Pool Simulation

Introduction:

To model the physics involved in a game of pool, we must combine many math and physics principles. In the pool simulation, there are 3 main calculations in which these principles are applied: impact between two balls, impact between a bumper and a ball and the resistance between the table and the balls. In this document, we will give an in-depth explanation of how each of these calculations are performed.

Impact Calculations for Collisions between Balls

Initial Velocity and Angle:
The initial velocities of the balls in the *x* and *y* directions are used to calculate the initial velocity and initial angle of the balls. The following equations can be used to show this step:

$$\theta_i = \tan^{-1}\left(\frac{v_{iy}}{v_{ix}}\right)\circ$$

$$v_i = \sqrt{{v_{ix}}^2 + {v_{iy}}^2}\ \mathrm{m/s}$$

where v_i is the initial velocity of the ball, v_{ix} is the initial velocity of the ball in the *x* direction, v_{iy} is the initial velocity of the ball in the *y* direction and θ_i is the initial angle from the positive x axis.

Initial Velocities on Rotated Line of Impact:

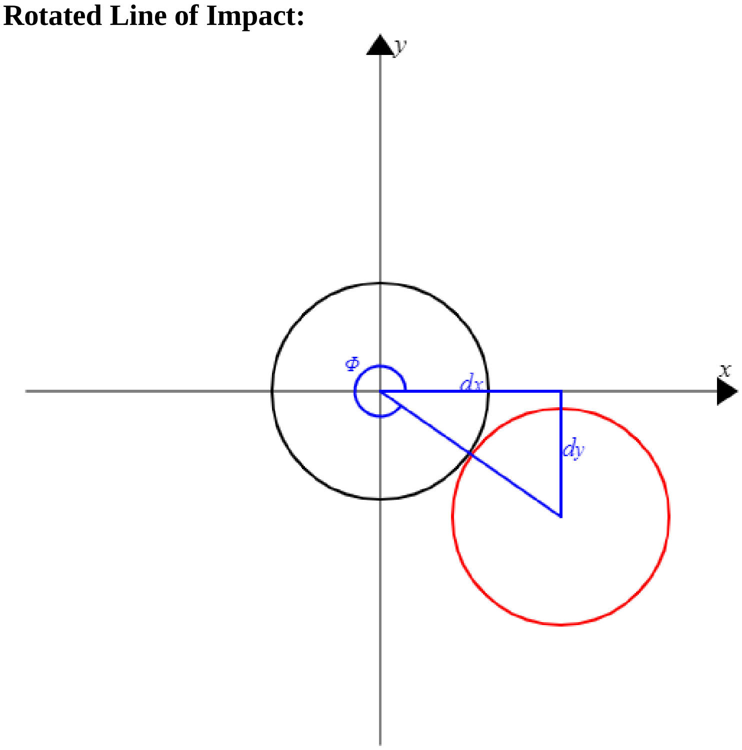

Figure 1: Diagram of the Line of Impact.

The system is rotated around the center of the first ball to make the line of impact horizontal. To do this, the angle ϕ is calculated using the following equation:

$$\phi = \tan^{-1}\left(\frac{d_y}{d_x}\right)^{\circ}$$

where d_y is the vertical distance between the balls, d_x is the horizontal distance between the balls and ϕ is the angle of collision from the positive *x* axis.

Now, the following equations are used to find the velocities in the *x* and *y* directions on the rotated line of impact:

$$v_{ix}' = v_i \cos(\theta_i - \phi) \text{ m/s}$$

$$v_{iy}' = v_i \sin(\theta_i - \phi) \text{ m/s}$$

where v_i is the initial velocity of the ball, v_{ix}' is the initial velocity of the ball in the *x* direction after the line of impact is rotated, v_{iy} is the initial velocity of the ball in the *y* direction after the line of impact is rotated, θ_i is the initial angle from the positive *x* axis and ϕ is the angle of collision from the positive *x* axis.

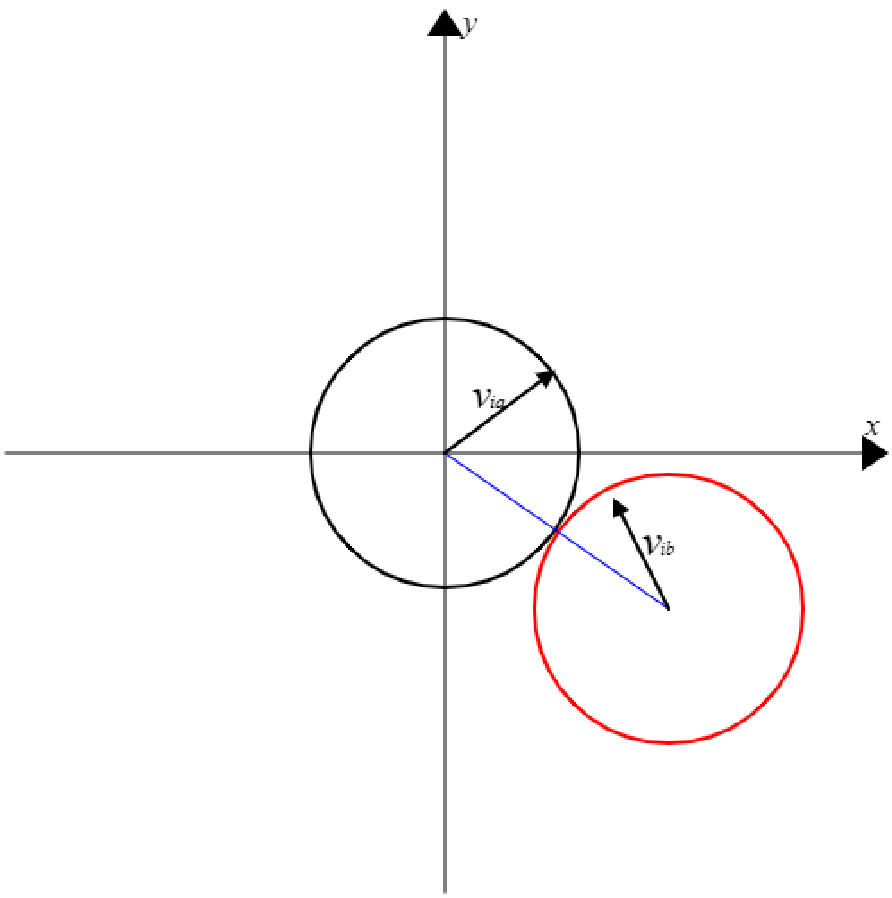

Figure 2a: Diagram showing the velocities before rotation of the line of impact.

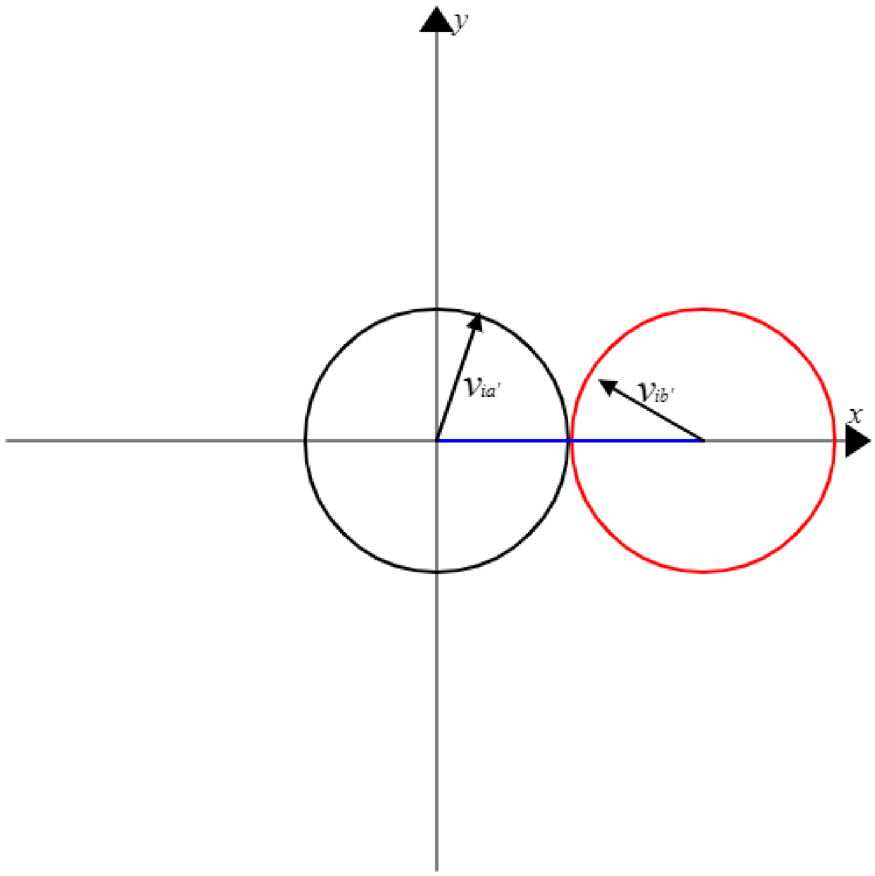

Figure 2b: Diagram showing the velocities after rotation of the line of impact.

Notice that the velocities in the *y* direction are now tangent to the line of impact meaning that they will not change after the collision.

Final Velocities on Rotated Line of Impact:
The next step is to derive an equation to calculate the velocities in the *x* direction after the collision. The following conservation of momentum and coefficient of restitution equations are used:

$$m_a v_{ia} + m_b v_{ib} = m_a v_{fa} + m_b v_{fb}$$

$$e = \frac{|v_{fb} - v_{fa}|}{|v_{ib} - v_{ia}|}$$

where m_a is the mass of the first ball, m_b is the mass of the second ball, v_{ia} is the initial velocity of the first ball, v_{ib} is the initial velocity of the second ball, v_{fa} is the final velocity of the first ball, v_{fb} is the final velocity of the second ball and e is the coefficient of restitution.

After rearranging the equations, we get:

$$v_{fa} = \frac{m_a v_{ia} + m_b v_{ib} - m_b v_{fb}}{m_a} \text{ m/s}$$

$$v_{fb} = e\,(v_{ib} - v_{ia}) + v_{fa} \text{ m/s}$$

Now, the second equation is substituted into the first and the variable v_{fa} is isolated. Since we are only using this equation for the velocity of the balls in the *x* direction, we can substitute those into the equation. These steps are shown below:

$$v_{fa} = \frac{m_a v_{ia} + m_b v_{ib} - m_b e\,(v_{ib} - v_{ia})}{m_a + m_b}\ \text{m/s}$$

$$v_{fax}{}' = \frac{m_a v_{iax}{}' + m_b v_{ibx}{}' - m_b e\,(v_{ibx}{}' - v_{iax}{}')}{m_a + m_b}\ \text{m/s}$$

$$v_{fbx}{}' = \frac{m_a v_{iax}{}' + m_b v_{ibx}{}' - m_a e\,(v_{iax}{}' - v_{ibx}{}')}{m_a + m_b}\ \text{m/s}$$

where $v_{fax}{}'$ is the final velocity of the first ball in the *x* direction after the line of impact is rotated, $v_{iax}{}'$ is the initial velocity of the first ball in the *x* direction before the line of impact is rotated, $v_{fbx}{}'$ is the final velocity of the second ball in the *x* direction after the line of impact is rotated and $v_{ibx}{}'$ is the initial velocity of the second ball in the *x* direction before the line of impact is rotated.

Final Velocities on Normal Line of Impact:

The next step is to rotate the line of impact back to its original form and get the velocities in the *x* and *y* directions in cartesian form. This is done using the following equations:

$$v_{fx} = v_{fx}{}' \cos\phi + v_{fy}{}' \cos(\phi + 90)$$

$$v_{fy} = v_{fx}{}' \sin\phi + v_{fy}{}' \sin(\phi + 90)$$

where v_{fx} is the final velocity of the ball in the *x* direction and v_{fy} is the final velocity of the ball in the *y* direction.

Final Velocity and Angle:

Lastly, we can use the final velocities in the *x* and *y* directions to calculate the final velocity and final angle of the balls. The following equations can be used to show this step:

$$\theta_f = \tan^{-1}\left(\frac{v_{fy}}{v_{fx}}\right)^{\circ}$$

$$v_f = \sqrt{v_{fx}{}^2 + v_{fy}{}^2}\ \text{m/s}$$

where v_f is the final velocity of the ball and θ_f is the final angle from the positive *x* axis.

Impact Calculations for Collisions between Bumpers and Balls

Top/Bottom Bumpers:

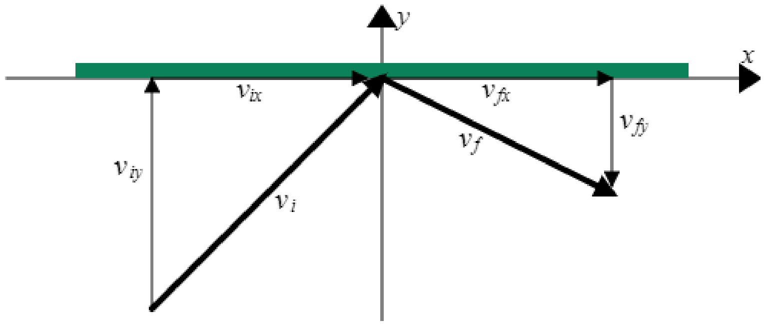

Figure 3: Diagram of the velocities before and after impact between a ball and the top bumper

During impact with the top/bottom bumpers, the velocity of the ball in the *x* direction is parallel to the plane of contact. This means it will not change after the collision. Therefore, we can apply the coefficient of restitution equation to the velocity of the ball in the *y* direction as shown below:

$$e = \frac{|v_{fy}|}{|v_{iy}|}$$

$$v_{fy} = -v_{iy}\, e$$

where v_{fy} is the velocity of the ball in the *y* direction after the collision, v_{iy} is the velocity of the ball in the *y* direction before the collision and e is the coefficient of restitution.

Left/Right Bumpers:

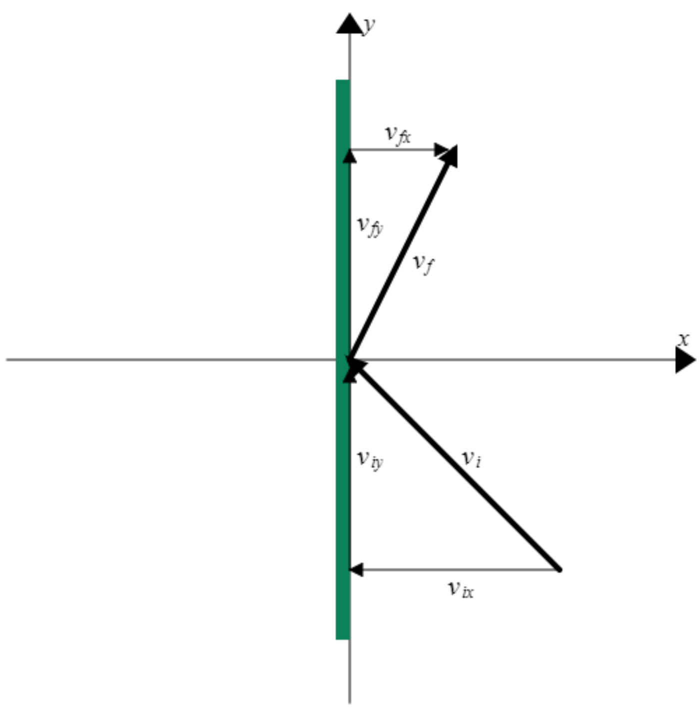

Figure 4: Diagram of the velocities before and after impact between a ball and the left bumper

During impact with the left/right bumpers, the velocity of the ball in the *y* direction is parallel to the plane of contact. This means it will not change after the collision. Therefore, we can apply the coefficient of restitution equation to the velocity of the ball in the *x* direction as shown below:

$$e = \frac{|v_{fx}|}{|v_{ix}|}$$

$$v_{fx} = -v_{ix}\, e$$

where v_{fx} is the velocity of the ball in the *x* direction after the collision, v_{ix} is the velocity of the ball in the *x* direction before the collision and e is the coefficient of restitution.

Calculation for Resistance between Tablecloth and Ball in Motion

Deceleration due to Force of Rolling Resistance:

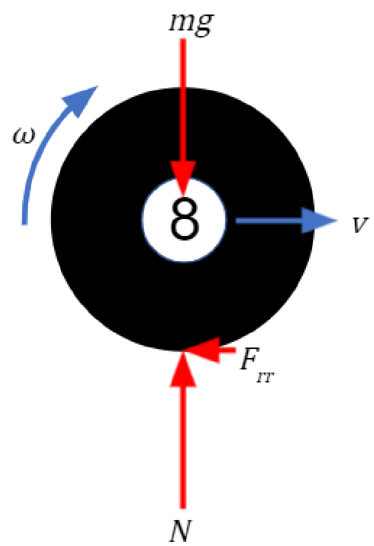

Figure 5: Free Body Diagram of a Pool Ball in Motion

When a pool ball is moving, there is a force of rolling resistance on the ball due to the tablecloth. Based off this, we can apply Newtons second law of motion and isolate a to derive an expression for the acceleration of the ball:

$$F_{net} = \sum F$$

$$ma = -F_{rr}$$

$$ma = -\mu_{rr}N$$

$$a = -\frac{\mu_{rr}(mg)}{m} \text{ m/s}^2$$

$$a = -\mu_{rr}g \text{ m/s}^2$$

where F_{rr} is the force of rolling resistance, *N* is the normal force, $\mu_{rr}$is the coefficient of rolling resistance between the ball and the tablecloth, *m* is the mass of the ball, *g* is the acceleration due to gravity and a is the acceleration of the ball.

Deceleration in *x* and *y* directions:

To calculate the deceleration of ball in the *x* and *y* directions, we need to calculate the angle at which the ball is travelling. The calculation is shown below:

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)^{\circ}$$

where v_y is the velocity of the ball in the *y* direction, v_x is the velocity of the ball in the *x* direction and θ is the angle at which the ball is travelling from the positive *x* axis.

We can now use this angle to calculate the deceleration of ball in the *x* and *y* directions using the following equations:

$$a_x = a \cos\theta \text{ m/s}^2$$

$$a_y = a \sin\theta \text{ m/s}^2$$

where a is the acceleration of the ball, a_x is the acceleration of the ball in the *x* direction and a_y is the acceleration of the ball in the *y* direction.

Conclusion:

In this document, we explained the physics involved in a game of pool. By applying the necessary physics principles, we showed the calculations and equations used to run the simulation.