

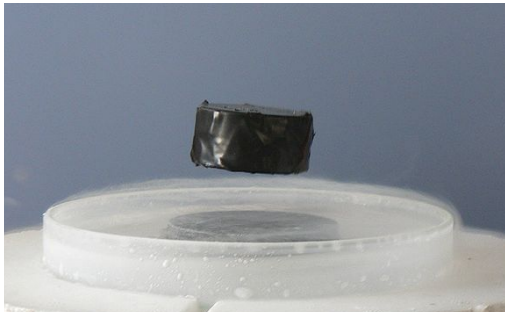


Ideas in Science

Funny Fractals in Phase Transitions

Divyesh Dagia

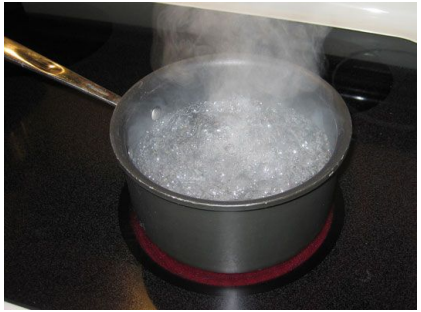
Phase Transitions

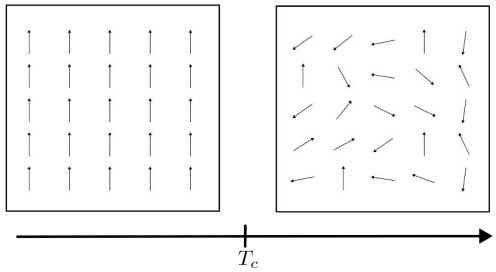


[1]



[2]

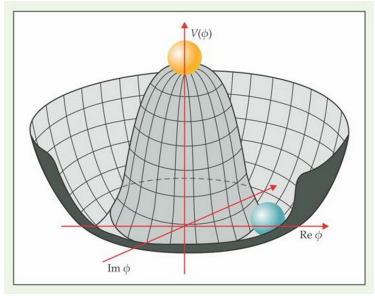




[3]



[4]



[5]

What is a Phase Transition?

- ★ One *kind* of an object turns to another
  - Ice to water to steam
  - Rock to Magma
  - Superconductor to Ordinary Conductor
- ★ Spontaneous Symmetry breaking
  - Higgs mechanism
- ★ Power law scaling
- ★ Divergence of Correlation length

All these are linked concepts… But we still need a rigorous way to define these.

Defining Phase Transitions

Helmholtz free energy:

$$\mathcal{F}(T,\vec{\phi}) \quad = U(\vec{\phi}) - TS(\vec{\phi}) = k_b T \ln(Z)$$ [6]

or other energy (like Gibbs, etc).

We would like to minimize $\mathcal{F}$ as a function of $\vec{\phi}$ for a given T.

Defining Phase Transitions

Phase transitions are where:

- Where the free energy $\mathcal{F}$ (T) doesn't have a good Taylor expansion.
- Where $\vec{\phi}$ (that minimizes $\mathcal{F}$) goes from a zero-value to a non-zero value. [7]
  - Or breaks some symmetry.

Most phase transitions are either

first-order, where $\left.\frac{\partial \mathcal{F}}{\partial T}\right|_{T=T_c}$ diverges or second-order, where $\left.\frac{\partial^2 \mathcal{F}}{\partial T^2}\right|_{T=T_c}$
(aka continuous)

For simplicity we'll look at continuous transitions. [7]

Ferromagnetic Transition

Let $\vec{\phi} = \langle \vec{\phi} \rangle = \phi \vec{n}$

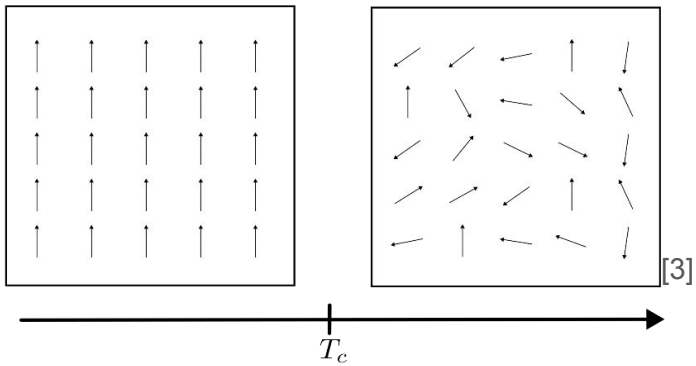
When hot, the spins are randomized:

disordered: $\phi = 0$

When cold, spins tend to align:

Ordered: $\phi \neq 0$

This is a continuous transition. [6]



[3]

Series Expansion

Define reduced temperature $t = \frac{T - T_c}{T_c}$. Then near T_c ($t = 0$)

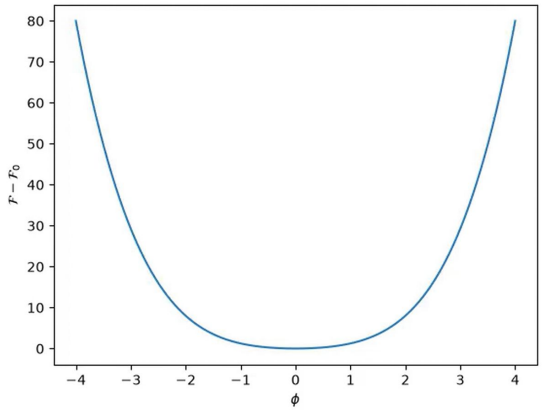
$\mathcal{F}$ can be approximated by:

$$\mathcal{F} - \mathcal{F}_0 = \frac{1}{2} r t \phi^2 + \frac{1}{4} u \phi^4$$

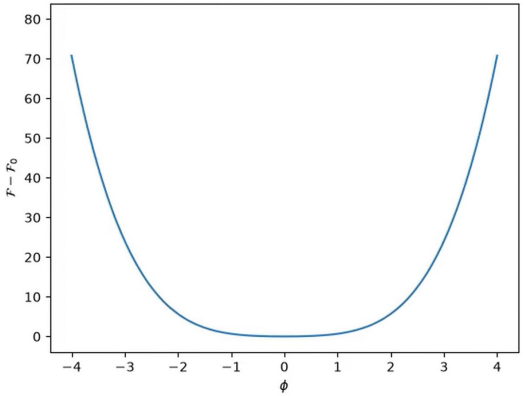
Where $r > 0, u > 0$

Varying t

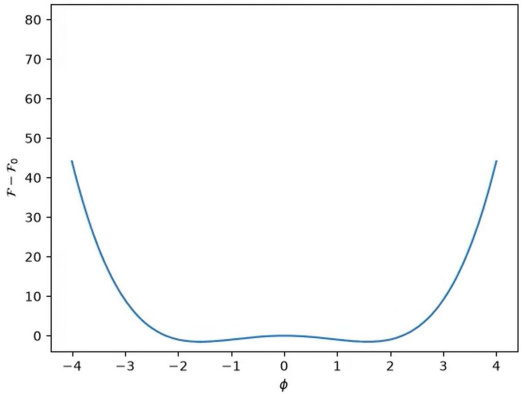
t > 0



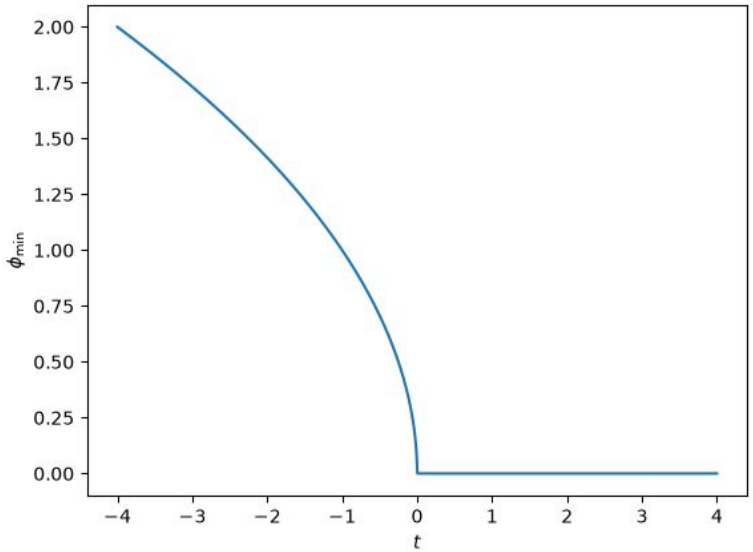
t = 0



t < 0



Mapping ϕ



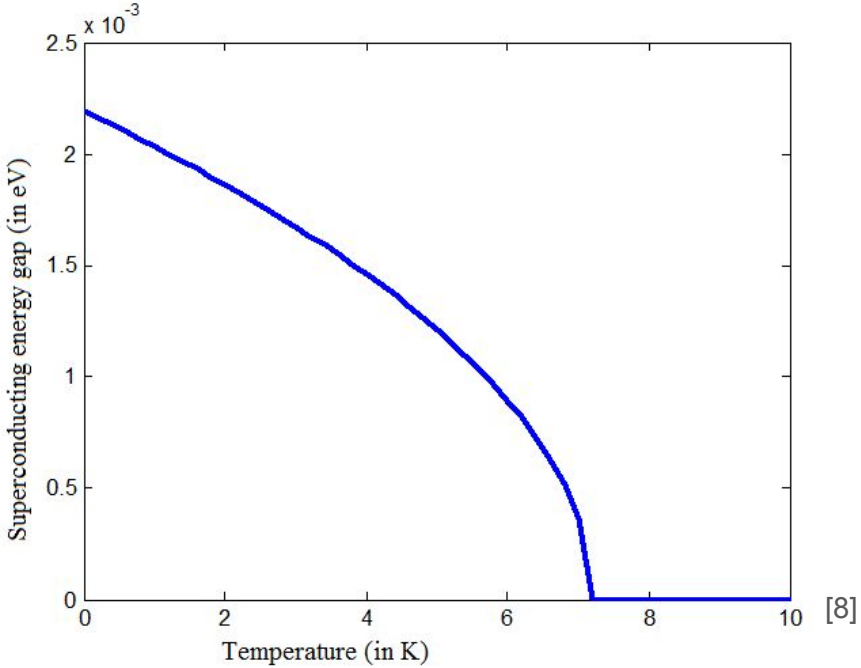
$t > 0$ gives

$$\phi = \sqrt{\frac{-rt}{u}}$$

There is power-law scaling of ϕ.

When t<0: $\phi \propto (-t)^{\beta}$ with $\beta = \frac{1}{2}$

Coincidentally



[8]

Introducing dimensionality

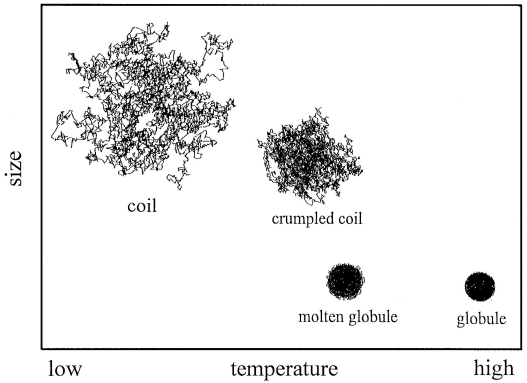
$$S = K(\partial_\mu \vec{\phi})^2 + \frac{rt}{2}\vec{\phi}^2 + \frac{u}{4}\vec{\phi}^4$$

$$\mathcal{F} = \frac{1}{\beta}\ln \int \mathcal{D}\vec{\phi} e^{-\int d^d x\, S[\vec{\phi}]}$$

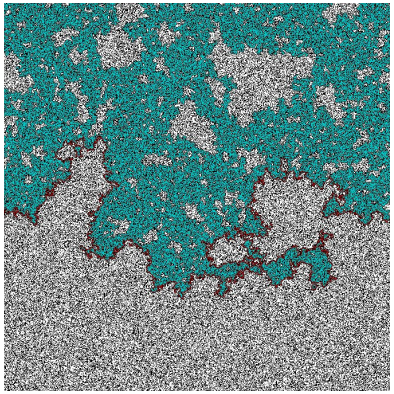
gives $\beta = \frac{d}{4} - \frac{1}{2}$ [9]

Since I promised fractals

More examples



[10]



And there’s more

References

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