensive modifications. Let time 0 be the time of the last passage from state 2 to state 1. This is the only type of renewal point in this model. Define $k_n(v)$ $(n \geq 0)$ to be the p.d.f. of the time spent in the $(n+1)$th passage through state 0, after time 0 and before the next passage into state 2, multiplied by the probability that this is the $(n+1)$th such passage. Then the p.d.f. of the series up time is given by

$$k(v) = \frac{\sum_{n=0}^{\infty} k_n(v)}{\int_0^{\infty} \sum_{n=0}^{\infty} k_n(v)dv} .$$

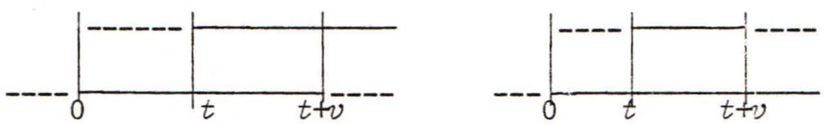


Figure 3.5.

For $k_0(v)$ the two possibilities are those illustrated above, and so

$$k_0(v) = \overline{F}(v) \int_0^{\infty} g(t)f(t+v)dt + f(v) \int_0^{\infty} g(t)\overline{F}(t+v)dt .$$

For $n \geq 1$ the general cases are illustrated in Fig. 3.6.

Figure 3.6.

The three dots indicate that n passages through state 0 occur in $[0,u]$. Upon recalling the definition of $\tilde{l}(t,u)$, we immediately get

$$k(v)\alpha k_0(v) + \overline{F}(v) \int_0^\infty \int_0^t \tilde{l}(t+v,u)g(t-u)dudt$$

$$+ f(v) \int_0^\infty \int_0^t \int_v^\infty \tilde{l}(t+w,u)g(t-u)dwdudt \quad . \tag{3.11}$$

If the operating times are exponentially distributed, the point t in Fig.'s 3.5 and 3.6 is a renewal point. Since no repairs occur in $[t,t+v]$ we would not expect k to be a function of g. To check this, and to illustrate a property of $l(t,u)$, let $f(t) = \lambda e^{-\lambda t}$. Then

$$\tilde{l}(t+v,u) = e^{-\lambda v}\tilde{l}(t,u)$$

since the only difference between $\tilde{l}(t+v,u)$ and $\tilde{l}(t,u)$ is that in the former an exponentially distributed operating phase continues for a further v units of time. Substituting into (3.11) and integrating out w in the last expression there gives

$$k(v)\alpha 2\lambda e^{-2\lambda v} \int_0^\infty g(t)e^{-\lambda t}dt + 2e^{-2\lambda v} \int_0^\infty \int_0^t g(t-u)l(t,u)dudt \quad . \tag{3.12}$$

From (3.6),

$$\tilde{l}_1(t,u) = \lambda e^{-\lambda t}(g*2\lambda e^{-\lambda(\cdot)})(u) \tag{3.13}$$

By (3.13), (3.8) and induction,

$$\tilde{l}_{n+1}(t,u) = \lambda e^{-\lambda t}(g*2\lambda e^{-\lambda(\cdot)})^{n+1}(u) \qquad (n \geq 0)$$

and hence

$$\tilde{l}(t,u) = \lambda e^{-\lambda t} \sum_{n=1}^{\infty} (g*2\lambda e^{-\lambda(\cdot)})^n(u) \ .$$

Substituting into (3.12),

$$k(v)\alpha 2\lambda e^{-2\lambda v}\int_0^\infty g(t)e^{-\lambda t}dt$$

$$+\ 2e^{-2\lambda v}\int_0^\infty\int_0^t g(t-u)\lambda e^{-\lambda t}\sum_{n=1}^\infty (g*2\lambda e^{-\lambda(\cdot)})^n(u)dudt$$

$$= 2\lambda e^{-2\lambda v}\hat{g}(\lambda)$$

$$+\ 2\lambda e^{-2\lambda v}\int_0^\infty e^{-\lambda t}(g * \sum_{n=1}^\infty (g*2\lambda e^{-\lambda(\cdot)})^n)(t)dt$$

$$= 2\lambda e^{-2\lambda v}\hat{g}(\lambda) + 2\lambda e^{-2\lambda v}\frac{\hat{g}^2(\lambda)}{1-\hat{g}(\lambda)}$$

$$= 2\lambda e^{-2\lambda v}\left(\frac{\hat{g}(\lambda)}{1-\hat{g}(\lambda)}\right)$$

and so

$$k(v) = 2\lambda e^{-2\lambda v} \quad . \tag{3.14}$$

The normalizing constant is the sum of the probabilities that state 0 is entered at least n times ($n > 0$) between passages through state 2. In the exponential case, state 0 is entered each time there is a repair before a failure, with probability

$$\int_0^\infty g(t)e^{-\lambda t}dt = \hat{g}(\lambda),$$

independently of the number of times this state has previously been entered. The required sum is thus

$$\sum_{n=1}^\infty (\hat{g}(\lambda))^n = \frac{\hat{g}(\lambda)}{1-\hat{g}(\lambda)} ,$$

as above.

This normalizing constant is equal both to $\int_0^\infty \int_0^t \tilde{l}(t,u)dudt$, and to the expected number of passages through state 0 between successive passages through state 2. That these three terms are equal is true in general. Define, for $i = 0,1$, the random variables

$N_i(\tilde{N}_i)$ = "number of passages through state i during the parallel up time, with two (one) repair facilities"

and put $E_i = E(N_i)$, $\tilde{E}_i = E(\tilde{N}_i)$. It follows from the definition of $l_n(t,u)$ that

$$\int_0^\infty \int_0^t l_n(t,u)dudt = P(N_0 \geq n)$$

so that

$$\int_0^\infty \int_0^t l(t,u)dudt = \sum_{n=1}^{\infty} P(N_0 \geq n) = E_0 \quad .$$

Similarly,

$$\int_0^\infty \int_0^t \tilde{l}(t,u)dudt = \tilde{E}_0 \quad . \tag{3.15}$$

Integrating the term on the right hand side of (3.11) verifies that $\tilde{E}_0$ is the normalizing constant for $k(v)$.

To obtain the series down time p.d.f. with one repair facility, define time u to be that of a passage from states 0 to 1, and time $t+u$ to be that of the next passage into state 0. Then $[u,u+t]$ is the down time. Define time 0 to be the time of the last passage, before u, from states 2 to 1. Let $d_n(t)$ be the p.d.f. of the down time after n entries into state 0 in $[0,u]$, multiplied by the probability that n passages through state 0 precede the down time. Then the total down time p.d.f. is

$$d(t) \propto \sum_{n=1}^{\infty} d_n(t).$$

For each n, we must consider two cases, depending upon whether or not state 2 is entered during $[u,u+t]$.

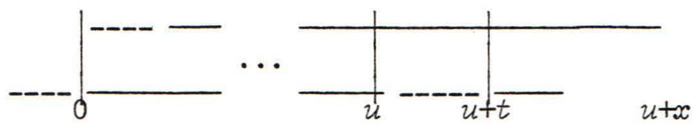


Figure 3.7.

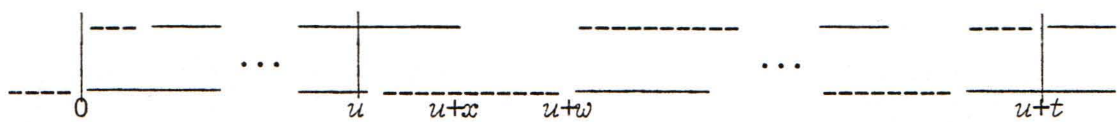


Figure 3.8.

In each figure, the three dots in $[0,u]$ indicate n passages through state 0. In Fig. 3.8, the dots in $[u+w,u+t]$ indicate arbitrarily many passages from states 2 to 1. Noting that these latter passages define renewal points, we set

$$\gamma(t) = \left(\sum_{i=0}^{\infty} (Fg)^{i} * \bar{F}g\right)(t),$$

where $*$ indicates convolution and exponentiation indicates iterated convolution. Then

$$d(t) \propto g(t) \int_0^{\infty}\int_t^{\infty} \mathcal{T}(u+x,u)\,dx\,du + \int_0^{\infty}\int_0^{t}\int_0^{w} \mathcal{T}(u+x,u)g(w)\gamma(t-w)\,dx\,dw\,du \quad . \qquad (3.16)$$

Now define

$$\psi(t) = g(t) \int_0^\infty \int_0^t \tilde{l}(u+x,u)\,dx\,du.$$

Using (3.15), (3.16) then becomes

$$d(t) \alpha\ g(t)\tilde{E}_0 - \psi(t) + (\gamma * \psi)(t). \tag{3.17}$$

The Laplace transform of the term above is

$$\hat{g}(s)\tilde{E}_0 - \hat{\psi}(s) + \hat{\gamma}(s)\hat{\psi}(s),$$

so that the normalizing constant is

$$\hat{g}(0)\tilde{E}_0 - \hat{\psi}(0)(1-\hat{\gamma}(0)) = \tilde{E}_0.$$

Thus

$$d(t) = g(t) - \frac{\psi(t)-(\gamma*\psi)(t)}{\tilde{E}_0} \tag{3.18}$$

with Laplace transform

$$\hat{d}(s) = \hat{g}(s) - \frac{\hat{\psi}(s)(1-\hat{g}(s))}{\tilde{E}_0\ (1-\widehat{Fg}(s))}. \tag{3.19}$$

The mean down time is

$$\hat{\hat{D}}(0) = \hat{\hat{G}}(0)\left[1 + \frac{\hat{\psi}(0)}{\tilde{E}_0(1-\widehat{Fg}(0))}\right].$$

But

$$\hat{\psi}(0) = \int_0^\infty g(t) \int_0^\infty \int_0^t \tilde{l}(u+x,u)\,dx\,du\,dt$$

$$= \int_0^\infty \int_0^v \tilde{l}(v,w)\overline{G}(v-w)\,dw\,dv$$

$$= \int_0^\infty \tilde{h}(v) - \tilde{h}_0(v)\,dv$$

$$= 1 - \widehat{Fg}(0) ,$$

so that the mean down time for this system is

$$\beta\left(1 + \frac{1}{\tilde{E}_0}\right) \quad . \tag{3.20}$$

The parallel down time with one repair facility is obtained from $l(t,u)$ in a similar manner. Let $[v,v+t]$ be this down time, so that there are passages from states 1 to 2 at v, and from 2 to 1 at $v+t$. Let 0 be the time of the last passage from states 2 to 1 before v. Define $p_n(t)$ to be the p.d.f. of the down time, given that exactly n (≥ 0) passages through state 0 precede time v, multiplied by the probability of this event. The relevant cases are illustrated below.

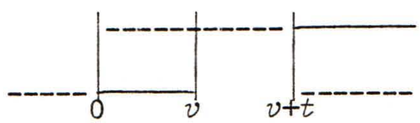


Figure 3.9.

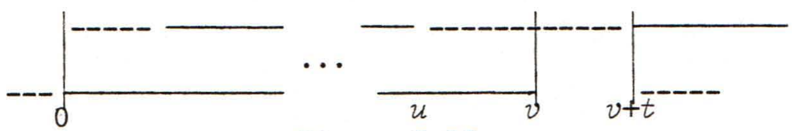


Figure 3.10.

The three dots in Fig. 3.10 indicate n (≥ 1) passages through state 0 in $[0,u]$. It follows that the down time p.d.f. is

$$p(t) = \sum_{n=0}^{\infty} p_n(t)$$

$$= \int_0^{\infty} g(v+t)f(v)dv + \int_0^{\infty}\int_0^{v} g(v+t-u)\tilde{l}(v,u)dudv$$

$$= \int_0^{\infty} g(v+t)[f(v)+\chi(v)]dv \quad , \tag{3.21}$$

where

$$\chi(v) = \int_0^{\infty} \tilde{l}(v+u,u)du \quad .$$

If there are two repair facilities, several of the mean times are independent of the forms of the distributions involved. To obtain these, we first require the distribution of the parallel down time with two repair facilities. Such a system may remain down in $[0,t]$ in only two possible ways, illustrated below.

Figure 3.11.

The down time p.d.f. is thus

$$\bar{G}(t)\,\frac{\bar{G}(t)}{\beta} + g(t)\int_t^{\infty} \frac{\bar{G}(u)}{\beta}\,du \tag{3.22}$$

with mean

$$E_D = \frac{\beta}{2} . \tag{3.23}$$

If $G(t)$ is thought of as the operating distribution function, then (3.22) gives the series up time p.d.f. with two repair facilities.

Given the availability of the parallel system, we may solve for the mean up time from

$$A = \frac{E_U}{E_U+E_D} = \frac{E_U}{E_U+\beta/2} . \tag{3.24}$$

Since there are two repairmen, the two one-unit sub-processes are independent of one another, and the proportion of time for which each unit is non-operative tends to $\frac{\beta}{\alpha+\beta}$ as $t \to \infty$. The proportion of time for which the system is down thus tends to $(\beta/(\alpha+\beta))^2$ as $t \to \infty$, and the availability is

$$A = 1 - \left(\frac{\beta}{\alpha+\beta}\right)^2 . \tag{3.25}$$

Equating (3.24) and (3.25) gives

$$E_U = \alpha + \frac{\alpha^2}{2\beta} . \tag{3.26}$$

Now define

D_i = "Expected duration of a passage through state i."

$(i = 0,1)$.

Then

$$E_U = E_0 D_0 + E_1 D_1 . \tag{3.27}$$

The proportion of the up time which is spent in state 0 is obtained in the same way as was the availability, and is

$$\frac{E_0 D_0}{E_0 D_0 + E_1 D_1} = \frac{\alpha^2}{\alpha^2 + 2\alpha\beta} . \tag{3.28}$$

Since each passage through state 0 during the up time is preceded and followed by a passage through state 1,

$$N_1 \equiv N_0 + 1 . \tag{3.29}$$

If the roles of the operating and repair distributions are reversed, D_0 becomes the expected down time for the parallel system. From (3.23),

$$D_0 = \frac{\alpha}{2} . \tag{3.30}$$

From (3.26) - (3.30),

$$E_0 = \frac{\alpha}{\beta} , \quad E_1 = 1 + \frac{\alpha}{\beta} , \quad D_1 = \frac{\alpha\beta}{\alpha+\beta} .$$

Thus the two summands of E_U in (3.26) are the expected times spent in states 1 and 0, respectively, during the up time. Also, the mean series down time with two repair facilities is now expressible as

$$\beta(1 + \frac{1}{2E_0}) .$$

(Compare this with (3.20)).

CHAPTER 4. ON THE DENSENESS OF TWO CLASSES OF DISTRIBUTIONS

In Chapters 5 and 6, expressions for the distribution of the parallel up time shall be obtained under the assumption that the operating time distribution is either Erlangian, or a linear combination of exponential distributions. In this chapter we show that any lifetime distribution function F may be approximated arbitrarily closely by mixtures of the first above-mentioned class, or by the members themselves of the second if $F(0) = 0$. In the latter case we exhibit a sequence of such functions converging vaguely to the given distribution function.

We shall require:

LEMMA 4.1: *Let $x \in R$ be fixed, $F_{\lambda,x}(t)$ a family of distribution functions with mean x, variance $\sigma_\lambda^2 \to 0$ as $\lambda \to \infty$. Let $g(t)$ be a bounded, measureable function on $(-\infty,\infty)$. Then*

$$\int_{-\infty}^{\infty} g(t)dF_{\lambda,x}(t) \to g(x) \text{ as } \lambda \to \infty,$$

provided g is continuous at x.

Proof: See Feller (1971).

Define the Erlangian distribution functions by

$$E_\lambda^k(t) = \int_0^t \frac{(\lambda x)^{k-1}\lambda e^{-\lambda x}}{(k-1)!}\,dx = e^{-\lambda t}\sum_{i=k}^{\infty}\frac{(\lambda t)^i}{i!} \qquad k = 1,2,3,\ldots;\ \lambda > 0$$

$$E_\lambda^0(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases} .$$

The proof, due to Schassberger, of the following result is not that which he originally used, and so we give it here.

THEOREM 4.1 (Schassberger, 1968): *Let* $F(t)$ *be any distribution function on* $[0,\infty)$, *and put*

$$F_\lambda(t) = \sum_{k=0}^{\infty} [F(\tfrac{k}{\lambda})-F(\tfrac{k-1}{\lambda})]E_\lambda^k(t).$$

Then F_λ *is a distribution function, and* $F_\lambda \overset{\nu}{\to} F$ *as* $\lambda \to \infty$.

Proof: Identify $F(t)$ with the function $g(t)$ of Lemma 4.1. Let $F_{\lambda,x}(t)$ be the distribution function concentrating on $\{0/\lambda, 1/\lambda, 2/\lambda, \ldots\}$ with mass $e^{-\lambda x}\frac{(\lambda x)^k}{k!}$ on k/λ. Then $F_{\lambda,x}(t)$ has mean x, variance $x/\lambda \to 0$ as $\lambda \to \infty$. By Lemma 4.1,

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} F(t)dF_{\lambda,x}(t) = F(x)$$

if F is continuous at x.

But

$$\int_{-\infty}^{\infty} F(t)dF_{\lambda,x}(t) = \sum_{k=0}^{\infty} F(\tfrac{k}{\lambda})e^{-\lambda x}\frac{(\lambda x)^k}{k!}$$

$$= \sum_{k=0}^{\infty} F(\tfrac{k}{\lambda})[1-e^{-\lambda x}-(\lambda x)e^{-\lambda x}-\ldots-\frac{(\lambda x)^{k-1}e^{-\lambda x}}{(k-1)!}]$$

$$- \sum_{k=0}^{\infty} F(\tfrac{k}{\lambda})[1-e^{-\lambda x}-(\lambda x)e^{-\lambda x}-\ldots-\frac{(\lambda x)^{k}e^{-\lambda x}}{k!}$$

$$= \sum_{k=0}^{\infty} F(\tfrac{k}{\lambda})[1-e^{-\lambda x}-\ldots-\frac{(\lambda x)^{k-1}e^{-\lambda x}}{(k-1)!}]$$

$$- \sum_{k=0}^{\infty} F(\tfrac{k-1}{\lambda})[1-e^{-\lambda x}-\ldots-\frac{(\lambda x)^{k-1}e^{-\lambda x}}{(k-1)!}]$$

$$= \sum_{k=0}^{\infty} (F(\tfrac{k}{\lambda})-F(\tfrac{k-1}{\lambda}))E_{\lambda}^{k}(x)$$

$$= F_{\lambda}(x).$$

COROLLARY 4.1: *Let $F(t)$ be any distribution function on $[0,\infty)$. There exists a sequence $\{G_n\}$ of distribution functions, each a finite mixture of Erlangian distribution functions, such that $G_n \overset{\nu}{\to} F$ as $n \to \infty$.*

Proof: Let t be a continuity point of F, and let $m = m(\lambda)$ be any integer valued function of λ such that $m/\lambda \to \infty$ as $\lambda \to \infty$. Define

$$G_{m,\lambda}(t) = \sum_{k=0}^{m} [F(\tfrac{k}{\lambda})-F(\tfrac{k-1}{\lambda})]E_{\lambda}^{k}(t) + [1-F(\tfrac{m}{\lambda})]E_{\lambda}^{m+1}(t) .$$

Then $G_{m,\lambda}$ is a distribution function, and

$$|G_{m,\lambda}(t)-F(t)| \leq |G_{m,\lambda}(t)-F_{\lambda}(t)| + |F_{\lambda}(t)-F(t)| ,$$

where the last term $\to 0$ as $\lambda \to \infty$ by Theorem 4.1. Upon expanding the middle term as a series and re-arranging the partial sums, we get

$$\lim_{\lambda\to\infty} |G_{m,\lambda}(t)-F_{\lambda}(t)|$$

$$= \lim_{\lambda\to\infty} \lim_{N\to\infty} |E_{\lambda}^{m+1}(t) - \sum_{k=m+1}^{N-1} F(\tfrac{k}{\lambda})[E_{\lambda}^{k}(t)-E_{\lambda}^{k+1}(t)]-F(\tfrac{N}{\lambda})E_{\lambda}^{N}(t)|$$

$$= \lim_{\lambda\to\infty} \lim_{N\to\infty} |e^{-\lambda t} \sum_{k=m+1}^{N-1} (1-F(\tfrac{k}{\lambda})) \frac{(\lambda t)^k}{k!} + (1-F(\tfrac{N}{\lambda}))e^{-\lambda t} \sum_{k=N}^{\infty} \frac{(\lambda t)^k}{k!}|$$

$$\leq \lim_{\lambda\to\infty} \lim_{N\to\infty} |[1-F(\tfrac{m+1}{\lambda})]e^{-\lambda t} \sum_{k=m+1}^{N-1} \frac{(\lambda t)^k}{k!} + (1-F(\tfrac{N}{\lambda}))e^{-\lambda t} \sum_{k=N}^{\infty} \frac{(\lambda t)^k}{k!}|$$

$$= 0,$$

due to the condition imposed upon m.

Put $G_n = G_{m(n),n}$, then $G_n \overset{v}{\to} F$ as $n \to \infty$.

We note:

i) If the limit function F is continuous, vague convergence is uniform. See, e.g. Chung (1974) for a proof.

ii) If $F(0) = 0$, the summation in the definitions of F_{λ} and $G_{m,\lambda}$ may begin at $k = 1$.

LEMMA 4.2: *Let k be a fixed positive integer. There exists a sequence*

$\{E_n\}$ of distribution functions, each of the form

$$\int_0^t \sum_{i=1}^{k} a_i e^{-\lambda_i x}\,dx,$$

such that $E_n \rightrightarrows E_\lambda^k$ as $n \to \infty$.

Proof: If

$$a_i = \lambda_1\lambda_2\cdots\lambda_k \prod_{\substack{j=1\\ j\neq i}}^{k} (\lambda_j-\lambda_i)^{-1},$$

and

$$G(\lambda_1,\lambda_2,\ldots,\lambda_k;t) = \int_0^t \sum_{i=1}^{k} a_i e^{-\lambda_i x}\,dx\ ,$$

then the Laplace-Stieltjes transform

$$G(\lambda_1,\lambda_2,\ldots,\lambda_k;s) = \prod_{i=1}^{k} \left(\frac{\lambda_i}{\lambda_i+s}\right)$$

$$\to \left(\frac{\lambda}{\lambda+s}\right)^k = E_\lambda^k(s)$$

as $\lambda_1,\lambda_2,\ldots,\lambda_k \to \lambda$ through distinct values.

Put

$$\lambda_{j,n} = \lambda + \frac{j}{n^3} \qquad (j = 1,2,\ldots,k;n = 1,2,\ldots)$$

and define

$$E_n(t) = G(\lambda_{1,n},\lambda_{2,n},\ldots,\lambda_{k,n};t)\ .$$

By the convergence theorem for Laplace transforms,

$$E_n \overset{v}{\to} E_\lambda^k$$

as $n \to \infty$. The uniformity follows by i) above.

By Lemma 4.2, any E_λ^k for $k > 0$, hence any mixture thereof, may be approximated by distribution functions of the form $\int_0^t \sum_i a_i e^{-\lambda_i x} dx$. This, combined with Corollary 4.1, establishes

THEOREM 4.2: *Let $F(t)$ be any distribution function on $[0,\infty)$ with $F(0) = 0$. There exists a sequence $\{D_n\}$ of distribution functions, each of the form*

$$D_n(t) = \int_0^t \sum_{i=1}^m a_i e^{-\lambda_i x} dx$$

such that $D_n \overset{\nu}{\to} F$. It is sufficient for $m = m(n)$ to satisfy $m/n \to \infty$ as $n \to \infty$, and for each m, $\sup_{1 \le i,j \le m} |\lambda_i - \lambda_j|$ may be chosen to be arbitrarily small. If F is continuous the convergence is uniform.

Even if $F_n \rightrightarrows F$ and the corresponding densities f_n, f exist and are continuous, it is not necessarily true that $f_n \to f$. If $f(t)$ is continuous, it is however an easy but non-constructive consequence of the Stone-Weierstrass theorem that f is the uniform limit of a sequence of densities, each of the form $\sum_{i=1}^n a_i e^{-\lambda_i t}$.

An Explicit Construction

If $F(0) = 0$ and t is a continuity point of F then, in the notation of Corollary 4.1 if we put $G_{m,\lambda} = G_{n^2,n}$ we have

$$F(t) = \lim_{n\to\infty} \sum_{k=1}^{n^2} [F(\tfrac{k}{n})-F(\tfrac{k-1}{n})]E_n^k(t) + [1-F(n)]E_n^{n^2+1}(t)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n^2} F(\tfrac{k}{n})(E_n^k-E_n^{k+1})(t) + E_n^{n^2+1}(t)$$

$$= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n^2} F(\tfrac{k}{n}) \int_0^t \sum_{i=1}^{k} \prod_{j=1}^{k} (n+\tfrac{j}{n^3}) \prod_{\substack{j=1\\ j\neq i}}^{k} (\tfrac{j-i}{n^3})^{-1} e^{-(n+\frac{i}{n^3})x} \right.$$

$$- \sum_{i=1}^{k+1} \prod_{j=1}^{k+1} (n+\tfrac{j}{n^3}) \prod_{\substack{j=1\\ j\neq i}}^{k+1} (\tfrac{j-i}{n^3})^{-1} e^{-(n+\frac{i}{n^3})x} dx$$

$$\left. + \int_0^t \sum_{i=1}^{n^2+1} \prod_{j=1}^{n^2+1} (n+\tfrac{j}{n^3}) \prod_{\substack{j=1\\ j\neq i}}^{n^2+1} (\tfrac{j-i}{n^3})^{-1} e^{-(n+\frac{i}{n^3})x} dx \right\}$$

by Lemma 4.2. Putting $\mu_i = (n+\frac{i}{n^3})$ and simplifying, the above expression becomes

$$F(t) = \lim_{n\to\infty} \int_0^t \sum_{k=1}^{n^2} F(\tfrac{k}{n}) \left\{ \sum_{i=1}^{k+1} \frac{(-1)^i \prod_{j=1}^{k} (n^4+j)}{(i-1)!(k-i+1)!} \mu_i e^{-\mu_i x} \right\}$$

$$+ \sum_{i=1}^{n^2+1} \frac{(-1)^{i-1} \prod_{\substack{j=1\\ j\neq i}}^{n^2+1} (n^4+j)\mu_i e^{-\mu_i x} dx}{(i-1)!(n^2+1-i)!} .$$

Reversing the order of summation in the first term, matching coefficients of $e^{-\mu_i x}$, and simplifying, gives

$$F(t) = \lim_{n\to\infty} \int_0^t \sum_{i=1}^{n^2+1} (-1)^i a_i e^{-\mu_i x} dx$$

where

$$\mu_i = n + i/n^3 ,$$

$$a_i = \left\{\mu_i \sum_{k=i-1}^{n^2} \begin{pmatrix} n^4+k \\ n^4, i-1 \end{pmatrix} F(k/n)\right\} - \frac{i}{n^3} \begin{pmatrix} n^4+n^2+1 \\ n^4, i \end{pmatrix} ,$$

and

$$\begin{pmatrix} p \\ q, r \end{pmatrix} \equiv \frac{p!}{q!r!(p-q-r)!} .$$

The second component of a_i comes from the remainder term in $G_{n^2,n}$. If it is ignored the elements of the sequence become sub-distribution functions, but the convergence is not affected.

CHAPTER 5. SPECIAL CASES -I

Let the density of the operating times of the two units be of the form

$$f(t) = \sum_{i=1}^{m} a_i e^{-\lambda_i t}, \tag{5.1}$$

where

$$\sum_{i=1}^{m} \frac{a_i}{\lambda_i} = 1, \tag{5.2}$$

and assume that the repair times are identically but arbitrarily distributed with density $g(t)$. Unless otherwise specified, all summations are from 1 to m. We will show that the terms of the series (3.1) have a form which allows their Laplace transforms to be summed, with a subsequent reduction to a linear system. As in (3.6),

$$\rho(t,u) = \int_0^u \frac{\overline{G}(v)}{\beta} [f(t)f(u-v) + f(u)f(t-v)]dv \ .$$

Substituting (5.1),

$$\rho(t,u) = \sum_{i,j} a_i a_j e^{-\lambda_i t} \int_0^u \frac{\overline{G}(v)}{\beta} e^{-\lambda_j (u-v)} dv + \sum_{i,j} a_i a_j e^{-\lambda_j u} \int_0^u \frac{\overline{G}(v)}{\beta} e^{-\lambda_i (t-v)} dv$$

$$= \sum_{i,j} a_i a_j e^{-\lambda_i t - \lambda_j u} \int_0^u \frac{\overline{G}(v)}{\beta} [e^{\lambda_j v} + e^{\lambda_i v}] dv \quad .$$

The form of $\rho(t,u)$ is thus

$$\rho(t,u) = \sum_{i,j} a_i a_j e^{-\lambda_i t - \lambda_j u} \rho_{ij}(u) \tag{5.3}$$

where

$$\rho_{ij}(u) = \int_0^u \frac{\bar{G}(v)}{\beta} [e^{\lambda_j v} + e^{\lambda_i v}] dv \tag{5.4}$$

and thus, from (3.4),

$$h_1(t) = \int_0^t \bar{G}(t-u) \rho(t,u) du$$

has Laplace transform

$$\hat{h}_1(s) = \sum_{i,j} a_i a_j \hat{\bar{G}}(s+\lambda_i) \hat{\rho}_{ij}(s+\lambda_i+\lambda_j) \tag{5.5}$$

where

$$\hat{\rho}_{ij}(s+\lambda_i+\lambda_j) = \frac{\hat{\bar{G}}(s+\lambda_i) + \hat{\bar{G}}(s+\lambda_j)}{\beta(s+\lambda_i+\lambda_j)} \quad . \tag{5.6}$$

The form exhibited in (5.3) is preserved when operated upon by $*$ or $\otimes$. To see this, assume that

$$\sigma(t,u) = \sum_{i,j} a_i a_j e^{-\lambda_i t - \lambda_j u} \sigma_{ij}(u) .$$

Then

$$*\sigma(t,u) = \int_0^u f(u-v) \int_0^v g(v-w) \sigma(t,w) dw dv$$

$$= \int_0^u \sum_k a_k e^{-\lambda_k (u-v)} \int_0^v g(v-w) \sum_{i,j} a_i a_j e^{-\lambda_i t - \lambda_j w} \sigma_{ij}(w) dw dv .$$

Permuting the indices j and k gives

$$*\sigma(t,u) = \sum_{i,j} a_i a_j e^{-\lambda_i t - \lambda_j u} \sum_k a_k \int_0^u e^{\lambda_j v} \int_0^v g(v-w) e^{-\lambda_k w} \sigma_{ik}(w)\,dw\,dv$$

$$= \sum_{i,j} a_i a_j e^{-\lambda_i t - \lambda_j u} \bar{\sigma}_{ij}(u), \tag{5.7}$$

where

$$\bar{\sigma}_{ij}(u) = \sum_k a_k \int_0^u e^{\lambda_j v} \int_0^v g(v-w) e^{-\lambda_k w} \sigma_{ik}(w)\,dw\,dv \; . \tag{5.8}$$

Similarly,

$$\otimes\sigma(t,u) = \int_0^u f(t-v) \int_0^v g(v-w)\sigma(u,w)\,dw\,dv$$

$$= \int_0^u \sum_k a_k e^{-\lambda_k (t-v)} \int_0^v g(v-w) \sum_{i,j} a_i a_j e^{-\lambda_i u - \lambda_j w} \sigma_{ij}(w)\,dw\,dv \; .$$

Cyclically permuting all indices,

$$\otimes\sigma(t,u) = \sum_{i,j} a_i a_j e^{-\lambda_i t - \lambda_j u} \sum_k a_k \int_0^u e^{\lambda_i v} \int_0^v g(v-w) e^{-\lambda_k w} \sigma_{jk}(w)\,dw\,dv$$

$$= \sum_{i,j} a_i a_j e^{-\lambda_i t - \lambda_j u} \bar{\sigma}_{ji}(u) \; . \tag{5.9}$$

Recalling (3.6) and (3.8), we see that each $l_n(t,u)$ is of the form

$$l_n(t,u) = \sum_{i,j} a_i a_j e^{-\lambda_i t - \lambda_j u} \overset{n}{\sigma}_{ij}(u) \tag{5.10}$$

for some set of "components" $\{\overset{n}{\sigma}_{ij} | i,j = 1,2,\ldots,m\}$ so that

$$h_n(t) = \int_0^t \bar{G}(t-u) l_n(t,u)\,du$$

has

$$\hat{h}_n(s) = \sum_{i,j} a_i a_j \hat{\hat{G}}(s+\lambda_i)\hat{\sigma}^n_{ij}(s+\lambda_i+\lambda_j) \ . \tag{5.11}$$

From (5.7), (5.8) and (5.9),

$$l_{n+1}(t,u) = \sum_{i,j} a_i a_j e^{-\lambda_i t-\lambda_j u}\sigma^{n+1}_{ij}(u)$$

with

$$\hat{\sigma}^{n+1}_{ij}(s+\lambda_i+\lambda_j) = \frac{1}{s+\lambda_i+\lambda_j}\sum_k a_k \ [\hat{g}(s+\lambda_i)\hat{\sigma}^n_{ik}(s+\lambda_i+\lambda_k)$$

$$+ \hat{g}(s+\lambda_j)\hat{\sigma}^n_{jk}(s+\lambda_j+\lambda_k)] \tag{5.12}$$

and

$$\hat{h}_{n+1}(s) = \sum_{i,j} a_i a_j \hat{\hat{G}}(s+\lambda_i)\hat{\sigma}^{n+1}_{ij}(s+\lambda_i+\lambda_j) \ . \tag{5.13}$$

These equations may be expressed in matrix form, and the series

$$\hat{h}(s) = \sum_{n=0}^{\infty} \hat{h}_n(s)$$

summed, in two ways. The first method leads to a linear system requiring the inversion of an $(m\times m)$ symmetric matrix of a very general form. The second leads to the development of an $m^2\times m^2$ matrix, partitioned into $m\times m$ blocks, which is then reduced to tri-diagonal form. The individual blocks are easily invertible, and an algorithm is described by which the matrix may be inverted block by block.

Method 1.

Define $(m \times m)$ matrices

$$M = \text{diag}\left\{\sum_j \frac{a_j \hat{g}(s+\lambda_1)}{s+\lambda_1+\lambda_j}, \ldots, \sum_j \frac{a_j \hat{g}(s+\lambda_m)}{s+\lambda_m+\lambda_j}\right\}$$

$$N = (N_{ij}) = \left\{\frac{a_j \hat{g}(s+\lambda_j)}{s+\lambda_i+\lambda_j}\right\}_{i,j=1,\ldots,m}$$

$$P = (P_{ij}) = \left(\hat{\rho}_{ij}(s+\lambda_i+\lambda_j)\right)_{i,j=1,\ldots,m}$$

$$Q = \text{diag}\left(\hat{\bar{G}}(s+\lambda_1), \ldots, \hat{\bar{G}}(s+\lambda_m)\right)$$

and $(m\times 1)$ vectors

$$\underline{a} = (a_1, \ldots, a_m)^T$$

$$\underline{x} = Q\underline{a} = \left(a_1\hat{\bar{G}}(s+\lambda_1), \ldots, a_m\hat{\bar{G}}(s+\lambda_m)\right)^T$$

$$\underline{y} = P\underline{a} = \left(\sum_j a_j\hat{\rho}_{1j}(s+\lambda_1+\lambda_j), \ldots, \sum_j a_j\hat{\rho}_{mj}(s+\lambda_m+\lambda_j)\right)^T .$$

Then for $n \geq 1$,

$$[(M+N)^{n-1}\underline{y}]_i = \sum_j a_j\hat{\sigma}^n_{ij}(s+\lambda_i+\lambda_j) \tag{5.14}$$

so that

$$\underline{x}^T[M+N]^{n-1}\underline{y} = \sum_{i,j} a_i\hat{\bar{G}}(s+\lambda_i)a_j\hat{\sigma}^n_{ij}(s+\lambda_i+\lambda_j)$$

$$= \hat{h}_n(s) \tag{5.15}$$

by (5.11). We establish (5.14) by induction. For $n = 1$, the left hand side is just

$$(I\underline{y})_i = \sum_j a_j \hat{\rho}_{ij}(s+\lambda_i+\lambda_j)$$

in agreement with (5.5). Assume that (5.14) is true for some $n \geq 1$. Then for $n+1$,

$$[(M+N)^n \underline{y}]_i = [(M+N)(M+N)^{n-1}\underline{y}]_i$$

$$= \sum_j M_{ij} \sum_k a_k \hat{\sigma}^n_{jk}(s+\lambda_j+\lambda_k) + \sum_j N_{ij} \sum_k a_k \hat{\sigma}^n_{jk}(s+\lambda_j+\lambda_k)$$

by the induction hypothesis. Substituting the values of M_{ij} and N_{ij}, this becomes

$$\sum_j \frac{a_j \hat{g}(s+\lambda_i)}{s+\lambda_i+\lambda_j} \sum_k a_k \hat{\sigma}^n_{ik}(s+\lambda_i+\lambda_k) + \sum_j \frac{a_j \hat{g}(s+\lambda_j)}{s+\lambda_i+\lambda_j} \sum_k a_k \hat{\sigma}^n_{jk}(s+\lambda_j+\lambda_k)$$

$$= \sum_j a_j \hat{\sigma}^{n+1}_{ij}(s+\lambda_i+\lambda_j)$$

by (5.12), establishing (5.14). From (5.15), we have

$$\hat{h}(s) = \hat{h}_0(s) + \underline{x}^T \sum_{n=1}^{\infty} [M+N]^{n-1}\underline{y} \ . \tag{5.16}$$

Recalling the definition of $h_n(t)$ as the conditional p.d.f. of the up time given that state 0 is entered exactly n times, multiplied by the probability of this event, we have that for all s,

$$\hat{h}_n(s) \leq \hat{h}_n(0) = \int_0^{\infty} h_n(t)dt = p_n ,$$

where $p_n = P\{\text{State 0 is entered exactly } n \text{ times}\}$.

Thus each term on the left hand side of (5.15) is bounded above by p_n, where $p_n \to 0$ as $n \to \infty$ since $\sum_{n=0}^{\infty} p_n = 1$. This implies that $[M+N]^{n-1}$ tends to the zero matrix as $n \to \infty$. Equivalently, if $\|\cdot\|$ is any matrix norm with $\|I\| = 1$,

$$\|[M+N]^{n-1}\| \to 0 \qquad \text{as } n \to \infty,$$

so that

$$\lim_{n\to\infty} \|I-[I-M-N] \sum_{i=1}^{n} [M+N]^{i-1}\| = 0$$

and

$$I = \lim_{n\to\infty} [I-M-N] \sum_{i=1}^{n} [M+N]^{i-1}$$

i.e.

$$\sum_{i=1}^{\infty} [M+N]^{i-1} = [I-M-N]^{-1} .$$

Thus (5.16) becomes

$$\hat{h}(s) = \hat{h}_0(s) + \underline{x}^T [I-M-N]^{-1} \underline{y} \quad . \tag{5.17}$$

We can perform a slight reduction so that the matrix to be inverted is symmetric. Define $(m \times m)$ matrices

$$A = \text{diag}\left\{\frac{1}{a_1\hat{g}(s+\lambda_1)}, \ldots, \frac{1}{a_m\hat{g}(s+\lambda_m)}\right\}$$

$$B = \text{diag}\left\{\frac{1}{a_1\hat{g}(s+\lambda_1)} - \sum_j \frac{a_j}{a_1(s+\lambda_1+\lambda_j)}, \ldots, \frac{1}{a_m\hat{g}(s+\lambda_m)} - \sum_j \frac{a_j}{a_m(s+\lambda_m+\lambda_j)}\right\}$$

$$C = (C_{ij}) = \left(\frac{1}{s+\lambda_i+\lambda_j}\right)_{i,j=1,\ldots,m} .$$

Then

$$A[B-C]^{-1} = [BA^{-1}-CA^{-1}]^{-1}$$

$$= \left[\text{diag}\left(1-\hat{g}(s+\lambda_1) \sum_j \frac{a_j}{s+\lambda_1+\lambda_j}, \ldots, 1-\hat{g}(s+\lambda_m) \sum_j \frac{a_j}{s+\lambda_m+\lambda_j}\right) - \left(\frac{a_j\hat{g}(s+\lambda_j)}{s+\lambda_i+\lambda_j}\right)_{i,j=1,\ldots,m}\right]^{-1}$$

$$= [I-M-N]^{-1},$$

so that (5.17) becomes

$$\hat{h}(s) = \hat{h}_0(s) + \underline{z}^T[B-C]^{-1}\underline{y} \tag{5.18}$$

where

$$\underline{z}^T = \underline{x}^T A = \left(\frac{\hat{\bar{G}}(s+\lambda_1)}{\hat{g}(s+\lambda_1)}, \ldots, \frac{\hat{\bar{G}}(s+\lambda_m)}{\hat{g}(s+\lambda_m)}\right) ,$$

$B-C$ is symmetric, and, from (3.2),

$$\hat{h}_0(s) = \sum_i \frac{a_i}{s+\lambda_i} \left\{1 - \frac{\hat{\bar{G}}}{\beta} (s+\lambda_i)\right\} . \tag{5.19}$$

The series up and down, and parallel down p.d.f.'s with one repair facility are now obtainable (numerically) by evaluating the matrix at zero before inverting it. First replace $\overline{G}/\beta$ by g in $\underline{y}$, then substitute the form of $\tilde{l}(t,u) = \sum_{n=1}^{\infty} \tilde{l}_n(t,u)$, from (5.10), into the relevant expressions in Chapter Three. Then apply (5.14). Upon defining $(m\times 1)$ vectors $\underline{t}$, $\underline{u}$, $\underline{v}$, $\underline{w}$, $\underline{z}$, $\underline{1}$ and $\underline{\varphi}$ by

$$t_i = a_i e^{-\lambda_i v}, \qquad u_i = a_i \hat{G}(\lambda_i), \qquad v_i = a_i e^{-\lambda_i v}[\hat{g}(\lambda_i)\overline{F}(v)+\hat{G}(\lambda_i)f(v)]$$

$$w_i = a_i/\lambda_i, \qquad z_i = \frac{a_i[\hat{g}(s)-\hat{g}(s+\lambda_i)]}{\lambda_i}, \qquad 1_i = 1,$$

$$\underline{\varphi} = [I-M-N]^{-1}\underline{y}\Big|_{s=0},$$

the above described calculations yield

$$k(v) = \frac{\underline{v}^T(\underline{1}+\underline{\varphi})}{\underline{u}^T(\underline{1}+\underline{\varphi})} \qquad \left(= \frac{\underline{v}^T(\underline{1}+\underline{\varphi})}{\underline{w}^T\underline{\varphi}}\right) \tag{5.20}$$

$$\tilde{E}_0 = \underline{w}^T\underline{\varphi}, \qquad \hat{\psi}(s) = \underline{z}^T\underline{\varphi}, \qquad \chi(v) = \underline{t}^T\underline{\varphi} \tag{5.21}$$

$$\hat{d}(s) = \hat{g}(s) - \frac{1-\hat{g}(s)}{1-\widehat{Fg}(s)} \cdot \frac{\underline{z}^T\underline{\varphi}}{\underline{w}^T\underline{\varphi}} \tag{5.22}$$

$$p(t) = \int_0^{\infty} g(v+t)[f(v)+\underline{t}^T\underline{\varphi}]dv \;. \tag{5.23}$$

If we put $m = 1$, $a_1 = \lambda_1 = \lambda$, so that $f(t) = \lambda e^{-\lambda t}$, then (5.18) becomes, upon simplification,

$$\hat{h}(s) = \frac{\lambda}{s+\lambda}\left(1 - \frac{\hat{\bar{G}}}{\beta}(s+\lambda)\right) + \frac{2\lambda^2\hat{\bar{G}}^2(s+\lambda)}{\beta(s+2\lambda(s+\lambda)\hat{\bar{G}}(s+\lambda))} \tag{5.24}$$

with mean

$$\frac{1}{\lambda} + \frac{1}{2\lambda^2\beta} = \alpha + \frac{\alpha^2}{2\beta}$$

and variance

$$\frac{1}{\lambda^2}\left[1 + \frac{1}{\lambda\beta} - \frac{1}{4\lambda^2\beta^2} + \frac{1}{2\beta\lambda^2\hat{\bar{G}}(\lambda)}\right] .$$

As mentioned previously, the up time in the one-repairman model is obtained by replacing each occurence of $\bar{G}/\beta$ by g. Upon doing this, (5.24) becomes

$$\hat{h}(s) = \frac{\lambda(s+2\lambda)\hat{\bar{G}}(s+\lambda)}{s+2\lambda(s+\lambda)\hat{\bar{G}}(s+\lambda)}$$

with

$$\hat{\bar{H}}(s) = \frac{1+\lambda\hat{\bar{G}}(s+\lambda)}{s+2\lambda(s+\lambda)\hat{\bar{G}}(s+\lambda)} ,$$

which is (2.9) in the case $\lambda_1 = \lambda_2$.

For $m = 2$, put $a_1 = a$, $\lambda_1 = \lambda$, $\lambda_2 = \lambda_1+b$. Then (5.2) implies

$$a_2 = \frac{(\lambda+b)(\lambda-a)}{\lambda} .$$

The terms in (5.18) then become

$$\hat{h}_0(s) = \frac{a}{s+\lambda}\left(1 - \frac{\hat{\bar{G}}}{\beta}(s+\lambda)\right) + \frac{(\lambda+b)(\lambda-a)}{\lambda(s+\lambda+b)}\left(1 - \frac{\hat{\bar{G}}}{\beta}(s+\lambda+b)\right)$$

$$\underset{\sim}{z} = \left(\frac{\hat{\bar{G}}(s+\lambda)}{\hat{g}(s+\lambda)}, \frac{\hat{\bar{G}}(s+\lambda+b)}{\hat{g}(s+\lambda+b)}\right)^T$$

$$B-C = \begin{pmatrix} \frac{1}{a\hat{g}(s+\lambda)} - \frac{2}{s+2\lambda} - \frac{(\lambda+b)(\lambda-a)}{\lambda a(s+2\lambda+b)} & \frac{-1}{s+2\lambda+b}. \\ \frac{-1}{s+2\lambda+b} & \frac{\lambda}{(\lambda+b)(\lambda-a)\hat{g}(s+\lambda+b)} - \frac{\lambda a}{(\lambda+b)(\lambda-a)(s+2\lambda+b)} - \frac{2}{s+2\lambda+2b} \end{pmatrix}$$

$$\underline{y} = \left\{\frac{2a\hat{\overline{G}}(s+\lambda)}{\beta(s+2\lambda)} + \frac{(\lambda+b)(\lambda-a)}{\lambda}\left(\frac{\hat{\overline{G}}(s+\lambda)+\overline{G}(s+\lambda+b)}{\beta(s+2\lambda+b)}\right),\right.$$

$$\left.\frac{a(\hat{\overline{G}}(s+\lambda)+\hat{\overline{G}}(s+\lambda+b))}{\beta(s+2\lambda+b)} + \frac{2(\lambda+b)(\lambda-a)\hat{\overline{G}}(s+\lambda+b)}{\lambda\beta(s+2\lambda+2b)}\right\}^T$$

from which $\hat{h}(s)$ is easily obtained. In this notation,

$$\hat{f}(s) = \frac{a}{\lambda+s} + \frac{(\lambda+b)(\lambda-a)}{\lambda(\lambda+b+s)} \quad . \tag{5.25}$$

If we put

$$a = \frac{\lambda(\lambda+b)}{b} \tag{5.26}$$

then (5.25) becomes

$$\hat{f}(s) = \frac{\lambda(\lambda+b)}{(\lambda+s)(\lambda+b+s)} \to \left(\frac{\lambda}{\lambda+s}\right)^2$$

as $b \to 0$, hence

$$f(t) \to \lambda^2 t e^{-\lambda t}$$

as in Lemma 4.2. Substituting (5.26) into the linear system above yields

$$\hat{h}_0(s) = \frac{\lambda(\lambda+b)}{b}\left[\frac{1 - \frac{\hat{\overline{G}}}{\beta}(s+\lambda)}{s+\lambda} - \frac{1 - \frac{\hat{\overline{G}}}{\beta}(s+\lambda+b)}{s+\lambda+b}\right]$$

$$\to -\lambda^2 \frac{d}{ds}\,\frac{1 - \frac{\hat{\overline{G}}}{\beta}(s+\lambda)}{s+\lambda} \qquad \text{as } b \to 0,$$

and

$$B-C = \begin{pmatrix} \frac{b}{\lambda(\lambda+b)\hat{g}(s+\lambda)} - \frac{2}{s+2\lambda} + \frac{1}{s+2\lambda+b} & \frac{-1}{s+2\lambda+b} \\ \frac{-1}{s+2\lambda+b} & \frac{-b}{\lambda(\lambda+b)\hat{g}(s+\lambda+b)} + \frac{1}{s+2\lambda+b} - \frac{2}{s+2\lambda+2b} \end{pmatrix}$$

$$\underline{y} = \frac{\lambda(\lambda+b)}{\beta b}\left(\frac{2\hat{\bar{G}}(s+\lambda)}{s+2\lambda} - \frac{\hat{\bar{G}}(s+\lambda)+\hat{\bar{G}}(s+\lambda+b)}{s+2\lambda+b}, \frac{\hat{\bar{G}}(s+\lambda)+\hat{\bar{G}}(s+\lambda+b)}{s+2\lambda+b} - \frac{2\hat{\bar{G}}(s+\lambda+b)}{s+2\lambda+2b}\right)^T$$

$$\to \frac{\lambda^2}{\beta}\left[\frac{2\hat{\bar{G}}(s+\lambda)-(s+2\lambda)\hat{\bar{G}}'(s+\lambda)}{(s+2\lambda)^2}\right](1,1)^T \qquad \text{as } b \to 0.$$

Writing D for the transpose of the cofactor matrix of $B-C$, we have

$$\hat{h}(s) = -\lambda^2 \frac{d}{ds}\frac{1-\frac{\hat{\bar{G}}}{\beta}(s+\lambda)}{s+\lambda} + \lim_{b\to 0}\frac{\underline{z}^T D\underline{y}}{|B-C|} .$$

The limit is evaluated by two applications of l'Hopital's rule. The calculation yields, as the Laplace transform of the parallel up time given that the operating times are Erlangian in two stages,

$$\hat{h}(s) = -\lambda^2 \frac{d}{ds}\frac{1-\frac{\hat{\bar{G}}}{\beta}(s+\lambda)}{s+\lambda} + \frac{\frac{1}{\beta}\left[2\left(\frac{\lambda}{s+2\lambda}\right)^2\hat{\bar{G}}(s+\lambda) - \frac{\lambda^2\hat{\bar{G}}'(s+\lambda)}{s+2\lambda}\right]^2}{1 - \left(\frac{4\lambda^2\hat{g}(s+\lambda)}{(s+2\lambda)^2} - \frac{\lambda^2\hat{g}'(s+\lambda)}{s+2\lambda}\right)} . \quad (5.27)$$

This result will be obtained by direct methods in Chapter 6. Although the calculations would be extremely tedious, one could, in the same way as was done above, obtain $\hat{h}(s)$ when $f(t)$ is a mixture of k Erlangian densities, each in c_i stages $(i = 1,\ldots,k)$. This would necessitate the inversion of a matrix of order $\sum_{i=1}^{k} c_i$, and the evaluation of the associated bilinear form as k parameters tended to zero. Some simplification in the inversion of the matrix might be afforded by expressing the linear system in the way now to be described.

Method 2.

Let J be the $m^2 \times m^2$ identity matrix, and define $m^2 \times 1$ vectors

$$\underline{b} = (b_{11}, b_{12}, \ldots, b_{1m}, \ldots, b_{m1}, b_{m2}, \ldots, b_{mm})^T$$

where

$$b_{ij} = a_i a_j \hat{\bar{G}}(s+\lambda_i)$$

$$\underline{\rho} = (\rho_{11}, \rho_{12}, \ldots, \rho_{1m}, \ldots, \rho_{m1}, \rho_{m2}, \ldots, \rho_{mn})^T$$

where

$$\rho_{ij} = \hat{\rho}_{ij}(s+\lambda_i+\lambda_j)$$

$$= \frac{\hat{\bar{G}}(s+\lambda_i) + \hat{\bar{G}}(s+\lambda_j)}{\beta(s+\lambda_i+\lambda_j)}$$

$$\underline{\sigma}^n = (\sigma_{11}^n, \sigma_{12}^n, \ldots, \sigma_{1m}^n, \ldots, \sigma_{m1}^n, \sigma_{m2}^n, \ldots, \sigma_{mm}^n)^T \qquad (n \geq 2)$$

where

$$\sigma_{ij}^n = \hat{\sigma}_{ij}^n(s+\lambda_i+\lambda_j)$$

$\underline{\sigma}^1 = \underline{\rho}$.

Define the $m \times 1$ vector

$$\underline{G} = \left(\frac{1 - \frac{\hat{\underline{G}}}{\beta}(s+\lambda_1)}{s+\lambda_1}, \ldots, \frac{1 - \frac{\hat{\underline{G}}}{\beta}(s+\lambda_m)}{s+\lambda_m} \right)^T$$

and the $1\times m$ vector

$$\underline{a} = (a_1, \ldots, a_m) \ .$$

Define also

$$g_{ij}^k = \frac{\hat{g}(s+\lambda_k)}{s+\lambda_i+\lambda_j} \qquad i,j,k = 1,2,\ldots,m \ .$$

Then in this notation (5.19), (5.5) and (5.13) become

$$\hat{h}_0(s) = \underline{a}\ \underline{G} \tag{5.28}$$

$$\hat{h}_1(s) = \underline{b}^T J \underline{\rho} \tag{5.29}$$

$$\hat{h}_{n+1}(s) = \underline{b}^T \underline{\sigma}^{n+1} \ . \tag{5.30}$$

Now let Q be the $m^2 \times m^2$ matrix written below, where $g_{ij}^k \underline{a}$ denotes the string of m elements $g_{ij}^k a_1, \ldots, g_{ij}^k a_m$ and $\underline{0}$ denotes a string of m zeroes.

$$
Q = \left[\begin{array}{cccccc}
2g^{1}_{11}\underline{a} & \underline{0} & \underline{0} & \cdots & & \underline{0} \\
g^{1}_{12}\underline{a} & g^{2}_{12}\underline{a} & \underline{0} & & & \underline{0} \\
g^{1}_{13}\underline{a} & \underline{0} & g^{3}_{13}\underline{a} & & & \underline{0} \\
\vdots & & & & & \vdots \\
g^{1}_{1m}\underline{a} & \underline{0} & \underline{0} & \cdots & & g^{m}_{1m}\underline{a} \\
\hline
g^{1}_{21}\underline{a} & g^{2}_{21}\underline{a} & \underline{0} & \cdots & & \underline{0} \\
\underline{0} & 2g^{2}_{22}\underline{a} & \underline{0} & & & \underline{0} \\
\underline{0} & g^{2}_{23}\underline{a} & g^{3}_{23}\underline{a} & & & \underline{0} \\
\vdots & & & & & \vdots \\
\underline{0} & g^{2}_{2m}\underline{a} & \underline{0} & \cdots & & g^{m}_{2m}\underline{a} \\
\hline
\vdots & & & & & \vdots \\
\hline
g^{1}_{m1}\underline{a} & \underline{0} & \cdots & & & g^{m}_{m1}\underline{a} \\
\underline{0} & g^{2}_{m2}\underline{a} & & & & g^{m}_{m2}\underline{a} \\
\vdots & & & & g^{m-1}_{m,m-1}\underline{a} & g^{m}_{m,m-1}\underline{a} \\
\underline{0} & \cdots & & & \underline{0} & 2g^{m}_{m,m}\underline{a}
\end{array}\right]
$$

Equation (5.12) now becomes

$$\underline{\sigma}^{n+1} = Q\underline{\sigma}^{n} \tag{5.31}$$

so that by (5.30)

$$\hat{h}^{(s)}_{n+1} = \underline{b}^T Q \underline{\sigma}^n . \qquad (5.32)$$

But by (5.31), the definition of $\underline{\sigma}^1$, and induction,

$$Q\underline{\sigma}^n = Q^n \underline{\rho} \qquad (n \geq 1)$$

so (5.32) becomes

$$\hat{h}^{(s)}_{n+1} = \underline{b}^T Q^n \underline{\rho} \qquad (n \geq 0)$$

and so

$$\hat{h}(s) = \hat{h}_0(s) + \underline{b}^T \left(\sum_{n=0}^{\infty} Q^n \right) \underline{\rho} .$$

As in the previous method, the eventual termination, with probability one, of the process implies that

$$\sum_{n=0}^{\infty} Q^n = [J-Q]^{-1}$$

and so

$$\hat{h}(s) = \underline{a}\ \underline{G} + \underline{b}^T [J-Q]^{-1} \underline{\rho} . \qquad (5.33)$$

$J-Q$ may be partitioned as

$$J-Q = (Q_{ij})_{i,j=1,2,\ldots,m} .$$

Here, each Q_{ij} is an $m \times m$ matrix defined by

$$Q_{ii} = I_{m\times m} - \begin{pmatrix} g^i_{i1}\,\underline{a} \\ \cdot \\ \cdot \\ \cdot \\ g^i_{i,i-1}\underline{a} \\ 2g^i_{i,i}\,\underline{a} \\ g^i_{i,i+1}\underline{a} \\ \cdot \\ \cdot \\ \cdot \\ g^i_{i,m}\,\underline{a} \end{pmatrix}$$

$$\underset{(i\neq j)}{Q_{ij}} = -g^j_{ij}\underline{1}_j\underline{a} = \begin{pmatrix} \underline{0} \\ \cdot \\ \cdot \\ \cdot \\ \underline{0} \\ -g^j_{ij}\underline{a} \\ \underline{0} \\ \cdot \\ \cdot \\ \cdot \\ \underline{0} \end{pmatrix} \leftarrow j$$

where $\underline{1}_j$ is the $m\times 1$ vector with 1 in the j-th position and zeroes elsewhere.

Define $m\times m$ matrices

$$D_{i,j,k} = \text{diag}(0,0,\ldots,\underset{\substack{\uparrow\\(j-1)}}{0},s+\lambda_i+\lambda_j,s+\lambda_i+\lambda_{j+1},\ldots,s+\lambda_i+\lambda_k,0,\ldots,0)$$

for $i,j,k = 1,\ldots,m$; $j \le k$ and $m\times 1$ vectors

$$\underline{\varphi}_i = \hat{g}(s+\lambda_i)\begin{pmatrix}0\\ \cdot\\ \cdot\\ \cdot\\ 0\\ -1\\ 1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{pmatrix} \leftarrow i, \qquad \underline{\psi}_i = \hat{g}(s+\lambda_i)\begin{pmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\\ 2\\ 1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{pmatrix} \leftarrow i, \qquad \underline{\chi}_i = \hat{g}(s+\lambda_i)\begin{pmatrix}1\\ \cdot\\ \cdot\\ \cdot\\ 1\\ -1\\ 0\\ \cdot\\ \cdot\\ \cdot\\ 0\end{pmatrix} \leftarrow i$$

for $i = 1,2,\ldots,m$.

Define $m^2\times m^2$ matrices

$$R = \begin{bmatrix} D_{1,1,m} & -D_{2,3,m} & & & & \bigcirc \\ & D_{2,1,m} & -D_{3,4,m} & & & \\ & -D_{2,1,1} & D_{3,1,m} & \ddots & & \\ & & -D_{3,1,2} & \ddots & -D_{m-1,m,m} & \\ & & & \ddots & D_{m-1,1,m} & \\ & \bigcirc & & & -D_{m-1,1,m-2} & D_{m,1,m} \end{bmatrix}$$

$$T = \begin{bmatrix} D_{1,1,m}-\underline{\psi_1 a} & \underline{\varphi_2 a}-D_{2,3,m} & & & & & \bigcirc \\ \underline{\chi_1 a} & D_{2,1,m}-\underline{\psi_2 a} & \underline{\varphi_3 a}-D_{3,4,m} & & & & \\ & \underline{\chi_2 a}-D_{2,1,1} & D_{3,1,m}-\underline{\psi_3 a} & \ddots & & & \\ & & \underline{\chi_3 a}-D_{3,1,2} & \ddots & \ddots & & \\ & & & \ddots & \ddots & \underline{\varphi_{m-1} a}-D_{m-1,m,m} & \\ & & & & \ddots & D_{m-1,1,m}-\underline{\psi_{m-1} a} & \underline{\varphi_m a} \\ & \bigcirc & & & & \underline{\chi_{m-1} a}-D_{m-1,1,m-2} & D_{m,1,m}-\underline{\psi_m a} \end{bmatrix}.$$

Then

$$R[J-Q] = T \tag{5.34}$$

hence

$$[J-Q]^{-1} = T^{-1}R. \tag{5.35}$$

To see (5.34), first represent R as $(R_{ij})_{i,j=1,\ldots,m}$. Then

$$[R(J-Q)]_{ij} = R_{i,i-1}Q_{i-1,j} + R_{i,i}Q_{i,j} + R_{i,i+1}Q_{i+1,j}$$

where

$$R_{i,i-1} = -D_{i-1,1,i-2}$$

$$R_{i,i} = D_{i,1,m}$$

$$R_{i,i+1} = -D_{i+1,i+2,m}$$

and $D_{i,j,k}$ is defined to be $0_{m\times m}$ if any of i,j,k are ≤ 0 or $> m$.

Note that $R_{i,i-1}$ has non-zero elements only in the first i-2 diagonal positions, and $Q_{i-1,j}$ has non-zero elements only in row j if $j \neq i-1$. Hence if $j > i-1$; $R_{i,i-1}Q_{i-1,j} = 0_{m\times m}$. Similarly if $j < i+1$; $R_{i,i+1}Q_{i+1,j} = 0_{m\times m}$.

Case i): $j < i-1$.

$$[R(J-Q)]_{ij} = R_{i,i-1}Q_{i-1,j} + R_{i,i}Q_{i,j}$$

$$= (D_{i-1,1,i-2}g^{j}_{i-1,j} - D_{i,1,m}g^{j}_{i,j})\, \underline{1}_j\underline{a}$$

$$= 0$$

since $\underline{1}_j\underline{a}$ has non-zero elements only in the j-th row, and the j-th diagonal element of $(D_{i-1,1,i-2}g^{j}_{i-1,j}-D_{i,1,m}g^{j}_{i,j})$ is

$$(s+\lambda_{i-1}+\lambda_j)\,\frac{\hat{g}(s+\lambda_j)}{s+\lambda_{i-1}+\lambda_j} - (s+\lambda_i+\lambda_j)\,\frac{\hat{g}(s+\lambda_j)}{s+\lambda_i+\lambda_j} = 0.$$

Case ii): $j > i+1$, is similar to Case i).

Case iii): $j = i-1$.

$$[R(J-Q)]_{i,i-1} = R_{i,i-1}Q_{i-1,i-1} + R_{i,i}Q_{i,i-1}$$

$$= -D_{i-1,1,i-2}Q_{i-1,i-1} - D_{i,1,m}g^{i-1}_{i,i-1}\underline{1}_{i-1}\underline{a}\ .$$

But $Q_{i-1,i-1} = I - D^{-1}_{i-1,1,m}\underline{\psi}_{i-1}\underline{a}$, and so the above expression is

$$-D_{i-1,1,i-2} + [D_{i-1,1,i-2}D^{-1}_{i-1,1,m}\underline{\psi}_{i-1} - D_{i,1,m}g^{i-1}_{i,i-1}\underline{1}_{i-1}]\underline{a}$$

$$= -D_{i-1,1,i-2} + [\text{diag}(1,\ldots,\underset{\substack{\uparrow\\ i-2}}{1},0,\ldots,0)\underline{\psi}_{i-1}$$

$$- \frac{\hat{g}(s+\lambda_{i-1})}{s+\lambda_i+\lambda_{i-1}}(0,\ldots,0,\underset{\substack{\uparrow\\ i-1}}{s+\lambda_i+\lambda_{i-1}},0,\ldots,0)^T]\underline{a}$$

$$= -D_{i-1,1,i-2} + \hat{g}(s+\lambda_i)[(1,\ldots,\underset{\substack{\uparrow\\ i-2}}{1},0,\ldots,0)^T - (0,\ldots,0,\underset{\substack{\uparrow\\ i-1}}{1},0,\ldots,0)^T]\underline{a}$$

$$= -D_{i-1,1,i-2} + \underline{\chi}_{i-1}.$$

Case iv): $j = i+1$. As in Case iii),

$$[R(J-Q)]_{i,i+1} = \underline{\varphi}_{i+1} - D_{i+1,i+2,m}.$$

Case v): $j = i$.

$$[R(J-Q)]_{i,i} = R_{ii}Q_{ii}$$

$$= D_{i,1,m}[I - D^{-1}_{i,1,m}\underline{\psi}_i\underline{a}]$$

$$= D_{i,1,m} - \underline{\psi}_i\underline{a}.$$

Thus (5.34) is established. By (5.33) and (5.35),

$$\hat{h}(s) = \underline{a}\ \underline{G} + \underline{b}^T T^{-1}R\underline{\rho} \tag{5.36}$$

T may now be inverted block by block, by making use of the identity, valid for any $m\times m$ non-singular matrix A and $m\times 1$ vectors $\underline{x}$, $\underline{y}$:

$$[A-\underline{x}\,\underline{y}^T]^{-1} = A^{-1} + \frac{A^{-1}\underline{x}\,\underline{y}^T A^{-1}}{1-\underline{y}^T A^{-1}\underline{x}} \,. \tag{5.37}$$

In our case, A is diagonal. By repeatedly applying (5.37) one may obtain the inverse of a matrix of the form

$$A - \sum_{i=1}^{k} \underline{x}_i\,\underline{y}_i^T \,, \tag{5.38}$$

which form is preserved under the taking of sums, products and inverses.

Pre- and post-multiplying $T = (T_{ij})_{i,j=1,\ldots,m}$ by

$$T^{(1)} = \left[\begin{array}{cc|c} T_{11}^{-1} & 0 & 0 \\ -T_{21}T_{11}^{-1} & I & \\ \hline & 0 & I \end{array}\right] \quad \text{and} \quad T^{(2)} = \left[\begin{array}{cc|c} I & -T_{11}^{-1}T_{12} & 0 \\ 0 & I & \\ \hline & 0 & I \end{array}\right]$$

respectively yields

$$T^{-1} = T^{(2)}T_1^{-1}T^{(1)}$$

where

$$T_1 = \begin{bmatrix} I & 0 & 0 & \\ 0 & T_{22}-T_{21}T_{11}^{-1}T_{12} & T_{23} & \cdots \\ 0 & T_{32} & T_{33} & \cdots \\ \cdot & \cdot & T_{43} & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot \quad \cdot & \cdot & \\ \cdot & \cdot \quad \cdot & \cdot & \end{bmatrix}$$

This process may evidently be continued, since $T_{22}-T_{21}T_{11}^{-1}T_{12}$ is of the form (5.38) for some non-singular diagonal matrix A. The non-singularity follows from the fact that the product of the diagonal matrices involved in the definitions of T_{21} and T_{12} (more generally, $T_{i+1,i}$ and $T_{i,i+1}$) is $0_{m\times m}$, and hence the diagonal matrix in $T_{22}-T_{21}T_{11}^{-1}T_{12}$ is the same as that in T_{22}.

Continuing, then, one arrives at T^{-1} after m such calculations.

All of our results concerning the parallel up time p.d.f. are expressed in, or reducible to, the form

$$\hat{h}(s) = \hat{h}_0(s) + \underline{y}^T(s)[I-M(s)]^{-1}\underline{z}(s) \tag{5.39}$$

where

$$\hat{h}_n(s) = \underline{y}^T(s)M^{n-1}(s)\underline{z}(s) \qquad (n \geq 1) \;. \tag{5.40}$$

Although the application of numerical methods to the inversion of $[I-M(s)]$ would yield only finitely many values of $\hat{h}(s)$, various parameters of interest are obtainable upon inverting $[I-M(0)]$, a matrix of constants.

Writing T for the up time, $d\underline{y}^T(0)$ for $(\frac{d}{ds}y_i(s)|_{s=0})_i^T$, $dM(0)$ for $(\frac{d}{ds}M_{ij}(s)|_{s=0})_{i,j}$ etc., we have

$$E(T) = -\hat{h}'(0)$$

$$= -\{h_0'(0)+d\underline{y}^T(0)[I-M(0)]^{-1}\underline{z}(0)+\underline{y}^T(0)[I-M(0)]^{-1}[dM(0)][I-M(0)]^{-1}\underline{z}(0)$$

$$+ \underline{y}^T(0)[I-M(0)]^{-1}d\underline{z}(0)\} \tag{5.41}$$

with higher moments being obtained in a similar fashion.

The moments of the random variables N_i and $\tilde{N}_i$ may also be gotten from $[I-M(0)]^{-1}$. The probability generating function of N_0 is

$$E(x^{N_0}) = \sum_{n=0}^{\infty} x^n P(N_0=n)$$

$$= \hat{h}_0(0) + \sum_{n=1}^{\infty} x^n \hat{h}_n(0)$$

$$= \hat{h}_0(0) + x\underline{y}^T(0)[I-xM(0)]^{-1}z(0) \tag{5.42}$$

and so, e.g.,

$$E(N_0) = \underline{y}^T(0)[I-M(0)]^{-2}\underline{z}(0) \ . \tag{5.43}$$

CHAPTER 6. SPECIAL CASES -II.

In this chapter we obtain directly, without the necessity of taking limits, expressions for the Laplace transform of the parallel up time pdf given that the operating times are distributed with Erlangian density

$$f(t) = \frac{(\lambda t)^{m-1}\lambda e^{-\lambda t}}{(m-1)!} \qquad \lambda > 0;\ m = 1,2,\ldots \tag{6.1}$$

As in Chapter 5, we show that $\rho(t,u)$ has a form which is preserved when operated on by $*$ or by $\otimes$, hence which is inherited by $l(t,u)$. Substituting this form into the integral equation (3.9) and taking Laplace transforms provides a reduction to a linear system.

Substituting (6.1) into

$$\rho(t,u) = \int_0^u \frac{\overline{G}(v)}{\beta} \{f(u-v)f(t)+f(t-v)f(u)\}dv$$

gives

$$\rho(t,u) = \left(\frac{\lambda^m}{(m-1)!}\right)^2 e^{-\lambda t} \left\{t^{m-1} \int_0^u \frac{\overline{G}(v)}{\beta} (u-v)^{m-1} e^{-\lambda(u-v)} dv\right.$$

$$\left. + u^{m-1} \int_0^u \frac{\overline{G}(v)}{\beta} (t-v)^{m-1} e^{-\lambda(u-v)} dv\right\} .$$

Writing t^{m-1} as $((t-u)+u)^{m-1}$, $(t-v)^{m-1}$ as $((t-u)+(u-v))^{m-1}$ and expanding,

$$\rho(t,u) = \left(\frac{\lambda^m}{(m-1)!}\right)^2 e^{-\lambda t}\left\{\sum_{i=0}^{m-1}\binom{m-1}{i}(t-u)^i u^{m-1-i}\int_0^u \frac{\overline{G}(v)}{\beta}(u-v)^{m-1}e^{-\lambda(u-v)}dv\right.$$

$$\left.+\sum_{i=0}^{m-1}\binom{m-1}{i}(t-u)^i u^{m-1}\int_0^u \frac{\overline{G}(v)}{\beta}(u-v)^{m-1-i}e^{-\lambda(u-v)}dv\right\}$$

$$= e^{-\lambda t}\sum_{i=0}^{m-1}(t-u)^i \rho_i(u) \tag{6.2}$$

where

$$\rho_i(u) = \left(\frac{\lambda^m}{(m-1)!}\right)^2\left\{\binom{m-1}{i}u^{m-1-i}\int_0^u \frac{\overline{G}(v)}{\beta}(u-v)^{m-1}e^{-\lambda(u-v)}dv\right.$$

$$\left.+\binom{m-1}{i}u^{m-1}\int_0^u \frac{\overline{G}(v)}{\beta}(u-v)^{m-1-i}e^{-\lambda(u-v)}dv\right\} . \tag{6.3}$$

If $\varphi(t,u)$ has the form exhibited in (6.2):

$$\varphi(t,u) = e^{-\lambda t}\sum_{i=0}^{m-1}(t-u)^i\varphi_i(u)$$

then

$$*\varphi(t,u) = \int_0^u f(u-v)\int_0^v g(v-w)\varphi(t,w)\,dw\,dv$$

$$= \frac{\lambda^m}{(m-1)!}e^{-\lambda t}\sum_{i=0}^{m-1}\int_0^u (u-v)^{m-1}e^{-\lambda(u-v)}\int_0^v g(v-w)(t-w)^i\varphi_i(w)\,dw\,dv .$$

Writing $(t-w)^i$ as $((t-v)+(v-w))^i$ and expanding,

$$*\varphi(t,u) = \frac{\lambda^m}{(m-1)!} e^{-\lambda t} \sum_{i=0}^{m-1} \sum_{j=0}^{i} \binom{i}{j} \int_0^u (u-v)^{m-1}(t-v)^j e^{-\lambda(u-v)}$$

$$\int_0^v g(v-w)(v-w)^{i-j} \varphi_i(w)\,dw\,dv \quad .$$

Writing $(t-v)^j$ as $((t-u)+(u-v))^j$, expanding and changing the order of summation, this becomes

$$\frac{\lambda^m}{(m-1)!} e^{-\lambda t} \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{k=0}^{j} \binom{i}{j}\binom{j}{k}(t-u)^k \int_0^u (u-v)^{m-1+j-k} e^{-\lambda(u-v)}$$

$$\int_0^v g(v-w)(v-w)^{i-j} \varphi_i(w)\,dw\,dv$$

$$= \frac{\lambda^m}{(m-1)!} e^{-\lambda t} \sum_{k=0}^{m-1} (t-u)^k \sum_{i=k}^{m-1} \sum_{j=k}^{i} \binom{i}{j}\binom{j}{k} \int_0^u (u-v)^{m-1+j-k} e^{-\lambda(u-v)}$$

$$\int_0^v g(v-w)(v-w)^{i-j} \varphi_i(w)\,dw\,dv$$

and so

$$*\varphi(t,u) = e^{-\lambda t} \sum_{k=0}^{m-1} (t-u)^k \, \psi_k(u) \tag{6.4}$$

where

$$\psi_k(u) = \frac{\lambda^m}{(m-1)!} \sum_{i=k}^{m-1} \sum_{j=k}^{i} \binom{i}{j} \binom{j}{k} \int_0^u (u-v)^{m-1+j-k} e^{-\lambda(u-v)}$$

$$\int_0^v g(v-w)(v-w)^{i-j} \varphi_i(w)\,dw\,dv \quad . \qquad (6.5)$$

Similarly,

$$\otimes\varphi(t,u) = e^{-\lambda t} \sum_{k=0}^{m-1} (t-u)^k \, \overline{\psi}_k(u) \qquad (6.6)$$

where

$$\overline{\psi}_k(u) = \frac{\lambda^m}{(m-1)!} \sum_{i=0}^{m-1} \sum_{j=0}^{i} \binom{i}{j} \binom{m-1}{k} \int_0^u (u-v)^{m-1+j-k} e^{-\lambda(u-v)}$$

$$\int_0^v g(v-w)(v-w)^{i-j} \varphi_i(w)\,dw\,dv \quad . \qquad (6.7)$$

Thus this form is inherited by $l(t,u)$:

$$l(t,u) = e^{-\lambda t} \sum_{i=0}^{m-1} (t-u)^i \varphi_i(u) \quad . \qquad (6.8)$$

Substituting (6.8) and (6.2) into (3.9), and applying (6.4) and (6.6),

$$e^{-\lambda t}\sum_{i=0}^{m-1}(t-u)^i\varphi_i(u) = e^{-\lambda t}\sum_{i=0}^{m-1}(t-u)^i\rho_i(u) + e^{-\lambda t}\sum_{k=0}^{m-1}(t-u)^k\psi_k(u)$$

$$+ e^{-\lambda t}\sum_{k=0}^{m-1}(t-u)^k\,\overline{\psi}_k(u) \quad . \tag{6.9}$$

Letting $t \downarrow u$,

$$\varphi_0(u) = \rho_0(u) + \psi_0(u) + \overline{\psi}_0(u) \quad .$$

Repeatedly differentiating (6.9) with respect to t, then letting $t \downarrow u$ shows that

$$\varphi_i(u) = \rho_i(u) + \psi_i(u) + \overline{\psi}_i(u) \qquad (i = 0,1,\ldots,m-1) \tag{6.10}$$

or, from (6.5) and (6.7),

$$\varphi_i(u) = \rho_i(u) + \frac{\lambda^m}{(m-1)!}\sum_{j=i}^{m-1}\sum_{k=i}^{j}\binom{j}{k}\binom{k}{i}\int_0^u (u-v)^{m-1+k-i}e^{-\lambda(u-v)}$$

$$\int_0^v g(v-w)(v-w)^{j-k}\varphi_j(w)\,dw\,dv + \frac{\lambda^m}{(m-1)!}\sum_{j=0}^{m-1}\sum_{k=0}^{j}\binom{j}{k}\binom{m-1}{i}$$

$$\int_0^u (u-v)^{m-1+k-i}e^{-\lambda(u-v)}\int_0^v g(v-w)(v-w)^{j-k}\varphi_j(w)\,dw\,dv \ ,$$

$$i = 0,1,\ldots,m-1 \ . \tag{6.11}$$

Since we seek

$$h(t) = f(t) \int_t^\infty \frac{\overline{G}(u)}{\beta}\, dv + \int_0^t \overline{G}(t-u)\, l(t,u)\, dv$$

$$= \frac{\lambda^m t^{m-1}}{(m-1)!} e^{-\lambda t} \int_t^\infty \frac{\overline{G}(u)}{\beta}\, dv + e^{-\lambda t} \sum_{i=0}^{m-1} \int_0^t (t-u)^i\, \overline{G}(t-u)\varphi_i(u)\, dv$$

which has Laplace transform

$$\hat{h}(s) = (-1)^{m-1} \frac{\lambda^m}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \frac{1 - \frac{\hat{\overline{G}}}{\beta}(s+\lambda)}{s+\lambda}$$

$$+ \sum_{i=0}^{m-1} (-1)^i \hat{\varphi}_i(s+\lambda) \frac{d^i}{ds^i} \hat{\overline{G}}(s+\lambda) \ , \tag{6.12}$$

we take Laplace transforms in (6.11) with parameter $s+\lambda$:

$$\hat{\varphi}_i(s+\lambda) = \hat{\rho}_i(s+\lambda) + \frac{\lambda^m}{(m-1)!} \left\{ \sum_{j=i}^{m-1} \sum_{k=i}^{j} \frac{\binom{j}{k}\binom{k}{i}(-1)^{j-k}(m-1+k-i)!\,\hat{g}^{(j-k)}_{(s+\lambda)}}{(s+2\lambda)^{m+k-i}} \hat{\varphi}_j(s+\lambda) \right.$$

$$\left. + \sum_{j=0}^{m-1} \sum_{k=0}^{j} \frac{\binom{j}{k}\binom{m-1}{i}(-1)^{j-k}(m-1+k-i)!\,\hat{g}^{(j-k)}_{(s+\lambda)}}{(s+2\lambda)^{m+k-i}} \hat{\varphi}_j(s+\lambda) \right\} . \tag{6.13}$$

The identity

$$\frac{d^j}{ds^j} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^k} = \sum_{i=0}^{j} \frac{(-1)^i \binom{j}{i} (k+i-1)!\,\hat{g}^{(j-i)}_{(s+\lambda)}}{(k-1)!\,(s+2\lambda)^{k+i}} \tag{6.14}$$

applied to (6.13) yields

$$\hat{\varphi}_i(s+\lambda) = \hat{\rho}_i(s+\lambda) + \lambda^m \sum_{j=i}^{m-1} \binom{j}{i} (-1)^{j-i} \frac{d^{j-i}}{ds^{j-i}} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^m} \hat{\varphi}_j(s+\lambda)$$

$$+ \lambda^m \sum_{j=0}^{m-1} \frac{(-1)^j}{i!} \frac{d^j}{ds^j} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m-i}} \hat{\varphi}_j(s+\lambda) \ , \qquad (6.15)$$

a linear system which may be expressed in matrix form in the following manner.

Define $(m\times 1)$ vectors

$$\underline{\varphi} = (\hat{\varphi}_0(s+\lambda),\ldots,\hat{\varphi}_{m-1}(s+\lambda))^T$$

$$\underline{\delta} = (\hat{\hat{G}}(s+\lambda), -\frac{d}{ds}\hat{\hat{G}}(s+\lambda),\ldots,(-1)^{m-1}\frac{d^{m-1}}{ds^{m-1}}\hat{\hat{G}}(s+\lambda))^T$$

$$\underline{\gamma} = (\hat{\rho}_0(s+\lambda),\ldots,\hat{\rho}_{m-1}(s+\lambda))^T$$

where, by (6.3)

$$\hat{\rho}_i(s+\lambda) = \frac{1}{\beta}\left(\frac{\lambda^m}{(m-1)!}\right)^2 \binom{m-1}{i} \left[(-1)^{m-1-i} \frac{d^{m-1-i}}{ds^{m-1-i}} \frac{(m-1)!\hat{\hat{G}}(s+\lambda)}{(s+2\lambda)^m}\right.$$

$$\left. + (-1)^{m-1} \frac{d^{m-1}}{ds^{m-1}} \frac{(m-1-i)!\hat{\hat{G}}(s+\lambda)}{(s+2\lambda)^{m-i}}\right] \ .$$

Define $(m\times m)$ matrices $U = (u_{ij})_{i,j=0,\ldots,m-1}$; $V = (v_{ij})_{i,j=0,\ldots,m-1}$ by

$$u_{ij} = \begin{cases} \binom{j}{i}(-1)^{j-i} \dfrac{d^{j-i}}{ds^{j-i}} \dfrac{\hat{g}(s+\lambda)}{(s+2\lambda)^m} & j \geq i \\ 0 & j < i \end{cases}$$

$$v_{ij} = \frac{(-1)}{i!} \frac{d^j}{ds^j} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m-i}} \quad .$$

Then (6.15) becomes

$$\underline{\varphi} = \underline{\gamma} + \lambda^m [U+V]\underline{\varphi}$$

so that

$$\underline{\varphi} = [I-\lambda^m(U+V)]^{-1}\underline{\gamma} \tag{6.16}$$

and by (6.12),

$$\hat{h}(s) = \frac{(-1)^{m-1}\lambda^m}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \frac{1 - \frac{\hat{G}}{\beta}(s+\lambda)}{s+\lambda} + \underline{\delta}^T[I-\lambda^m(U+V)]^{-1}\underline{\gamma} \quad . \tag{6.17}$$

Although the general form of $[I-\lambda^m(U+V)]^{-1}$ has not been found, an interesting decomposition exists. The matrix V, corresponding to the operation $\otimes$, may be transformed into U, corresponding to $*$, by multiplication on the left by a lower triangular matrix L satisfying $L^2 = I$.

To see this, first write V as

$$V = V^{(0)} = \begin{bmatrix} v_0^{(0)} \\ \cdot \\ \cdot \\ \cdot \\ v_{m-1}^{(0)} \end{bmatrix}$$

where $\underline{v}_i^{(0)} = (v_{i0},\ldots,v_{i,m-1})$.

Define

$$V^{(k+1)} = \begin{bmatrix} \underline{v}_0^{(k+1)} \\ \cdot \\ \cdot \\ \cdot \\ \underline{v}_{m-1}^{(k+1)} \end{bmatrix}$$

where

$$\underline{v}_i^{(k+1)} = \begin{cases} \underline{v}_i^{(k)} & i = 0,1,\ldots,k \\ \underline{v}_i^{(k)} - \frac{s+2\lambda}{i}\,\underline{v}_{i-1}^{(k)} & i = k+1,\ldots,m-1 \end{cases} \tag{6.18}$$

i.e., to obtain $V^{(k+1)}$ from $V^{(k)}$, we subtract $\frac{s+2\lambda}{i}$ times row i-1 from row i for $i = k+1,\ldots,m-1$. We claim that

$$\underline{v}_i^{(k+1)} = \begin{cases} \underline{v}_i^{(k)} & i = 0,1,\ldots,k \\ \left(\underline{v}_i^{(k+1)},\ldots,v_{i,m-1}^{(k+1)}\right) & i = k+1,\ldots,m-1 \end{cases}$$

where

$$v_{ij}^{(k+1)} = \begin{cases} 0 & j = 0,1,\ldots,k \\ \frac{(-1)^j\, j!}{(j-k-1)!\,i!}\,\frac{d^{j-k-1}}{ds^{j-k-1}}\,\frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m+k-i+1}} & j = k+1,\ldots,m-1. \end{cases} \tag{6.19}$$

Hence, since each row i is affected only by the first i transformations,

$$V^{(m-1)} = \begin{bmatrix} \underline{v}_0^{(0)} \\ \underline{v}_1^{(1)} \\ \cdot \\ \cdot \\ \cdot \\ \underline{v}_{m-1}^{(m-1)} \end{bmatrix}$$

where

$$\underline{v}_{i,j}^{(i)} = \begin{cases} 0 & j = 0,1,\dots,i-1 \\ (-1)^j \binom{j}{i} \dfrac{d^{j-i}}{ds^{j-i}} \dfrac{\hat{g}(s+\lambda)}{(s+2\lambda)^m} & \end{cases} \tag{6.20}$$

i.e.
$$V^{(m-1)} = \mathrm{diag}(1,-1,\dots,(-1)^{m-1})U \quad . \tag{6.21}$$

We establish (6.19) by induction on k. The identity (6.14) is rewritten for convenience as

$$\frac{d^j}{ds^j} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m-i}} = \sum_{n=0}^{j} \frac{(-1)^n \binom{j}{n} (m-i+n-1)!\,\hat{g}^{(j-n)}_{(s+\lambda)}}{(m-i-1)!\,(s+2\lambda)^{m-i+n}} \quad . \tag{6.22}$$

The case $k = 0$ is essentially the same as the inductive step, and so we give only the latter here. Assume the truth of (6.19) for k-1. Then for k,

$$\underline{v}_i^{(k+1)} = \underline{v}_i^{(k)} \qquad\qquad i = 0,1,\dots,k$$

by definition. For $i > k$,

$$v_{ij}^{(k+1)} = v_{ij}^{(k)} - \frac{(s+2\lambda)}{i} v_{i-1,j}^{(k)} \tag{6.23}$$

which is 0 for $j = 0,1,\ldots,k-1$ by the induction hypothesis. For $j = k,\ldots,m-1$, (6.23) is, again by the induction hypothesis,

$$v_{ij}^{(k+1)} = \frac{(-1)^j j!}{(j-k)!i!} \frac{d^{j-k}}{ds^{j-k}} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m+k-i}}$$

$$- \frac{(s+2\lambda)}{i} \frac{(-1)^j j!}{(j-k)!(i-1)!} \frac{d^{j-k}}{ds^{j-k}} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m+k-i+1}}$$

$$= \frac{(-1)^j j!}{(j-k)!i!} \left[\frac{d^{j-k}}{ds^{j-k}} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m+k-i}} - (s+2\lambda) \frac{d^{j-k}}{ds^{j-k}} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m+k-i+1}}\right] \quad (6.24)$$

which is 0 for $j = k$. For $j > k$, (6.24) is, applying (6.22),

$$v_{ij}^{(k+1)} = \frac{(-1)^j j!}{(j-k)!i!} \left[\sum_{n=0}^{j-k} \left\{\frac{(-1)^n \binom{j-k}{n} (m+k-i+n-1)!\hat{g}^{(j-k-n)}_{(s+\lambda)}}{(m+k-i-1)!(s+2\lambda)^{m+k-i+n}}\right.\right.$$

$$\left.\left. - \frac{(-1)^n \binom{j-k}{n} (m+k-i+n)!\hat{g}^{(j-k-n)}_{(s+\lambda)}}{(m+k-i)!(s+2\lambda)^{m+k-i+n}}\right\}\right]$$

$$= \frac{(-1)^j j!}{(j-k)!i!} \sum_{n=1}^{j-k} \frac{(-1)^{n-1} \binom{j-k}{n} n(m+k-i+n-1)!}{(m+k-i)!(s+2\lambda)^{m+k-i+n}} \hat{g}^{(j-k-n)}_{(s+\lambda)}$$

$$= \frac{(-1)^j j!}{(j-k-1)!i!} \sum_{n=1}^{j-k} \frac{(-1)^{n-1} \binom{j-k-1}{n-1} (m+k-i+n-1)!}{(m+k-i)!(s+2\lambda)^{m+k-i+n}} \hat{g}^{(j-k-n)}_{(s+\lambda)}$$

$$= \frac{(-1)^j\, j!}{(j-k-1)!\,i!} \sum_{n=0}^{j-k-1} \frac{(-1)^n \binom{j-k-1}{n} (m+k-i+n)!}{(m+k-i)!\,(s+2\lambda)^{m+k-i+n+}} \hat{g}^{(j-k-n-1)}_{(s+\lambda)}$$

$$= \frac{(-1)^j\, j!}{(j-k-1)!\,i!} \frac{d^{j-k-1}}{ds^{j-k-1}} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m+k-i+1}}$$

by (6.22), thus proving (6.19).

Each of the transformations $V^{(k)} \to V^{(k+1)}$ is effected by the multiplication of $V^{(k)}$ on the left by $L^{(k+1)}$, where

$$L^{(k+1)}_{i,j} = \begin{cases} -\dfrac{(s+2\lambda)}{i} & i = k+1,\ldots,m-1;\ j = i-1 \\ 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

By (6.21),

$$U = \text{diag}\,(1,-1,\ldots,(-1)^{m-1})L^{(m-1)}L^{(m-2)}\ldots L^{(1)}V\,. \qquad (6.25)$$

A straightforward but tedious induction establishes that for $k = 1,2,\ldots,m-1$;

$$(L^{(k)}L^{(k-1)}\ldots L^{(1)})_{ij} = \begin{cases} 0 & i=0,1,\ldots,k;j>i \\ \dfrac{(-1)^{i-j}(s+2\lambda)^{i-j}}{(i-j)!} & i=0,1,\ldots,k;j\le i \\ 0 & i=k+1,\ldots,m-1;j<i-k,j>i \\ \dfrac{(-1)^{i-j}(s+2\lambda)^{i-j}j!\binom{k}{i-j}}{i!} & i=k+1,\ldots,m-1;j=i-k,\ldots,i. \end{cases}$$

hence

$$(L^{(m-1)} \cdots L^{(1)})_{ij} = \begin{cases} \dfrac{(-1)^{i-j}(s+2\lambda)^{i-j}}{(i-j)!} & j \le i \\ 0 & j > i. \end{cases}$$

By (6.25), putting $L = \operatorname{diag}(1,-1,\ldots,(-1)^{m-1})L^{(m-1)}\ldots L^{(1)}$,

$$U = LV$$

where

$$L_{ij} = \begin{cases} \dfrac{(-1)^{j}(s+2\lambda)^{i-j}}{(i-j)!} & j \le i \\ 0 & j > i \,. \end{cases}$$

That $L^2 = I$ is easily verified. Hence

$$V = LU = UV^{-1}U \tag{6.26}$$

and

$$(I-\lambda^m(U+V))^{-1} = (I-\lambda^m(I+L)U)^{-1}. \tag{6.27}$$

One can also express (6.27) as

$$I + \lambda^m[I+L][I-\lambda^m U(I+L)]^{-1}U \ ,$$

hoping that $[I-\lambda^m U(I+L)]$ might be easier to invert. This seems unlikely. By considering the sequence of column operations corresponding to right-multiplication by L, one can show that

$$(UL)_{ij} = \frac{(-1)^j\, m!}{(m-1-j)!} \sum_{k=\max(0,i-j)}^{m-1-j} \frac{\binom{m-1-j}{k}\binom{j+k}{i}}{(j+k+1)!} (-1)^{j+k-i} \frac{d^{j+k-i}}{ds^{j+k-i}} \frac{\hat{g}(s+\lambda)}{(s+2\lambda)^{m-k}}$$

$$= (-1)^j \sum_{k=\ell}^{m-1} \binom{m}{k+1}\binom{k}{i} (LU)_{k-j,k-i} \qquad (6.28)$$

where $\ell = \max(i,j)$.

Substituting (6.8) into the relevant expressions in Chapter Three yields the series up and down, and parallel down, one repair p.d.f.'s. After some manipulation, we get

$$k(v) = \frac{1}{\tilde{E}_0} [k_0(v) + \sum_{i=0}^{m-1} (-1)^i \hat{\phi}_i(\lambda)\{\overline{F}(v) \frac{d^i}{d\lambda^i} e^{-\lambda v}\hat{g}(\lambda)$$

$$+ f(v) \frac{d^i}{d\lambda^i} e^{-\lambda v}\hat{G}(\lambda)\}] , \qquad (6.29)$$

where

$$\tilde{E}_0 = \sum_{i=0}^{m-1} \frac{\hat{\phi}_i(\lambda) i!}{\lambda^{i+1}} . \qquad (6.30)$$

The $\hat{\phi}_i(\lambda)$ may be gotten from (6.16), upon replacing $\overline{G}/\beta$ by g and setting $s = 0$. From

$$\hat{\psi}(s) = \sum_{i=0}^{m-1} (-1)^i \hat{\phi}_i(\lambda) \frac{d^i}{d\lambda^i} \frac{\hat{g}(s)-\hat{g}(s+\lambda)}{\lambda} \qquad (6.31)$$

$$\chi(v) = e^{-\lambda v} \sum_{i=0}^{m-1} v^i \hat{\phi}_i(\lambda) \qquad (6.32)$$

we get $\hat{d}(s)$ and $p(t)$ as well.

Some special cases.

For $m = 1$, we get after simplification

$$\hat{h}(s) = \frac{\lambda}{s+\lambda}\left(1 - \frac{\hat{\bar{G}}}{\beta}(s+\lambda)\right) + \frac{2\lambda^2\hat{\bar{G}}^2(s+\lambda)}{\beta(s+2\lambda(s+\lambda)\hat{\bar{G}}(s+\lambda))} \tag{6.33}$$

in agreement with (5.24).

For $m = 2$,

$$\underline{\gamma} = \frac{\lambda^4}{\beta(s+2\lambda)^3}\left(2\hat{\bar{G}}(s+\lambda)-(s+2\lambda)\hat{\bar{G}}'(s+\lambda)\right)\left(\frac{2}{s+2\lambda},1\right)^T$$

$$\underline{\delta} = \left(\hat{\bar{G}}(s+\lambda),-\hat{\bar{G}}'(s+\lambda)\right)^T \quad .$$

$$(I-\lambda^2(U+V))^{-1} = \frac{\begin{pmatrix} 1 - \dfrac{2\lambda^2\hat{g}(s+\lambda)}{(s+2\lambda)^2} + \dfrac{\lambda^2\hat{g}'(s+\lambda)}{s+2\lambda} & \dfrac{4\lambda^2\hat{g}(s+\lambda)}{(s+2\lambda)^3} - \dfrac{2\lambda^2\hat{g}'(s+\lambda)}{(s+2\lambda)^2} \\ \dfrac{\lambda^2\hat{g}(s+\lambda)}{s+2\lambda} & 1 - \dfrac{2\lambda^2\hat{g}(s+\lambda)}{(s+2\lambda)^2} \end{pmatrix}}{1 - \left(\dfrac{4\lambda^2\hat{g}(s+\lambda)}{(s+2\lambda)^2} - \dfrac{\lambda^2\hat{g}'(s+\lambda)}{s+2\lambda}\right)}$$

and so

$$\hat{h}(s) = -\lambda^2 \frac{d}{ds} \frac{1 - \frac{\hat{\bar{G}}}{\beta}(s+\lambda)}{s+\lambda} + \frac{\frac{1}{\beta}\left[2\left(\frac{\lambda}{s+2\lambda}\right)^2 \hat{\bar{G}}(s+\lambda) - \frac{\lambda^2\hat{\bar{G}}'(s+\lambda)}{s+2\lambda}\right]^2}{1 - \left\{\frac{4\lambda^2\hat{g}(s+\lambda)}{(s+2\lambda)^2} - \frac{\lambda^2\hat{g}'(s+\lambda)}{s+2\lambda}\right\}} \tag{6.34}$$

agreeing with (5.27).

For $m = 3$ the calculations are already extremely cumbersome:

$$\hat{h}(s) = \frac{\lambda^3}{2} \frac{d^2}{ds^2} \frac{1 - \frac{\hat{\bar{G}}}{\beta}(s+\lambda)}{s+\lambda} + \underline{\delta}^T (I - \lambda^3(U+V))^{-1} \underline{\gamma}$$

where, writing g for $\hat{g}(s+\lambda)$, $\bar{G}$ for $\hat{\bar{G}}(s+\lambda)$:

$$\underline{\delta} = (\bar{G}, -\bar{G}', \bar{G}'')^T$$

$$\underline{\gamma} = \frac{\lambda^6}{4\beta} \begin{pmatrix} \frac{48\bar{G}}{(s+2\lambda)^5} - \frac{24\bar{G}'}{(s+2\lambda)^4} + \frac{4\bar{G}''}{(s+2\lambda)^3} \\ \frac{24\bar{G}}{(s+2\lambda)^4} - \frac{12\bar{G}'}{(s+2\lambda)^3} + \frac{2\bar{G}''}{(s+2\lambda)^2} \\ \frac{4\bar{G}}{(s+2\lambda)^3} - \frac{2\bar{G}'}{(s+2\lambda)^2} + \frac{\bar{G}''}{(s+2\lambda)} \end{pmatrix}$$

$$I-\lambda^3(U+V) = \begin{pmatrix} 1-\dfrac{2\lambda^3 g}{(s+2\lambda)^3} & \dfrac{-6\lambda^3 g}{(s+2\lambda)^4}+\dfrac{2\lambda^3 g'}{(s+2\lambda)^3} & \dfrac{-24\lambda^3 g}{(s+2\lambda)^5}+\dfrac{12\lambda^3 g'}{(s+2\lambda)^4}-\dfrac{2\lambda^3 g''}{(s+2\lambda)^3} \\ \dfrac{-\lambda^3 g}{(s+2\lambda)^2} & 1-\dfrac{3\lambda^3 g}{(s+2\lambda)^3}+\dfrac{\lambda^3 g'}{(s+2\lambda)^2} & \dfrac{-12\lambda^3 g}{(s+2\lambda)^4}+\dfrac{6\lambda^3 g'}{(s+2\lambda)^3}-\dfrac{\lambda^3 g''}{(s+2\lambda)^2} \\ \dfrac{-\lambda^3 g}{2(s+2\lambda)} & \dfrac{-\lambda^3 g}{2(s+2\lambda)^2}+\dfrac{\lambda^3 g'}{2(s+2\lambda)} & 1-\dfrac{2\lambda^3 g}{(s+2\lambda)^3}+\dfrac{\lambda^3 g'}{(s+2\lambda)^2}-\dfrac{\lambda^3 g''}{2(s+2\lambda)} \end{pmatrix}$$

$$(I-\lambda^3(U+V))^{-1} = \frac{[\underline{c}_0,\underline{c}_1,\underline{c}_2]}{D}$$

where

$$\underline{c}_0 = \begin{pmatrix} 1-\dfrac{4\lambda^3 g}{(s+2\lambda)^3}+\dfrac{\lambda^3 g'}{(s+2\lambda)^2}-\dfrac{\lambda^3 g''}{2(s+2\lambda)}+\dfrac{4\lambda^6 gg'}{(s+2\lambda)^5}+\dfrac{\lambda^6 gg''}{(s+2\lambda)^4}-\dfrac{2\lambda^6(g')^2}{(s+2\lambda)^4} \\ \dfrac{\lambda^3 g}{(s+2\lambda)^2}+\dfrac{4\lambda^6(g)^2}{(s+2\lambda)^5}-\dfrac{2\lambda^6 gg'}{(s+2\lambda)^4} \\ \dfrac{\lambda^3 g}{2(s+2\lambda)}-\dfrac{\lambda^6(g)^2}{(s+2\lambda)^4} \end{pmatrix}$$

$$\underline{c}_1 = \begin{bmatrix} \dfrac{6\lambda^3 g}{(s+2\lambda)^4} - \dfrac{2\lambda^3 g'}{(s+2\lambda)^3} + \dfrac{4\lambda^6 (g')^2}{(s+2\lambda)^5} - \dfrac{8\lambda^6 g g'}{(s+2\lambda)^6} - \dfrac{2\lambda^6 g g''}{(s+2\lambda)^5} \\[2ex] 1 - \dfrac{4\lambda^3 g}{(s+2\lambda)^3} + \dfrac{\lambda^3 g'}{(s+2\lambda)^2} - \dfrac{\lambda^3 g''}{2(s+2\lambda)} - \dfrac{8\lambda^6 (g)^2}{(s+2\lambda)^6} + \dfrac{4\lambda^6 g g'}{(s+2\lambda)^5} \\[2ex] \dfrac{\lambda^3 g}{2(s+2\lambda)^2} - \dfrac{\lambda^3 g'}{2(s+2\lambda)} + \dfrac{2\lambda^6 (g)^2}{(s+2\lambda)^5} \end{bmatrix}$$

$$\underline{c}_2 = \begin{bmatrix} \dfrac{24\lambda^3 g}{(s+2\lambda)^5} - \dfrac{12\lambda^3 g'}{(s+2\lambda)^4} + \dfrac{2\lambda^3 g''}{(s+2\lambda)^3} \\[2ex] \dfrac{12\lambda^3 g}{(s+2\lambda)^4} - \dfrac{6\lambda^3 g'}{(s+2\lambda)^3} + \dfrac{\lambda^3 g''}{(s+2\lambda)^2} \\[2ex] 1 - \dfrac{5\lambda^3 g}{(s+2\lambda)^3} + \dfrac{\lambda^3 g'}{(s+2\lambda)^2} \end{bmatrix}$$

$$D = 1 - \left[\frac{7\lambda^3 g}{(s+2\lambda)^3} - \frac{2\lambda^3 g'}{(s+2\lambda)^2} + \frac{\lambda^3 g''}{2(s+2\lambda)} + \frac{8\lambda^6 (g)^2}{(s+2\lambda)^6} - \frac{8\lambda^6 g g'}{(s+2\lambda)^5} + \frac{2\lambda^6 (g')^2}{(s+2\lambda)^4} - \frac{\lambda^6 g g''}{(s+2\lambda)^4} \right]$$

Expanding the bilinear form offers no simplification.

CHAPTER 7. NUMERICAL APPROXIMATIONS

In the previous chapters, we have obtained several of the mean times related to the parallel and series systems, and have obtained the higher moments and the Laplace transforms of the relevant densities in two fairly broad classes of special cases. As interesting as these may be, the complexity of many of the expressions renders all but negligible their intuitive content.

In this chapter, we exhibit graphs of the (approximate) parallel up time pdf in four cases. We evaluate, numerically, the first one to five terms of the series of integrals defining h. The program is entered in the Appendix. As a criterion of convergence, we use the ratio of the mean operating time to the mean repair time, seen in Chapter 3 to be equal to E_0, the mean number of passages through state 0 during the up time. If E_0 is large, one must evaluate a large number of terms of the series before capturing a significant portion of the mass of h. This, together with the large amounts of computing time required to perform numerical integration, dictated our choices of E_0.

The results are quite surprising. In the mathematical analysis, h_0 has always been treated as a separate entity, playing as it does no role in the inductive process defining h. This distinctiveness appears to be a major determining factor in the shape of h as E_0 becomes relatively large. Denote $h(t)-h_0(t)$ by $h_R(t)$. If E_0 is small, the dominant term of the series is h_0, whose shape is in turn largely determined by that of the operating pdf. As E_0 increases, so does the mass of h_R. This mass is concentrated to the right of that of h_0 which, in the examples considered here, remains sufficiently significant to impart a bimodal shape to h.

This bimodality is perhaps best accounted for by considering the hazard rate, defined by

$$r(t) = \lim_{\Delta t \to 0} \frac{\text{P(System fails in } (t, t+\Delta t) \,|\, \text{System survives until } t)}{\Delta t}$$

$$= \frac{h(t)}{\overline{H}(t)} \,.$$

The hazard rate is a measure of the susceptibility of the system to failure at time t, and ought to increase at a faster rate when the system is in state 1 than when it is in state 0. Suppose then that $E_0 > 1$, so that the system ought to enter state 0 at least once before failure. In the interval between $t = 0$ and the expected time required for the system to be leaving state 0 for the first time, $r(t)$ could be expected to be concave. For t small, $h(t) = r(t)\overline{H}(t) \approx r(t)$, and this concavity may well be inherited by h when h_1 is added to h_0. The addition of the rest of the mass of h_R, less affected by $r(t)$ for t large, would then yield a second, major mode. Indeed, the graphs show that when the bimodality exists ($E_0 = 2, 4$) it is almost entirely created by the addition of h_1 to h_0.

In each case, we use as the operating pdf $f(t)$ a linear combination of exponential pdf's, and as the repair pdf $g(t)$ an Erlangian density. In all cases, f has a convenient factorization in terms of hyperbolic sines and cosines. In the fourth case, the coefficients of f are chosen in such a way that f itself very closely approximates an Erlangian density. The quality of the approximation is illustrated in Figures 7.1 and 7.2.

In these figures, $f(t) = a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t}$, with

$$\lambda = 2, \qquad \lambda_1 = \lambda+\varepsilon, \qquad \lambda_2 = \lambda-\varepsilon$$

$$a_1 = \frac{\lambda_1 \lambda_2}{\lambda_2-\lambda_1}, \qquad a_2 = -a_1.$$

As $\varepsilon \to 0$, $f(t) = \frac{(\lambda^2-\varepsilon^2)}{\varepsilon} e^{-\lambda t} \sinh \varepsilon t \to \lambda^2 t e^{-\lambda t} = g(t)$. In Figure 7.1, $\varepsilon = 1$ and the approximation is poor. In Figure 7.2, however, $\varepsilon = .01$ and the two functions are so close as to be virtually indistinguishable. Below them is plotted the absolute value of their difference, increased in magnitude by a factor of 10,000.

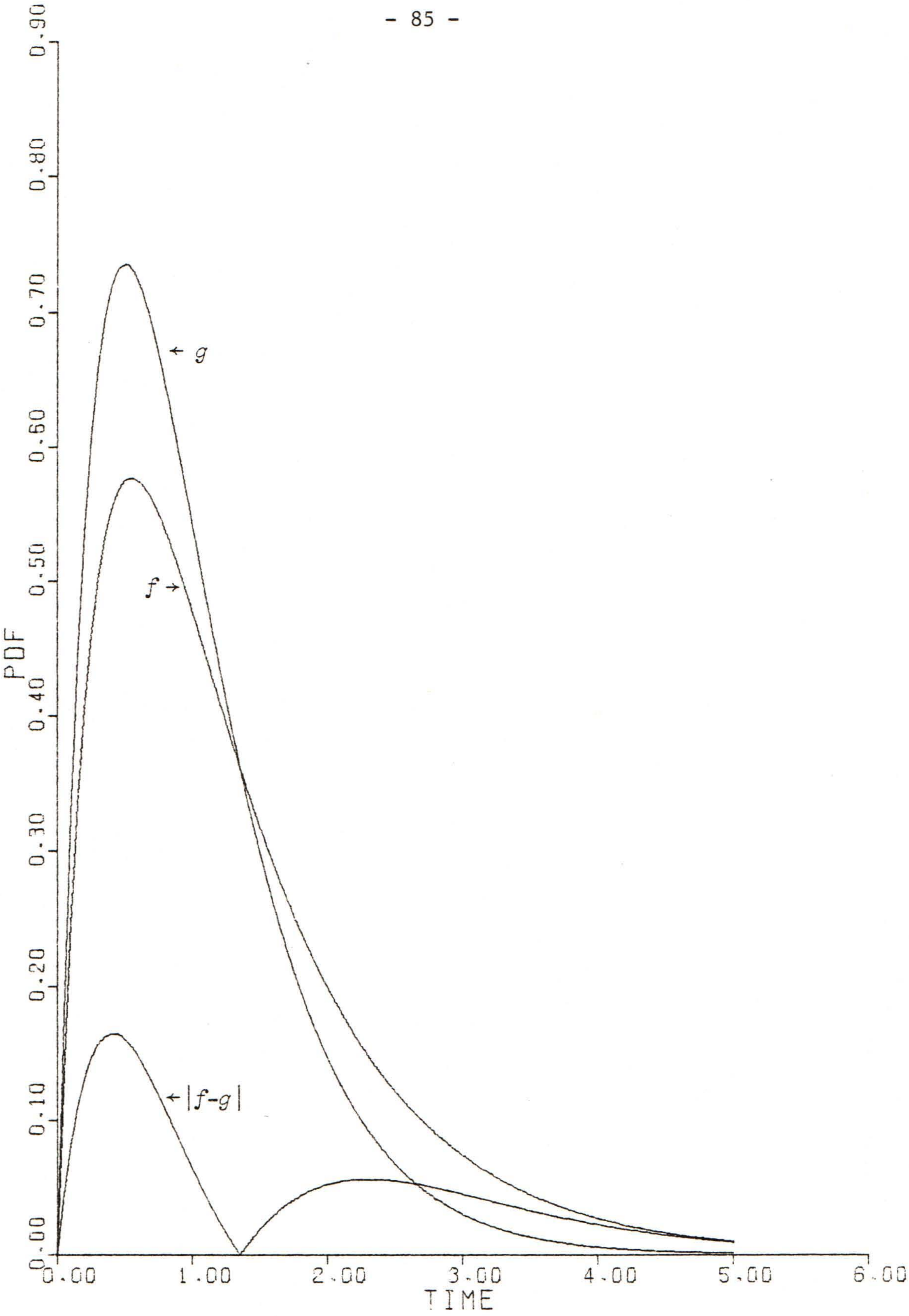

0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
← g
f →
← |f-g|
0.00
1.00
2.00
3.00
4.00
5.00
6.00
TIME

Figure 7.1

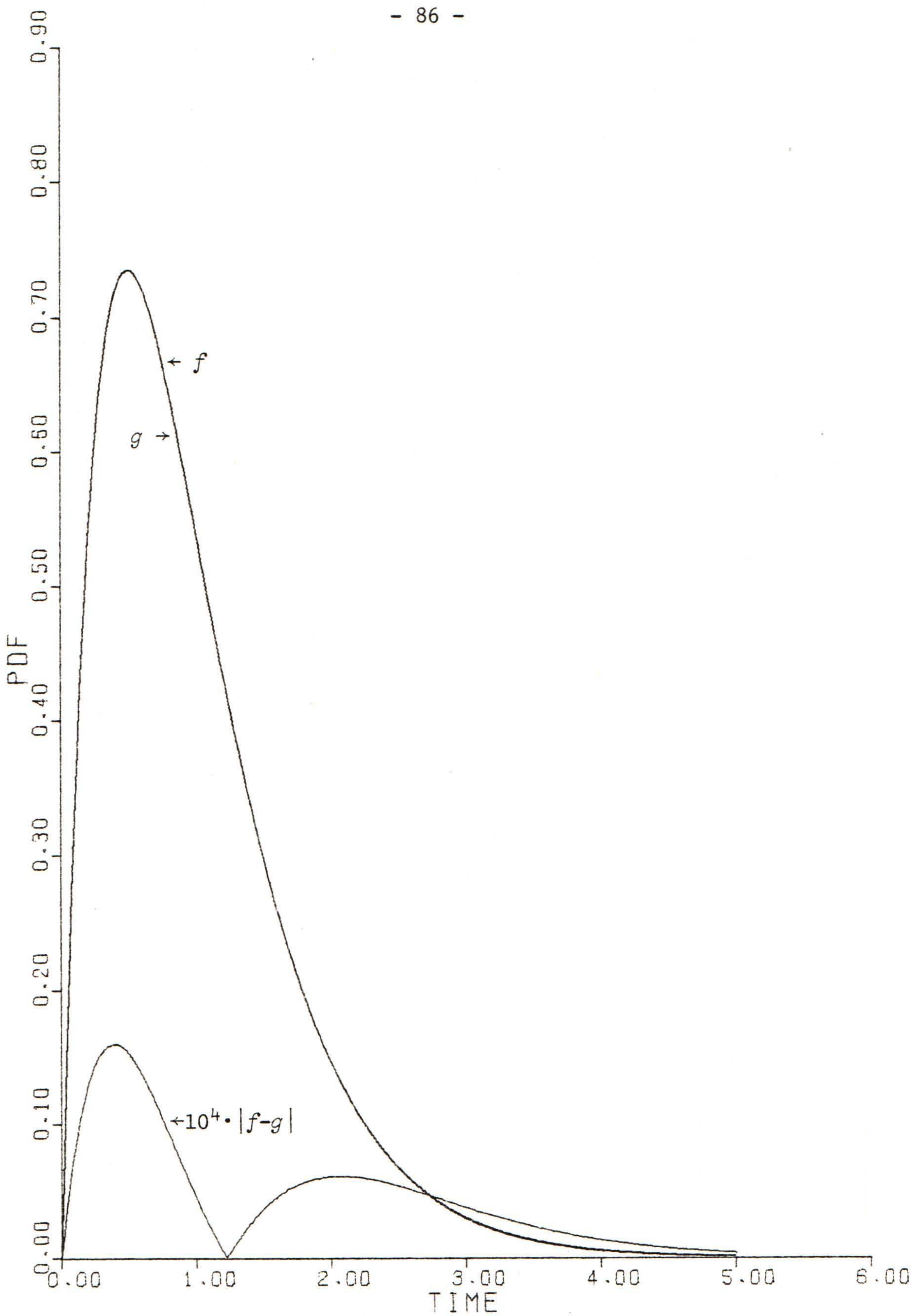

0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
← f
g →
←10⁴·|f-g|
0.00
1.00
2.00
3.00
4.00
5.00
6.00
TIME

Figure 7.2

Figures 7.3 and 7.4

In these figures, $g(t)$ is Erlangian in 3 stages, with mean 2 and variance 4/3; $f(t) = 50.4e^{-8t}\sinh^3 1.6t\cosh 1.6t$, with mean .992 and variance .447, approximately. Thus $E_0 \approx .446$ and one would expect h_0 and h_1 to be the dominant terms of the series. In Figure 7.3 we have used only these 2 terms, while in Figure 7.4 the third term is incorporated.

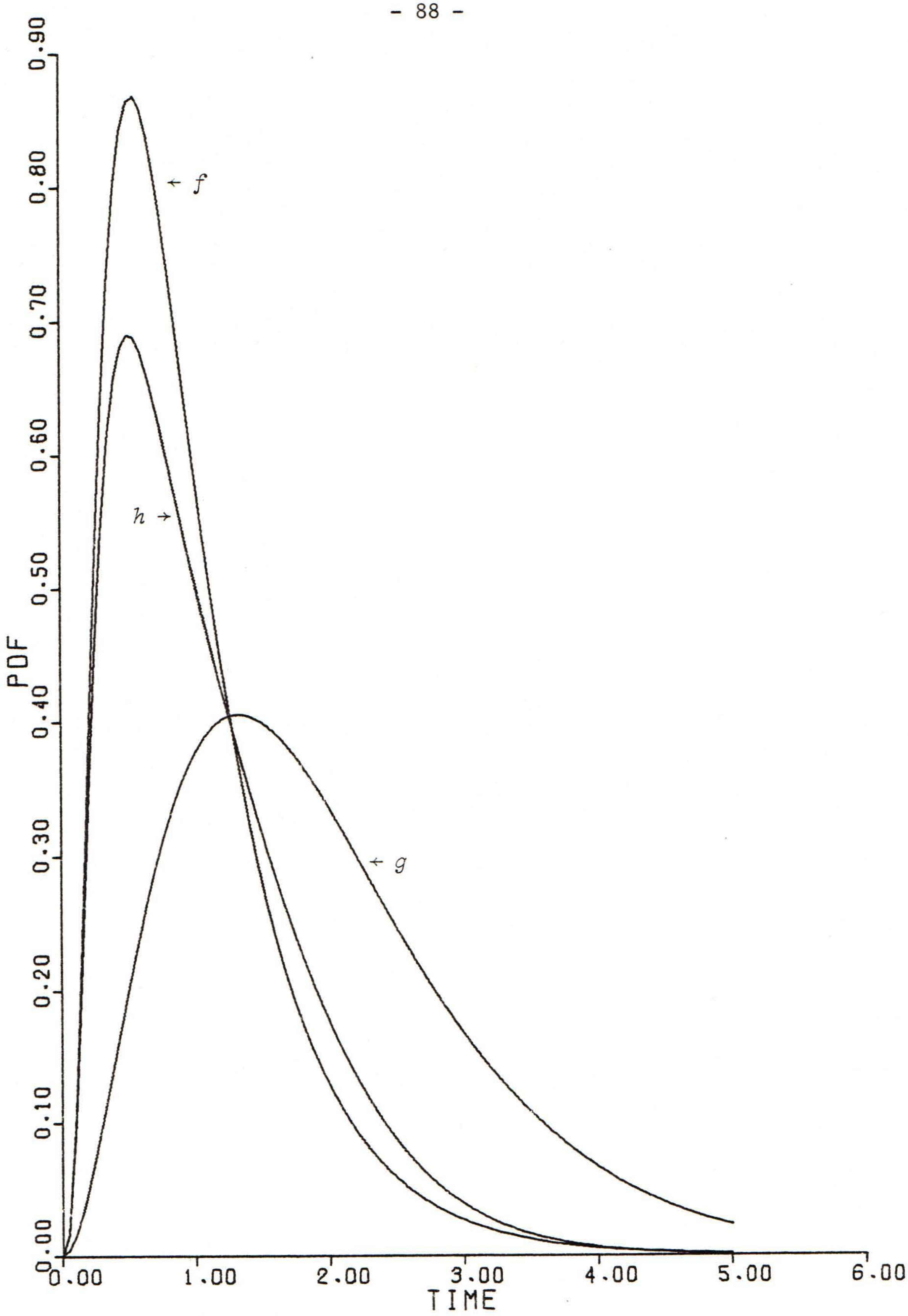
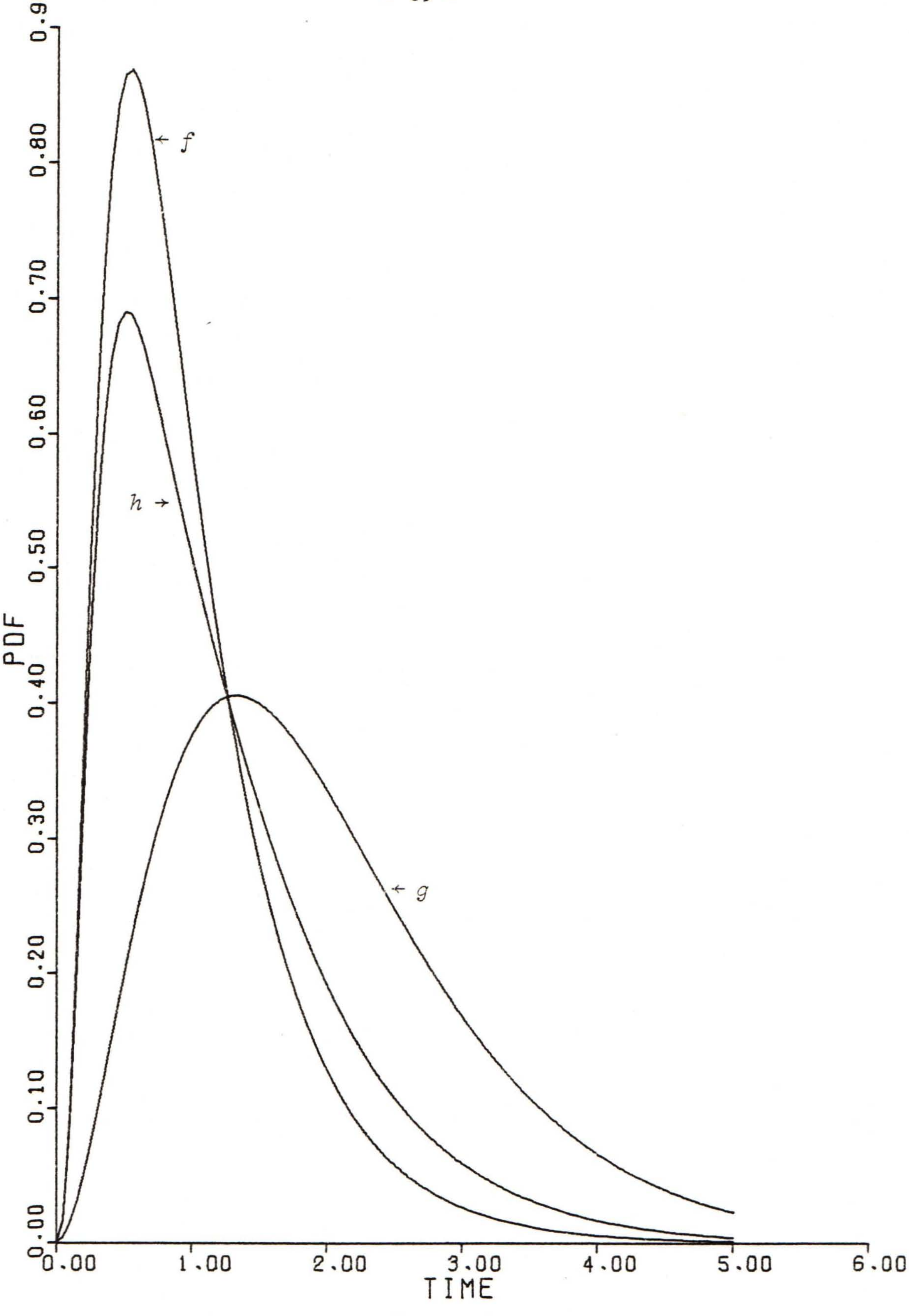

0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
← f
h →
← g
0.00
1.00
2.00
3.00
4.00
5.00
6.00
TIME

Figure 7.3



0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
f
h
g
0.00
1.00
2.00
3.00
4.00
5.00
6.00
TIME

Figure 7.4

Figures 7.5 and 7.6

Here, $g(t)$ is Erlangian in 2 stages, mean .25, variance .03125. The operating pdf factors as

$$\frac{400-\varepsilon^2}{\varepsilon} e^{-20t} \sinh \varepsilon t, \qquad \text{with } \varepsilon = 15.493,$$

$$= 10.325 e^{-20t} \sinh 15.493t, \text{ approximately.}$$

The operating time has mean .25005 and variance .05002, so that $E_0 = 1.0002$. We again used two and three terms of the series. As seen in Figure 7.6, the contribution of the third term to the tail is somewhat more significant than it was in the previous case.

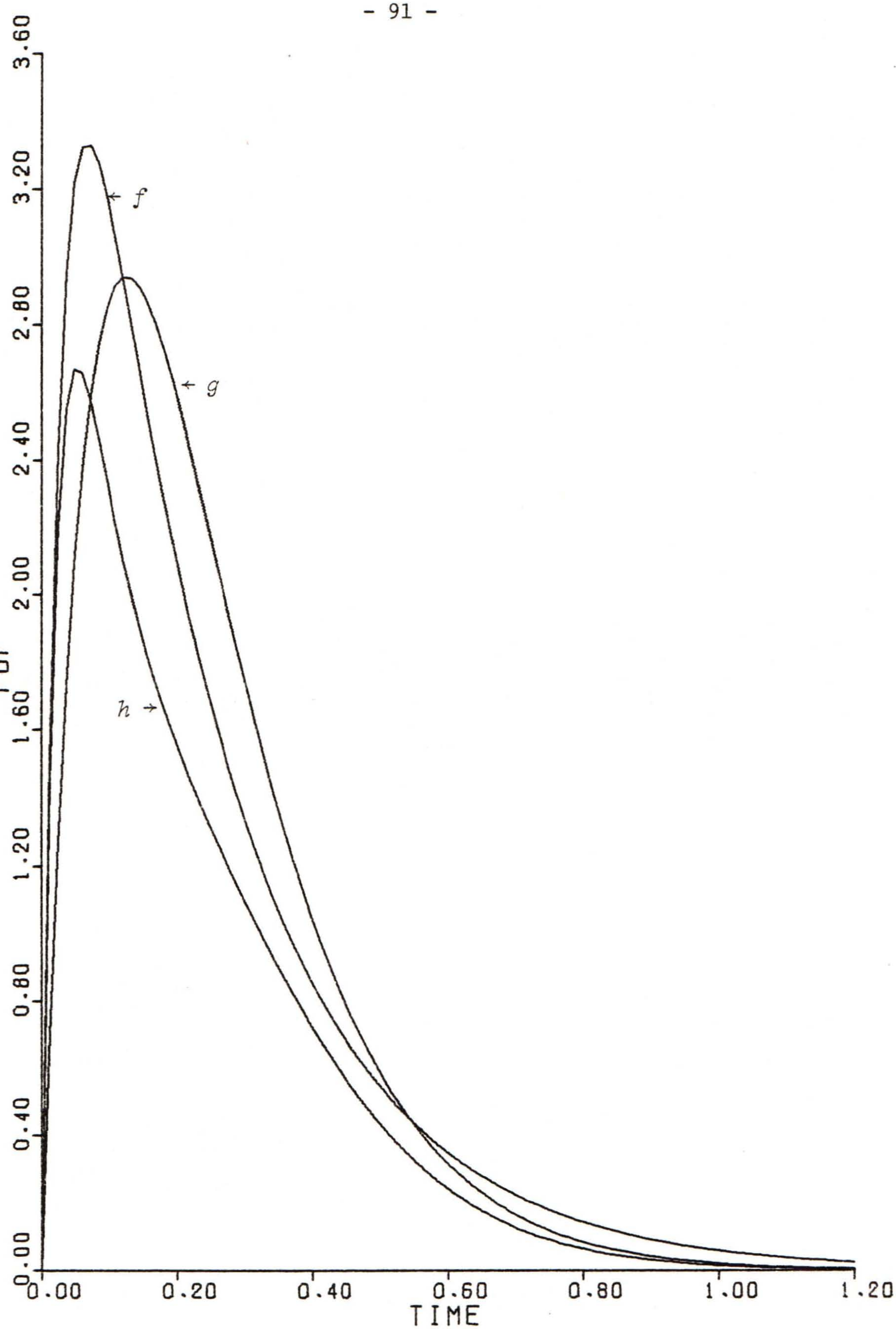

3.60
3.20
2.80
2.40
2.00
1.60
1.20
0.80
0.40
0.00
PDF
f
g
h
0.00
0.20
0.40
0.60
0.80
1.00
1.20
TIME

Figure 7.5

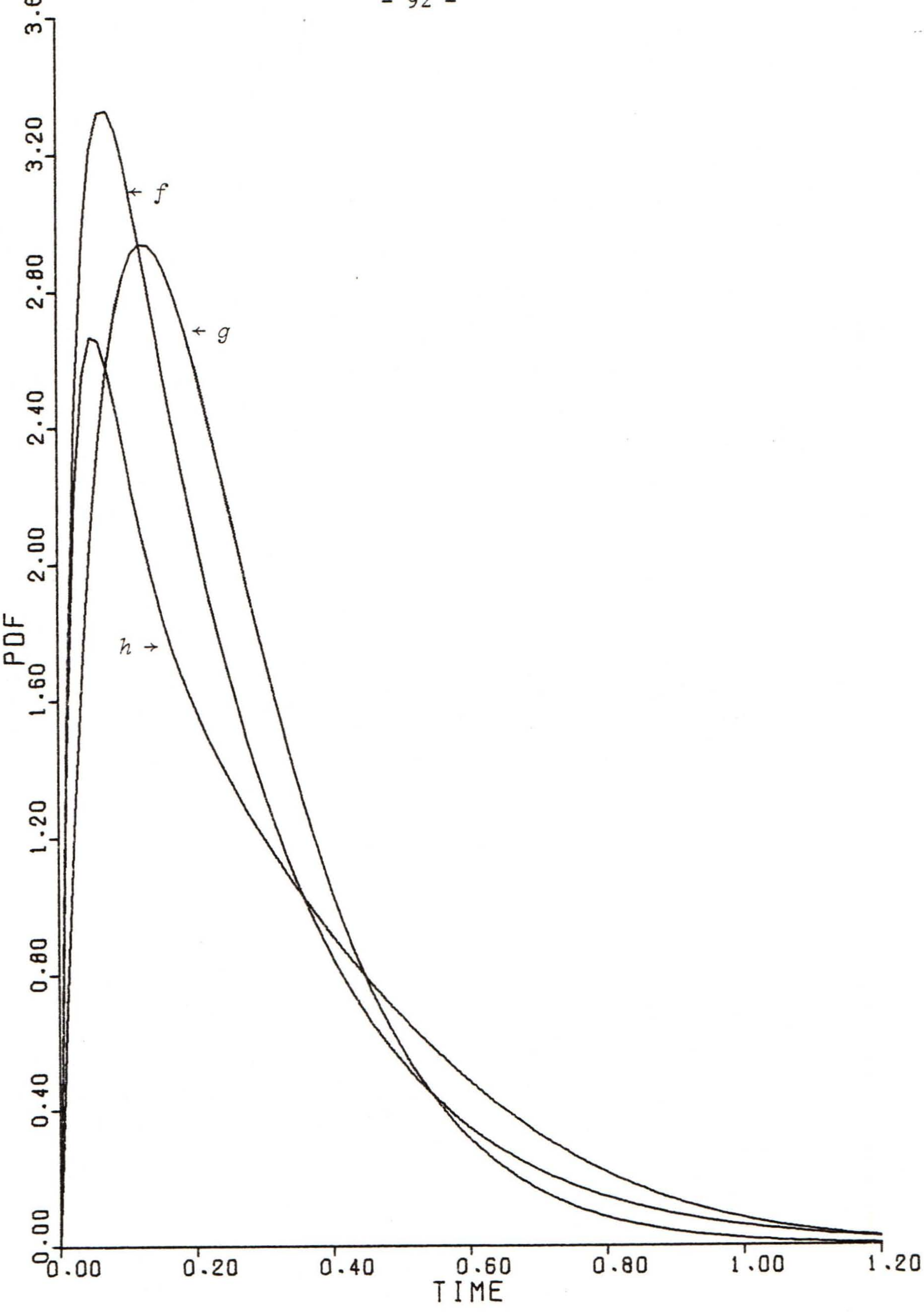
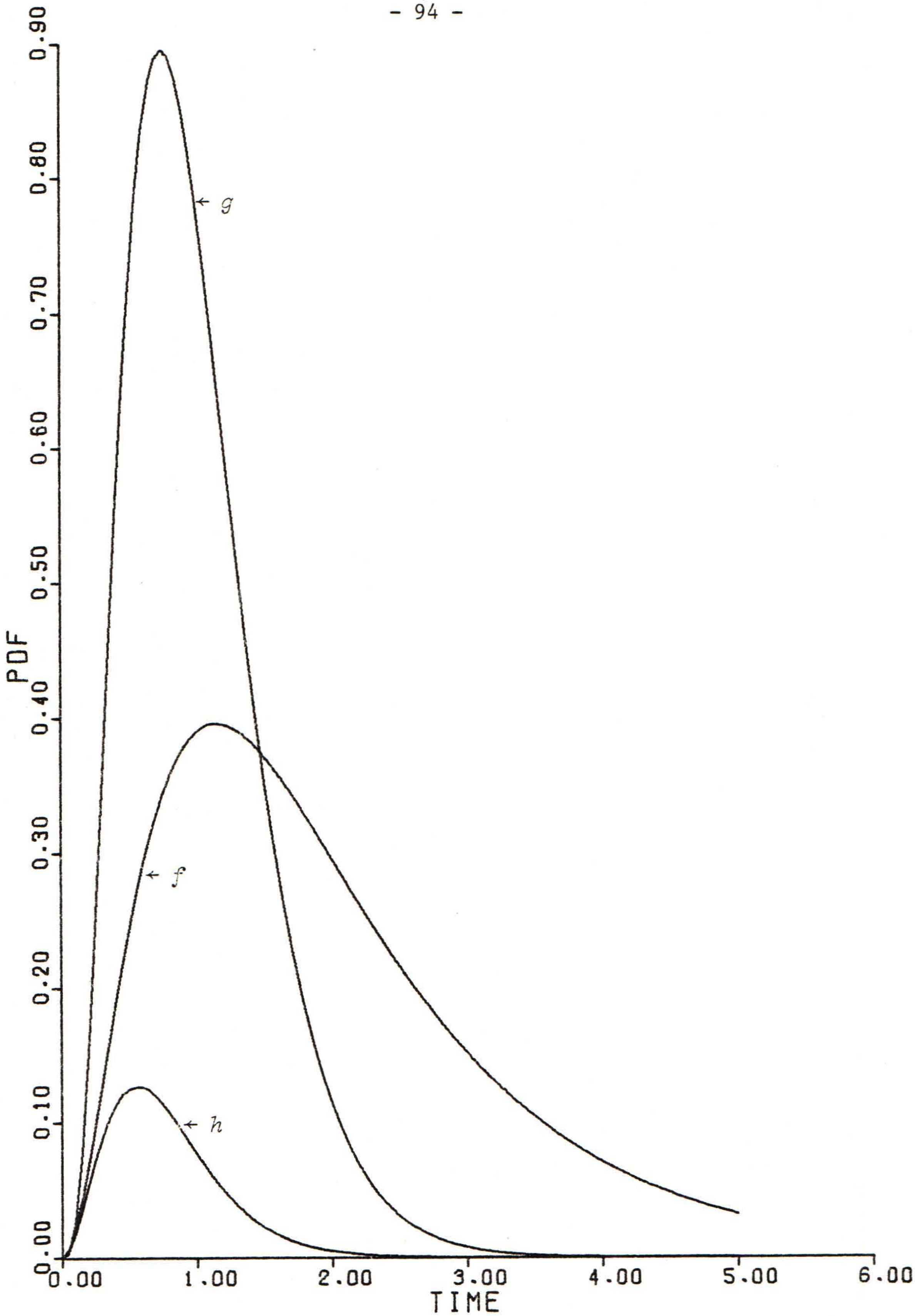

3.60
3.20
2.80
2.40
2.00
1.60
1.20
0.80
0.40
0.00
PDF
0.00
0.20
0.40
0.60
0.80
1.00
1.20
TIME
f
g
h

Figure 7.6

Figures 7.7 - 7.10

Here, the bimodality of h begins to emerge. The repairs are Erlangian in 4 stages, with mean 1 and variance .25. The operating pdf factors as $f(t) = \left(\frac{8e^{-t}\sinh .6t}{3}\right)^2$, with mean 2.0625 (= E_0) and variance 1.9102. The graphs of h use one, two, three and four terms of the series, respectively.



0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
PDF
g
f
h
0.00
1.00
2.00
3.00
4.00
5.00
6.00
TIME

Figure 7.7

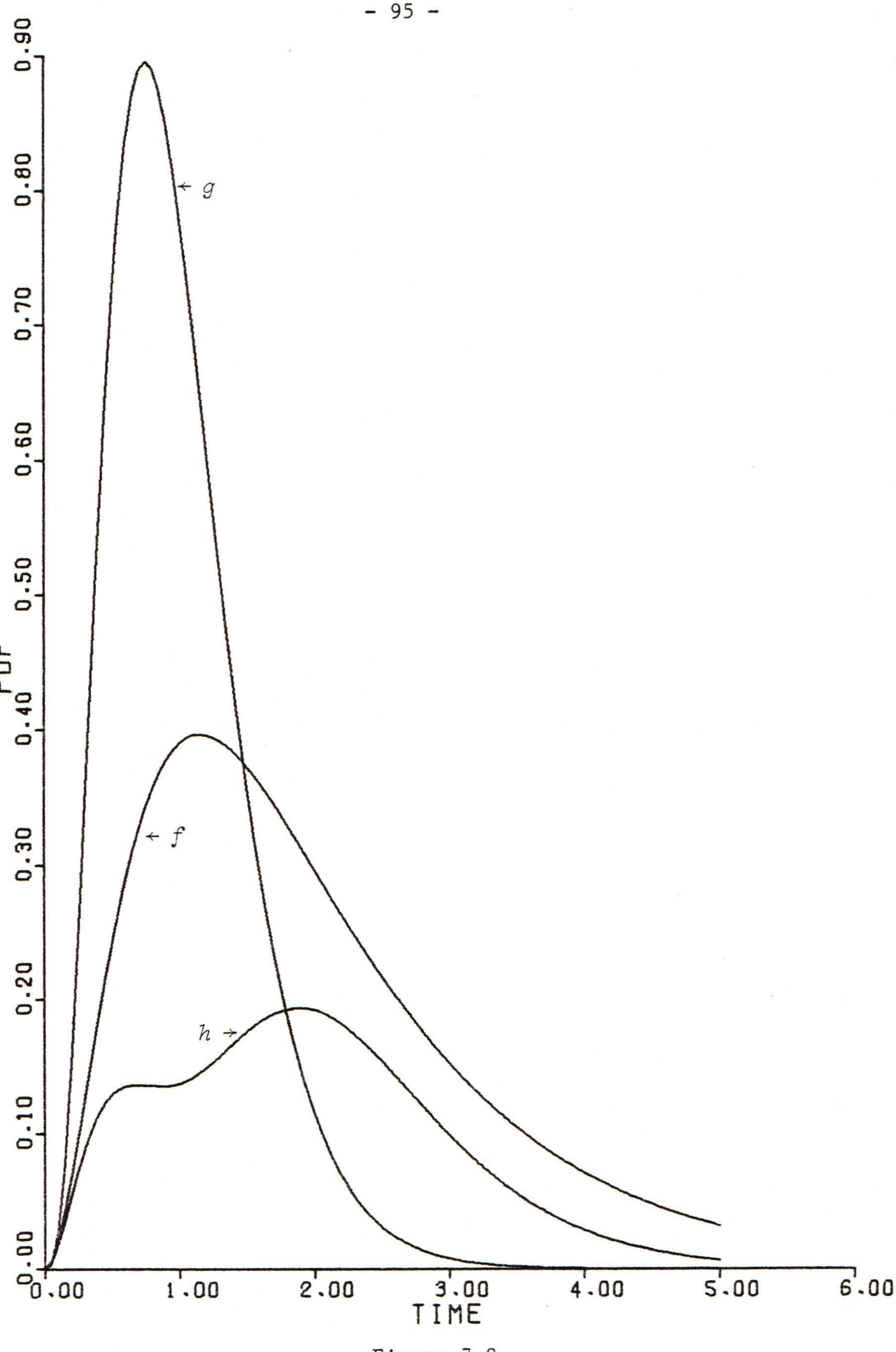

PDF
TIME
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
0.00
1.00
2.00
3.00
4.00
5.00
6.00
g
f
h

Figure 7.8

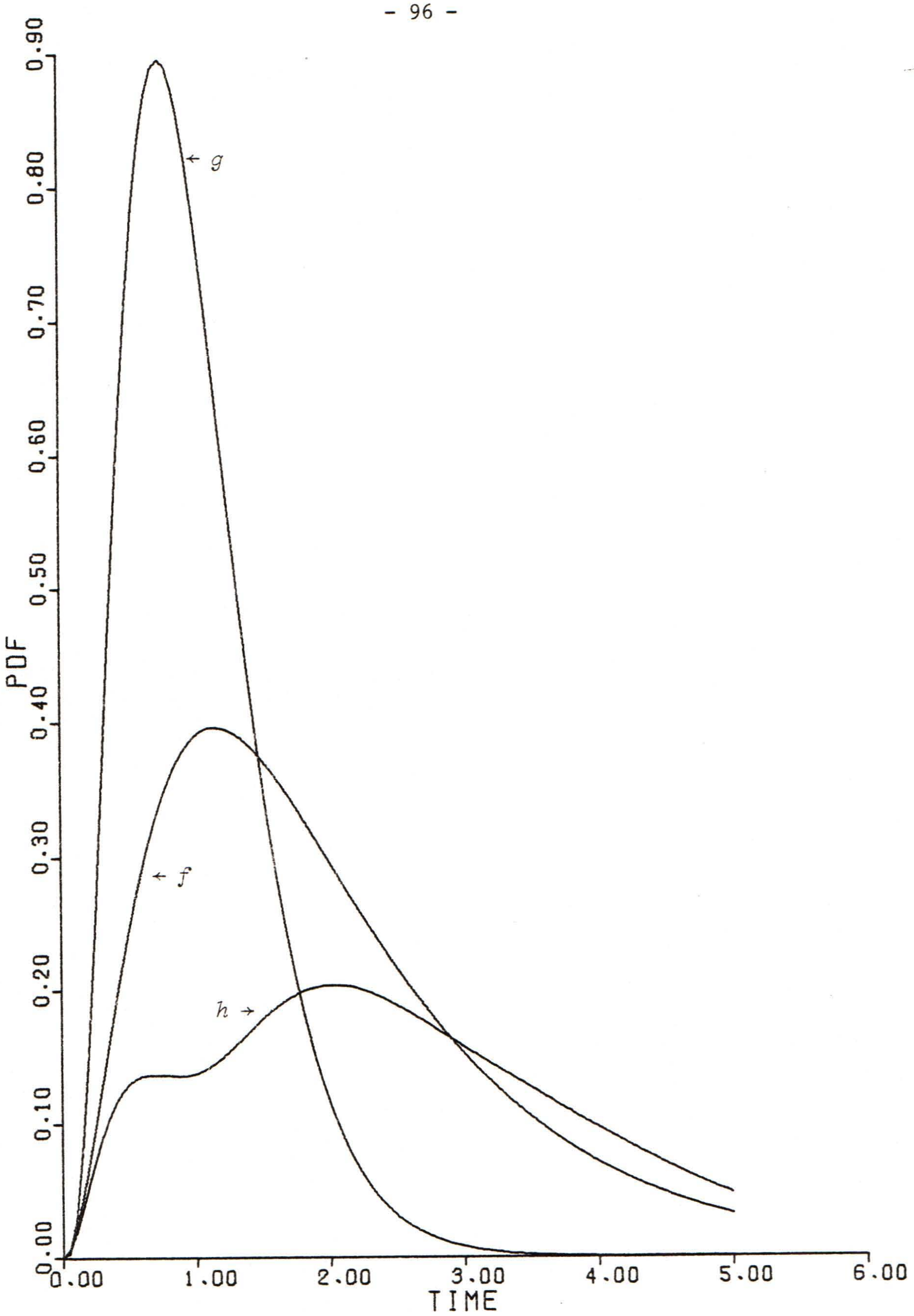

0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
g
f
h
0.00
1.00
2.00
3.00
4.00
5.00
6.00
TIME

Figure 7.9

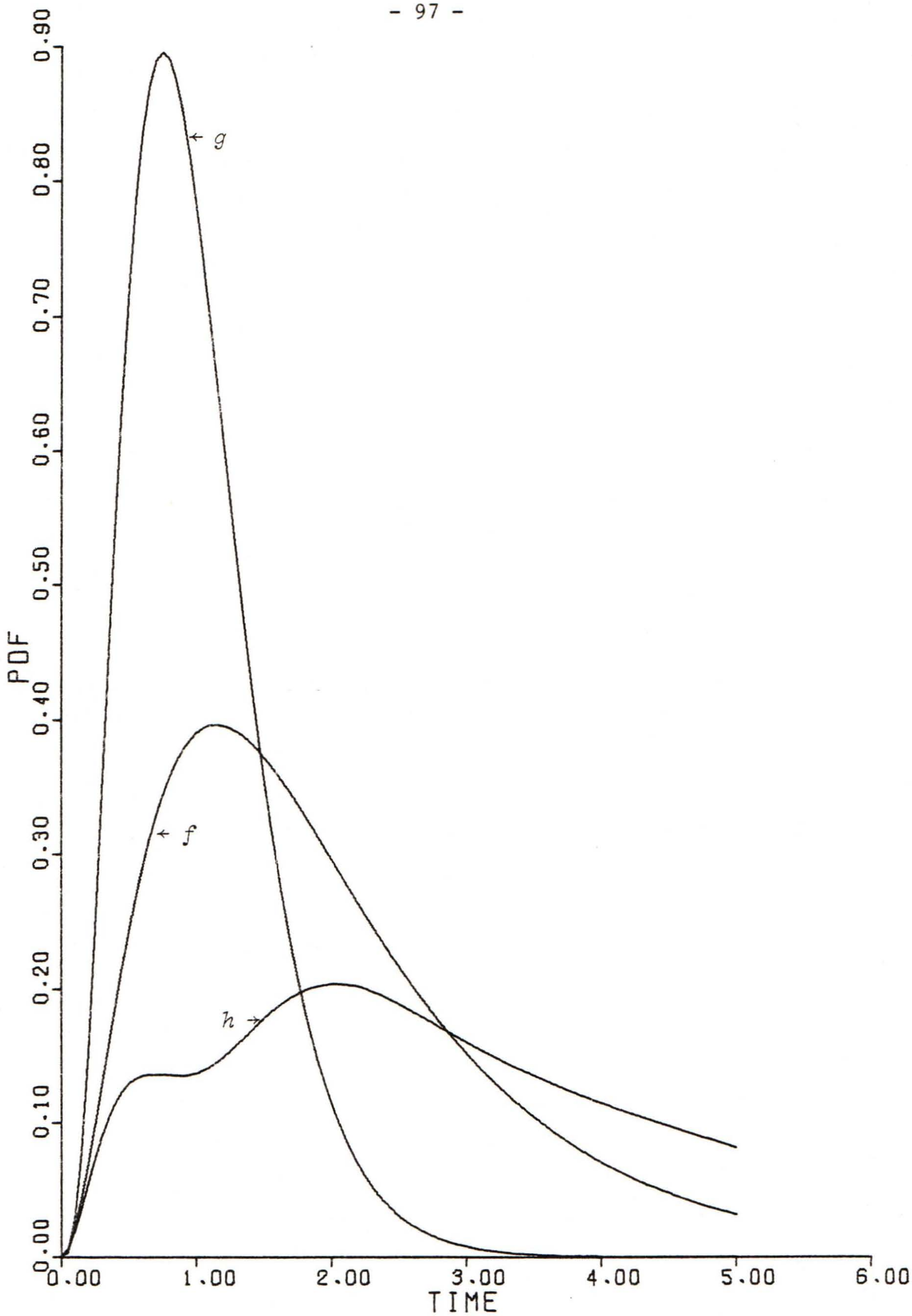

0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
g
f
h
0.00
1.00
2.00
3.00
4.00
5.00
6.00
TIME

Figure 7.10

Figures 7.11 - 7.14

In these figures, $g(t)$ is Erlangian in 3 stages, with mean .25 and variance 1/48. The operating pdf is that illustrated in Figure 7.2, i.e. $f(t) = 399e^{-2t}\sinh(t/100) \approx 4te^{-2t}$, with mean 1.000025 and variance .5000375. Thus $E_0 = 4.0001$. The larger values attained by g, which has a maximum of approximately 3.24, are not plotted. We again use one, two, three and four terms to approximate h. A comparison of the four graphs shows clearly the rightward shift, and relative significance, of the mass of each additional term. In particular, h_0 and h_R appear to have the bulk of their mass concentrated in disjoint intervals. The contribution of the fifth term was analysed at three points: at the second and third critical points and at $t = 3.0$. The results were:

t	$\sum_{i=0}^{3} h_i(t)$	$\sum_{i=0}^{4} h_i(t)$
.32	.1607324169486	.1607324169622
.92	.2757641092654	.2758122120488
3.00	.05695070662497	.1153781393879

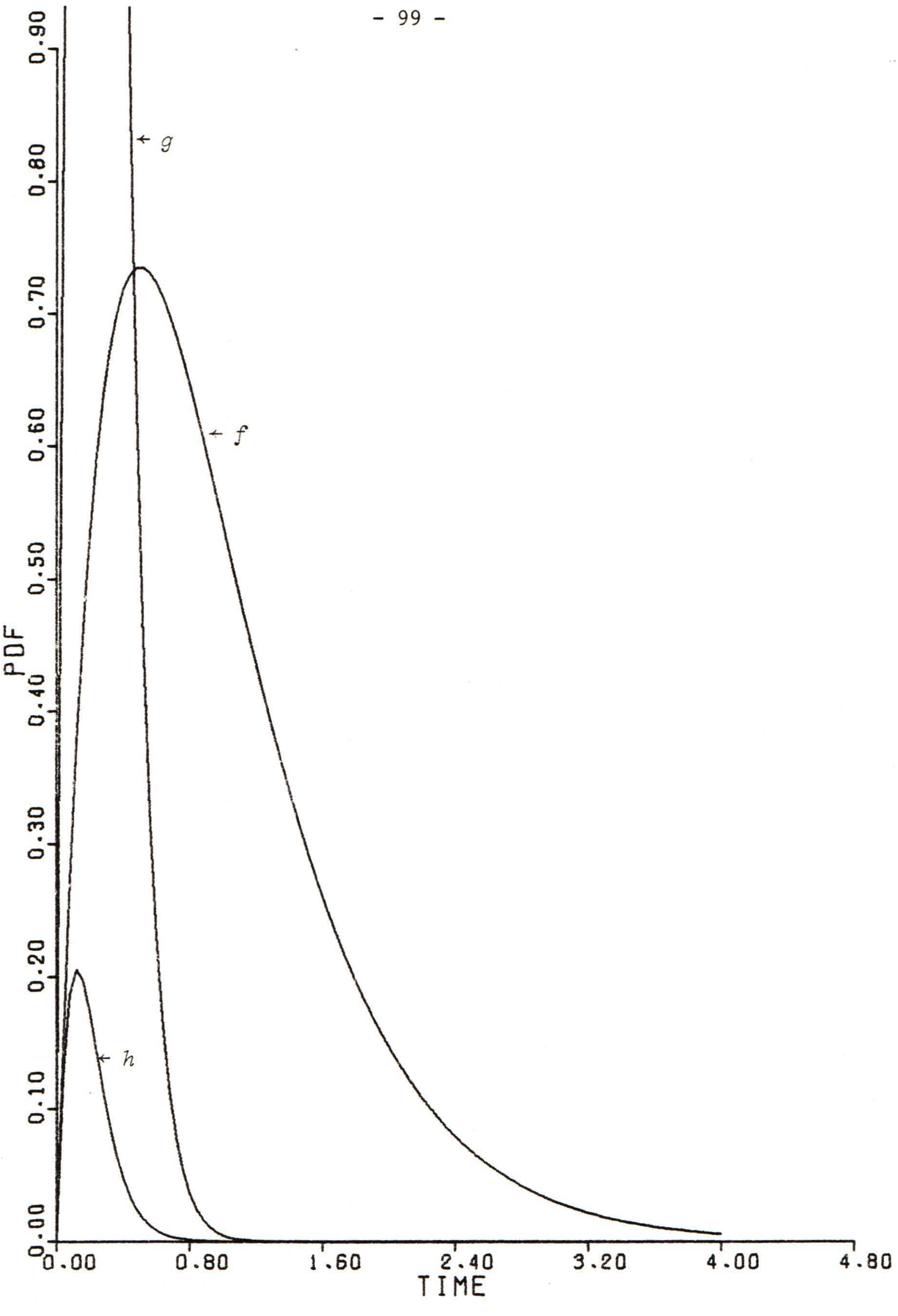

0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
g
f
h
0.00
0.80
1.60
2.40
3.20
4.00
4.80
TIME

Figure 7.11

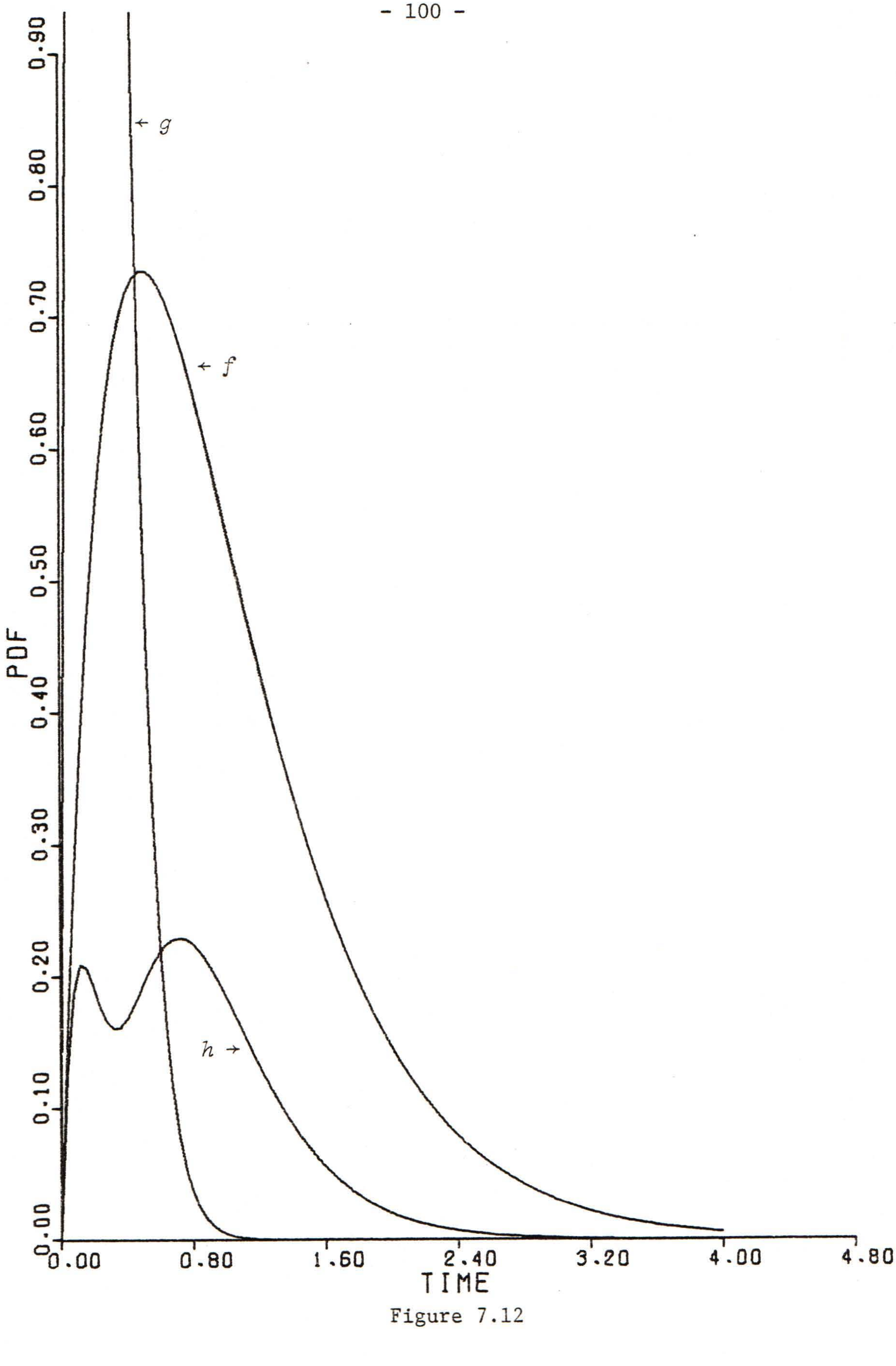

0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
g
f
h
0.00
0.80
1.60
2.40
3.20
4.00
4.80
TIME

Figure 7.12

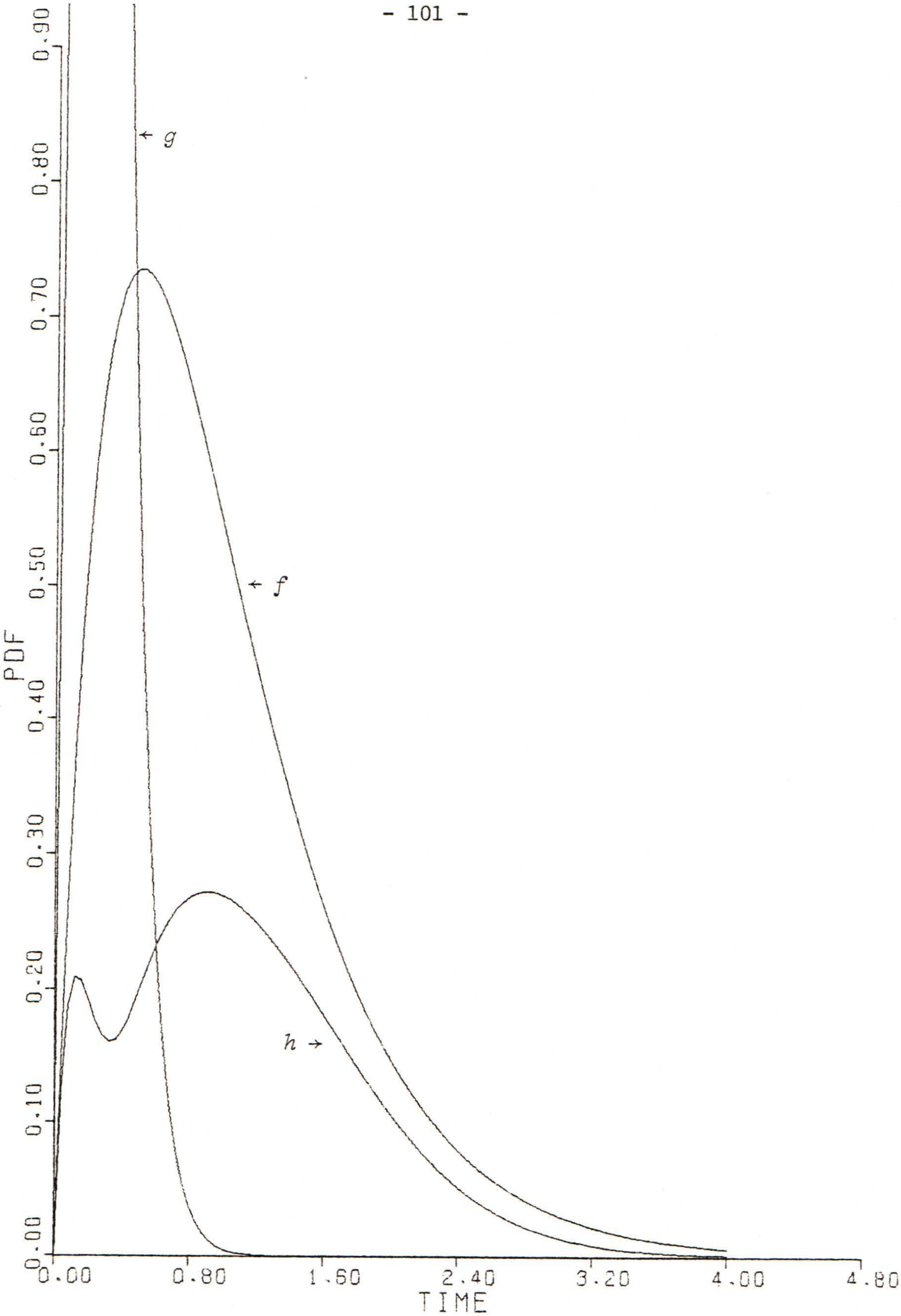

0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
g
f
h
0.00
0.80
1.60
2.40
3.20
4.00
4.80
TIME

Figure 7.13

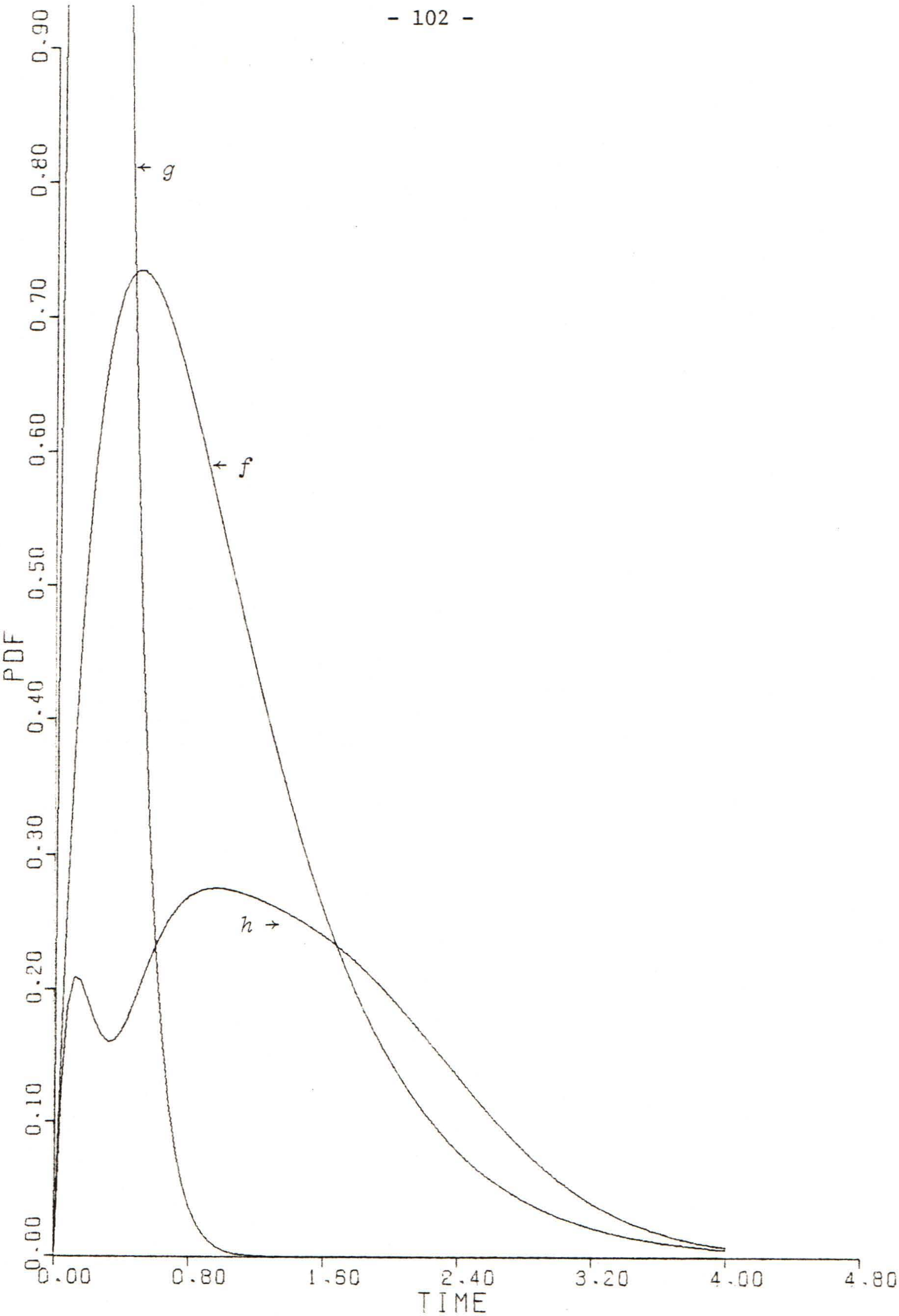

0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
PDF
g
f
h
0.00
0.80
1.60
2.40
3.20
4.00
4.80
TIME

Figure 7.14

SUMMARY

Some of the results obtained for the Active Standby models are summarized below, with the relevant items numbers or page numbers. An (l) indicates that the expressions are given as functions of $l(t,u)$.

	Parallel Arrangement	Series Arrangement
Up Time	1 Repair: Completely analagous to parallel up, 2 repairs -- see note, p. 21	1 Repair: pdf - (3.11)(l), (3.15)(l) $f(t) = \sum a_i e^{-\lambda_i t}$ - (5.20) $f(t)$ Erlangian - (6.29), (6.30)
	2 Repairs: pdf - (3.7)(l) Mean - (3.26) $f(t) = \sum a_i e^{-\lambda_i t}$ - (5.17), (5.18), (5.36) $f(t)$ Erlangian - (6.17)	2 Repairs: Completely analagous to parallel down -- see note, p. 30
Down Time	1 Repair: pdf - (3.21)(l) $f(t) = \sum a_i e^{-\lambda_i t}$ - (5.23) $f(t)$ Erlangian - (6.32)	1 Repair: pdf - (3.18)(l), (3.19)(l) Mean - (3.20)(l) $f(t) = \sum a_i e^{-\lambda_i t}$ - (5.22) $f(t)$ Erlangian - (6.30), (6.31)
	2 Repairs: pdf - (3.22) Mean - (3.23)	2 Repairs: Completely analagous to parallel up, two repairs -- see note, p. 21

APPENDIX

The program by which the numerical approximations in Chapter 7 were carried out is here entered and described. The algorithm is basically that developed in Chapter 3, with the exception that the order of integration is reversed in the definitions of * and ⊗. They then become

$$*\sigma(t,u) + \otimes\sigma(t,u) = \int_0^u \sigma(t,z)k(u,u,z) + \sigma(u,z)k(t,u,z)dz$$

where

$$k(w,u,z) = \int_0^{u-z} g(y)f(w-y-z)dy.$$

To reduce the computing time required, $k(w,u,z)$, $\rho(t,u)$ and $h_0(t)$ are calculated explicitly, rather than being obtained by integration.

The Gaussian quadrature method is used to perform the integrations, with the refinements of the quadratures varying from term to term. Altering the order in which the various quadratures are applied makes no significant difference to the results.

```
 PROGRAM WWWW(INPUT,OUTPUT)
C    THIS PROGRAM CALCULATES AN APPROXIMATION TO THE UP TIME
C    PDF H(T),GIVEN THE OPERATING PDF F(T) AND REPAIR PDF G(T),
C    WITH COMPLEMENTARY D.F. GB(T).  THE FIRST 'NT' TERMS OF
C    THE INTEGRAL SERIES DEFINING H ARE CALCULATED AND SUMMED.
C    THE RESULTING FUNCTION IS THEN PLOTTED AT 'NP' POINTS
C    BETWEEN THE 'INIT' AND 'FINAL' TIMES.
 DOUBLE PRECISION T,MU,INIT,FINAL,X
 DOUBLE PRECISION ALPHA,SECMOM,EPS,LIM,BETA
 DOUBLE PRECISION F,G,GB,H
 DOUBLE PRECISION LAM(20),A(20)
 REAL FP(303),HP(303),GP(303),TP(303)
 INTEGER M,N,NP,I,NT,J,PRINT,CALC,IP,IPP1,MORE
 COMMON/AREA1/M,N,LAM,MU,A
 COMMON/AREA2/T,X,NT
 CALL PLOTS
 200 PRINT*,'ENTER INITIAL AND FINAL TIMES,NUMBER OF SUBDIVISIONS,'
 PRINT*,'AND NUMBER OF TERMS OF THE INTEGRAL SERIES TO BE USED.',
 READ*,INIT,FINAL,NP,NT
 PRINT*,'ENTER 0 IF THE PARAMETERS OF F AND G '
 PRINT*,'ARE NOT TO BE PRINTED, AND 0 IF H IS NOT TO BE CALCULATED.',
 READ*,PRINT,CALC
 PRINT*,'ENTER M,BETA,N,LIM,EPS'
 READ*,M,BETA,N,LIM,EPS
C    THE REPAIR PDF IS ERLANGIAN IN M STAGES, MEAN BETA.
C    THE OPERATING PDF IS A SUM OF N EXPONENTIAL TERMS.
C    THE COEFFICIENTS AND RATE PARAMETERS ARE ARBITRARY.
C    HERE, THEY ARE CALCULATED IN SUCH A WAY THAT AS 'EPS'
C    TENDS TO ZERO, F(T) TENDS TO AN ERLANGIAN DENSITY
C    IN N STAGES, RATE PARAMETER 'LIM'.
 MU=M/BETA
 IP=INT(N/2.0)
 IPP1=IP+1
 IF(2*IP.EQ.N)GO TO 320
C    SET THE RATE PARAMETERS LAM(I).
 DO 300 I=1,IPP1
 LAM(I+IP)=LIM-(I-1)*EPS
 LAM(IP+2-I)=LIM+(I-1)*EPS
 300 CONTINUE
 GO TO 340
 320 DO 330 I=1,IP
 LAM(I+IP)=LIM-I*EPS
 LAM(IP+1-I)=LIM+I*EPS
 330 CONTINUE
 340 DO 335 I=1,N
 IF(LAM(I).NE.MU)GO TO 335
 PRINT*,'LAM',I,' = MU, P CANNOT BE CALCULATED.  '
 335 CONTINUE
C    SET THE COEFFICIENTS A(I), AND CALCULATE THE
C    FIRST TWO MOMENTS OF F(T).
 ALPHA=SECMOM=0.0D0
 IF(PRINT.EQ.0)GO TO 350
 PRINT 5
 5 FORMAT('1')
 PRINT*,'OPERATING PDF A LINEAR COMBINATION OF ',N,' EXPONENTIAL TERMS,'
 PRINT*,'WITH COEFFICIENTS AND RATE PARAMETERS '
 350 DO 500 I=1,N
 A(I)=1.0D0
```

```
DO 400 J=1,N
IF(J.EQ.I)GO TO 400
A(I)=A(I)*LAM(J)/(LAM(J)-LAM(I))
400 CONTINUE
A(I)=A(I)*LAM(I)
IF(PRINT.EQ.0)GO TO 450
PRINT*,'A',I,' = ',A(I),'   LAM',I,' = ',LAM(I)
450 ALPHA=ALPHA+A(I)/(LAM(I)**2.0D0)
SECMOM=SECMOM+2.0D0*A(I)/(LAM(I)**3.0D0)
500 CONTINUE
IF(PRINT.EQ.0)GO TO 600
PRINT*,'                    '
PRINT*,'                    '
PRINT*,'MEAN OPERATING TIME = ',ALPHA
PRINT*,'VARIANCE = ',SECMOM-(ALPHA**2.0D0)
PRINT*,'REPAIRS ERLANGIAN IN ',M,' STAGES,MEAN ',M/MU
PRINT*,' VARIANCE ',M/(MU**2.0D0)
C    GET F,G AND H FROM THE FUNCTION ROUTINES
C    FOLLOWING THE MAIN PROGRAM.
600 T=INIT
I=1
PRINT*,'              '
PRINT*,'             '
PRINT 10
10 FORMAT('0',6X,'T',13X,'F',20X,'G',20X,'H')
1000 TP(I)=T
FP(I)=F(T)
GP(I)=G(T)
HP(I)=0.0D0
IF(CALC.EQ.0)GO TO 1500
HP(I)=H(T)
1500 PRINT 20,TP(I),FP(I),GP(I),HP(I)
20 FORMAT('0',2X,F7.4,2X,E19.13,3X,E19.13,3X,E19.13)
IF(T.GE.FINAL) GO TO 2000
T=T+((FINAL-INIT)/NP)
I=I+1
GO TO 1000
2000 IF(CALC.EQ.0)GO TO 2200
PRINT 15
15 FORMAT('0')
PRINT*,'THE FIRST ',NT,' TERMS OF THE SERIES OF INTEGRALS'
PRINT*,'WERE USED IN THE CALCULATION OF H(T).',
2200 IF(CALC.NE.0)GO TO 2300
PRINT*,'MORE',
READ*,MORE
C    FOR CALCULATING F AND G ONLY, A "1" ENTERED
C    HERE RETURNS THE PROGRAM TO THE BEGINNING.
IF(MORE.EQ.1)GO TO 200
2300 CALL SCALE(TP,6.0,NP+1,1)
CALL SCALE(GP,9.0,NP+1,1)
FP(NP+2)=GP(NP+2)
HP(NP+2)=GP(NP+2)
FP(NP+3)=GP(NP+3)
HP(NP+3)=GP(NP+3)
CALL AXIS(0.,0.,6H TIME ,-6,6.0,0.,TP(NP+2),TP(NP+3))
CALL AXIS(0.,0.,5H PDF ,5,9.0,90.,GP(NP+2),GP(NP+3))
CALL LINE(TP,GP,NP+1,1,0,42)
CALL LINE(TP,FP,NP+1,1,0,42)
CALL LINE(TP,HP,NP+1,1,0,42)
CALL PLOT(10.0,0.0,999)
STOP
END
DOUBLE FUNCTION H(D1)
DOUBLE PRECISION HZ,D1,Y,T,X,LAM(20),MU,A(20)
INTEGER NT
COMMON/AREA1/M,N,LAM,MU,A
```

```
COMMON/AREA2/T,X,NT
EXTERNAL R
Y=0.0D0
IF(NT.EQ.1)GO TO 2400
IF(D1.LE.0.0D0)GO TO 2400
CALL DQG4(0.0D0,D1,R,Y)
2400 H=Y+HZ(D1)
RETURN
END
DOUBLE FUNCTION HZ(D2)
C   HZ IS THE FIRST TERM OF THE INTEGRAL SERIES.
C   IT IS CALCULATED EXPLICITLY, RATHER THAN
C   BEING OBTAINED BY INTEGRATION, IN ORDER
C   TO REDUCE THE COMPUTING TIME REQUIRED.
DOUBLE PRECISION D2,SUM,MU,T,X,LAM(20),A(20)
DOUBLE PRECISION F
INTEGER J,M
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
HZ=F(D2)
IF(D2.LE.0.0D0)GO TO 3500
SUM=0.0D0
DO 3000 J=1,M
SUM=SUM+(1+M-J)*((MU*D2)**(J-1))/MAC(J-1)
3000 CONTINUE
HZ=HZ*SUM*DEXP(-MU*D2)/M
3500 RETURN
END
DOUBLE FUNCTION R(D3)
DOUBLE PRECISION T,D3,X,LAM(20),A(20),MU
DOUBLE PRECISION GB,L3,P
INTEGER NT
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
X=D3
IF(NT.EQ.2)GO TO 3700
R=L3(T,D3)*GB(T-D3)
RETURN
3700 R=P(T,D3)*GB(T-D3)
RETURN
END
DOUBLE FUNCTION L3(E3,F3)
DOUBLE PRECISION E3,F3,Y3,T,X,MU,LAM(20),A(20)
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
DOUBLE PRECISION P
EXTERNAL Q3
Y3=0.0D0
IF(F3.LE.0.0D0)GO TO 3800
CALL DQG8(0.0D0,F3,Q3,Y3)
3800 L3=Y3+P(E3,F3)
RETURN
END
DOUBLE FUNCTION Q3(G3)
DOUBLE PRECISION G3,T,X,MU,LAM(20),A(20)
DOUBLE PRECISION L4,K,P
INTEGER NT
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
Q3=0.0D0
IF(G3.GT.T.OR.G3.GT.X)GO TO 3900
IF(NT.EQ.3)GO TO 3600
Q3=L4(T,G3)*K(X,X,G3)+L4(X,G3)*K(T,X,G3)
RETURN
3600 Q3=P(T,G3)*K(X,X,G3)+P(X,G3)*K(T,X,G3)
3900 RETURN
```

```
END
DOUBLE FUNCTION L4(E4,F4)
DOUBLE PRECISION E4,F4,Y4,T,X,MU,LAM(20),A(20),X4,T4
DOUBLE PRECISION P
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
COMMON/AREA3/X4,T4
EXTERNAL Q4
T4=E4
X4=F4
Y4=0.0D0
IF(F4.LE.0.0D0)GO TO 4000
CALL DQG12(0.0D0,F4,Q4,Y4)
4000 L4=Y4+P(E4,F4)
RETURN
END
DOUBLE FUNCTION Q4(G4)
DOUBLE PRECISION G4,T,X,MU,LAM(20),A(20),X4,T4
DOUBLE PRECISION P,K,L5
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
COMMON/AREA3/X4,T4
Q4=0.0D0
IF(G4.GT.T4.OR.G4.GT.X4)GO TO 4500
IF(NT.EQ.4)GO TO 4200
Q4=L5(T4,G4)*K(X4,X4,G4)+L5(X4,G4)*K(T4,X4,G4)
RETURN
4200 Q4=P(T4,G4)*K(X4,X4,G4)+P(X4,G4)*K(T4,X4,G4)
4500 RETURN
END
DOUBLE FUNCTION L5(E5,F5)
DOUBLE PRECISION E5,F5,Y5,T,X,MU,LAM(20),A(20),X5,T5
DOUBLE PRECISION P
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
COMMON/AREA4/X5,T5
EXTERNAL Q5
T5=E5
X5=F5
Y5=0.0D0
IF(F5.LE.0.0D0)GO TO 4700
CALL DQG16(0.0D0,F5,Q5,Y5)
4700 L5=Y5+P(E5,F5)
RETURN
END
DOUBLE FUNCTION Q5(G5)
DOUBLE PRECISION G5,T,X,MU,LAM(20),A(20),X5,T5
DOUBLE PRECISION P,K,L6
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
COMMON/AREA4/X5,T5
Q5=0.0D0
IF(G5.GT.T5.OR.G5.GT.X5)GO TO 4900
IF(NT.EQ.5)GO TO 4800
Q5=L6(T5,G5)*K(X5,X5,G5)+L6(X5,G5)*K(T5,X5,G5)
RETURN
4800 Q5=P(T5,G5)*K(X5,X5,G5)+P(X5,G5)*K(T5,X5,G5)
4900 RETURN
END
DOUBLE FUNCTION L6(E6,F6)
DOUBLE PRECISION E6,F6,Y6,T,X,MU,LAM(20),A(20),X6,T6
DOUBLE PRECISION P
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
COMMON/AREA5/X6,T6
EXTERNAL Q6
```

```
09070 T6=E6
09080 X6=F6
09090 Y6=0.0D0
09100 IF(F6.LE.0.0D0)GO TO 4905
09110 CALL DQG24(0.0D0,F6,Q6,Y6)
09120 4905 L6=Y6+P(E6,F6)
09130 RETURN
09140 END
09500 DOUBLE FUNCTION Q6(G6)
09510 DOUBLE PRECISION G6,T,X,MU,LAM(20),A(20),X6,T6
09520 DOUBLE PRECISION P,K,L7
09530 COMMON/AREA1/M,N,LAM,MU,A
09540 COMMON/AREA2/T,X,NT
09550 COMMON/AREA5/X6,T6
09560 Q6=0.0D0
09570 IF(G6.GT.T6.OR.G6.GT.X6)GO TO 4915
09580 IF(NT.EQ.6)GO TO 4910
09590 Q6=L7(T6,G6)*K(X6,X6,G6)+L7(X6,G6)*K(T6,X6,G6)
09600 RETURN
09610 4910 Q6=P(T6,G6)*K(X6,X6,G6)+P(X6,G6)*K(T6,X6,G6)
09620 4915 RETURN
09630 END
10000 DOUBLE FUNCTION L7(E7,F7)
10010 DOUBLE PRECISION E7,F7,Y7,T,X,MU,LAM(20),A(20),X7,T7
10020 DOUBLE PRECISION P
10030 COMMON/AREA1/M,N,LAM,MU,A
10040 COMMON/AREA2/T,X,NT
10050 COMMON/AREA6/X7,T7
10060 EXTERNAL Q7
10070 T7=E7
10080 X7=F7
10090 Y7=0.0D0
10100 IF(F7.LE.0.0D0)GO TO 4920
10110 CALL DQG32(0.0D0,F7,Q7,Y7)
10120 4920 L7=Y7+P(E7,F7)
10130 RETURN
10140 END
10500 DOUBLE FUNCTION Q7(G7)
10510 DOUBLE PRECISION G7,T,X,MU,LAM(20),A(20),X7,T7
10520 DOUBLE PRECISION P,K,L8
10530 COMMON/AREA1/M,N,LAM,MU,A
10540 COMMON/AREA2/T,X,NT
10550 COMMON/AREA6/X7,T7
10560 Q7=0.0D0
10570 IF(G7.GT.T7.OR.G7.GT.X7)GO TO 4930
10580 IF(NT.EQ.7)GO TO 4925
10590 Q7=L8(T7,G7)*K(X7,X7,G7)+L8(X7,G7)*K(T7,X7,G7)
10600 RETURN
10610 4925 Q7=P(T7,G7)*K(X7,X7,G7)+P(X7,G7)*K(T7,X7,G7)
10620 4930 RETURN
10630 END
12500 DOUBLE FUNCTION L8(E8,F8)
12510 DOUBLE PRECISION E8,F8
12520 DOUBLE PRECISION P
12530 L8=P(E8,F8)
12540 RETURN
12550 END
12600C   IF MORE THAN 7 TERMS OF THE SERIES ARE REQUIRED,
12610C   REPLACE LINES 10000 TO 12550 WITH ONE COPY OF SAME
12620C   FOR EACH ADDITIONAL TERM.  INCREASE THE
12630C   INDICES 6,7 AND 8 BY ONE IN EACH COPY.
13000 DOUBLE FUNCTION K(D4,D5,D6)
13010C   K AND P ('RHO') ARE CALCULATED EXPLICITLY TO
13020C   REDUCE THE COMPUTING TIME REQUIRED.
13030 DOUBLE PRECISION D4,D5,D6,MU,T,X,LAM(20),A(20)
13040 DOUBLE PRECISION SUM2(20),SUM1,K1,K2,K2P
```

```
INTEGER I,J,M,N,NT
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
K=0.0D0
IF(D4.LT.D5.OR.D5.LE.D6)GO TO 5200
SUM1=0.0D0
DO 4950 I=1,N
SUM1=SUM1+A(I)*DEXP(LAM(I)*(D6-D4))/((MU-LAM(I))**M)
4950 CONTINUE
K1=(MU**M)*SUM1
K2P=0.0D0
DO 5150 J=1,M
SUM2(J)=0.0D0
DO 5100 I=1,N
SUM2(J)=SUM2(J)+A(I)*DEXP(LAM(I)*(D5-D4))/((MU-LAM(I))**(M-J+1))
5100 CONTINUE
K2P=K2P+SUM2(J)*((D5-D6)**(J-1))/MAC(J-1)
5150 CONTINUE
K2=DEXP(MU*(D6-D5))*(MU**M)*K2P
K=K1-K2
5200 RETURN
END
DOUBLE FUNCTION P(D7,D8)
DOUBLE PRECISION D7,D8,MU,T,X,Z1,Z2,Z3,LAM(20),A(20),SUM(20)
DOUBLE PRECISION Z4,Z5,Z6,Z7
INTEGER I,J,M,N,NT
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
P=0.0D0
IF(D7.LT.D8)GO TO 5500
DO 5350 J=1,N
IF(D8.NE.0.0D0)GO TO 5250
SUM(J)=0.0D0
GO TO 5350
5250 Z2=0.0D0
Z1=MU/(MU-LAM(J))
DO 5300 I=1,M
Z2=Z2+((MU*D8)**(I-1))*((Z1**(M-I+1))-1)/MAC(I-1)
5300 CONTINUE
SUM(J)=((Z1**M)-1-Z2*DEXP((LAM(J)-MU)*D8))/LAM(J)
5350 CONTINUE
Z4=Z5=Z6=Z7=0.0D0
DO 5400 I=1,N
Z4=Z4+A(I)*DEXP(-LAM(I)*D7)
Z5=Z5+A(I)*DEXP(-LAM(I)*D8)
Z6=Z6+A(I)*DEXP(-LAM(I)*D8)*SUM(I)
Z7=Z7+A(I)*DEXP(-LAM(I)*D7)*SUM(I)
5400 CONTINUE
P=(Z6*Z4+Z5*Z7)*MU/M
5500 RETURN
END
DOUBLE FUNCTION F(D9)
DOUBLE PRECISION D9,MU,LAM(20),A(20),T,X
INTEGER M,N,NT,I
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
F=0
IF(D9.LE.0.0D0.AND.N.NE.1)GO TO 5750
DO 5700 I=1,N
F=F+A(I)*DEXP(-LAM(I)*D9)
5700 CONTINUE
5750 RETURN
END
DOUBLE FUNCTION G(D10)
DOUBLE PRECISION D10,MU,LAM(20),A(20),T,X
INTEGER M,N,NT
```

```
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
G=0
IF(D10.LE.0.0D0.AND.M.NE.1)GO TO 5800
G=((MU*D10)**(M-1))*MU*DEXP(-MU*D10)/MAC(M-1)
5800 RETURN
END
DOUBLE FUNCTION GB(D11)
DOUBLE PRECISION D11,MU,LAM(20),A(20),T,X
COMMON/AREA1/M,N,LAM,MU,A
COMMON/AREA2/T,X,NT
INTEGER I,M,N,NT
SUM=0.0D0
GB=1.0D0
IF(D11.LE.0.0D0)GO TO 6300
DO 6000 I=1,M
SUM=SUM+((MU*D11)**(I-1))/MAC(I-1)
6000 CONTINUE
GB=DEXP(-MU*D11)*SUM
6300 RETURN
END
INTEGER FUNCTION MAC(I1)
INTEGER I1,J1
MAC=1
IF(I1.EQ.0)GO TO 7000
DO 6500 J1=1,I1
MAC=MAC*J1
6500 CONTINUE
7000 RETURN
END
```

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