Ideas in Science



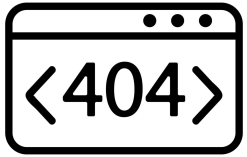
What’s In My Wallet?

Davidson Noby Joseph

Fun fact

Back in 2018 ...





What's In My Wallet?



Manali, India

Wallet guess

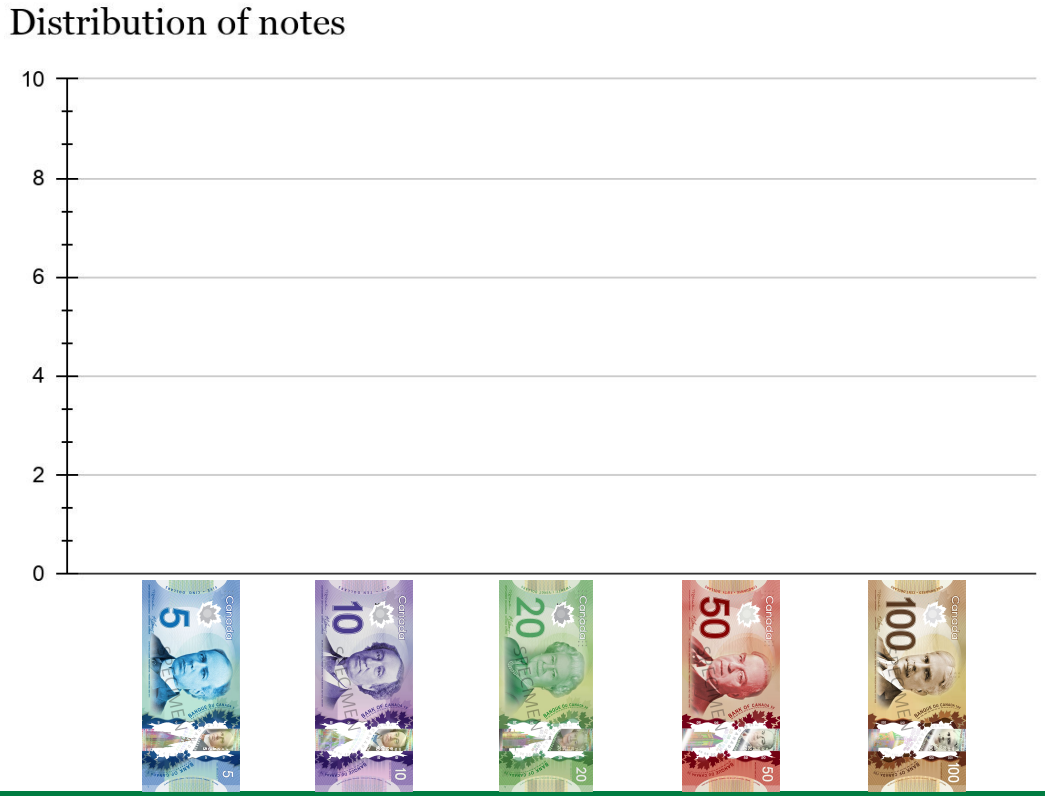
Given:

Number of notes $=10$

Guess:

$(N_{5\$}, N_{10\$}, N_{20\$}, N_{50\$}, N_{100\$})$

What's *your* most **reasonable** guess?



Wallet guess



P. S. Laplace

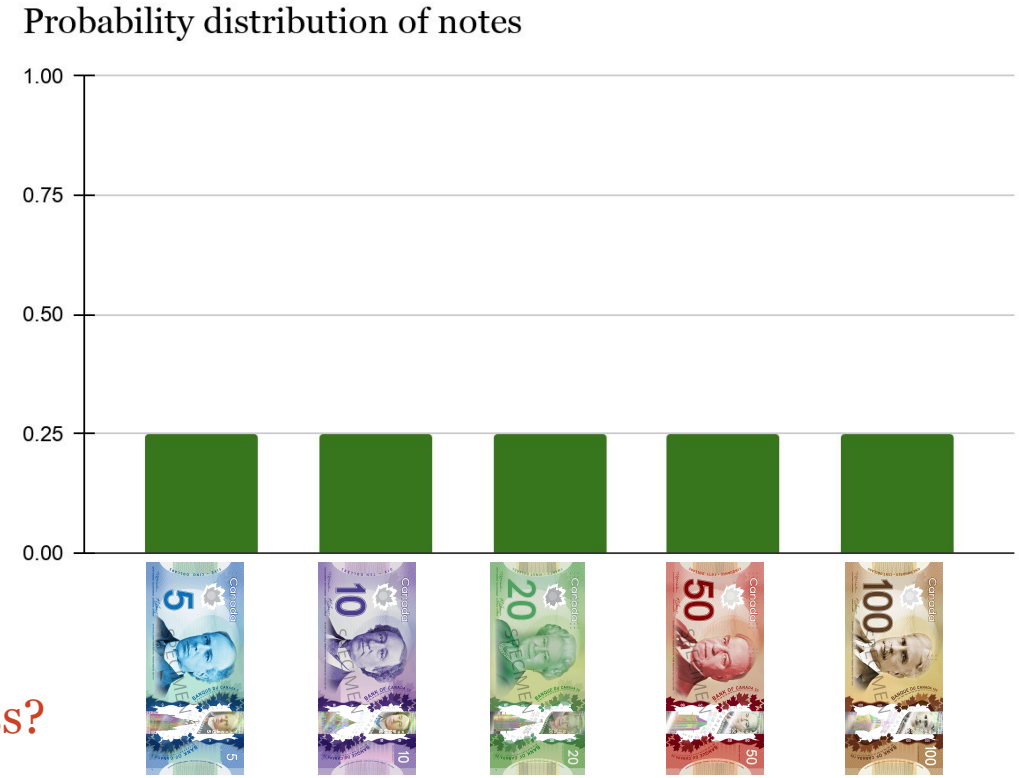
"The principle of indifference"

"Reasonable guess"

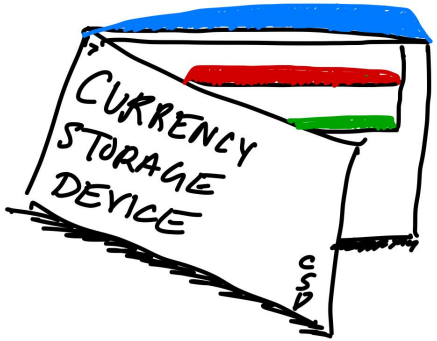
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All equally likely

Any way to quantify "reasonable" guess?



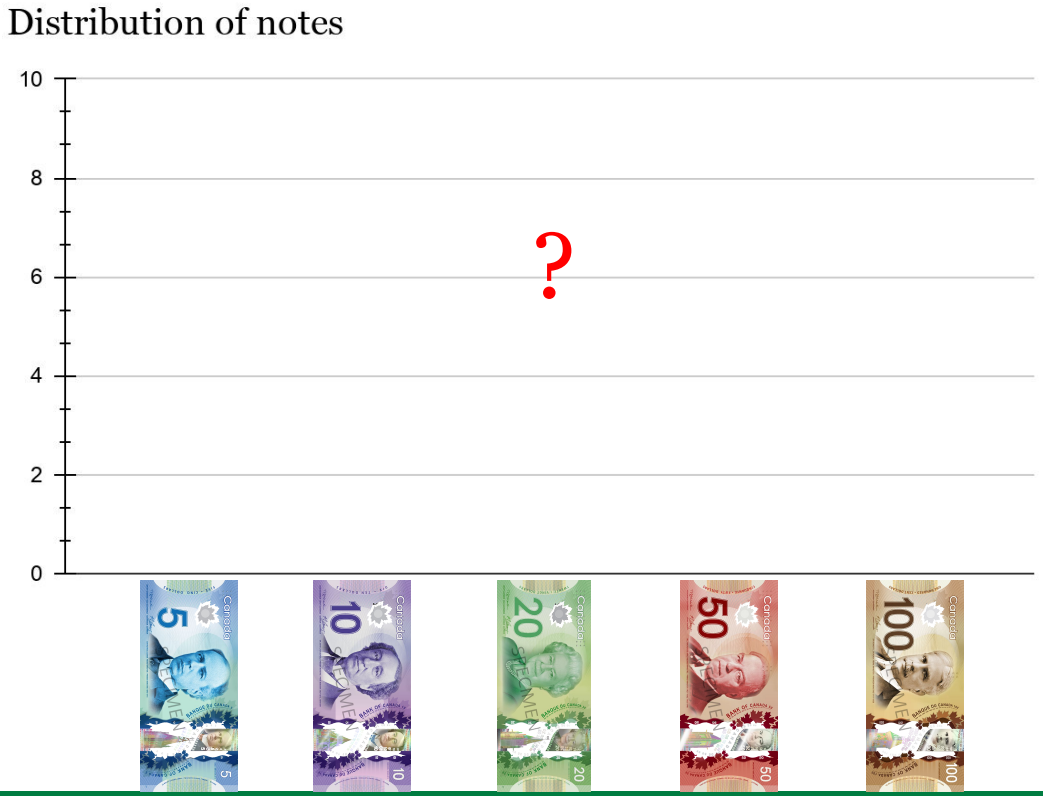
Connor's wallet



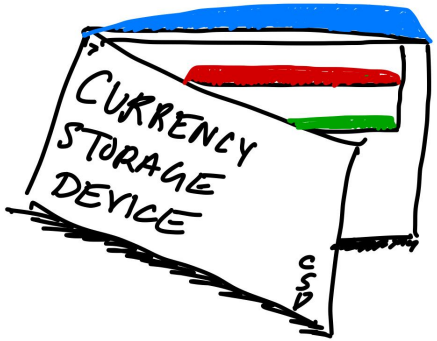
Given:

Number of notes =10

Average cash = 22$



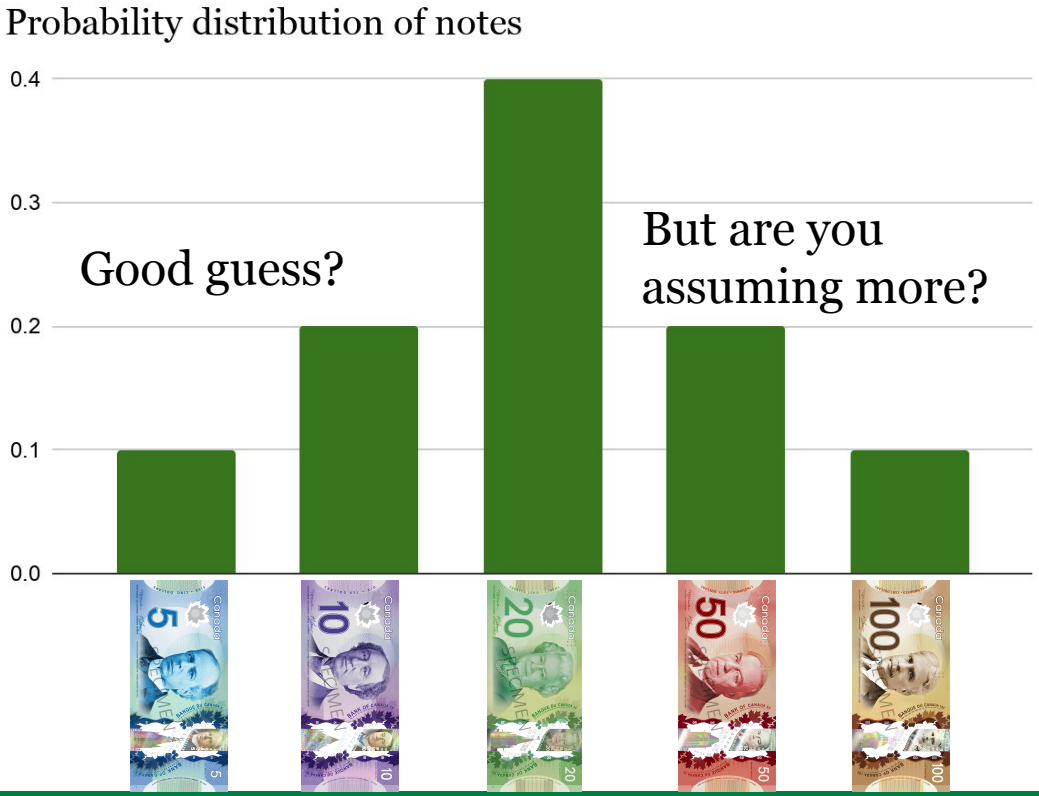
Connor's wallet



Given:

Number of notes =10

Average cash = 22$



How to assume reasonably?

But are you assuming more?

How to assume reasonably?

But are you assuming more?

How to assume reasonably?

But are you assuming more?

How to assume reasonably?

But are you assuming more?

How to assume reasonably?



E. T. Jaynes

The Principle of Maximum Entropy

Recall: Entropy = Average number of yes/no questions to deduce an outcome ⇔ Information

High entropy → high ignorance or least assumptions

Assume reasonably → assume the least → maximize entropy

Additional information → constraints on entropy

Maximize Entropy given constraints

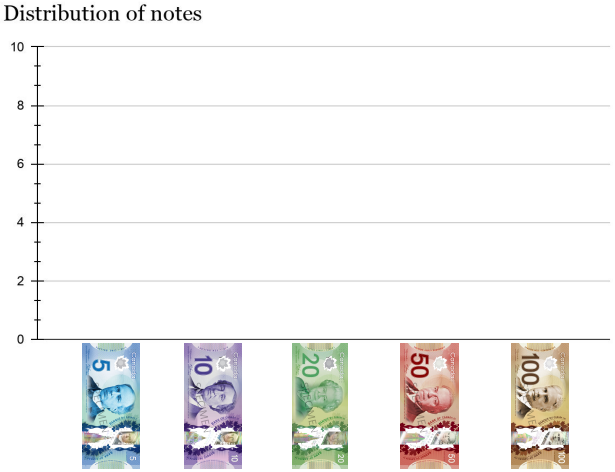
Principle of maximum entropy

Start: Discrete (probability) function $p_{i\$} = p$



Maximize Entropy $$S(p) = \sum_i p_i \ln\left(\frac{1}{p_i}\right)$$

Subject to $$\sum_{i\$} p_i = 1 \qquad \sum_{i\$} i p_i = 22\$$$



Lagrange multipliers!

$$\max \mathcal{L} = S(p) + \alpha\left(\sum_i p_i - 1\right) + \beta\left(\sum_i i p_i - 22\right)$$

Principle of maximum entropy

$$\mathcal{L} := S(p) + \alpha\left(\sum_i p_i - 1\right) + \beta\left(\sum_i i p_i - 22\right) \qquad S(p) = -\sum_i p_i \ln(p_i)$$

$$\frac{\partial \mathcal{L}}{\partial p_i} = 0 \implies \frac{\partial \mathcal{L}}{\partial p_i} = -\ln p_i - 1 - \alpha - \beta i = 0$$

$$\ln p_i = -1 - \alpha - \beta i \implies p_i = e^{-(1+\alpha)}\, e^{-\beta i}$$

$$\sum_i p_i = 1 \implies \sum_i p_i = e^{-(1+\alpha)} e^{-\beta_i} \implies p_i = \frac{e^{-\beta_i}}{\sum_i e^{-\beta_i}}$$

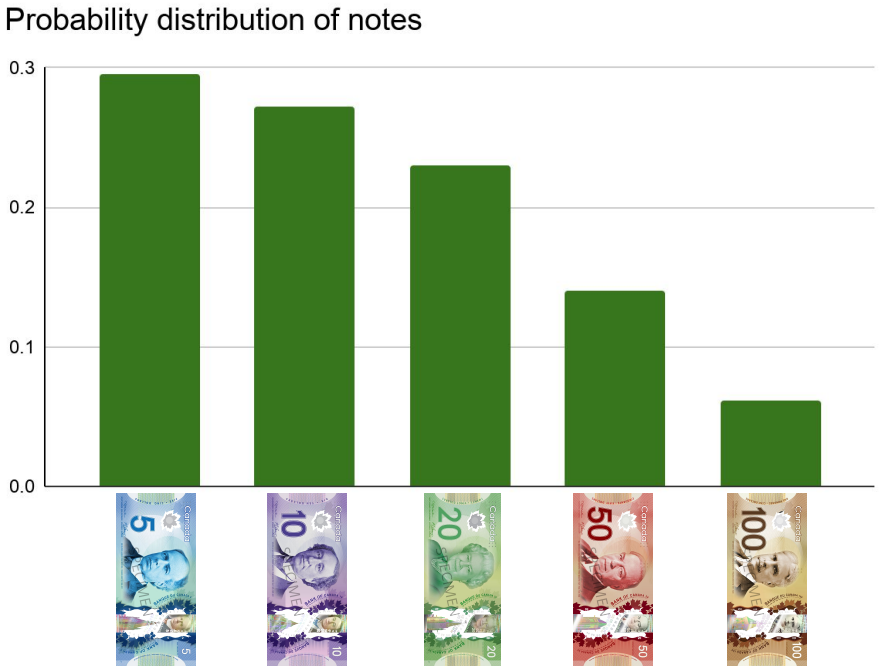
Principle of maximum entropy

$$p_i = \frac{e^{-\beta i}}{\sum_i e^{-\beta i}} \qquad i \in \{5\$, 10\$, 20\$, 50\$, 100\$\} \qquad \beta = ?$$

$$\sum_i i p_i = 22\$ \implies \frac{\sum_i i e^{-\beta i}}{\sum_i e^{-\beta i}} = 22\$$$

Numerically solve $\beta \approx 0.0164$

"maxEnt"



Remarks: Familiar friends

$$p_i = \frac{e^{-\beta i}}{\sum_i e^{-\beta i}}$$

'Boltzmann distribution'

1) $$i = E_i \implies p(E_i) = \frac{e^{-\beta E_i}}{\sum_i e^{-\beta E_i}}$$

'Soft-max'

$= 0$

Given:

2) Number of notes =10

~~Average cash = 22$~~

$$\mathcal{L} := S(p) + \alpha\left(\sum_i p_i - 1\right) + \beta\left(\sum_i i p_i - 22\right)$$

$$p_i \to p_i' = \frac{1}{\#\text{ types of notes}}$$

Laplace!

Extras: Continuous case

$$p_i \to p(x) \quad x \in \mathbb{R}$$

Entropy

$$S[p] = -\int_{-\infty}^{\infty} p(x) \ln\left(p(x)\right)$$

Constraints

$$\int_{-\infty}^{\infty} x p(x) = \mu \quad \int_{-\infty}^{\infty} (x-\mu)^2 p(x) = \sigma^2$$

$$\mathcal{L}[p] = -\int_{-\infty}^{\infty} p \ln p \, dx - \alpha\left(\int_{-\infty}^{\infty} p \, dx - 1\right) - \beta\left(\int_{-\infty}^{\infty} x \, p \, dx - \mu\right) - \gamma\left(\int_{-\infty}^{\infty} (x-\mu)^2 p \, dx - \sigma^2\right)$$

Using *functional derivatives:*

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

Normal distribution!