3. General Random Variables
Part III: Normal (Gaussian) Random Variable

ECE 302 Spring 2012

Purdue University, School of ECE

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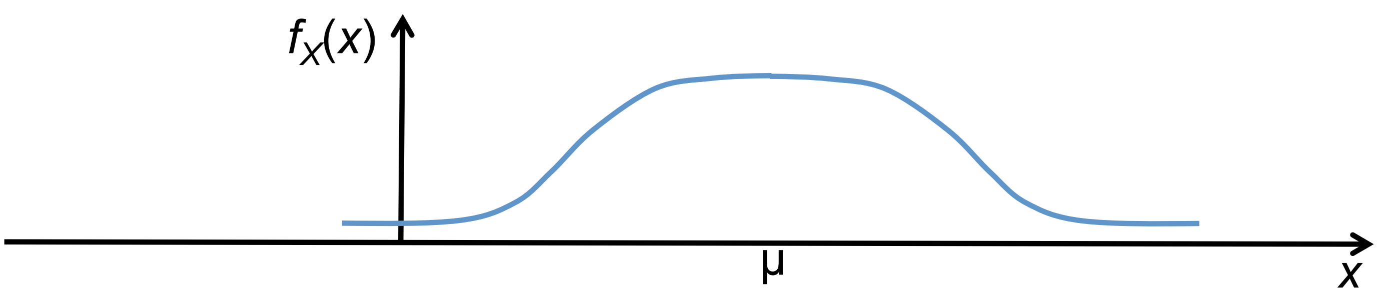
Normal (aka Gaussian) r.v.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Here, $\sigma > 0$ and μ are two parameters characterizing the PDF.

Sometimes denoted $X \sim N(\mu, \sigma^2)$



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Problem 3.14: show that $\int_{-\infty}^{\infty} f_X(x)dx = 1$.

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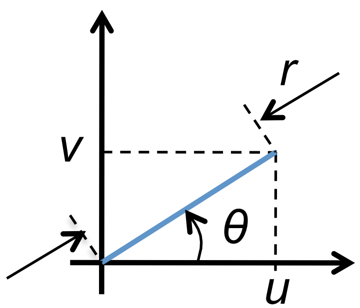
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{u^2+v^2}{2}} dudv$$

Normalization, continued

To compute $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-\frac{u^2+v^2}{2}}dudv$, use polar coordinates $r=\sqrt{u^2+v^2}$, $\theta=\tan^{-1}\frac{v}{u}$



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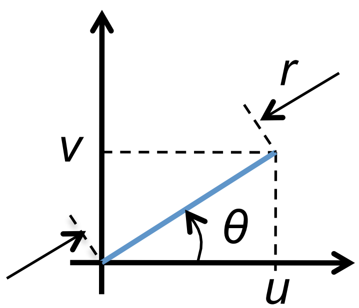
Recall: $rdrd\theta = dudv$.



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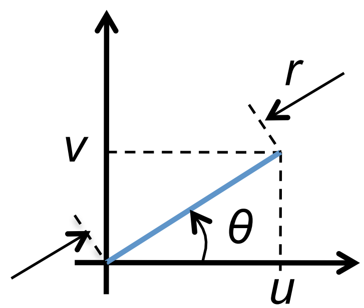

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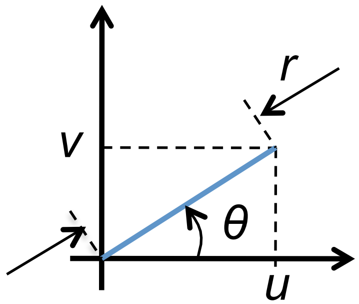

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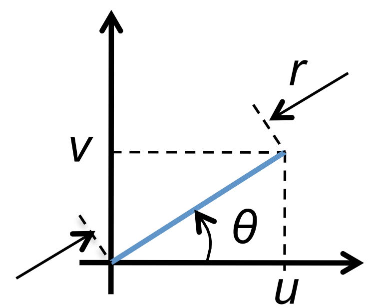

Another change of variable, $y=\frac{r^2}{2}$.

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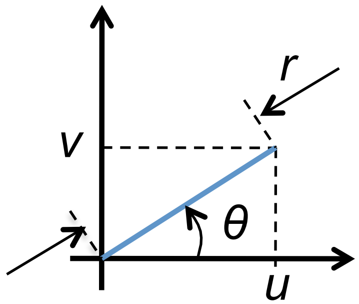

Another change of variable, $y=\frac{r^2}{2}$. Then $dy = rdr$, and

$$\int_0^{\infty}e^{-\frac{r^2}{2}}r\,dr=\int_0^{\infty}e^{-y}\,dy$$

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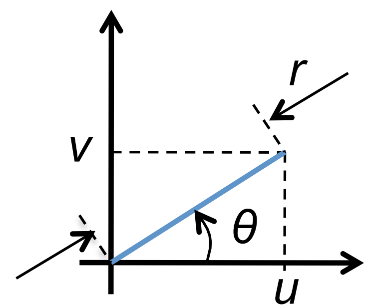

Another change of variable, $y=\frac{r^2}{2}$. Then $dy = rdr$, and

$$\int_0^{\infty}e^{-\frac{r^2}{2}}r\,dr = \int_0^{\infty}e^{-y}\,dy = -e^{-u}\Big|_0^{\infty} = 1$$

Mean of a normal r.v.

Again, use a change of variable $s = \dfrac{x-\mu}{\sigma}$, $x = s\sigma + \mu$, $dx = \sigma ds$:

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

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$$= \sigma \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} s ds + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$$

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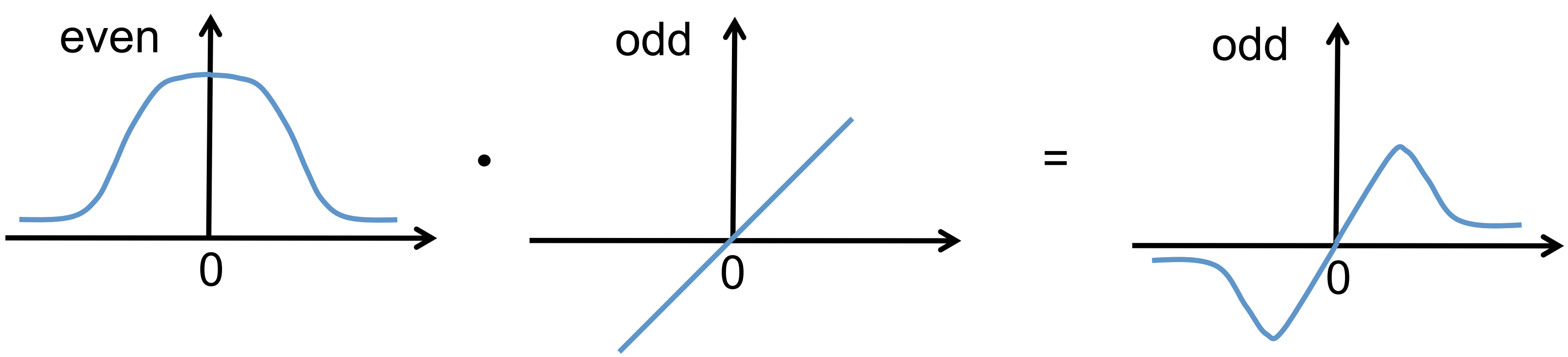

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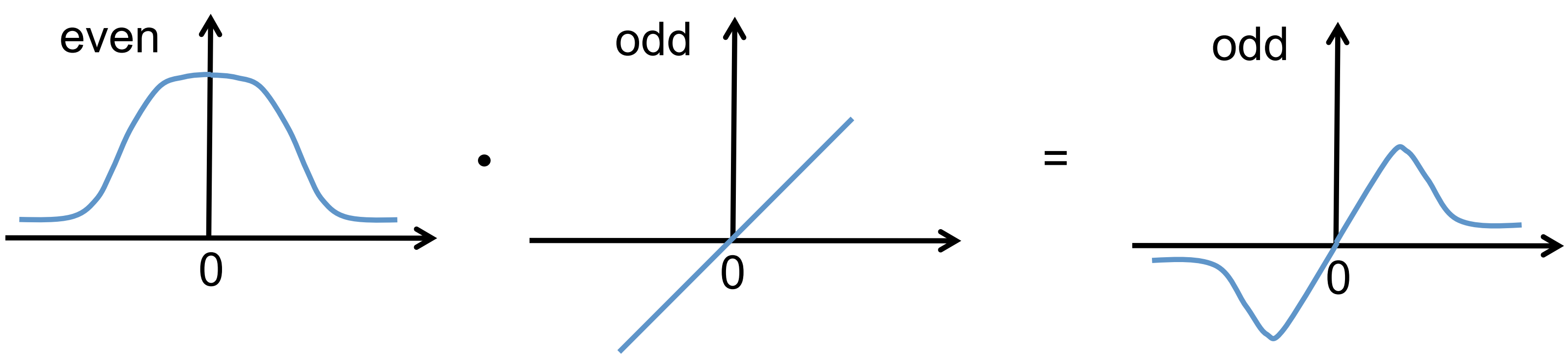

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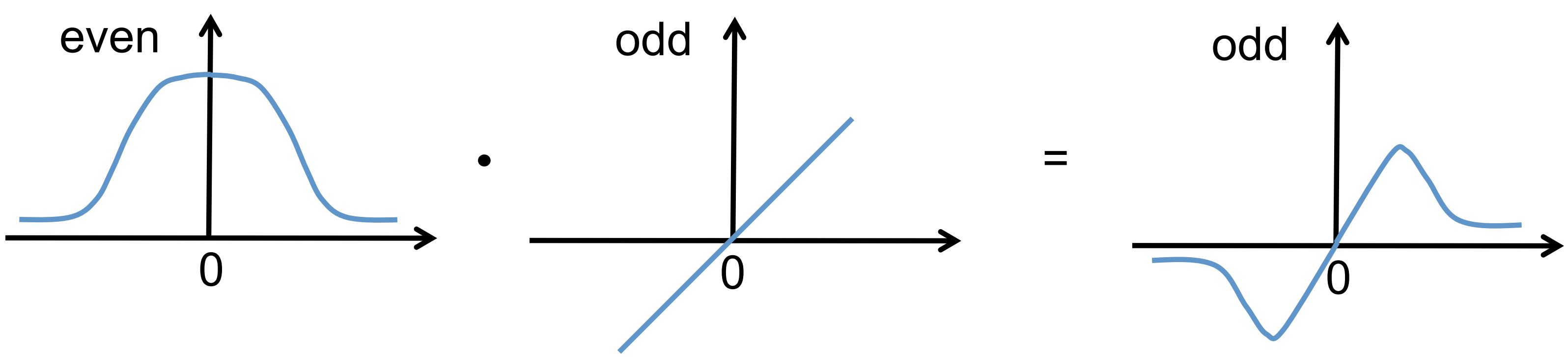

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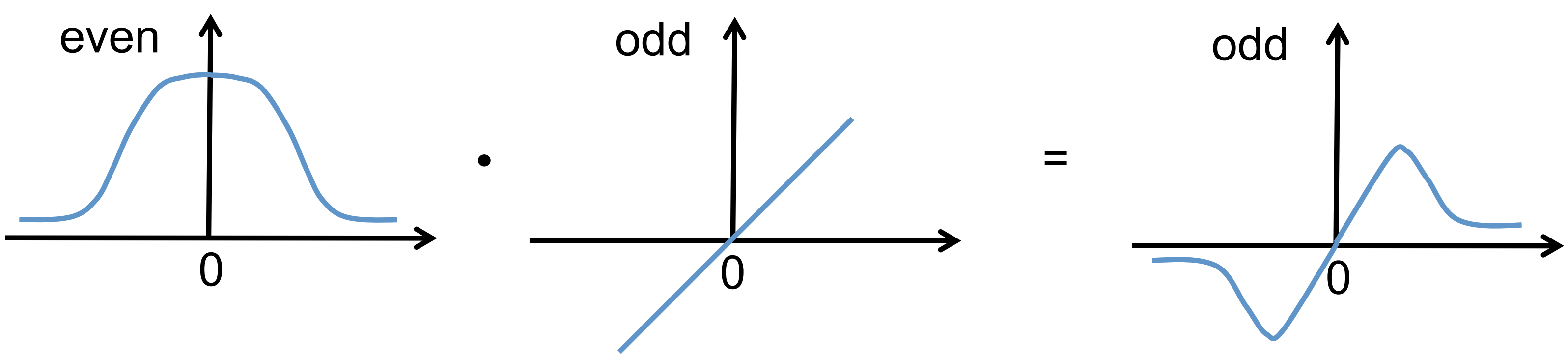

Normal PDF is symmetric around the mean

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f_X(\mu+u) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{u^2}{2\sigma^2}} = f_X(\mu-u)$$

The maximum of a normal PDF is at its mean

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f_X'(x) = -\frac{x-\mu}{\sigma^2} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

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Unique extremum is at $x = \mu$.

Since $f_X'(x) > 0$ for $x < \mu$

and $f_X'(x) < 0$ for $x > \mu$,

it's a maximum.

The variance and standard deviation of a normal r.v.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Variance = σ^2
- Standard deviation = σ
- Exercise: integration by parts

A linear function of a normal r.v. is a normal r.v.

- Suppose X is normal with mean μ and variance σ^2, and $a \neq 0$, b are real numbers

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- $\text{var}(Y) = a^2\sigma^2$

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$$= \mathbf{P}\left(X \le \frac{y-b}{a} \right)$$

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$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}(aX + b \leq y)$$

$$= \mathbf{P}\left(X \leq \frac{y-b}{a} \right)$$ --- not quite!

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$$F_Y(y) = \mathbf{P}(Y \le y) = \mathbf{P}(aX + b \le y)$$

$$= \begin{cases} \mathbf{P}\left(X \le \dfrac{y-b}{a} \right), & a > 0 \\ \mathbf{P}\left(X \ge \dfrac{y-b}{a} \right), & a < 0 \end{cases}$$

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- Mean $\mu=0$
- Standard deviation $\sigma=1$

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- Mean μ=0
- Standard deviation σ=1
- Its CDF is denoted by Φ

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$\Phi(y) = \mathbf{P}(Y \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}} dt$$

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  – if $X \sim N(\mu,\sigma^2)$ and $Y = (X-\mu)/\sigma$,
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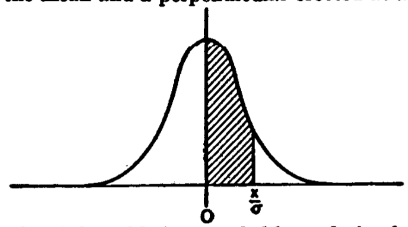

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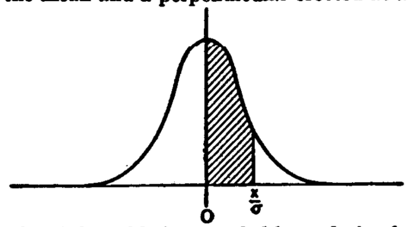

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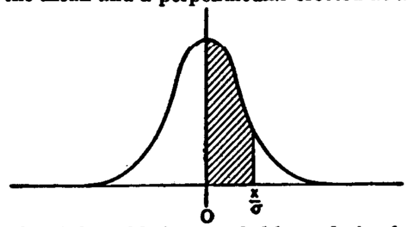

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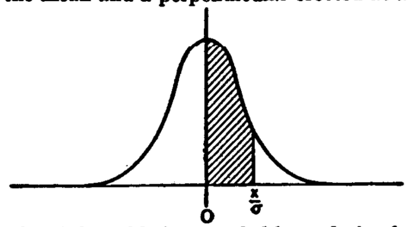

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Then there is Wolfram Alpha

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Then there is Wolfram Alpha and other software packages

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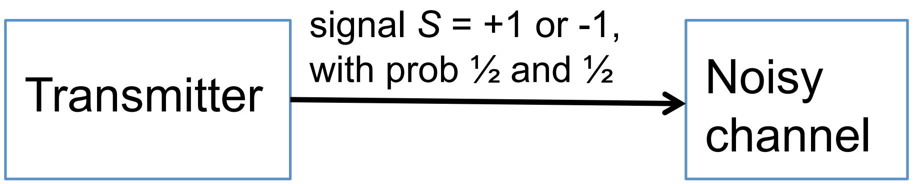
- Most scientific software packages (e.g., Matlab) have normal CDF.

Example 3.8: Signal detection



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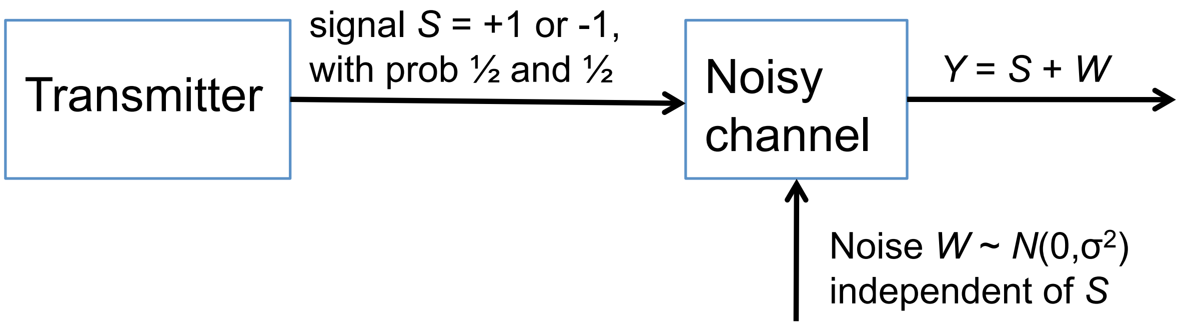

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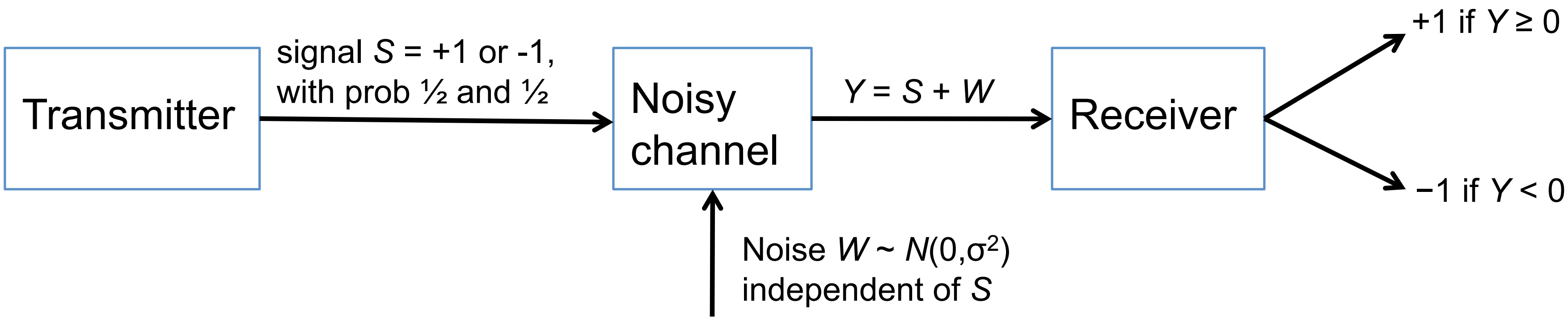

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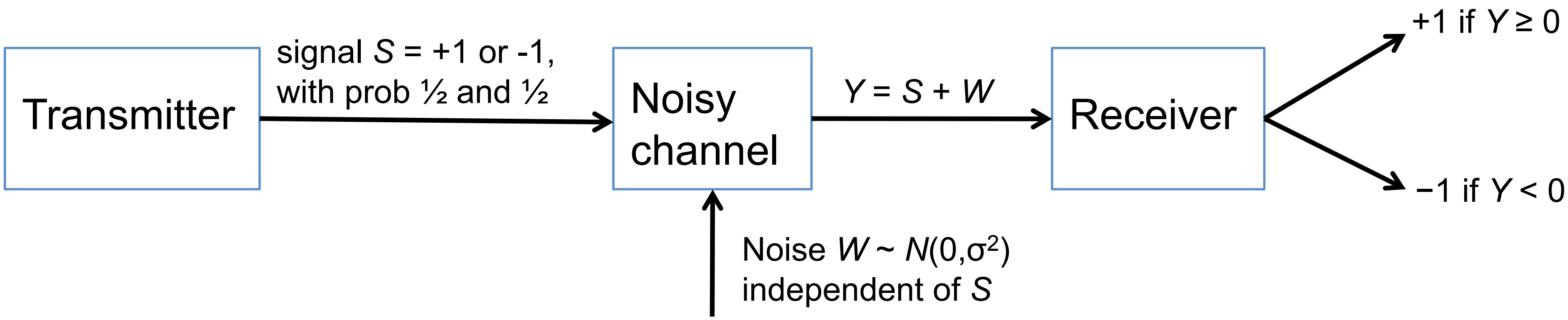

$\mathbf{P}(\text{error}) = ?$

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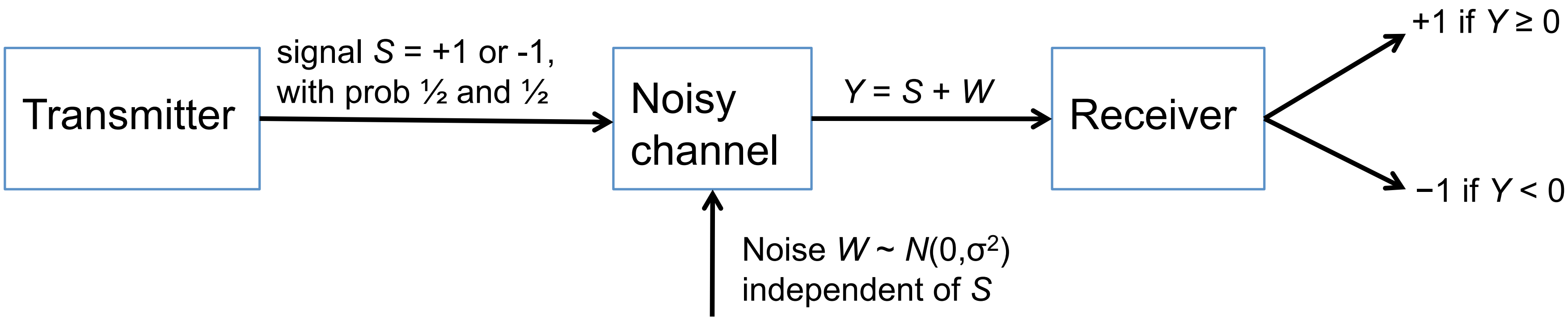

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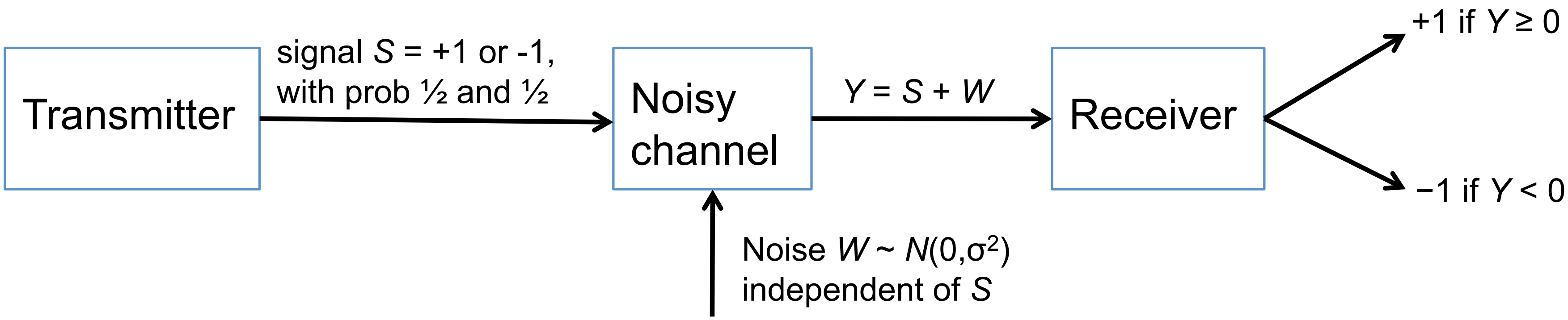

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$$= \mathbf{P}(W \geq 1 \mid S=-1)\cdot\frac{1}{2} + \mathbf{P}(W < -1 \mid S=1)\cdot\frac{1}{2}$$

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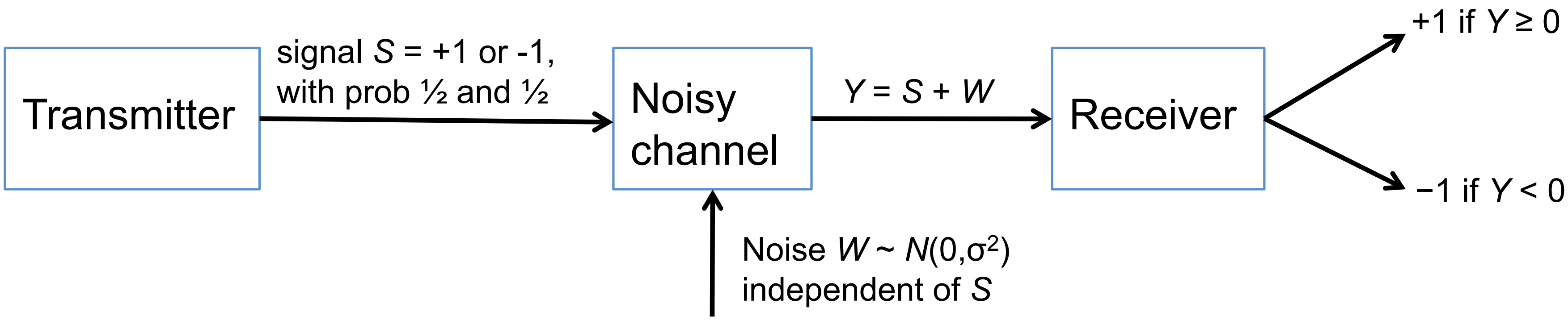

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\begin{aligned}
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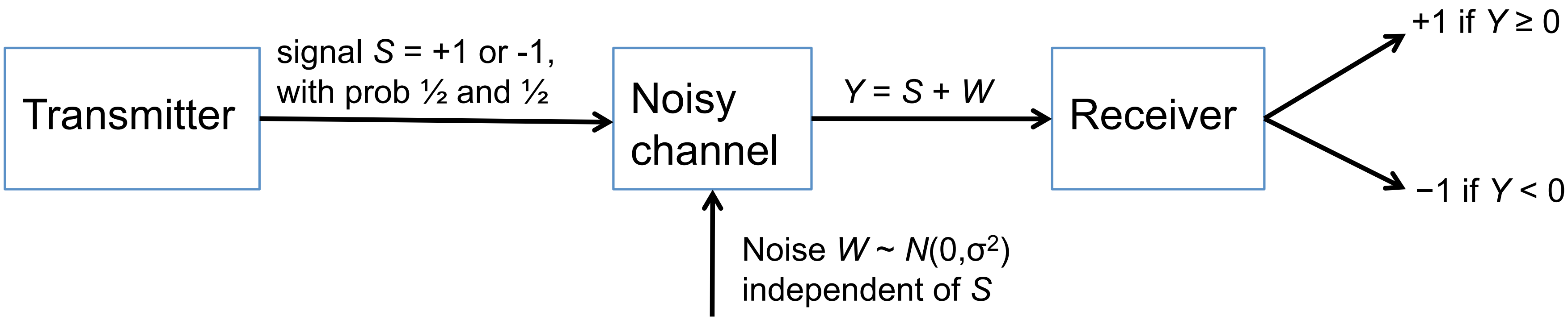
\end{aligned}
$$

Example 3.8: Signal detection



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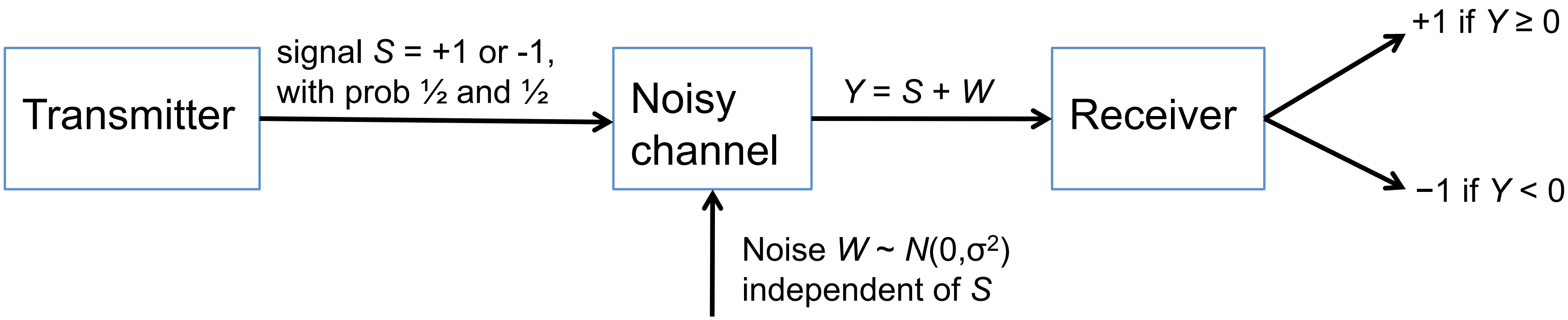

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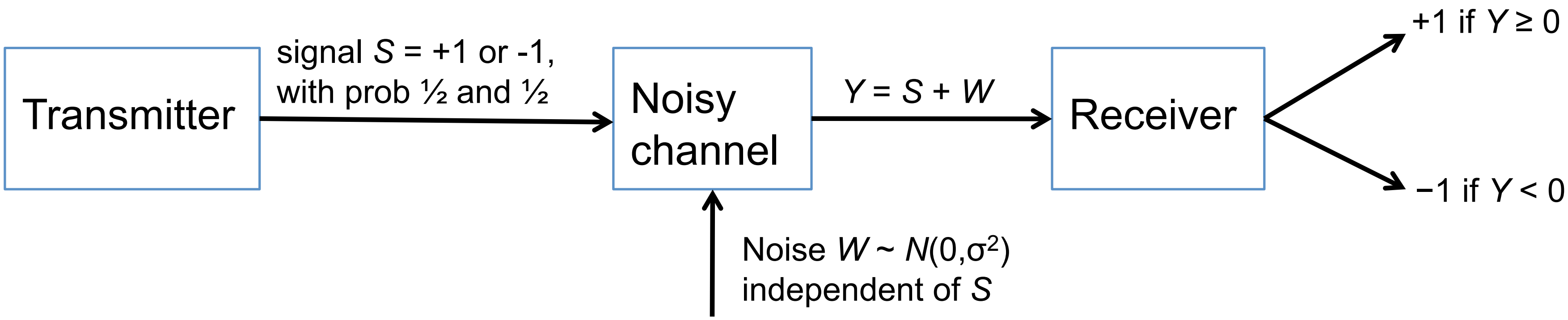

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E.g., if $\sigma = 1$, this is $1 - 0.8413 \approx 0.1587$

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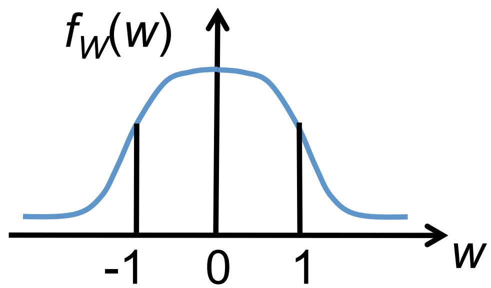

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Error function (erf) as an alternative to Φ function

The error function, erf, is built in to Matlab, Google, and Wolfram Alpha, and can be called using erf

$$\operatorname{erf}(x) \triangleq \int_{-x}^{x} \frac{1}{\sqrt{\pi}} e^{-t^2}\, dt$$

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$$\Phi(x) = \frac{1+\operatorname{erf}(x/\sqrt{2})}{2}$$