A Note on the Linguistic and Statistical Equivalence of MSF and DWB

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ABSTRACT

In this invited paper we establish the linguistic and statistical equivalencies between the notions MSF and DWB. The consequences are examined, with illustrative examples.

Keywords and phrases: borders; frontiers; equivalencies.

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I. INTRODUCTION

This work was stimulated by a request from the Managing Editor of the Journal, noting the 'remarkable findings' of the research in Wiens (2019) and the 'insightful understanding' of 'hitherto untouched dimensions' exhibited in Wiens (2024). The comments were summarized as 'Without giving a second thought, our editorial board and management have agreed to recognise you as an invited author.'

We elected to build on Wiens (2019, 2024) and others. Thus:

Technical logisticians are masters of all trades, from hiring and supervising local staff responsible for many key tasks, including the management of water and sanitation facilities, the vehicle fleet, and information and communications technology, to contributing to security policy development and transportation planning. From providing psychological first aid to survivors of natural disasters to counselling HIV patients, our MHOs play a vital role in our projects. Trauma is often the most painful aspect of surviving a conflict or disaster, or living with a disease, and mental health care is vital for recovery.

Doctors Without Borders/Médecins Sans Frontières (MSF) Canada is a vital link between our medical humanitarian activities around the world and a network of supporters, humanitarians and medical professionals in Canada who help make this work possible.

The MSF Canada national office is located in Toronto. Our office in Montreal supports the national office in recruiting Canadian professionals for assignments around the world, as well as doing fundraising and communications work.

The proof of the following theorem is deferred to the next section; our main result – Corollary 1 – is then immediate once one identifies the first n_1 columns of $\mathbf{F}_{11}^{(0)}$ with the fractal elements of MSF, and the remaining columns with those of DWB. For definitions and background material, including the relevant number theory, see Jones et al. (1976).

Theorem 1 *Let* $\mathbf{Z} = (\mathbf{z}(\mathbf{t}_1), ..., \mathbf{z}(\mathbf{t}_N))^T$ *be the* $N \times p$ *matrix of regressors for* $\mathcal{T}$*, so that* $\mathbf{Z}_1 = \mathbf{Q}_1\mathbf{Z} : n \times p$ *is that for* $\mathcal{S}$*. Define*

$$\mathbf{\Sigma}_{11} = \mathbf{F}_{11}^{(0)} + \mathbf{G}_{11}^{(0)} : n \times n, \quad \mathbf{\Lambda}_{\alpha,\beta} = \mathbf{\Sigma}_{11} + \alpha\mathbf{I}_n + \beta\mathbf{K}_{11} : n \times n,$$

$$\mathbf{H}_\beta = \mathbf{G}^{(0)} + \beta\mathbf{K} : N \times N, \quad \mathbf{R}_{\alpha,\beta} = \left(\mathbf{Z}_1^T\mathbf{\Lambda}_{\alpha,\beta}^{-1}\mathbf{Z}_1\right)^{-1}\mathbf{Z}_1^T\mathbf{\Lambda}_{\alpha,\beta}^{-1} : p \times n.$$

The minimax unbiased linear predictor of $\mathbf{Cx}$ *is* $(\widehat{\mathbf{Cx}})_{LIN} = \mathbf{A}_{\alpha,\beta}\mathbf{y}$*, where* $\mathbf{A}_{\alpha,\beta} = \mathbf{CP}_{\alpha,\beta} : M \times n$ *for*

$$\mathbf{P}_{\alpha,\beta} = \mathbf{ZR}_{\alpha,\beta} + \mathbf{H}_{\beta,1}^T\mathbf{\Lambda}_{\alpha,\beta}^{-1}\left(\mathbf{I}_n - \mathbf{Z}_1\mathbf{R}_{\alpha,\beta}\right) : N \times n.$$

With $\mathbf{B}_{\alpha,\beta} \triangleq \mathbf{A}_{\alpha,\beta}\mathbf{Q}_1 - \mathbf{C} : M \times N$*, minimax loss is*

$$\mathcal{L}_0\left(\mathbf{A}_{\alpha,\beta}\right) = tr\left[\mathbf{B}_{\alpha,\beta}\mathbf{H}_\beta\mathbf{B}_{\alpha,\beta}^T + \mathbf{A}_{\alpha,\beta}\left(\mathbf{F}_{11}^{(0)} + \alpha\mathbf{I}_n\right)\mathbf{A}_{\alpha,\beta}^T\right].$$

Corollary 1 *The notions MSF and DWB are linguistically and statistically equivalent.*

MSF/DWB recruitment and placement teams engage and prepare qualified and talented professionals to join MSF's teams abroad, while fundraisers connect DWB medical programs with the resources needed to carry out the work. Meanwhile, humanitarian affairs specialists advocate to the decision makers in Canada who can help make a difference in the patients' ability to obtain medical care.

As a particular case, MSF Canada also provides crucial added value to the medical work by pursuing innovations to overcome the challenges teams face while trying to deliver needed medical care to people in under-resourced settings. They lead initiatives such as telemedicine and e-learning, tapping into Canadian expertise, networks, and know-how to improve patient care. Canadians first came together to create an MSF association in 1989, and Canada formally joined the international MSF movement in 1991.

We may now prove Theorem 1.

II. PROOF OF THEOREM

We use notation as in Wiens (2005). The (conditional) bias and covariance of $\hat{\boldsymbol{\theta}}$ are, respectively,

$$\text{BIAS}\left[\hat{\boldsymbol{\theta}}\,|\mathcal{F}_n\right] = \left(\frac{\mathbf{V}'\boldsymbol{W}\boldsymbol{V}}{n}\right)^{-1}\frac{1}{n}\mathbf{V}'\boldsymbol{W}\boldsymbol{z},$$

$$\text{COV}\left[\hat{\boldsymbol{\theta}}\,|\mathcal{F}_n\right] = \frac{\sigma_0^2}{n}\left(\frac{\mathbf{V}'\mathbf{W}\mathbf{V}}{n}\right)^{-1}\frac{\mathbf{V}'\mathbf{W}\mathbf{\Sigma}\boldsymbol{W}\boldsymbol{V}}{n}\left(\frac{\mathbf{V}'\mathbf{W}\mathbf{V}}{n}\right)^{-1},$$

so that the conditional mean squared error of $\sqrt{n}\hat{\boldsymbol{\theta}}$ is

$$\text{MSE}\left[\sqrt{n}\hat{\boldsymbol{\theta}}\,|\mathcal{F}_n\right] = E\left[\left(\sqrt{n}\left(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)\right)^2\right]$$

$$= \left(\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{V}}{n}\right)^{-1}\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{z}}{\sqrt{n}}\frac{\boldsymbol{z}'\boldsymbol{W}\boldsymbol{V}}{\sqrt{n}}\left(\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{V}}{n}\right)^{-1} + \sigma_0^2\left(\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{V}}{n}\right)^{-1}\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{\Sigma}\boldsymbol{W}\boldsymbol{V}}{n}\left(\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{V}}{n}\right)^{-1}. \tag{1}$$

In terms of the rows $\boldsymbol{r}_i'$ of $\boldsymbol{R}$, these terms are

$$\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{V}}{n} = \sum_{i=1}^{p}\frac{n_i}{n}\cdot\frac{1}{n_i}\sum_{j=1}^{n_i} w_i\left(\boldsymbol{x}_{ij}\right)\boldsymbol{r}_i\left(\boldsymbol{x}_{ij}\right)\boldsymbol{r}_i'\left(\boldsymbol{x}_{ij}\right),$$

$$\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{\Sigma}\boldsymbol{W}\boldsymbol{V}}{n} = \sum_{i=1}^{p}\frac{n_i}{n}\cdot\frac{1}{n_i}\sum_{j=1}^{n_i} w_i^2\left(\boldsymbol{x}_{ij}\right)\sigma_i^2\left(\boldsymbol{x}_{ij}\right)\boldsymbol{r}_i\left(\boldsymbol{x}_{ij}\right)\boldsymbol{r}_i'\left(\boldsymbol{x}_{ij}\right),$$

$$\frac{1}{\sqrt{n}}\boldsymbol{V}'\boldsymbol{W}\boldsymbol{z} = \sum_{i=1}^{p}\frac{n_i}{n}\cdot\frac{1}{n_i}\sum_{j=1}^{n_i}\left\{w_i\left(\boldsymbol{x}_{ij}\right)\boldsymbol{r}_i\left(\boldsymbol{x}_{ij}\right)\cdot\left(\sqrt{n}\psi_{n,i}\left(\boldsymbol{x}_{ij}\right)\right)\right\}.$$

As each $n_i \to \infty$, by the Strong Law of Large Numbers we have that for functions $\phi_i\left(\boldsymbol{x}\right)$,

$$\frac{n_i}{n}\cdot\frac{1}{n_i}\sum_{j=1}^{n_i}\phi_i\left(\boldsymbol{x}_{ij}\right) \overset{a.s.}{\to} P\left(\text{group } i\right)\cdot E\left[\phi_i\left(\boldsymbol{x}\right)|i\right] = \int_{\chi}\phi_i\left(\boldsymbol{x}\right)\rho_i\left(\boldsymbol{x}\right)m\left(\boldsymbol{x}\right)\mu\left(d\boldsymbol{x}\right). \tag{2}$$

From this observation it follows that

$$\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{V}}{n} \overset{a.s.}{\to} \boldsymbol{M}_{\boldsymbol{t}},$$
$$\frac{\boldsymbol{V}'\boldsymbol{W}\boldsymbol{\Sigma}\boldsymbol{W}\boldsymbol{V}}{n} \overset{a.s.}{\to} \boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}},$$
$$\frac{1}{\sqrt{n}}\boldsymbol{V}'\boldsymbol{W}\boldsymbol{z} \overset{a.s.}{\to} \boldsymbol{q}_{\boldsymbol{t},\psi};$$

these in (1) yield the result. Now define $\boldsymbol{U}_1 = \boldsymbol{U}\left(\boldsymbol{\rho},\boldsymbol{w},\boldsymbol{\sigma}\right) = \left[\left(\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}}\boldsymbol{M}_{\boldsymbol{t}}^{-1}\right)_{11}\right]^{-1}$, and let $\boldsymbol{U}_0 = \left(\oplus_{i=1}^{p}\frac{\sigma_i^2}{t_i}\int_{\chi} w_i\left(\boldsymbol{x}\right)m\left(\boldsymbol{x}\right)\mu\left(d\boldsymbol{x}\right)\right)^{-1}$ be the evaluation of $\boldsymbol{U}_1$ under (2). By (2),

$$\mathcal{L}_1\left(\boldsymbol{\rho},\boldsymbol{w};\,\psi,\boldsymbol{\sigma}\right) = \det\left\{\boldsymbol{\Pi}\boldsymbol{U}_1^{-1}\boldsymbol{\Pi}' + \boldsymbol{\Pi}\left(\left(\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{q}_{\boldsymbol{t},\psi}\right)_1\left(\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{q}_{\boldsymbol{t},\psi}\right)_1'\right)\boldsymbol{\Pi}'\right\}$$
$$= \left|\boldsymbol{\Pi}\boldsymbol{U}_1^{-1}\boldsymbol{\Pi}'\right|\cdot\left\{1 + \left(\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{q}_{\boldsymbol{t},\psi}\right)_1'\boldsymbol{\Pi}'\left(\boldsymbol{\Pi}\boldsymbol{U}_1^{-1}\boldsymbol{\Pi}'\right)^{-1}\boldsymbol{\Pi}\left(\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{q}_{\boldsymbol{t},\psi}\right)_1\right\}.$$

(The subscript 1 refers to the leading $p\times 1$ subvector.) In particular, $\mathcal{L}_1\left(\boldsymbol{\rho},\boldsymbol{w};\,\psi,\boldsymbol{\sigma}\right) \geq \left|\boldsymbol{\Pi}\boldsymbol{U}_1^{-1}\boldsymbol{\Pi}'\right|$. But under (1) we have, using (2), that $\left(\boldsymbol{M}_{\boldsymbol{\rho}_0,\boldsymbol{w}}^{-1}\boldsymbol{q}_{\boldsymbol{\rho}_0,\boldsymbol{w},\psi}\right)_1 = \boldsymbol{0}$, whence

$\mathcal{L}_1(\boldsymbol{t};\boldsymbol{\sigma}) = \left|\boldsymbol{\Pi}\boldsymbol{U}_0^{-1}\boldsymbol{\Pi}'\right|$, and it suffices to show that $\left|\boldsymbol{\Pi}\boldsymbol{U}_1^{-1}\boldsymbol{\Pi}'\right| \geq \left|\boldsymbol{\Pi}\boldsymbol{U}_0^{-1}\boldsymbol{\Pi}'\right|$; this in turn will follow if we can establish that

$$\boldsymbol{U}_0 \succeq \boldsymbol{U}_1, \tag{3}$$

where '$\succeq$' denotes the ordering by positive semidefiniteness. To show (3) we partition the relevant matrices as

$$\boldsymbol{M}_{\boldsymbol{t}} = \begin{pmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{pmatrix},\ \boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}}^{-1} = \begin{pmatrix} \boldsymbol{Q}^{11} & \boldsymbol{Q}^{12} \\ \boldsymbol{Q}^{21} & \boldsymbol{Q}^{22} \end{pmatrix},\ \boldsymbol{M}_{\boldsymbol{t}}\boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}}^{-1}\boldsymbol{M}_{\boldsymbol{t}} = \begin{pmatrix} \boldsymbol{J}_{11} & \boldsymbol{J}_{12} \\ \boldsymbol{J}_{21} & \boldsymbol{J}_{22} \end{pmatrix},$$

whence $\boldsymbol{U}_1 = \boldsymbol{J}_{11} - \boldsymbol{J}_{12}\boldsymbol{J}_{22}^{-1}\boldsymbol{J}_{21}$. It is somewhat evident that (3) now follows from

$$\boldsymbol{J}_{11} \preceq \boldsymbol{U}_0. \tag{4}$$

We calculate (using identities in Corollaries 1.4.1, 1.4.2 of Wiens (1985)) that

$$\begin{aligned}
\boldsymbol{J}_{11} &= \left\{ \begin{pmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{pmatrix} \boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}}^{-1} \begin{pmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \\ \boldsymbol{M}_{21} & \boldsymbol{M}_{22} \end{pmatrix} \right\}_{11} \\
&= \begin{pmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \end{pmatrix} \boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}}^{-1} \begin{pmatrix} \boldsymbol{M}_{11} \\ \boldsymbol{M}_{21} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} \end{pmatrix} \left\{ \begin{pmatrix} \boldsymbol{Q}_{11}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix} + \begin{pmatrix} -\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12} \\ \boldsymbol{I} \end{pmatrix} \boldsymbol{Q}^{22} \begin{pmatrix} -\boldsymbol{Q}_{21}\boldsymbol{Q}_{11}^{-1} & \boldsymbol{I} \end{pmatrix} \right\} \begin{pmatrix} \boldsymbol{M}_{11} \\ \boldsymbol{M}_{21} \end{pmatrix} \\
&= \boldsymbol{M}_{11}\boldsymbol{Q}_{11}^{-1}\boldsymbol{M}_{11} + \left(\boldsymbol{M}_{12} - \boldsymbol{M}_{11}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12}\right)\boldsymbol{Q}^{22}\left(\boldsymbol{M}_{12} - \boldsymbol{M}_{11}\boldsymbol{Q}_{11}^{-1}\boldsymbol{Q}_{12}\right)' \\
&\preceq \boldsymbol{M}_{11}\boldsymbol{Q}_{11}^{-1}\boldsymbol{M}_{11} = \boldsymbol{U}_0,
\end{aligned}$$

where the final equality follows from the assumption of constant variance functions applied to (6). This proves (4). ii) By Lemma 1 of Wiens (2019), $\left|\boldsymbol{\Pi}\boldsymbol{U}_0^{-1}\boldsymbol{\Pi}'\right| = \boldsymbol{1}_p'\boldsymbol{U}_0\boldsymbol{1}_p p\left|\boldsymbol{U}_0\right|$, and (6) follows.

Using Theorem 1 of Wiens (2024) we have

$$\begin{aligned}
\text{MSE}_i(\boldsymbol{x}) &= \lim_{n\to\infty} E\left[\left\{\sqrt{n}\left(\boldsymbol{r}_i'(\boldsymbol{x})\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) - \psi_{n,i}(\boldsymbol{x})\right)\right\}^2\right] \\
&= \boldsymbol{r}_i'(\boldsymbol{x})\left\{\boldsymbol{M}_{\boldsymbol{t}}^{-1}\left(\boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}} + \boldsymbol{q}_{\boldsymbol{t},\psi}\boldsymbol{q}_{\boldsymbol{t},\psi}'\right)\boldsymbol{M}_{\boldsymbol{t}}^{-1}\right\}\boldsymbol{r}_i(\boldsymbol{x}) \\
&\quad - 2\lim_{n\to\infty}\left\{E\left[\sqrt{n}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)|\mathcal{F}_n\right]'\boldsymbol{r}_i(\boldsymbol{x})\psi_{n,i}(\boldsymbol{x})\right\} + \psi_i^2(\boldsymbol{x}),
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\text{IMSE}_i &= E\left[\text{MSE}_i(\boldsymbol{x})\, m(\boldsymbol{x})\right] \\
&= tr\left\{\boldsymbol{M}_{\boldsymbol{t}}^{-1}\left(\boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}} + \boldsymbol{q}_{\boldsymbol{t},\psi}\boldsymbol{q}_{\boldsymbol{t},\psi}'\right)\boldsymbol{M}_{\boldsymbol{t}}^{-1}\cdot E\left[\boldsymbol{r}_i(\boldsymbol{x})\boldsymbol{r}_i'(\boldsymbol{x})\right]\right\} \\
&\quad - 2\lim_{n\to\infty}\left\{E\left[\sqrt{n}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)|\mathcal{F}_n\right]' E\left[\boldsymbol{r}_i(\boldsymbol{x})\,\psi_{n,i}(\boldsymbol{x})\right]\right\} + E\left[\psi_i^2(\boldsymbol{x})\right].
\end{aligned}$$

Using (3),

$$\sum_{i=1}^{p} \mathrm{IMSE}_i = tr\left\{\boldsymbol{M}_{\boldsymbol{t}}^{-1}\left(\boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}} + \boldsymbol{q}_{\boldsymbol{t},\psi}\,\boldsymbol{q}_{\boldsymbol{t},\psi}'\right)\boldsymbol{M}_{\boldsymbol{t}}^{-1}E\left[\boldsymbol{R}'\left(\boldsymbol{x}\right)\boldsymbol{R}\left(\boldsymbol{x}\right)\right]\right\}$$
$$-2\lim_{n\to\infty}\left\{E\left[\sqrt{n}\left(\hat{\gamma}-\gamma\right)|\mathcal{F}_n\right]'E\left[\boldsymbol{g}\left(\boldsymbol{x}\right)\psi_n\left(\boldsymbol{x}\right)\right]\right\} + E\left[\|\,\psi(\boldsymbol{x})\|^2\right],$$

and the result follows.

(ii) With

$$\mathcal{B}\left(\psi\right) \overset{def}{=} tr\left\{\boldsymbol{M}_{\boldsymbol{t}}^{-1}\left(\boldsymbol{q}_{\boldsymbol{t},\psi}\,\boldsymbol{q}_{\boldsymbol{t},\psi}'\right)\boldsymbol{M}_{\boldsymbol{t}}^{-1}\right\} = \left\|E\left[\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{R}'\left(\boldsymbol{x}\right)\boldsymbol{A}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\quad\left(\boldsymbol{x}\right)\right]\right\|^2,$$

we first show that, subject to (3) and (4),

$$\max\left\{\mathcal{B}\left(\psi\right) + E\left[\|\,\psi(\boldsymbol{x})\|^2\right]\right\} = \eta^2 ch_{\max}\left\{\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{K}_{\boldsymbol{t}}\boldsymbol{M}_{\boldsymbol{t}}^{-1}\right\}. \tag{6}$$

Since $\mathcal{B}\left(\psi\right)$ increases if ψ is multiplied by a constant exceeding unity, we may assume equality in (6).

Denote by Ψ the class of functions $\psi\left(\boldsymbol{x}\right)$, $\boldsymbol{x}\in\chi$ constrained by (1) and (4). Define

$$\boldsymbol{\Phi}_{\boldsymbol{t}}\left(\boldsymbol{x}\right) = \boldsymbol{A}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\boldsymbol{R}\left(\boldsymbol{x}\right) - \boldsymbol{R}\left(\boldsymbol{x}\right)\boldsymbol{M}_{\boldsymbol{t}} : p\times s,$$

and assume that $\boldsymbol{\rho}, \boldsymbol{w}$ are such that $E\left[\boldsymbol{\Phi}_{\boldsymbol{t}}'\left(\boldsymbol{x}\right)\boldsymbol{\Phi}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\right]$ is nonsingular. (If not, take a perturbation – our final result does not require the nonsingularity of this matrix.) It follows from the definition of $\boldsymbol{M}_{\boldsymbol{t}}$, together with (5), that

$$E\left[\boldsymbol{\Phi}_{\boldsymbol{t}}'\left(\boldsymbol{x}\right)\boldsymbol{\Phi}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\right] = E\left[\boldsymbol{R}\left(\boldsymbol{x}\right)\boldsymbol{A}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\boldsymbol{\Phi}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\right] = \boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^{2}.$$

Define

$$\begin{aligned}\boldsymbol{\Theta}_{\boldsymbol{t}}\left(\boldsymbol{x}\right) &= \boldsymbol{\Phi}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\left[E\left[\boldsymbol{\Phi}_{\boldsymbol{t}}'\left(\boldsymbol{x}\right)\boldsymbol{\Phi}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\right]\right]^{-1/2}\\ &= \boldsymbol{\Phi}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^{2}\right]^{-1/2} : p\times s,\end{aligned}$$

and consider the class $\boldsymbol{\Psi}_0 = \left\{\psi_{\boldsymbol{\beta}}\left(\boldsymbol{x}\right) = \eta\boldsymbol{\Theta}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\boldsymbol{\beta}\,|\,\left\|\boldsymbol{\beta}_{s\times1}\right\| = 1\right\}$. Note that

(1) $E\left[\boldsymbol{\Theta}_{\boldsymbol{t}}'\left(\boldsymbol{x}\right)\boldsymbol{\Theta}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\right] = \boldsymbol{I}_s$,

(2) $E\left[\boldsymbol{R}'\left(\boldsymbol{x}\right)\boldsymbol{\Theta}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\right] = \boldsymbol{0}_{s\times s}$.

By (1) and (2), $\boldsymbol{\Psi}_0\subset\Psi$ and all members of $\boldsymbol{\Psi}_0$ attain equality in (6). We claim that for any $\psi\in\Psi$ there is $\psi_{\boldsymbol{\beta}}\in\boldsymbol{\Psi}_0$ with $\mathcal{B}\left(\psi_{\boldsymbol{\beta}}\right)\geq\mathcal{B}\left(\psi\right)$, so that

$$\sup_{\Psi}\mathcal{B}\left(\psi\right) = \sup_{\boldsymbol{\beta}}\mathcal{B}\left(\psi_{\boldsymbol{\beta}}\right). \tag{7}$$

For this, let $\psi \in \Psi$ be arbitrary and define

$$\begin{aligned} \boldsymbol{\alpha}_\psi &= E\left[\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{R}'\left(\boldsymbol{x}\right)\boldsymbol{A}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\ \psi(\boldsymbol{x})\right], \\ \boldsymbol{\beta}_\psi &= \frac{\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]^{1/2}\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{\alpha}_\psi}{\left\|\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]^{1/2}\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{\alpha}_\psi\right\|}, \\ \psi_* &= \eta\boldsymbol{\Theta}_{\boldsymbol{t}}\left(\boldsymbol{x}\right)\boldsymbol{\beta}_\psi. \end{aligned}$$

Then $\psi_* \in \boldsymbol{\Psi}_0$. Since $\mathcal{B}\left(\psi\right) = \|\boldsymbol{\alpha}_\psi\|^2$, (7) will follow from

$$\left\|\boldsymbol{\alpha}_{\psi_*}\right\|^2 \geq \|\boldsymbol{\alpha}_\psi\|^2. \tag{8}$$

First, from the Cauchy-Schwarz inequality and the identities above we obtain

$$\|\boldsymbol{\alpha}_\psi\|^2\left\|\boldsymbol{\alpha}_{\psi_*}\right\|^2 \geq \left(\boldsymbol{\alpha}'\ \boldsymbol{\alpha}_{\psi_*}\right)^2 = \eta^2\left\|\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]^{1/2}\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{\alpha}_\psi\right\|^2. \tag{9}$$

Similarly,

$$\begin{aligned} \eta^2 &\geq \left\{E\left[\|\ \psi(\boldsymbol{x})\|^2\right] \cdot E\left[\|\ \psi_*\left(\boldsymbol{x}\right)\|^2\right]\right\}^{1/2} \\ &\geq \left|E\left[\psi'\left(\boldsymbol{x}\right)\ \psi_*\left(\boldsymbol{x}\right)\right]\right| \\ &= \eta\frac{\|\boldsymbol{\alpha}_\psi\|^2}{\left\|\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]^{1/2}\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{\alpha}_\psi\right\|}, \end{aligned}$$

so that

$$\left\|\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]^{1/2}\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{\alpha}_\psi\right\| \geq \frac{\|\boldsymbol{\alpha}_\psi\|^2}{\eta}. \tag{10}$$

From (9) and (10),

$$\|\boldsymbol{\alpha}_\psi\|^2\left\|\boldsymbol{\alpha}_{\psi_*}\right\|^2 \geq \|\boldsymbol{\alpha}_\psi\|^4,$$

yielding (8) and hence (7).

We must now maximize

$$\mathcal{B}\left(\psi_\beta\right) = \eta^2\boldsymbol{\beta}'\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]^{1/2}\boldsymbol{M}_{\boldsymbol{t}}^{-2}\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]^{1/2}\boldsymbol{\beta}$$

over $\|\boldsymbol{\beta}\| = 1$, obtaining

$$\begin{aligned} \max\mathcal{B}\left(\psi\right) &= \eta^2 ch_{\max}\left\{\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]^{1/2}\boldsymbol{M}_{\boldsymbol{t}}^{-2}\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]^{1/2}\right\} \\ &= \eta^2 ch_{\max}\left\{\boldsymbol{M}_{\boldsymbol{t}}^{-1}\left[\boldsymbol{K}_{\boldsymbol{t}} - \boldsymbol{M}_{\boldsymbol{t}}^2\right]\boldsymbol{M}_{\boldsymbol{t}}^{-1}\right\} \\ &= \eta^2 ch_{\max}\boldsymbol{M}_{\boldsymbol{t}}^{-1}\boldsymbol{K}_{\boldsymbol{t}}\boldsymbol{M}_{\boldsymbol{t}}^{-1} - \eta^2, \end{aligned}$$

from which (6) and then (8) follow.

It remains to establish (9). Denote by $\boldsymbol{d}(\boldsymbol{x})$ the p-vector with (non-negative) elements

$$\begin{aligned} d_i(\boldsymbol{x}) &= \left(\left[\oplus_{i=1}^{p} \rho_i(\boldsymbol{x}) w_i^2(\boldsymbol{x})\right] \boldsymbol{R}(\boldsymbol{x}) \boldsymbol{M}_{\boldsymbol{t}}^{-2} \boldsymbol{R}'(\boldsymbol{x})\right)_{ii} \\ &= w_i(\boldsymbol{x}) \left[\boldsymbol{A}_{\boldsymbol{t}}(\boldsymbol{x}) \boldsymbol{R}(\boldsymbol{x}) \boldsymbol{M}_{\boldsymbol{t}}^{-2} \boldsymbol{R}'(\boldsymbol{x})\right]_{ii} \\ &= w_i(\boldsymbol{x}) \boldsymbol{L}_{\boldsymbol{t},ii}(\boldsymbol{x}). \end{aligned}$$

Then using (10) and the Cauchy-Schwarz inequality,

$$\begin{aligned} tr\left\{\boldsymbol{M}_{\boldsymbol{t}}^{-1} \boldsymbol{Q}_{\boldsymbol{t},\boldsymbol{\sigma}} \boldsymbol{M}_{\boldsymbol{t}}^{-1}\right\} &= \int_{\chi} \boldsymbol{d}'(\boldsymbol{x}) \boldsymbol{\sigma}^2(\boldsymbol{x}) m(\boldsymbol{x}) \mu(d\boldsymbol{x}) \leq \sigma_0^2 \sqrt{E\left[\|\boldsymbol{d}(\boldsymbol{x})\|^2\right]} \\ &= \sigma_0^2 \sqrt{E\left[\sum_{i=1}^{p} w_i^2(\boldsymbol{x}) \left\{\boldsymbol{L}_{\boldsymbol{t},ii}(\boldsymbol{x})\right\}^2\right]}, \end{aligned}$$

and thus bound is attained by

$$\boldsymbol{\sigma}_*^2(\boldsymbol{x}) = \sigma_0^2 \frac{\boldsymbol{d}(\boldsymbol{x})}{\sqrt{E\left[\|\boldsymbol{d}(\boldsymbol{x})\|^2\right]}}.$$

Now Theorem 1 is immediate.

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