Torque balance, Taylor’s constraint and torsional oscillations in a numerical model of the geodynamo

Mathieu Dumberry*, Jeremy Bloxham
Department of Earth and Planetary Sciences, Harvard University, Cambridge, MA 02138, USA

Accepted 11 July 2003

Abstract

Theoretical considerations and observations suggest that, to a first approximation, the Earth’s dynamo is in a quasi-Taylor state, where the axial Lorentz torque on cylindrical surfaces co-axial with the rotation axis vanishes, except for the part involved in torsional oscillations. The latter are rigid azimuthal accelerations of cylindrical surfaces which oscillate with typical periods of decades. We present a solution of a numerical model of the geodynamo in which rigid accelerations of cylinder surfaces are observed. The underlying dynamic state in the model is not a Taylor state because the Reynolds stresses and viscous torque remain large and provide an effective way to balance a large Lorentz torque. This is a consequence of the limited parameter regime which can be attained numerically. Nevertheless, departures in the torque equilibrium are primarily counterbalanced by rigid accelerations of cylindrical surfaces, which, in turn, excite rigid azimuthal oscillations of the surfaces. We show that the azimuthal motion is indeed quasi-rigid, though the torsional oscillations that are produced in the model probably differ from those in the Earth’s core because of the large influence of the Reynolds stresses on their dynamics. We also show that the continual excitation of rigid cylindrical accelerations is produced by the advection of the non-axisymmetric structure of the fields by a mean differential rotation of the cylindrical surfaces which produces disconnections and reconnections and continual fluctuations in the Lorentz torque and Reynolds stresses. We propose that the torque balance in Earth’s core may evolve in a similar chaotic fashion, except that the influence of the Reynolds stresses is probably weaker. If this is the case, the Lorentz torque on a cylindrical surface is continually fluctuating, even though its time-averaged value vanishes and satisfies Taylor’s constraint. Rigid accelerations of cylindrical surfaces are continually excited by the fluctuations in the Lorentz torque, and the torsional oscillations observed in the geomagnetic data are a mixture of forced and free oscillations.

Keywords: Torsoal oscillations; Geodynamo; Magnetohydrodynamics; Core

1. Introduction

We believe that the magnetic field of the Earth is generated and maintained against decay by convective motions in the fluid outer core, a process known as the geodynamo. However, the exact details of the dynamics involved are unknown. This is partly because, although we have a several 100-year record of the global geomagnetic field at the core–mantle boundary (CMB) from which we can build flow maps, we have little information on the fluid velocity and magnetic field inside the core.

Indeed, the only part of the flow in the interior of the core that can be inferred from observation is a time-dependent zonal wind component which consists of
azimuthal motions of rigid cylindrical surfaces aligned with the rotation axis. Because of its simple geometry, this rigid azimuthal flow is easily obtained from flow maps at the CMB: it represents the part of the axisymmetric azimuthal velocity that is symmetric about the equator. Since the angular momentum of the whole Earth must be conserved, changes in angular momentum of the core calculated from the time-dependent rigid azimuthal flows should correspond with changes in the angular momentum of the mantle measured in terms of variations in length-of-day. The agreement between the two is particularly good for the last few decades (Jault et al., 1988; Jackson et al., 1993), indicating that the time-varying part of the recovered rigid azimuthal flow, which has the form of a wave with decade periods propagating in the direction perpendicular to the rotation axis (Zatman and Bloxham, 1997, 1998; Pais and Hulot, 2000; Hide et al., 2000), is well determined.

These decadal oscillations of rigid cylindrical surfaces place a constraint on theoretical models of the geodynamo. One model that indeed predicts the occurrence of these time-dependent rigid azimuthal motions is that of a “Taylor state dynamo” (Taylor, 1963) in which the dynamics are controlled by a morphological constraint on the magnetic field: the azimuthal component of the Lorentz force must vanish when integrated on cylindrical surfaces coaxial with the rotation axis. When this constraint is not satisfied, rigid accelerations of cylindrical surfaces are produced in order to reestablish a Taylor state. In the process, rigid accelerations excite azimuthal oscillations of these rigid surfaces, which are referred to as torsional oscillations (Braginsky, 1970).

Observations thus suggest that the current dynamic state of the Earth’s core is that of a “quasi-Taylor state”, where the magnetic field is organized in such a way that the azimuthal Lorentz force integrated over cylindrical surfaces vanishes, except for the part involved in torsional oscillations. Important information about the dynamics in the core can then be extracted from the torsional oscillations. For instance, since the restoring force for the torsional oscillations is provided by the component of the magnetic field perpendicular to the rotation axis, we can then obtain information about the magnetic field strength inside the core for which we otherwise have no direct observation (Zatman and Bloxham, 1997, 1998). Similarly, the dissipation of the oscillating motion has revealed information about coupling between the core and the mantle at the CMB (Buffett, 1998; Zatman and Bloxham, 1999).

However, many questions remain unanswered. For instance, are the currently observed torsional oscillations being excited at their typical amplitudes, or are they at a maximum or minimum? A recent flow inversion suggests that the amplitude of higher wavenumber torsional oscillations has been growing for the last 50 years (Bloxham et al., 2002). A more general question is whether torsional oscillations are a permanent feature of the dynamo or if they are simply a transient between two quasi-equilibrium Taylor states. For many such questions, observations alone may be insufficient to provide answers. This is in part because our record of geomagnetic field data from which core flows can be built covers only the last 160 years, which corresponds to 2 or perhaps 3 full periods of the fundamental mode of torsional oscillations. Therefore, it is difficult to assess whether the observed torsional oscillations are representative of the average state of the geodynamo.

Additionally, the excitation of torsional oscillations remains an open question. An efficient excitation mechanism would appear to be required since their damping time may be short (Zatman and Bloxham, 1997). Mechanisms that have been proposed include instabilities in MAC waves (Braginsky, 1970), mean torques generated by the nonlinear interaction of turbulent fluctuations in the flow (Braginsky, 1984), and coupling of the main flow in the core to a stratified surface layer at the CMB (Braginsky, 1993, 1999), in which instabilities could excite natural oscillations resembling MAC waves, which could, in turn, excite torsional oscillations in the bulk of the core. An alternative mechanism is axial gravitational coupling between the inner core and the mantle (Buffett, 1996): torsional oscillations couple the mantle, the inner core and the outer core, forming a coupled oscillating system. The natural frequencies of this system are excited by fluctuating axial electromagnetic torques on the inner core.

A different class of mechanisms is in situ excitation caused by the changes of the magnetic field in the core as originally envisaged by Taylor (1963). If the core is in a Taylor state, then the axial Lorentz torque is large everywhere on a cylindrical surface but cancels when
averaged over the entire surface of the cylinder. Small local changes in the magnetic field due to the underlying evolution of the magnetostrophic balance could then produce a large Lorentz torque. The latter induces a rigid acceleration of the cylindrical surface which initiates torsional oscillations. Whether torsional oscillations are excited by this mechanism is a difficult question to address with observations since we do not have access to the convective flow and magnetic field morphology inside the core.

Perhaps a more promising avenue to study torsional oscillations is through three-dimensional self-consistent numerical models of the geodynamo. Since the pioneering work of Glatzmaier and Roberts (1995), there exists now a large number of numerical models (for a recent review, see Dormy et al. (2000)). These models provide direct access to all fields at any location and thus offer a means to examine the details of the dynamics. Moreover, simulations can extend in time far longer than the time span of the historical record. Of course, the danger is that the numerical models might provide a correct solution to a mathematical problem which is not representative of the dynamics of the core. This is to some extent the current situation in geodynamo modeling as computational limitations force the use of a parameter regime that is different from the Earth’s core (Glatzmaier and Roberts, 1995; Kuang and Bloxham, 1997; Dormy et al., 2000; Glatzmaier, 2002). However, as computing resources and numerical methods improve, our confidence in the results provided by these models will also improve.

In this work, we present a study of torsional oscillations in one such numerical model, the Kuang–Bloxham model (Kuang and Bloxham, 1997, 1999). This model was built specifically in order to produce a dynamo which captures the dynamics of a Taylor state and is, therefore, well suited for our investigation. In a recent study, Kuang (1999) showed that the azimuthal force balance on cylindrical surfaces is such that the Lorentz force is mostly balanced by inertial forces, both of which are twice as large as viscous forces, hence suggesting the presence of strong torsional oscillations. Here, we investigate this question in more detail.

The primary goal of the present study is to demonstrate that the numerical model captures the essence of an inertial state, with rigid accelerations being produced in response to unbalanced axial torques on cylindrical surfaces. We will show that this is indeed the case. A second goal is then to investigate how the ensuing azimuthal oscillations of rigid surfaces may resemble or differ from torsional oscillations observed in the Earth’s core. Additionally, we want to understand the way in which the torque balance is disrupted and, hence, how rigid accelerations are excited. Our hope is that the dynamics observed in the model may help to illuminate some aspects of a Taylor state dynamo and torsional oscillations. In the present work, we are only investigating one model calculation with a specific set of parameters. This is sufficient for a first study on torsional oscillations in numerical models since our primary concern is to demonstrate that they are indeed emerging as part of the solution. A more detailed analysis of the way that the torque balance is altered by the parameters of the calculation is left for future study. We also focus our investigation in this work on the region of the core outside the cylinder tangential to the inner core (the tangent cylinder). This is the region in which torsional oscillations are observed in the data and where it is most certain that the quasi-Taylor state applies.

One of the important results emerging from the model is that the nature of the equilibrium force balance on the cylinders is such that the contributing forces are continually fluctuating about their mean. One cause of these fluctuations is a large differential rotation between cylinders, which is continually shearing the non-axisymmetric velocity and magnetic field structures. This causes local changes in the Lorentz torque and Reynolds stresses and a chaotic evolution of the total torque on cylindrical surfaces. Continual chaotic rigid accelerations are then required to balance this torque. Results of the model, therefore, support the excitation of rigid azimuthal oscillations by in situ dynamics.

While we recognize the danger of relating the results of a numerical model operating with different parameters from the Earth’s core, we argue that torsional oscillations in the core may result from a similar process. Under this scenario, the geometry of the magnetic field in the core is such that the time-averaged Lorentz torque vanishes (i.e. satisfies Taylor’s constraint), but the instantaneous Lorentz torque is continually fluctuating in a chaotic way
about this time-averaged Taylor state due to interactions between the non-axisymmetric part of the field and the mean differential rotation flow. The chaotic evolution of the Lorentz torque provides a continual excitation for free oscillations which propagate on top of this dynamic background. The observed torsional oscillations would then represent a mixture of free and forced oscillations.

This work is organized as follows. In the next section we revisit the dynamical constraints on cylindrical surfaces that lead to a quasi-Taylor state. This section provides the theoretical basis for the interpretation of the numerical results. Similar treatments can be found in the literature, for instance in reviews by Roberts and Soward (1992), Fearn (1994, 1998), and Bloxham (1998). In Section 3, we discuss the limitations of the numerical model and how these limitations affect the prospect of capturing the dynamics of torsional oscillations. We then investigate the dynamics of the torque balance that emerges from the numerical model. In Section 4, we focus on the rigid azimuthal oscillations, first by showing that the motion is quasi-rigid and then by testing if these rigid oscillations correspond to torsional oscillations in the model. Finally, we discuss in Section 5 how the results of the numerical model may be relevant for the Earth’s core and consider future strategies to produce torsional oscillations in the numerical model.

2. Dynamical torque balance on cylindrical surfaces

2.1. Taylor’s constraint

As first demonstrated by Taylor (1963), when one integrates the azimuthal component of the momentum equation over a cylindrical surface aligned with the rotation axis, some terms vanish from the force balance. The pressure gradient is identically zero, the Coriolis force vanishes if the fluid is assumed incompressible, and the buoyancy force is also zero because it has no azimuthal component. There remains a balance between inertial forces, Lorentz forces and viscous forces, given, in non-dimensional form, by

$$R_e \int_S \left( \frac{\mathbf{u} \cdot \nabla \mathbf{u}}{\eta} \right)_\phi d\Sigma + R_o \int_S (\mathbf{u} \cdot \nabla \mathbf{u})_\phi d\Sigma = \int_S \left( (\nabla \times \mathbf{B}) \times \mathbf{B} \right)_\phi d\Sigma + E \int_S (\nabla^2 \mathbf{u})_\phi d\Sigma \quad (1)$$

Here, $R_o$ is the Rossby number, $E$ is the Ekman number, $\mathbf{u}$ is the velocity field, $\mathbf{B}$ is the magnetic field, $t$ is time measured in units of magnetic diffusion timescale, and $d\Sigma = \phi d\phi dz$, where $(\phi, \theta, z)$ are the usual cylindrical coordinates. The azimuthal force on a cylindrical surface is proportional to the axial torque. For convenience, we will refer to (1) as the torque balance. The Rossby and Ekman numbers represent respectively the importance of inertial forces and viscous forces relative to the Coriolis force, and are defined by

$$R_o = \frac{\Omega r_o^2}{2D^2}, \quad E = \frac{\nu}{2D^2} \quad (2)$$

where $r_o$ is the radius of the core, $\Omega$ is the frequency of Earth’s rotation, $\eta$ is the magnetic diffusivity and $\nu$ is the kinematic viscosity. The typical scales for length, time and magnetic field in the above non-dimensionalization are respectively, $L = r_o$, $\tau_o = r_o^2/\eta$ and $B = \sqrt{\eta \mu_0\rho}$, where $\rho$ is the mean density and $\mu$ is the magnetic permeability. Typical values for the Rossby and Ekman numbers are $R_o \approx 10^{-9}$ and $E \approx 10^{-15}$ (for molecular values of the viscosity). If the scale of the magnetic field inside the core is $B(1)$, which corresponds to a field of about 2 mT, then the inertial and viscous term in (1) are both vanishingly small with respect to the Lorentz term and,

$$\int_S \left( (\nabla \times \mathbf{B}) \times \mathbf{B} \right)_\phi d\Sigma = 0. \quad (3)$$

The morphology of the magnetic field inside the core is such that the torque from the Lorentz force must vanish when integrated over a cylindrical surface coaxial with the rotation axis. This result is known as Taylor’s constraint (Taylor, 1963) and solutions that satisfy (3) are said to represent a Taylor state.

2.2. Ekman state

Evidently, the Lorentz torque at any given time is not required to vanish identically but rather to have a magnitude on the order of the largest of the small terms in (1). One limiting case is obtained by balancing the
The viscous forces are most important in thin boundary layers at the CMB, and produce a secondary poloidal flow through the whole core as a result of Ekman pumping (Greenspan, 1968; Pedlosky, 1987). The viscous drag at the boundaries is thus communicated through the whole cylinder. The force balance in this case is given by

$$\frac{1}{4\pi s} \int \left( (\nabla \times B) \times B \right) d\Sigma = \frac{E^{1/2}}{(1 - x^2)^{3/4}} V_\phi(s),$$

where $V_\phi(s)$ is the rigid azimuthal velocity of the cylinder surface (Roberts and Soward, 1992; Fearn, 1994; Hollerbach, 1996). In the dynamo theory literature, $V_\phi(s)$ is usually referred to as the geostrophic flow and we adopt this terminology throughout this paper. (We note that this should not be confused with the terminology used in core flow modeling, where “geostrophic flow” is employed in its more general sense to imply any velocity field which satisfies a balance between the Coriolis force and pressure gradients.) Eq. (4) is no longer valid for $s \approx 1$. When (4) applies, the geodynamo is said to be in an Ekman state and changes in the magnetic field are balanced by a viscous readjustment of the geostrophic flow. Theoreticalexaminations of the Ekman and viscous torque are only possible with a magnetic field amplitude of $O(E^{3/4})$ (Fearn, 1994). Even if turbulent values of the viscosity are adopted ($E \approx 10^{-3}$), this scaling implies a magnetic field amplitude of about 0.1 G in dimensional units, two orders of magnitude smaller than the observed field at the CMB (e.g. Bloxham and Gubbins, 1985). The Ekman state is, therefore, not the appropriate balance for the Earth’s core because the viscous torque alone cannot provide the balance for a non-zero Lorentz torque while maintaining a magnetic field of $O(1)$.

2.3. Model Z

Braginsky (1975, 1978, 1994) showed an alternative magnetic field configuration for which the Lorentz torque is balanced by dissipation at the CMB. In this model, called “Model Z”, $B_z$ and $B_\phi$ are both $O(1)$, while $B_r$ is much smaller. The Lorentz torque in the bulk of the fluid is very small simply because $B_r$, on which the torque depends, is very small. The small Lorentz torque in the bulk of the core is balanced by viscous friction at the boundary, just as above, or by “magnetic friction” (Braginsky, 1970, 1988) if there is an electrically conducting layer at the base of the mantle. The advection of poloidal magnetic field lines by a geostrophic flow $V_\phi(s)$ is resisted by the finite time the field takes to diffuse through the layer. This produces a Lorentz force at the boundary which opposes $V_\phi(s)$ similar to a viscous friction effect.

It has been shown that if one keeps the acceleration term in the force balance, the scaling of Model Z breaks down when the spin-up timescale is longer than the timescale for inertial adjustment (Jault, 1995). For the Earth’s core, the viscous spin-up timescale is on the order of $10^4$ years (Gubbins and Roberts, 1987), while the magnetic spin-up timescale may be as short as 30 years if the conductance of the layer is as high as $10^8$ S (Buffett, 1998). Hence, the decade timescale variations of the rigid azimuthal velocity observed in the flow may represent this adjustment through magnetic friction at the CMB in a Model Z dynamo. However, the amplitude of $V_\phi(s)$ that is required is an order of magnitude larger than the largest observed flow velocities (Braginsky, 1994). In addition, other observations also suggest that the Model Z is not the appropriate dynamic state of the core, as we argue in the next section.

2.4. Quasi-Taylor state and torsional oscillations

If the unbalanced part of the Lorentz torque in (3) is equilibrated by the acceleration term, then we have

$$R_z(c^2)^{1/2} \frac{d V_\phi(s, t)}{d\theta} = \frac{1}{4\pi s} \int \left( (\nabla \times B) \times B \right) d\Sigma.$$  (5)

The evolution of the magnetic field that leads to changes in the Lorentz torque produces a rigid azimuthal acceleration of the cylindrical surface at a rate proportional to the unbalanced part of the Lorentz torque. The rigid acceleration has a feedback on the Lorentz torque: it shears the $s$-component of the magnetic field, which induces azimuthal secondary magnetic fields and a Lorentz torque in the opposite direction. When the initial changes in the Lorentz
torque are exactly balanced by this induced torque, the Taylor state is reestablished. However, the acceleration carries the cylindrical surface past the equilibrium position, which then builds an excess of Lorentz force (Braginsky, 1970). The observed amplitude of the time-dependent geostrophic velocity of different cylindrical surfaces is large compared with the changes of the total angular momentum of the core (Jackson et al., 1993; Jault et al., 1996), suggesting that large exchanges of angular momentum occur within the core and that the time variations of the geostrophic flow are likely to be torsional oscillations with relatively little exchange of angular momentum at the CMB. Indeed, the radial structure and time dependency of the geostrophic flow indicates a superposition of waves (Zatman and Bloxham, 1997, 1998; Pais and Hulot, 2000; Hide et al., 2000). Several lines of evidence suggest that the dynamics since the inner core conductivity is large. A dissipation, either through viscous or magnetic friction, certainly plays a role, but the first order dynamical state of the core is that of torsional oscillations about a Taylor state.

We note that the above description of torsional oscillations propagating about a Taylor state is convenient conceptually, but that it is most probably incorrect. The reason is that the changes in the Lorentz torque which lead to torsional oscillations in the first place would need to be driven by a mechanism which acts suddenly and then plays no further role in the dynamics. It is more likely that the changes in the Lorentz torque are gradual and, in order to excite efficiently torsional oscillations, that the timescale of the changes is similar to the periods of the natural modes of oscillations. Hence, the background Lorentz torque is itself being driven away from a Taylor state on the same timescale as the torsional oscillations. In other words, torsional oscillations do not propagate about a Taylor state, but about a dynamic underlying Lorentz torque which, on a time average, satisfies Taylor’s constraint.

We note that Eq. (6) is no longer valid for the part providing the restoring force to the torsional oscillations. Damping of these oscillations will leave the system in a Taylor state. The natural periods of torsional oscillations depend on the strength of the steady $B_t$ field. When viscous effects and magnetic coupling at the boundaries are neglected, the normal modes of oscillations of $V^s(s)$ satisfy the non-dimensional relation

$$\frac{d}{ds}\left(R^2(1-s^2)(B^s_t) + \frac{1}{2} s \frac{d}{ds} (V^s(s)) \right) = - R_o \sigma^2 (1-s^2)^2 V^s(s), \quad (6)$$

where $\sigma$ is the non-dimensional frequency of oscillations such that $V^s(s) = V^s(s) \exp(-i \omega t)$, and $B^s_t$ and $B^s$ are the steady axisymmetric and non-axisymmetric parts of the $s$-field ($\bar{B}$ denotes the average over $z$ and the over-bar the average over $\phi$ of the cylindrical surface). We note that Eq. (6) is no longer valid for $s \approx 0$ or $s \approx 1$.

Additionally, the general case includes a contribution from the magnetic coupling with the mantle at the CMB which produces “magnetic friction”, as we described above in the context of Model Z. For the cylinders inside the tangent cylinder, there is also a similar contribution term from the magnetic coupling with the inner core which is non-negligible in the dynamics since the inner core conductivity is large. A simple order of magnitude estimate of the period of the fundamental mode is given by

$$\tau_o \approx \frac{\Phi^2}{2 R^2}, \quad (7)$$

where $\Phi$ is the “rms field”. Using $R_o = 2 \times 10^{-9}$, $B_o = 0.28 \times 5$ G, we find $\tau_o \approx 1.7 \times 10^{-4}$, which is about 25 years, and corresponds to the timescale of the observed variations in the geostrophic flow.

Several lines of evidence suggest that the dynamical torque balance in the Earth’s core is more likely to be that of torsional oscillations about a Taylor state as opposed to an Ekman state or Model Z. First, the observed amplitude of the time-dependent geostrophic velocity of different cylindrical surfaces is large compared with the changes of the total angular momentum of the core (Jackson et al., 1993; Jault et al., 1996), suggesting that large exchanges of angular momentum occur within the core and that the time variations of the geostrophic flow are likely to be torsional oscillations with relatively little exchange of angular momentum at the CMB. Indeed, the radial structure and time dependency of the geostrophic flow indicates a superposition of waves (Zatman and Bloxham, 1997, 1998), the only part of the magnetic field that we can directly observe. This suggests that, contrary to a Model Z type dynamo, $R_t \sim R_t$. Thirdly, the recovered wave motion suggests that the inertial acceleration exceeds “friction” on decade timescales (Zatman and Bloxham, 1999), indicating that the dynamics follow oscillatory behavior as in (5), rather than a diffusive readjustment as in (4) or as in Model Z. Dissipation, either through viscous or magnetic friction, certainly plays a role, but the first order dynamical state of the core is that of torsional oscillations about a Taylor state.

We note that the above description of torsional oscillations propagating about a Taylor state is convenient conceptually, but that it is most probably incorrect. The reason is that the changes in the Lorentz torque which lead to torsional oscillations in the first place would need to be driven by a mechanism which acts suddenly and then plays no further role in the dynamics. It is more likely that the changes in the Lorentz torque are gradual and, in order to excite efficiently torsional oscillations, that the timescale of the changes is similar to the periods of the natural modes of oscillations. Hence, the background Lorentz torque is itself being driven away from a Taylor state on the same timescale as the torsional oscillations. In other words, torsional oscillations do not propagate about a Taylor state, but about a dynamic underlying Lorentz torque which, on a time average, satisfies Taylor’s constraint.
2.5. Reynolds stresses

In the above discussion, the role of the advective torque has been omitted. This torque is usually assumed to be smaller than the friction torque and the rigid accelerations because it is multiplied by the small quantity $R_m$ and typical values of the velocity are also small. However, as we show in section 3 of this study, the advective torque, through the action of Reynolds stresses (Pedlosky, 1987), plays a leading role in providing a balance to the Lorentz torque in the numerical model. In the context of convection in a rapidly rotating spherical shell, large Reynolds stresses can be produced by a correlation between the non-axisymmetric components of the radial (in a cylindrical sense) and azimuthal flow (Busse and Hood, 1982; Zhang, 1992). This produces an axisymmetric axial torque on the cylindrical surfaces which can lead to a large amplitude zonal flow. In this paper, we use the term "Reynolds stresses" to describe the "axisymmetric torque produced by the Reynolds stresses." While the large Reynolds stresses that are produced in the numerical model may be a consequence of the parameter regime of the calculation, as we explain in the next section, this finding warrants further consideration of the Reynolds stresses, at the very least for the purpose of providing a framework in which we can cast our results.

From (1), the amplitude of the Reynolds stresses averaged over a cylindrical surface ($T_A$) scales as

$$T_A \approx \frac{R_m U^2}{L_u} \quad \text{where} \quad U' \text{ is the rms non-axisymmetric velocity over a cylindrical surface and } L_u \text{ is the length scale associated with the flow eddies. Using } R_m = 10^{-9}, U' = 400 \approx 10 \text{ km per year and } L_u = 1, \text{ the Reynolds stresses have an amplitude of } O(10^{-2}), \text{ still very much smaller than the Lorentz torque of } O(1) \text{ if no cancellations were to occur, Taylor’s constraint still applies.}

However, it is not immediately clear that the Reynolds stresses play a secondary role to the viscous torque ($T_v$) and/or the rigid inertial acceleration ($A_o$) in the torque balance. The amplitude ratios between the Reynolds stresses and the viscous torque, and between the Reynolds stresses and the inertial acceleration are given by

$$\frac{T_A}{T_v} \approx \frac{R_m U^2}{E U^2 V_o L_u} \quad \text{and} \quad \frac{T_A}{A_o} \approx \frac{U'^2 \tau_o}{V_o L_u} \quad (9)$$

Using $R_m = 10^{-9}$, $E = 10^{-15}$, $U' = 400$, $V_o = 400$, $L_u \sim 1$ and $\tau_o = \sqrt{R_m}/B$ with $B = 0.25$, we obtain

$$\frac{T_A}{T_v} \approx 10 \quad \text{and} \quad \frac{T_A}{A_o} \approx 0.05. \quad (10)$$

This suggests that the Reynolds stresses are, at the very least, as important as the viscous torque, unless a turbulent value of the viscosity is used. On the other hand, the prevailing balance between the Lorentz torque and the rigid acceleration appears to be maintained. However, we note that the above scaling of the Reynolds stresses rests on quantities that we do not know well. The rms non-axisymmetric value of $U'$ = 400 is based on typical values from flow maps at the CMB. The latter might represent attenuated values of the interior velocities, for instance due to the presence of an outer stably stratified layer (Takehiro and Lister, 2001). Similarly, we cannot infer flows at a length scale smaller than the resolved part of the geomagnetic field, which corresponds roughly to spherical harmonic degree 14 (Bloxham and Jackson, 1991). The actual length scale of some of the flow eddies may be smaller by an order of magnitude. Hence, it is not impossible that the Reynolds stresses may be of similar magnitude to the rigid accelerations in the core.

If this is the case, or if the Reynolds stresses are actually greater than the rigid accelerations, one direct consequence is that they may provide an effective way to balance part of the Lorentz torque, thus reducing the morphological constraint on the magnetic field. The system could then be in a state where it is the sum of the Lorentz and advective torque that nearly cancels over cylindrical surfaces. A second consequence is that the Reynolds stresses may then alter the dynamics of torsional oscillations. If departures from this modified Taylor state are balanced by rigid accelerations, the latter will induce changes in both the Lorentz torque and the Reynolds stresses. These changes will produce a feedback on the rigid accelerations in a way that may be more complicated than the restoring nature of the Lorentz force taken alone. In the numerical model, where we find that the Reynolds stresses are large, both of these effects are observed.
3. The torque balance in the geodynamo model

3.1. The parameter regime of the model

It is presently too computationally expensive to solve the geodynamo problem using Earth-like values of the Rossby and Ekman numbers. However, since the main force balance in the core is thought to be between the Coriolis force, the Lorentz force, pressure gradients, and buoyancy, the so-called magnetostrophic balance, then as long as \( R_o \) and \( E \) are sufficiently small, there is hope that the solution is in the asymptotic regime that corresponds to that of the Earth.

However, even if the leading order force balance is magnetostrophic, the larger values of \( R_o \) and \( E \) may alter the dynamical regime of the torque balance. As we discussed in section 2, observations suggest that the dynamical state of the core is one where departures from Taylor’s constraint are balanced preferentially by rigid accelerations instead of viscous torques. The effect of a larger \( E \) is obviously to increase the viscous drag on the cylindrical surfaces and move the system towards an Ekman state. Similarly, one consequence of a larger \( R_o \) is to shift the spectrum of torsional oscillations to longer periods (see (7)). If the natural periods of torsional oscillations approach the viscous spin-up timescale, the system is again displaced towards an Ekman state. An additional effect of a larger \( R_o \) is that the advective torque may become more important than the rigid accelerations and the viscous torques. This has the effect of weakening Taylor’s constraint and alters the dynamics of torsional oscillations, as we have described in the previous section. We note also that to derive the torsional oscillation equation in (6), Braginsky (1970) neglected the effects of magnetic field diffusion in the induction equation. This is justified for the Earth since the decade period of oscillations is short compared to the magnetic diffusion timescale. However, this is probably no longer true in the model where the natural periods of oscillations are much longer. The larger \( E \) and \( R_o \) used in the numerical simulations of the geodynamo, therefore, have important consequences for the dynamical regime of the solution, as it moves it away from a Taylor state with torsional oscillations. In order to alleviate some of these consequences, several strategies have been used in the Kuang-Bloxham geodynamo model, including the following two. First, in order to decrease the amplitude of the viscous torque, stress-free boundary conditions on tangential velocity are used at the spherical boundaries. The intended effect is to produce a viscous torque which depends only on the shear between cylindrical shells in the core, of \( \mathcal{O}(vqE) \) (see Eq. (1)), as opposed to one dominated by the Ekman pumping at the spherical boundaries, of \( \mathcal{O}(vqE^{1/2}) \) (see Eq. (4)). Second, only the axisymmetric part of the inertial term (including acceleration and advection) is retained in the force balance. This allows the system to capture the dynamics associated with rigid accelerations while maintaining a predominant magnetostrophic balance.

Despite these strategies, the viscous torque remains large, in part because of the use of hyperviscosity, but also simply because \( E \) remains too large. Similarly, the part of the advective force that is retained, the axisymmetric part, produces Reynolds stresses that are sufficiently large that they play a major role in the dynamics. As we will make clear in Sections 3.2 and 3.3, the dynamical regime in the numerical model (for the set of parameters chosen) is not one where torsional oscillations propagate about a mean Taylor state.

All results presented in this work were obtained with the following set of parameters for the numerical model: \( R_o = E = 2 \times 10^{-5} \); Rayleigh number \( R_A = \alpha g o \gamma^2 r^2 \Omega^2 \eta = 15000 \); Prandtl number \( P_I = \nu/\kappa = 1 \); magnetic Prandtl number \( P_m = \nu/\eta = 1 \); and Roberts number \( q = \kappa/\eta = 1 \), where \( \kappa \) is the thermal diffusivity, \( \alpha \) is the thermal expansion coefficient, \( g_0 \) is the gravitational acceleration at the CMB and \( h_Y \) is the heat flux at the inner core boundary.

3.2. Time-averaged torque equilibrium

In a Taylor state, the magnetic field is of \( \mathcal{O}(1) \) on cylindrical surfaces but it is organized in such a way that the Lorentz torque integrated over the whole surface vanishes. When the magnetic field changes, torsional oscillations are excited about the Taylor state. We therefore expect that the time variations of the Lorentz torque on a cylindrical surface will show large amplitude oscillations about a weak time-averaged value. We also expect that rigid accelerations of the cylindrical surface will counterbalance the fluctuations in the Lorentz torque. Hence by investigating the time-averaged torque balance and the fluctuations...
Fig. 1. Time-averaged azimuthal force on cylindrical surfaces as a function of cylinder radius. The solid line is the Lorentz torque \( T_L \); the dashed line is the advective torque \( T_A \); and the dashed-dotted line is the viscous torque \( T_V \). All quantities are dimensionless.

of this balance (in the next section) we can establish whether the numerical model is in a Taylor state.

In Fig. 1 we show the time-averaged torque balance as a function of cylinder radius. The Lorentz torque is \( O(1) \). For this particular simulation, the non-dimensional amplitude of the magnetic field in the core is also \( O(1) \), which indicates that little cancellation over the cylindrical surface occurs. This is because the advective and viscous torques are also \( O(1) \), a result of the large \( R_o \) and \( E \) used in the model. It is immediately clear that the time-averaged state deviates from a true Taylor state. Instead, the balance involves all three torques,

\[
T_A + T_L + T_V = 0,
\]

where

\[
T_A = \frac{-R_o}{4\pi(1 - s^2)^{1/2}} \int \Sigma (u \cdot \nabla u) \phi \, d\Sigma,
\]

\[
T_L = \frac{1}{4\pi(1 - s^2)^{1/2}} \int \Sigma ((\nabla \times B) \times B) \phi \, d\Sigma,
\]

\[
T_V = \frac{E}{4\pi(1 - s^2)^{1/2}} \int \Sigma (\nabla^2 u) \phi \, d\Sigma.
\]

A more comprehensive view of the dynamics that maintains this time-average balance is obtained by expanding \( T_A, T_L \) and \( T_V \) in the following way,

\[
T_A = -R_o \{ \bar{u}_s \bar{u}_s - \{ \bar{u}_0 \bar{u}_s \} + \{ \bar{u}_t \bar{u}_t \} - \{ \bar{u}_0 \bar{u}_t \} \},
\]

\[
T_L = \{ \bar{J}_t \bar{B}_t \} - \{ \bar{J}_s \bar{B}_s \} + \{ \bar{J}_z \bar{B}_z \} - \{ \bar{J}_t \bar{B}_t \},
\]

\[
T_V = E \left( \frac{\partial}{\partial s} \bar{u}_z - \frac{\partial}{\partial \phi} \bar{u}_z \right).
\]

Fig. 2 shows the contribution of each of the terms in the expressions for \( T_A, T_L \) and \( T_V \) in Eqs. (15)–(17), as a function of cylindrical radius. All plots show time-averaged values. The term that dominates the advective torque is \( \{ \bar{u}_t \bar{u}_t \} \), which represents the mean torque that arises from the interaction between the \( s \)- and \( \phi \)-component of the non-axisymmetric part of the flow,

\[
\{ \bar{u}_t \bar{u}_t \} = \frac{1}{s^2} \int \Sigma (u_t' u_t' \phi) \, d\Sigma.
\]

This torque is what we refer to as the Reynolds stresses, and it dominates the other terms in (15) because the convection takes the form of columnar rolls aligned with the rotation axis (Busse, 1970). When viewed in a plane perpendicular to the rotation axis, the convection cells are elongated in the prograde direction (Zhang, 1992), as in Fig. 3 where we show a snapshot of the non-axisymmetric part of the flow on the plane \( \phi = 0.3 \). The azimuthal tilting of the cells produces a correlation between \( u_t' \) and \( u_\phi' \) and, as a result, a mean torque on the cylindrical surfaces.

\[
T_V = E \left( \frac{\partial}{\partial s} \bar{u}_z - \frac{\partial}{\partial \phi} \bar{u}_z \right).
\]
Fig. 2. Upper left plot: Time-averaged azimuthal advective force on cylindrical surfaces of radius \( r \). Thick solid line = \(-Ro\{\omega'zu'\}_s\); thick dashed line = \(-Ro\{\omega'zu'\}_s\); thin solid line = \(-Ro\{\omega'zu'\}_s\); thin dashed line = \(-Ro\{\omega'zu'\}_s\). Upper right plot: Time-averaged azimuthal Lorentz force. Thick solid line = \(\{J'zB'\}_s\); thick dashed line = \(-\{J'sB'\}_z\); thin solid line = \(\{JzBs\}_s\); thin dashed line = \(-\{JsBz\}_s\). Lower left: Time-averaged azimuthal viscous force. Solid line = \(E\{\partial\omega_z/\partial s\}\); dashed line = \(-E\{\partial\omega_s/\partial z\}\). Lower right: Time-averaged rigid velocity \( V_\phi \) (solid line) and rigid angular velocity \( V_\phi/s \) (dashed line).
Similarly, the largest contribution to the Lorentz torque in Fig. 2 comes from the $\{J_z'B_s\}$ term, which represents the mean torque from the interaction of the $s$- and $\phi$-component of the non-axisymmetric magnetic field,

$$\langle J_z'B_s \rangle = \left\{ \frac{1}{2} \frac{\partial}{\partial s} \left( s'B_s B_s \right) \right\}. \tag{19}$$

The magnetic field is advected by the columnar convection flow and magnetic field lines tend to wrap around the convection cells (Olson et al., 1999). When viewed on a plane whose normal is parallel to the rotation axis, the morphology of the horizontal component ($s$- and $\phi$-component) of the field is similar to that of the flow, with irregular cells tilted in the prograde direction. For illustration, we show in Fig. 3 a snapshot of the non-axisymmetric magnetic field on the $z$-level plane $z = 0.3$. The magnetic field pattern at other levels is similar to that shown in Fig. 3 but its magnitude varies with $z$ (and reverses direction across the equator). Overall, a mean torque arises which is similar to the Reynolds stresses, but which involves the magnetic field. By analogy, we refer to this torque as the magnetic Reynolds stresses. The three other terms in (16) also participate in the Lorentz torque, but with smaller amplitude.

The viscous force shown in Fig. 2 is almost entirely a result of the $\{\partial \omega_z / \partial s\}$ term which represents the part due to the shear of the time-averaged differential geostrophic flow $V_\phi$,

$$\left\{ \frac{\partial \omega_z}{\partial s} \right\} = \left\{ \frac{\partial}{\partial s} \frac{1}{2} \frac{\partial}{\partial s} \left( s'' B_s V_\phi \right) \right\} \approx \frac{1}{h^2} \frac{\partial}{\partial s} \left( \frac{1}{h^2} \frac{\partial}{\partial s} V_\phi \right), \tag{20}$$

where $h = (1 - s^2)^{1/2}$ is the semi-height of the cylinder and where we have used the stress-free boundary conditions on velocity. The time-averaged differential geostrophic flow is also shown in Fig. 2. It is the $z$-averaged part of the time-averaged azimuthal flow shown in Fig. 3. The differential geostrophic flow is maintained by the Reynolds stresses, as it is observed in thermal convection (Busse and Hood, 1982; Zhang, 1992), and also by the magnetic Reynolds stresses, which are as important as the Reynolds stresses in the Kuang–Bloxham model. For this particular calculation, the amplitude of the geostrophic flow is roughly equivalent to the largest non-axisymmetric velocities. We note that the use of hyperviscosity is responsible for a factor 5 to 10 increase in the viscous torque.

Hence, the mean torque balance in the simulation is principally one between the Reynolds stresses, the magnetic Reynolds stresses and the viscous shear associated with a mean differential geostrophic flow. A mean Taylor state is not achieved because the large viscous forces and the Reynolds stresses provide an efficient way to balance an $O(1)$ mean Lorentz torque.
3.3. Time-dependent torque balance

If the model is in a viscous state, then we expect that
\[ T_A(t) + T_L(t) + T_V(t) = 0 \] (21)
is respected at all times, i.e. the variations in the advective and Lorentz torque are balanced by a viscous readjustment of the geostrophic flow. On the other hand, in the case of an inertial adjustment, when the sum of the advective torque, Lorentz torque and viscous torque does not vanish on a cylindrical surface, we expect a rigid acceleration of the surface proportional to the unbalanced part,
\[ T_A(t) + T_L(t) + T_V(t) = R_o 4 \pi s (1 - s^2)^{1/2} \int \Sigma \frac{\partial u \phi(t)}{\partial t} \, d \Sigma = R_o \frac{\partial \nu \phi(t)}{\partial t}. \] (22)

In Fig. 4 we show the evolution of \( T_A \), \( T_L \) and \( T_V \) for one particular cylindrical surface, \( s = 0.5 \). Each torque fluctuates about its time-averaged value. We also show in Fig. 5 the time dependence of the Reynolds stresses, the magnetic Reynolds stresses and the viscous shear, in relation to the total advective torque, Lorentz torque and viscous torque. This shows that the terms that are the most important in the time-averaged balance are also the terms which carry most of the fluctuations about the mean. Results at different cylinder radii are not qualitatively different, except for \( s \) very close to 1.

We also show in Fig. 4 the rigid accelerations of the cylindrical surface \( s = 0.5 \), which fluctuate with amplitudes comparable to the fluctuations in the Reynolds stresses and the Lorentz torque. This indicates that at least part of the departures in the combined torque from the Reynolds stresses and Lorentz torque are balanced by rigid accelerations. The remaining part is balanced by the viscous torque. Given that the amplitude of the fluctuations in the viscous torque are similar to those of the rigid accelerations, the dynamical regime of the simulation presented is a combination of a viscous and inertial state.

In a related study of the dynamics of the Kuang–Bloxham geodynamo model, Kuang (1999) showed similar results to those reported here, for the same set of model parameters. In particular, it was shown that the Lorentz torque was mainly balanced by the inertial torque (including both acceleration and advective parts) and that the viscous torque was smaller than the latter two by a factor 2 or 3, in agreement with our present study. As a consequence, Kuang (1999) suggested that strong torsional oscillations would develop in the model, being slightly damped by viscous dissipation. By separating the inertial torque, we have been able to obtain a more precise picture of the dynamics: a large part of the total inertial torque is carried by the Reynolds stresses and while the rigid acceleration

Fig. 4. Variations in time of the azimuthal forces for cylinder radius \( s = 0.5 \). The thick solid line is the Lorentz torque; the dashed line is the advective torque; the dashed-dotted line is the viscous torque; and the thin solid line is the rigid acceleration.
Fig. 5. Top plot: variations in time of the total azimuthal advective force (solid line) and the the Reynolds stresses ($-\mathcal{R}_{\omega z/\partial s}$) (dashed line). Middle plot: variations in time of the total azimuthal Lorentz force (solid line) and the the magnetic Reynolds stresses ($\mathcal{J}_zB_s$) (dashed line). Bottom plot: variations in time of the total azimuthal viscous force (solid line) and the viscous shear due to the differential geostrophic flow ($E(\mathcal{G}_{\partial z/\partial s})$) (dashed line).
in general remains larger than the fluctuations in the viscous torque, the latter are often of comparable magnitude. Hence, our results suggest that while free oscillations of cylindrical surfaces may be produced, they would be damped somewhat rapidly. In the next section, we investigate explicitly whether the oscillations observed in Fig. 4 represent torsional oscillations.

4. Oscillations of rigid cylindrical surfaces

4.1. Rigid accelerations

From a purely mathematical point of view, Eq. (22) does not require that the azimuthal acceleration is rigid, but only that the axisymmetric acceleration averaged over $z$ balances the total torque on the left-hand side. However, rapid rotation tends to prevent large variations in velocity parallel to the rotation axis, a constraint known as the Proudman–Taylor theorem (Proudman, 1916; Taylor, 1917). Indeed, the nearly two-dimensional columnar convection observed in the model is a consequence of this constraint. Even in the presence of a strong magnetic field, the latter organizes itself such that the Proudman–Taylor constraint is largely satisfied (Zhang, 1995; Olson et al., 1999). Hence, we may expect that azimuthal velocities produced by a disequilibrium in the torques will also satisfy this constraint and be rigid. We now show explicitly that the rapid rotation dynamics are well captured by the numerical simulation and that azimuthal accelerations are almost rigid. In Fig. 6A, we present the axisymmetric part of the azimuthal velocity $u_\phi$ (vertical axis) as a function of height (depth axis) and time (horizontal axis), for the cylindrical surface at $s = 0.5$. At any given time, the structure along the $z$ direction is almost symmetric about the equator and represents the combined effects of a thermal and magnetic wind. Over time, this structure remains more or less intact, while the whole length of the cylinder accelerates azimuthally as if it were rigid. In Fig. 6B, the average azimuthal velocity at every height has been subtracted from $u_\phi$, which shows that the motion is quasi-rigid. We have verified that these results hold for all cylinder radii outside the tangent cylinder.

The fact that unbalanced torques result in accelerations that are rigid can also be understood with the following considerations. The Proudman–Taylor constraint is obtained by taking the azimuthal component of the curl of the of the momentum balance in the absence of body forces. When the timescale considered is much longer than the rotation period, as is the case here, inertial accelerations can be neglected and we have $2\Omega u_\phi/|c| = 0$, i.e. a rigid azimuthal flow. This constraint can be broken by the effects of additional forces such as Lorentz forces, advective forces, viscosity or buoyancy, in which case we have, $2\Omega u_\phi/|c| = -(\nabla \times F)_\phi$. We are interested here in the changes in $u_\phi$ that result from torques on cylinders. However, the forces that are responsible for the changes in the torques are entirely in the azimuthal direction. Hence, the curl of these forces has no azimuthal component and they cannot maintain a vertical gradient in $u_\phi$. Therefore, the azimuthal velocity that is produced in response to the torques must satisfy the Proudman–Taylor constraint. The small departures from a rigid velocity that are observed in Fig. 6B, have to be attributed to non-azimuthal time-dependent forces. But as it is well illustrated by Fig. 6A, the forces that can support an axial velocity gradient (i.e. those that produce the magnetic and thermal wind) must be in a nearly steady-state balance, since the axial vertical gradient remains nearly constant. Hence the non-azimuthal forces do not change significantly on the timescale of the azimuthal oscillations. The largest time-dependent forces are in the azimuthal direction, therefore indicating that the time-dependent dynamics in the model are largely controlled by the torque balance.

For cylindrical surfaces inside the tangent cylinder, we have observed a similar behavior with rigid accelerations propagating on a nearly steady thermal and magnetic wind background. However, we have also observed in this region occasional sudden episodes of large variations in the $z$-gradient of $u_\phi$ that were of the same amplitude as the rigid oscillations of $u_\phi$. Hence, even though the meridional force balance remains nearly steady for long periods of time, it does participate in the time-dependent dynamics during brief episodes.

4.2. Torsional oscillations in the geodynamo model?

We now investigate whether the fluctuations of the rigid accelerations shown in Fig. 4 are torsional oscillations. This particular calculation uses $R_\text{m} = 2 \times 10^{-5}$
Fig. 6. (A) Axisymmetric component of $u_\phi$ as a function of vertical position ($z$) and time for the cylindrical surface at $s = 0.5$. (B) Idem as (A), but minus the time-averaged value at each $z$.

and the rms magnetic field strength over the cylindrical surfaces is about 0.5 ($\sim 10$ G). According to (7) we expect the fundamental period of torsional oscillations to be on the order of a thousand years, in agreement with the periodicity of Fig. 4. However, the fluctuations in the rigid acceleration in Fig. 4 do not seem to be correlated strongly with those of the Lorentz torque, which is what we would expect for torsional oscillations. Instead, it appears that the rigid accelerations result from the combination of changes in the Lorentz torque and the Reynolds stresses. Moreover, the viscous torque is often almost as large as the rigid accelerations, which suggests that any free oscillations will be damped over one or two periods of oscillation and that the maintenance of the fluctuations has to be attributed to a forced oscillation component. We note in addition that because of the longer periods of free oscillations in the model, the effects of magnetic diffusion are certainly much stronger than they would be in the Earth’s core.

Whether torsional oscillations are observed in the numerical model (with the current set of parameters)
comes down to a matter of terminology. If one restricts “torsional oscillations” to refer only to the oscillations that arise as a balance between rigid accelerations and Lorentz forces, then the oscillations of Fig. 4 are not torsional oscillations because the Reynolds stresses participate in the dynamics. On the other hand, these fluctuations do represent inertial oscillations of rigid cylindrical surfaces and one may choose still to refer to them as “torsional oscillations”, just ones that are also influenced by Reynolds stresses. In any case, a large part of the oscillations in the rigid accelerations are forced oscillations.

4.3. Radial and time dependency of the rigid azimuthal velocities

In order to get a sense of the spatial dependency of the fluctuations in the geostrophic flow produced in the model, in Fig. 7 we show $V_\phi$ as a function of cylindrical radius and time. The mean differential geostrophic flow has been subtracted. The general pattern shows many oscillations with typical timescales of a thousand years that are propagating in the radial direction. The amplitude of the oscillations is on the same order as the amplitude of the mean geostrophic flow. The radial propagation of the waves further confirms the ability of the numerical model to capture the dynamics of inertial oscillations of rigid cylindrical surfaces.

It is interesting to note that, in specific time intervals, the wave pattern observed in Fig. 7 is similar in many ways to the inverted flow which is inferred to represent torsional oscillations in the core (Zatman and Bloxham, 1997, 1998; Hide et al., 2000). Hence, even if the dynamics of these waves is influenced by the Reynolds stresses and viscous forces, their radial and time dependency in small intervals reproduces a similar behavior to torsional oscillations. This issue will be reexamined in the discussion in Section 5.

4.4. Generation of rigid accelerations in the model

In Section 4.2, we argued that a large component of the oscillation in the rigid accelerations had to be attributed to a forced oscillation because of the large viscous dissipation. This allows for an investigation of the mechanisms that change the fields in such a way as to destroy the balance between the torques. In other words, it permits an investigation of the mechanism responsible for exciting the free oscillations in the model and by extension, torsional oscillations in the Earth’s core.

One way in which the rigid accelerations are produced is through the chaotic nature of the convection. As we showed in Section 3.2, the time-averaged torque equilibrium involves a mean differential geostrophic velocity which is maintained by the Reynolds stresses and the magnetic Reynolds stresses. Yet, the continual shearing of the non-axisymmetric structures of the velocity field and the magnetic field by this differential geostrophic flow produces continual changes.
in the Reynolds stresses and the magnetic Reynolds stresses. In order to maintain the torque balance at all times, rigid accelerations are continually produced and modify the differential geostrophic flow. Overall, a time-averaged equilibrium is established with continual chaotic fluctuations of the torques and of the differential geostrophic flow about their mean values. The equilibrium described above is a well established balance observed in numerical models of non-magnetic thermal convection in a chaotic regime (Sun et al., 1993; Cardin and Olson, 1994; Christensen, 2002) and has also been observed in numerical models of the geodynamo (Grote and Busse, 2001).

A demonstration of the advective action of the differential rotation on the magnetic field is shown in Fig. 8. We present snapshots at 75 years intervals of the non-axisymmetric magnetic field on a $z$-level plane ($z = 0.3$). The contours are those of a stream function $\psi_B$ of the in-plane magnetic field such that $B_h = \nabla \times \psi_B \hat{z}$. The mean differential geostrophic flow profile is that of Fig. 2, with cylinders near $s \approx 0.5$ rotating at the fastest rate (in a retrograde direction). While these results are shown at a particular $z$-level, the average over $z$ produces a similar picture because of the strong two-dimensionality of the convective structures. We identify three structures (labeled 1, 2 and 3 on the figure) that we follow in time. Structure 1 is a filament that extends from the tangent cylinder to $s \approx 1$ in snapshot a. The differential rotation shears the filament, which disconnects into two distinct magnetic field vortices. The disconnection is completed by snapshot f. Structure 2 is similar to structure 1 in a, and is likewise sheared in two separate structures. The disconnection in this case is accelerated by the intrusion from structure 3, which benefits from a faster retrograde rotation than the tail-end of structure 2. In snapshot d and e, structure 3 reconnects with the part that had previously disconnected from structure 1.

The differential geostrophic flow creates changes in the $s$- and $\phi$-component of the non-axisymmetric magnetic field (and hence in the magnetic Reynolds stresses) in two ways. First, it changes the tilt angle of the non-axisymmetric contours. Secondly, the constant advection of the non-axisymmetric structures by the differential rotation is physically displacing contours past one-another, which leads to disconnection and reconnection and local changes in the Lorentz torque. This is the crucial role played by the non-axisymmetric
field in making the Lorentz torque fluctuate constantly in the presence of a mean differential geostrophic flow. If the field were purely axi-symmetric, advection by a steady differential geostrophic flow would shear a steady radial magnetic field until it is balanced exactly by diffusion, at which point a steady Lorentz torque would be established. The relative azimuthal displacements of non-axi-symmetric structures at different radii prevents the establishment of such a steady-state. In addition, as is also obvious from Fig. 8, instabilities which draw energy from the differential geostrophic flow also play a role in altering the local Lorentz torque. The Reynolds stresses are altered by the differential geostrophic flow in a similar way.

We note that the gradient in $z$ of the mean zonal flow (shown in Fig. 3) produces a shear of the vertical non-axi-symmetric structure of the velocity and magnetic field. However, as shown in Fig. 5, most of the changes in time in the torques are carried by the shear of the $z$-plane structures as described above. The shear of the vertical structures represents then only a small contribution to the total time-dependent azimuthal torque. Hence, it is the differential geostrophic component of the total mean zonal flow that is mostly responsible for the chaotic evolution of the torque balance.

The dynamical regime of the geodynamo model is not one where free oscillations of rigid cylindrical surfaces are propagating with respect to a slowly varying background torque equilibrium. Instead, the nature of the torque equilibrium is such that rigid accelerations are continually produced in order to balance the chaotic evolution of the Reynolds stresses and the magnetic Reynolds stresses. In other words, the background torque equilibrium is varying on the same timescale as the periods of the free oscillations. The resulting rigid accelerations provide a continual and efficient excitation of the free oscillations.

5. Discussion and conclusion

5.1. The dynamical state of the numerical geodynamo model

In this paper we have investigated the torque balance on cylindrical surfaces in the Kuang–Blobsham numerical geodynamo model. For the parameters of our investigation, we find that the time-averaged torque balance is not in a Taylor state, but one that consists in large part of a balance between the Reynolds stresses, the magnetic Reynolds stresses, and the viscous shear associated with a mean differential geostrophic flow. When a departure from this balance occurs, it is accommodated by a rigid azimuthal acceleration of the cylindrical surface which excites free oscillations of these rigid surfaces that propagate in the direction perpendicular to the rotation axis. We have shown that the inertial acceleration of the cylindrical surfaces is indeed almost rigid, with little variations along the rotation axis. We have, therefore, confirmed that the model captures the dynamics of an inertial state dynamo, where rigid azimuthal accelerations of cylindrical surfaces are produced in response to a disequilibrium in the torque balance.

The rigid azimuthal oscillations that occur about the time-averaged state are not torsional oscillations in their conventional description because the Reynolds stresses play an important role in their dynamics. In addition, a large component of the observed fluctuations results from the changes in the underlying torque balance, which is continually fluctuating about its mean, a consequence of the advective action of the mean differential geostrophic flow which continually alters the Reynolds stresses and magnetic Reynolds stresses. Without this continual forcing, the free oscillating part of the fluctuations would be attenuated rapidly by viscous dissipation.

5.2. Reynolds stresses in the Earth’s core

It is clear that one main reason why the numerical model fails to reproduce a dynamic state in which the unbalanced part of the Lorentz torque is counterbalanced by rigid accelerations of cylindrical surfaces is because of the large Reynolds stresses. Yet, the two conventional conceptual pictures of the torque balance in the core—a friction controlled state or torsional oscillations about a Taylor state—do not involve Reynolds stresses. The role of the Reynolds stresses in the torque balance of the Earth’s core deserves closer examination.

In the model, the Reynolds stresses are of similar amplitude to the magnetic Reynolds stresses and provide a very efficient way of balancing a large Lorentz torque. This should not come as a surprise considering...
that they have an identical mathematical form but with opposite signs:

\[ T_s \approx -R_s \left( \frac{1}{\tau_0} \left( \delta \nu \cdot \nu' \right) \right), \]  

\[ T_L \approx \left[ \frac{1}{\tau_0^2} \left( \partial \phi \cdot \partial B \right) \right]. \]  

The convective motion in the form of spiraling rolls is directly responsible for each of these torques and the amplitudes of \( \nu' \) and \( B' \) are dynamically adjusted in the model so that they have an identical mathematical form but with opposite signs: that they have an identical mathematical form but with opposite signs:

\[ T_s \approx -R_s \left( \frac{1}{\tau_0} \left( \delta \nu \cdot \nu' \right) \right), \]  

\[ T_L \approx \left[ \frac{1}{\tau_0^2} \left( \partial \phi \cdot \partial B \right) \right]. \]  

The typical values that emerge from the model are \( B' \sim 0.5, \nu' \sim 100 \) and \( L_{nu}/L_B \sim 1 \), which gives \( T_L/T_s \sim 1 \). Preliminary results for calculations using smaller values of the Rossby and Ekman numbers \( (E = R_s = 10^{-5} \text{ and } E = R_s = 5 \times 10^{-6} \) ) supports the above analysis: \( B' \) decreases while \( \nu' \) increases so that \( T_L/T_s \sim 1 \) is maintained. (We note that in these simulations, as a consequence of the decrease in the magnetic field strength, the system is more in a viscous state than an inertial state.)

However, it is uncertain whether the above balance can be extrapolated to the parameter regime of the Earth’s core. As we have argued in Section 2.5, in order for the Reynolds stresses to provide a balance to an \( O(1) \) Lorentz torque, the combined increase in velocity and decrease in length scale of the flow eddies (both inferred from the flow inversions) has to be about 4 orders of magnitude. This is questionable and the above equilibrium between the Reynolds and the magnetic Reynolds stresses may no longer hold in the parameter regime of the Earth, otherwise the amplitude of \( B' \) would have to be much smaller than its observed value at the CMB. Our main point here is that the Reynolds stresses are necessarily important in the Earth’s core, but that a proper assessment of their role in the torque balance and torsional oscillations may be desirable. We recall that we do not know the length scale and amplitude of the flow eddies in the core, and that an important contribution from the Reynolds stresses in the dynamics, such as we observe in the model, remains a possibility. One way to assess this question in a more quantitative way using the numerical model is to investigate how the Lorentz torque is altered by changing the parameters that enter the calculation. Asymptotic laws can perhaps be derived and extrapolating these to core parameters may help provide a better understanding of the torque balance in the Earth. We leave such investigation to a future study.

### 5.3. Excitation of torsional oscillations in the Earth’s core and the dynamic torque balance of the geodynamo

The dynamic torque balance observed in the numerical model may not be directly applicable to the Earth’s core. This is for obvious reasons: the parameters used for the numerical calculation are different from those of the Earth’s core and the results of the model may also be highly dependent on some of the approximations employed (such as retaining only the axisymmetric part in the momentum equation and using hyperdiffusivity). Nevertheless, some aspects of the dynamics of the torque balance observed in the model may apply to the Earth’s core in a qualitative sense.

We propose here that a chaotic evolution of the torques on cylindrical surfaces, such as is observed in the model, may also occur in the Earth’s core and would be responsible for exciting torsional oscillations. In the model, this chaotic evolution involves a large contribution from the Reynolds stresses, which may be much weaker in the Earth’s core. However, even if the Reynolds stresses are unimportant, we expect that the torque balance will evolve in a similar chaotic fashion if the convective dynamics in the core involves a mean differential geostrophic flow. One can imagine a scenario in which, in concert with this mean differential geostrophic flow, the time-averaged torque balance involves only the Lorentz torque. In other words the time-averaged torque balance satisfies Taylor’s constraint. Indeed, kinematic studies have shown that for a magnetic field configuration that satisfies Taylor’s constraint, there exists a geostrophic flow that shears the field in the precise way so that it continues to satisfy Taylor’s constraint (Taylor, 1963; Fearn and Proctor, 1987; Jault, 1995). The inertial effects and
that which is observed in Fig. 8. Therefore, even in the absence of the magnetic field would be dynamically similar to potential geostrophic flow on the non-axisymmetric part of the magnetic field tends to prevent a large scale geostrophic flow. The question is whether a sufficiently large mean differential geostrophic flow would be present if the Reynolds stresses were much weaker, as it is probably the case in the core. The answer to that question is unknown and re-emphasizes the need for a systematic parameter study of the torque balance. However, tentative support for a mean differential geostrophic flow comes from the kinetic considerations of a Taylor state as mentioned above (Taylor, 1963; Fearn and Proctor, 1987; Jault, 1995), which suggest that a differential geostrophic flow of 5 km per year is consistent with Earth-like magnetic field strengths (Jault, 1995).

A second line of evidence supporting a mean differential geostrophic flow comes from inversions of the flow at the CMB. Over the last 160 years, most of the secular variations of the geomagnetic field can be explained with a steady flow which contains a rigid differential geostrophic component (Bloxham, 1992) with a typical amplitude of 5 km per year (Zatman and Bloxham, 1999). Although it is possible that this inverted steady differential rigid rotation flow is incorrect (Gubbins and Kelly, 1996), its typical amplitude is consistent with the amplitude of the observed torsional oscillations that would result from the scenario presented above.

The dynamical torque balance scenario that we have described here differs from the conventional picture of the quasi-Taylor state, where torsional oscillations are free oscillations propagating about a slowly evolving Taylor state. Rather, we suggest an alternative option in which the time-averaged torque satisfies Taylor’s constraint and a mean geostrophic flow acting on the non-axisymmetric structures of the magnetic field leads to chaotic fluctuations of the Lorentz torque about this mean Taylor state. The background torque equilibrium on which the free oscillations propagate is then not quasi-static but dynamic and provides a continual and efficient excitation for the free oscillations. Thus, a quasi-Taylor state where the Lorentz torque vanishes except for the part involved in torsional oscillations remains a valid description, provided that the torsional oscillations consist of a mixture of free oscillations and chaotically forced accelerations.

We note that if the Reynolds stresses are important in the Earth’s core, this dynamical description remains valid. However, the presence of a magnetic field tends to prevent a large scale geostrophic flow. The question is whether a sufficiently large mean differential geostrophic flow would be present if the Reynolds stresses were much weaker, as it is probably the case in the core. The answer to that question is unknown and re-emphasizes the need for a systematic parameter study of the torque balance. However, tentative support for a mean differential geostrophic flow comes from the kinetic considerations of a Taylor state as mentioned above (Taylor, 1963; Fearn and Proctor, 1987; Jault, 1995), which suggest that a differential geostrophic flow of 5 km per year is consistent with Earth-like magnetic field strengths (Jault, 1995).

A second line of evidence supporting a mean differential geostrophic flow comes from inversions of the flow at the CMB. Over the last 160 years, most of the secular variations of the geomagnetic field can be explained with a steady flow which contains a rigid differential geostrophic component (Bloxham, 1992) with a typical amplitude of 5 km per year (Zatman and Bloxham, 1999). Although it is possible that this inverted steady differential rigid rotation flow is incorrect (Gubbins and Kelly, 1996), its typical amplitude is consistent with the amplitude of the observed torsional oscillations that would result from the scenario presented above.

The dynamical torque balance scenario that we have described here differs from the conventional picture of the quasi-Taylor state, where torsional oscillations are free oscillations propagating about a slowly evolving Taylor state. Rather, we suggest an alternative option in which the time-averaged torque satisfies Taylor’s constraint and a mean geostrophic flow acting on the non-axisymmetric structures of the magnetic field leads to chaotic fluctuations of the Lorentz torque about this mean Taylor state. The background torque equilibrium on which the free oscillations propagate is then not quasi-static but dynamic and provides a continual and efficient excitation for the free oscillations. Thus, a quasi-Taylor state where the Lorentz torque vanishes except for the part involved in torsional oscillations remains a valid description, provided that the torsional oscillations consist of a mixture of free oscillations and chaotically forced accelerations.

We note that if the Reynolds stresses are important in the Earth’s core, this dynamical description remains
valid, except that the time-averaged torque balance would differ from a Taylor state, as we see in the model, and the Reynolds stresses would be involved in the chaotic fluctuations of the rigid accelerations and also participate in the dynamics of free oscillations.

Finally, as we pointed out in section 4.3, the rigid flow oscillations produced in the model shown in Fig. 7 resemble in many respects the recovered rigid velocities of the Earth’s core. One may then suggest that this result argues in favor of a large influence of the viscous torque and the Reynolds stresses in the observed torsional oscillations in the core. However, this is not necessarily the case because the dynamical role of the viscous torque and Reynolds stresses can both be accomplished by the Lorentz torque alone. As explained above, we expect chaotic rigid accelerations even if Reynolds stresses are weak. Additionally, the attenuation of the free oscillations by the viscous torque can be substituted by a “magnetic friction” if there exists a conducting layer at the base of the mantle (Braginsky, 1970, 1988). Indeed, a layer with a conductance of $10^8$ S, which is required to explain some of the discrepancies between the observed and predicted forced nutations of the Earth (Buffett, 1992), would be quite efficient at attenuating the torsional oscillations in one or two periods (Buffett, 1998), as we observe in the model. The resulting rigid oscillations would then contain a large component of forced oscillations in order to compensate for the magnetic “friction”, and would appear similar to that of Fig. 7.

5.4. Future numerical work

Finally, we wish to discuss future strategies for modeling the dynamical torque balance and torsional oscillations. Evidently, the chaotic torque balance observed in the model is obtained for one specific set of parameters. We have argued that this result may apply to the Earth in a qualitative way. However, it is certainly also possible that this chaotic torque balance is no longer in effect for the parameter regime of the Earth’s core. Therefore, additional numerical work is required to confirm our hypothesis. In order to better approximate the Earth’s core, the influence of the viscous torque and Reynolds stresses have to be much weaker. Additionally, the timescale of the oscillations has to be shorter to limit the effects of magnetic diffusion. This can be achieved by reducing the Ekman and Rossby numbers. However, limits in computing resources do not permit drastic reductions of these parameters. Hence, as we have mentioned already, one strategy is to vary systematically the different parameters with the hope of extracting asymptotic laws for the torque balance. The calculations at smaller values of $E$ and $R_o$, will be computationally expensive although we recall that to study the torque dynamics, it is only necessary to cover a time period that contains a few rigid oscillations, which is much shorter than the magnetic decay timescale. In reducing $E$ and $R_o$, one has to be careful to maintain an inertial state. In other words, the inertial acceleration has to be at least of comparable magnitude and preferably larger than the viscous torque.

Under the assumption that Reynolds stresses are unimportant in the torque balance of the core, additional strategies in the numerical model might permit a better approximation of a Taylor state. One strategy is to neglect the Reynolds stresses altogether, even the axisymmetric part. This is certainly not consistent with the current parameter regime of the simulation because, as we have seen in the present study, the Reynolds stresses are indeed very large. However, subtracting their effect from the torque balance removes one possible way in which a Lorentz torque of $O(1)$ can be balanced. Hence, it can lead to a closer representation of a Taylor state and perhaps provide a more accurate view of the dynamics in the core. Similarly, another strategy is to use a smaller Ekman number for the axisymmetric equation, or equivalently a lower degree of hyperviscosity. This will reduce the viscous torque of cylindrical shells without being prohibitively expensive numerically.

One last point we wish to discuss is whether the chaotic fluctuations and free oscillations of the cylindrical surfaces have an important effect on the long term evolution of the magnetic field. In other words, when one is interested in phenomena such as dipole field generation or reversals, whether it is appropriate to simply filter out the short timescale torsional oscillations, or, on the contrary, whether they are crucial for the long term equilibrium because they allow the instantaneous torque balance to relax towards its mean state? The above question may be addressed by comparing numerical solutions that have a common set of parameters, but different step sizes in time. We expect that solutions with different increments in time...
will have different time-histories of the torques. However, if the time-averaged torque balance remains unchanged, it suggests that the chaotic fluctuations and free oscillations have little influence on the long term torque equilibrium.

Acknowledgements

The authors would like to thank Steven Zatman, Philippe Cardin and Weijia Kuang for useful comments at various stages of the work presented in this paper and to Sabine Stanley for comments on the manuscript. Mathieu Dumberry is partly supported by NSERC/CRSNG and FCAR scholarships. This work was also supported by NSF Awards EAR-0073988 and EAR-0112469.

References


