Part I. The Lie bialgebra \( g[v] \)

1. Review

1.1. Quick review of September 26\(^{th}\) and October 3\(^{rd}\). In the first two meetings Matt introduced the basic definitions relevant to the theory of Lie bialgebras. I will not provide a comprehensive review below, but will recall some of the main definitions. The content we covered can mostly be found Sections 3.2–3.4, 3.6 and 4 of [ES]. The definition of a Lie bialgebra is in [ES, 2.2.1]. The first 45 pages of [ES] are available freely online. The appropriate definitions may also be found in [CP].

Definition 1.1. A Lie bialgebra is a pair \((g, \delta)\), where \(g\) is a Lie algebra and \(\delta\) is a linear map \(\delta : g \rightarrow g \wedge g \subset g \otimes g\), called the cocommutator of \(g\), satisfying

\[
\begin{align*}
(1) & \quad \delta^*|_{g^* \otimes g^*} : g^* \otimes g^* \rightarrow g^* \text{ defines a Lie bracket on the vector space } g^*, \\
(2) & \quad \delta \text{ is a 1-cocycle with values in } g \wedge g:
\end{align*}
\]

\[
(1.1) \quad \delta([X, Y]) = X \cdot \delta(Y) - Y \cdot \delta(X) \quad \forall \, X, Y \in g,
\]

where \(g\) acts on \(g \otimes g\) via the adjoint action.

One should think of a Lie bialgebra as a Lie algebra structure on both the vector space \(g\) and its dual \(g^*\), subject to a certain compatibility relation (the cocycle relation (1.1)) which makes their structure sufficiently nice.

Example 1.2. The most important example for us will be the polynomial current algebra \(g[v]\), where \(g\) is a finite-dimensional simple Lie algebra. As a Lie algebra, this is equal to the vector space of polynomial maps \(f : \mathbb{C} \rightarrow g\), with Lie structure given pointwise. Equivalently,

\[
g[v] = g \otimes_{\mathbb{C}} \mathbb{C}[v], \quad [Xv^p, Yv^q] = [X, Y]v^{p+q} \quad \forall \, X, Y \in g, \, p, q \in \mathbb{N}.
\]

Here \(Xv^p = X \otimes v^p\), and \(\mathbb{N}\) includes zero. This is an \(\mathbb{N}\)-graded Lie algebra:

\[
g[v] = \bigoplus_{p \in \mathbb{N}} g[v]_p, \quad [g[v]_p, g[v]_q] \subset g[v]_{p+q}, \quad \text{where } g[v]_p = g \otimes \mathbb{C}v^p.
\]

To describe the cocommutator \(\delta\) of \(g[v]\), let \(\Omega \in g \otimes g\) be the Casimir 2-tensor with respect to the Killing form \((, )\) or any other fixed non-degenerate, symmetric and \(g\)-invariant bilinear form. If \(\{x_b\}_{b \in B}\) is an orthonormal basis of \(g\) with respect to this form, then

\[
\Omega = \sum_{b \in B} x_b \otimes x_b \in g \otimes g.
\]
Then \( \delta : g[v] \to g[v] \otimes g[u] \cong (g \otimes g)[v,u] \) is defined by
\[
\delta(Xv) = \left( \frac{v^p - u^p}{v - u} \right) [X \otimes 1, \Omega] = \sum_{n=0}^{p-1} [X, x_n] v^n \otimes x_{n+1} u^{p-n-1} \quad \forall X \in g \text{ and } p \in \mathbb{N}.
\]
Since \( [\Delta(X), \Omega] = 0 \), this is equivalent to
\[
\delta(f(v)) = \left[ f(v) \otimes 1 + 1 \otimes f(u), \frac{\Omega}{v - u} \right] \quad \forall f(v) \in g[v],
\]
where for now \( \Omega/(v - u) \) should be just viewed as a purely formal rational function. We will soon see a better way of interpreting it.

**Remark 1.3 (On the Casimir \( \Omega \)).** A better way of defining \( \Omega \), which is “coordinate free”, is as follows. Since the form \( (, ) \) is non-degenerate, it induces an isomorphism \( \nu : g \to g^* \), \( \nu(X)(Y) = (X,Y) \quad \forall X,Y \in g \).

On the other hand, we have a canonical isomorphism \( \kappa : g \otimes g^* \to \text{End}_C(g) \) given by
\[
\kappa(X \otimes \varphi)(Y) = \varphi(X) \quad \forall X,Y \in g, \varphi \in g^*.
\]
Composing \( \kappa \) with \( \text{id}_g \otimes \nu \), we obtain an isomorphism of vector spaces
\[
\kappa \circ (\text{id}_g \otimes \nu) : g \otimes g \cong \text{End}_C(g).
\]
The Casimir 2-tensor \( \Omega \) is precisely the preimage of \( \text{id}_g \) under this isomorphism. It is an easy exercise to check that the previous definition is equivalent to this one.

**Definition 1.4.** Let \( (g, \delta) \) be a Lie bialgebra.

1. \( (g, \delta) \) is **coboundary** if there is \( r \in g \otimes g \) such that
   \[
   \delta(X) = [\Delta(X), r] = [X \otimes 1 + 1 \otimes X, r] \quad \forall X \in g.
   \]
2. \( (g, \delta) \) is **triangular** if is coboundary with \( r \in g \wedge g \) and
   \[\quad r_{12} + r_{13} + r_{23} = 0 \quad \text{in} \quad g \otimes g \otimes g.\]
3. \( (g, \delta) \) is **quasitriangular** if is coboundary with \( r + r_{21} \) a \( g \)-invariant element of \( g \otimes g \) satisfying the relation (1.2).

**Remark 1.5.**

1. The relation (1.2) is called the classical Yang-Baxter equation, and can be expressed as \( \text{CYB}(r) = 0 \), where \( \text{CYB} : g \otimes g \to g \otimes g \otimes g \) is the map which sends \( r \) to the left-hand side of (1.2).
2. It is a theorem of Drinfeld that, given \( r \in g \otimes g \), the assignment \( \delta(X) = [\Delta(X), r] \) for all \( X \in g \) determines a Lie bialgebra structure on \( g \) if and only if a) \( r + r_{21} \) is \( g \)-invariant and b) \( \text{CYB}(r) \) is \( g \)-invariant.

The easiest way for these to be satisfied is by demanding
\[
r + r_{21} = 0 \quad \text{and} \quad \text{CYB}(r) = 0.
\]
This is what it means to be a triangular Lie bialgebra. Often, this is too strong a requirement. A weaker requirement is that \( r + r_{21} \in g \otimes g \) is \( g \)-invariant and \( \text{CYB}(r) = 0 \). This is precisely what it means to be quasitriangular.
The problems of finding solutions of the classical Yang-Baxter equation and constructing quasitriangular Lie bialgebras are thus closely linked. The latter is actually more approachable as Drinfeld’s double construction gives a systematic way of producing quasitriangular Lie bialgebras.

**Theorem 1.6.** Let \((\mathfrak{g}, \delta_\mathfrak{g})\) be a finite-dimensional Lie bialgebra.

1. The vector space \(\text{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*\) can be given a Lie algebra structure with bracket
   \[ [X + \varphi, Y + \phi] = [X, Y]_\mathfrak{g} + ([X, \phi] - [Y, \varphi]) + [\varphi, \phi]_\mathfrak{g}^* \quad \forall \ X, Y \in \mathfrak{g} \text{ and } \varphi, \phi \in \mathfrak{g}^*, \]
   where \([\varphi, \phi]_\mathfrak{g}^* = \delta_\mathfrak{g}^*(\varphi \otimes \phi)\) and the term \(([X, \phi] - [Y, \varphi])\) is determined by
   \[ [X, \phi] = \text{ad}^*(\phi)(X) - \text{ad}^*(X)(\phi), \]
   with \(\text{ad}^*(\phi)(X)\) (resp. \(\text{ad}^*(X)(\phi)\)) the coadjoint action of \(\phi\) on \(X\) (resp. \(X\) on \(\phi\)).

2. Let \(\{x_\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{g}\) and \(\{x^\lambda\}_{\lambda \in \Lambda} \subset \mathfrak{g}^*\) be dual bases. Then
   \[ r = \sum_{\lambda \in \Lambda} x_\lambda \otimes x^\lambda \in \mathfrak{g} \otimes \mathfrak{g}^* \subset \text{D}(\mathfrak{g}) \otimes \text{D}(\mathfrak{g}) \]
   defines a quasitriangular Lie bialgebra structure on \(\text{D}(\mathfrak{g})\).

3. \(\text{D}(\mathfrak{g})\) naturally contains \((\mathfrak{g}, \delta_\mathfrak{g})\) and \((\mathfrak{g}^*, \delta^*_\mathfrak{g})\), where \(\delta^*_\mathfrak{g} = -\delta\), as Lie sub-bialgebras.

The quasitriangular Lie bialgebra \(\text{D}(\mathfrak{g})\) defined by this theorem is called the Drinfeld double of \((\mathfrak{g}, \delta_\mathfrak{g})\).

**Remark 1.7.**

1. The construction of \(\text{D}(\mathfrak{g})\) is closely related to the notion of Manin triples. Matt discussed this, but I won’t say more about this here.

2. The construction of the Drinfeld double relies on the fact that the dual of a finite-dimensional Lie bialgebra is itself a Lie bialgebra. This symmetry breaks in the infinite-dimensional setting. Nonetheless, there are still suitable modifications of the procedure used to construct the double in the infinite-dimensional setting, but one must be more careful.

**1.2. Quick review of October 10th.** Let us now fix \(\mathfrak{g}\) to be finite-dimensional simple Lie algebra over the complex numbers. Since our focus is the infinite-dimensional Lie algebra \(\mathfrak{g}[v]\), we should think about how the definitions and constructions of the previous lectures fit \(\mathfrak{g}[v]\).

**Question:** Is \((\mathfrak{g}[v], \delta)\) (defined in Example 1.2) a quasitriangular Lie bialgebra?

The answer is no. In particular, there is no way to interpret \(\frac{\Omega}{v-u}\) as an element of \(\mathfrak{g}[v] \otimes \mathfrak{g}[u]\).

To get a quasitriangular Lie bialgebra, we could take the Drinfeld double of \(\mathfrak{g}[v]\). Since it is infinite-dimensional, this must be done carefully. Consider the Lie algebra \(\mathfrak{g}((v^{-1})) = \mathfrak{g} \otimes \mathbb{C}((v^{-1}))\). Recall\(^1\) that this Lie algebra comes equipped with a non-degenerate, symmetric, invariant bilinear form
\[ \langle \cdot, \cdot \rangle : \mathfrak{g}((v^{-1})) \otimes \mathfrak{g}((v^{-1})) \rightarrow \mathbb{C}, \quad \langle f(v), g(u) \rangle = -\text{Res}_v(f(v), g(v)) \]
where
\[^1\text{This has not been typed, but was discussed in one of our first meetings when we constructed } \delta \text{ using the Manin triple formalism.}\]
where \( D((v^{-1})) = g \) as a Lie algebra, and has quasitriangular Lie bialgebra structure given by Definition 1.8.

The form \( \langle , \rangle \) induces an isomorphism \( g[v]^* \cong v^{-1}g[v^{-1}] \) and we have

\[
\mathfrak{g}((v^{-1})) \cong \mathfrak{g}[v] \oplus v^{-1}g[v^{-1}].
\]

Moreover, the element \( \frac{\Omega}{v-u} \) can naturally be viewed as an element of \( g((v^{-1})) \otimes g((u^{-1}))^2 \) by expanding \((v-u)^{-1}\) as a geometric series in \( u^{-1} \):

\[
\frac{\Omega}{v-u} = -\sum_{b \in B, p \geq 0} x_b v^p \otimes x_b u^{-p-1} \in g[v] \otimes u^{-1}g[u^{-1}].
\]

As the set \( \{ -x_b u^{-p-1} \}_{b \in B, p \in \mathbb{N}} \) is dual to \( \{ x_b v^p \}_{b \in B, p \in \mathbb{N}} \) under \( \langle , \rangle \), it is tempting to declare that \( g((v^{-1})) \) should be called the Drinfeld double of \( g[v] \).

However, the full dual of \( g[v] \) is too big and this would lead to several problems. In particular, \( g((v^{-1})) \) is not even itself a Lie bialgebra. If it were a Lie bialgebra with coboundary structure given by \( \Omega/(v-u) \), then we would have

\[
\delta \left( \sum_{q \geq 0} X v^{-q-1} \right) \subset \mathfrak{g}((v^{-1})) \otimes \mathfrak{g}((u^{-1})) \quad \forall \, X \in \mathfrak{g},
\]

but we have

\[
\delta \left( \sum_{q \geq 0} X v^{-q-1} \right) = \sum_{q \geq 0} \left( \frac{v^{-q-1} - u^{-q-1}}{v-u} \right) [X \otimes 1, \Omega] = \sum_{q \geq 0} \sum_{p=0}^{q} v^{-p-1} u^{-p-1} [X \otimes 1, \Omega].
\]

Since \( \sum_{q \geq 0} \sum_{p=0}^{q} v^{-p-1} u^{-p-1} \) does not belong to \( \mathfrak{C}((v^{-1})) \otimes \mathfrak{C}((u^{-1})) \), the above element does not belong to \( g((v^{-1})) \otimes g((u^{-1})) \).

A much better option is to exploit the fact that \( g[v] \) is graded and replace * by the graded dual \( \circ \). We have

\[
g[v] = \bigoplus_{p \in \mathbb{N}} (g[v]_p)^* \cong \bigoplus_{p \in \mathbb{N}} g \otimes \mathbb{C} v^{-p-1} \cong v^{-1}g[v^{-1}].
\]

**Definition 1.8.** The graded Drinfeld double of \( g[v] \) is equal to

\[
D(g[v]) = g[v] \oplus v^{-1}g[v^{-1}] = g[v^\pm 1]
\]
as a Lie algebra, and has quasitriangular Lie bialgebra structure given by

\[
\delta(f(v)) = [f(v) \otimes 1 + 1 \otimes f(u), r_g] \quad \forall \, f(v) \in D(g[v]),
\]

where

\[
r_g = -\sum_{b \in B, p \geq 0} x_b v^p \otimes x_b u^{-p-1} \in g[v] \otimes u^{-1}g[u^{-1}] \subset D(g[v]) \otimes D(g[u]).
\]

**Remark 1.9.** \( D(g[v]) \otimes D(g[u]) \) can be defined as follows. Set \( D(g[v])_k = g \otimes \mathbb{C} v^k \). Then

\[
D(g[v]) \otimes D(g[u]) = \left\{ \sum_{k \in \mathbb{Z}} A_k \in \prod_{k \in \mathbb{Z}} (D(g[v])_k \otimes D(g[u])) : A_m = 0 \quad \forall \, m \gg 0 \right\}.
\]

\(^2\)Here \( g((v^{-1})) \otimes g((u^{-1})) \) denotes a completion of \( g((v^{-1})) \otimes g((u^{-1})) \). It’s precise definition will not be needed (see Remark 1.9).
In conclusion, we have seen that $\mathfrak{g}[v]$ is not itself quasi-triangular, but it embeds in the topologically quasi-triangular Lie bialgebra $D(\mathfrak{g}[v]) = \mathfrak{g}[v^{\pm 1}]$, which is equal to its graded Drinfeld double. Here the word “topologically” just indicates that $r_\mathfrak{g}$ belongs to a completion of $D(\mathfrak{g}[v]) \otimes D(\mathfrak{g}[u])$ and not the genuine tensor square of $D(\mathfrak{g}[v])$.

In the next section, we will see that $\mathfrak{g}[v]$ has what is called a pseudo-quasi-triangular Lie bialgebra structure, which in many ways is much more interesting than an honest quasi-triangular structure.

October 17th 2018

2. Pseudo quasi-triangular structure on $\mathfrak{g}[v]$

2.1. Shift automorphisms and their formal analogues.

**Lemma 2.1.** Let $c \in \mathbb{C}$. Then

$$\tau_c : f(v) \mapsto f(v + c) \quad \forall f(v) \in \mathfrak{g}[v]$$

defines a Lie algebra automorphism of $\mathfrak{g}[v]$.

These are called *shift automorphisms* and they play a critical role in the representation theory of $\mathfrak{g}[v]$ (and the Yangian!). We will give one interesting application in Proposition 2.3 below, but first we will need some terminology.

For each $c \in \mathbb{C}$, set

$$V(c) = \tau_c^*(V).$$

Consider the *evaluation homomorphism* $\text{ev}_\mathfrak{g}$ defined by

$$\text{ev}_\mathfrak{g} : \mathfrak{g}[v] \rightarrow \mathfrak{g}, \quad f(v) \mapsto f(0) \quad \forall f(v) \in \mathfrak{g}[v].$$

Of course, $\text{ev}_\mathfrak{g}$ uniquely extends to $U(\mathfrak{g}[v]) \rightarrow U(\mathfrak{g})$. We will not distinguish between these two epimorphisms.

**Definition 2.2.** Given a $U(\mathfrak{g})$-module $V$, we define $V_{ev}$ to be the $U(\mathfrak{g}[z])$-module

$$V_{ev} = \text{ev}_\mathfrak{g}^*(V).$$

Any $U(\mathfrak{g}[z])$-module of the form $V_{ev}(c)$ is called an *evaluation module*.

**Proposition 2.3.** Let $V, W$ be finite-dimensional irreducible $U(\mathfrak{g})$-modules. The $U(\mathfrak{g}[v])$-module

$$V_{ev}(c) \otimes W_{ev}(b)$$

is irreducible whenever $c - b \in \mathbb{C} \setminus \mathbb{C}^\times$.

**Proof.** We begin by reducing the proof to the case where $b$ is taken to be zero.

**Claim:** It suffices to prove the assertion of the proposition for $b = 0$.

To see why this is true, note that $\tau_c$ is a coalgebra morphism (or more precisely, its extension to $U(\mathfrak{g}[v])$ is). This means that

$$\tau_c \otimes \tau_c \circ \Delta = \Delta \circ \tau_c,$$

(2.1)
where $\Delta$ is the standard coproduct on $U(g[v])$. Moreover, since $\tau_{-b}$ is an automorphism, $V_{ev}(c) \otimes W_{ev}(b)$ is irreducible if and only if $\tau_{-b}^*(V_{ev}(c) \otimes W_{ev}(b))$ is. By (2.1) and the fact that $\tau_\lambda \circ \tau_\gamma = \tau_{\lambda + \gamma}$, we have

$$\tau_{-b}(V_{ev}(c) \otimes W_{ev}(b)) = V_{ev}(c - b) \otimes W_{ev},$$

which proves the claim.

Hence, we can assume that $b = 0$ and $c \in \mathbb{C}^\times$. Let $\pi_V : U(g) \rightarrow \text{End}_\mathbb{C}(V)$ be the homomorphism equipping $V$ with a $U(g)$-module structure and define $\pi_W$ similarly. Let

$$\pi : U(g[v]) \rightarrow \text{End}(V \otimes W)$$

be the algebra homomorphism associated to $V_{ev}(c) \otimes W_{ev}$, so

$$\pi = (\pi_V \circ \tau_c) \otimes \pi_W.$$

To prove the proposition it suffices to show that $\pi$ is surjective. By Jacobson’s Density Theorem, both $\pi_V$ and $\pi_W$ are themselves surjective. Thus, $\text{End}_\mathbb{C}(V \otimes W) \cong \text{End}_\mathbb{C}(V) \otimes \text{End}_\mathbb{C}(W)$ is generated by elements of the form

$$\pi_V(X) \otimes \pi_W(Y) = (\pi_V(X) \otimes \text{id}_W)(\text{id}_V \otimes \pi_W(Y)) \quad \forall X, Y \in g.$$

By definition,

$$\pi(c^{-1}Xv) = \pi_V(X) \otimes \text{id}_W \quad \text{and} \quad \pi(X - c^{-1}Xv) = \text{id}_V \otimes \pi_W(X) \quad \forall X \in g. \quad \square$$

**Definition 2.4.** The formal shift homomorphism is the Lie algebra injection

$$\tau_z : g[v] \hookrightarrow (g[v])[z], f(v) \mapsto f(v + z) \quad \forall f(v) \in g[v].$$

Note that $\tau_z$ is a $\mathbb{N}$-graded homomorphism, provided we view $(g[v])[z] \cong g[v, z]$ as a graded Lie algebra with $\text{deg} z = \text{deg} v = 1$. It satisfies

$$\tau_c = \text{ev}_c \circ \tau_z \quad \forall c \in \mathbb{C},$$

where $\text{ev}_c : (g[v])[z] \rightarrow g[v]$ is given by $z \mapsto c$.

**Goal A.** Show $\tau_z$ extends to $\Phi_z : D(g[v]) \hookrightarrow (g[v])((z^{-1}))$ which has image in a $\mathbb{Z}$-graded subalgebra of $(g[v])((z^{-1}))$, and is itself a $\mathbb{Z}$-graded homomorphism.

The hidden subgoal is that we want to use $\Phi_z$ to “project” the quasitriangular structure of $D(g[v])$ onto $g[v]$. Realizing the above goal is in fact not at all difficult, but let us first consider a simpler version of this story.

Consider the shift homomorphism

$$\gamma_z : \mathbb{C}[w] \rightarrow \mathbb{C}[v][z], \quad f(w) \mapsto f(v + z) \quad \forall f(w) \in \mathbb{C}[w].$$

Note that $\gamma_z = \text{id}_g \otimes \gamma_z$.

**Lemma 2.5.**

1. $\gamma_z$ extends uniquely to $\Gamma_z : \mathbb{C}[w^{\pm 1}] \rightarrow \mathbb{C}[v]((z^{-1}))$.
2. $\Gamma_z$ has image in $\bigoplus_{n \in \mathbb{Z}} z^n \mathbb{C}[v]$ with $\mathbb{C}[v][z] = \prod_{k \in \mathbb{N}} \mathbb{C}v^k z^{-k}$.
3. $\Gamma_z$ is $\mathbb{Z}$-graded.

**Proof.** To prove (1), note that for each $p \geq 0$,

$$\gamma_z(w^p) = (v + z)^p \in \mathbb{C}[v]((z^{-1}))^\times.$$

Indeed, $\sum_{k \geq 0} (-v)^k z^{-k-1}$ is the inverse of $\gamma_z(w)$, and its $p$-th power is the inverse of $\gamma_z(w^p)$. Since

$$\mathbb{C}[w^{\pm 1}] = S^{-1}(\mathbb{C}[w]), \quad \text{where} \quad S = \{w^p : p \geq 0\},$$
the universal property of localization implies that $\gamma_z$ extends uniquely to $\Gamma_z$, as desired. Parts 2 and 3 follow from the fact that

\begin{equation}
\tag{2.2}
\Gamma_z(w^p) = z^p \sum_{k=0}^{p} \binom{p}{k} w^k z^{-k}, \quad \Gamma_z(w^{-p-1}) = z^{-p-1} \sum_{k=0}^{p+1} \binom{p+1}{k} (-1)^k w^k z^{-k}. \quad \square
\end{equation}

As a corollary, we get Goal A for free by setting $\Phi_z = \text{id}_g \otimes \Gamma_z$.

**Proposition 2.6.** $\Phi_z = \text{id}_g \otimes \Gamma_z : g[\pm 1] \rightarrow g \otimes \mathbb{C}[v](\langle z^{-1} \rangle) \subset (g[v])(\langle z^{-1} \rangle)$ is a Lie algebra injection satisfying

1. $\Phi_z$ is $\mathbb{Z}$-graded with image in $\bigoplus_{n \in \mathbb{Z}} z^n g_\sim[v]$, where $g_\sim[v] = \prod_{p \in \mathbb{N}} g[v]_k z^{-k}$.
2. $\Phi_z|_{g[w]} = \tau_z$ and $\Phi_z(w^{-1}g[w^{-1}]) \subset (g[v])[\langle z^{-1} \rangle]$.

There are a few nice ways of writing down $\Phi_z$ explicitly. One such way is to use (2.2). Another is as follows: for each $X \in g$, set

\[X(u) = \sum_{p \in \mathbb{Z}} Xw^p u^{-p-1} \in (g[\pm 1])[u, u^{-1}]\]

Then we have

\[\Gamma_z(X(u)) = \sum_{n \in \mathbb{N}} X^{n} \partial_z^{(n)}(u^{-1} \delta(z/u)) \]

\[= \exp(v\partial_z)(u^{-1}\delta(z/u)X) = u^{-1} \delta\left(\frac{z+v}{u}\right) X,\]

Where $\delta(x) = \sum_{p \in \mathbb{Z}} x^p$ and the above equality is understood to mean that the $u^{-p-1}$ coefficient of $X(u)$ gets sent to the $u^{-p-1}$ coefficient of the right-hand side for each $p \in \mathbb{Z}$.

The following exercise is not difficult, but it is important and I unfortunately don’t have time to discuss it.

**Exercise.** Let $\lambda \in \mathbb{C}^\times$.

1. Show that there is a Lie algebra homomorphism $\Phi_\lambda : g[\pm 1] \rightarrow g[v] = g \otimes \mathbb{C}[v]$ given by composing $\Phi_z$ with the evaluation map at $z = \lambda$.
2. Show that $\Phi_\lambda$ is injective.
3. Let $g[\pm 1] = \varprojlim_n (g[\pm 1]/\mathfrak{J}_n)$, where $\mathfrak{J}_n = (z-1)^n g[z]$. Show that $\Phi = \Phi_\lambda|_{\lambda=1}$ induces an isomorphism

\[\hat{\Phi} : g[\pm 1] \rightarrow g[v].\]

As a hint for the last part, I would recommend rephrasing the problem just in terms of $\mathbb{C}[\pm 1]$ and $\mathbb{C}[v]$ and using that $\mathbb{C}[v] = \varprojlim_n (\mathbb{C}[v]/v^n \mathbb{C}[v])$.

**References**


DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA.

E-mail address: cwendlan@ualberta.ca