Part I. The Lie bialgebra $g[v]

1. Review
1.1. Quick review of September 26th and October 3rd

1.2. Quick review of October 10th

2. Pseudo quasitriangular structure on $g[v]

2.1. Shift automorphisms and their formal analogues

2.2. The classical universal $r$-matrix

3. Rationality properties of the current algebra

3.1. Rationality of generating series

3.2. Rationality of the universal $r$-matrix

Part II. The Yangian $Y_{\hbar}(g)$

4. Quantization

4.1. Quantizations over $\mathbb{C}[\hbar]

4.2. Homogeneous quantizations

5. The Yangian defined

5.1. Chevalley-Serre presentation for $g[v]

5.2. Definition of $Y_{\hbar}(g)

5.3. Hopf structure and quantization

5.4. Generating series and automorphisms

6. Filtered algebras and Rees algebras

6.1. Filtered algebras

6.2. Rees algebras

References

Part I. The Lie bialgebra $g[v]

1. Review

1.1. Quick review of September 26th and October 3rd. In the first two meetings Matt introduced the basic definitions relevant to the theory of Lie bialgebras. I will not provide a comprehensive review below, but will recall some of the main definitions. The content we covered can mostly be found Sections 3.2–3.4, 3.6 and 4 of [ES]. The definition of a Lie bialgebra is in [ES, 2.2.1]. The first 45 pages of [ES] are available freely online. The appropriate definitions may also be found in [CP].

Definition 1.1. A Lie bialgebra is a pair $(g, \delta)$, where $g$ is a Lie algebra and $\delta$ is a linear map

$$\delta : g \rightarrow g \wedge g \subset g \otimes g,$$

called the cocommutator of $g$, satisfying

(1) $\delta^*|_{g^* \otimes g^*} : g^* \otimes g^* \rightarrow g^*$ defines a Lie bracket on the vector space $g^*$,

(2) $\delta$ is a 1-cocycle with values in $g \wedge g$:

(1.1) $\delta([X,Y]) = X \cdot \delta(Y) - Y \cdot \delta(X) \quad \forall X, Y \in g$,
where \( \mathfrak{g} \) acts on \( \mathfrak{g} \otimes \mathfrak{g} \) via the adjoint action.

One should think of a Lie bialgebra as a Lie algebra structure on both the vector space \( \mathfrak{g} \) and its dual \( \mathfrak{g}^* \), subject to a certain compatibility relation (the cocycle relation (1.1)) which makes their structure sufficiently nice.

**Example 1.2.** The most important example for us will be the *polynomial current algebra* \( \mathfrak{g}[v] \), where \( \mathfrak{g} \) is a finite-dimensional simple Lie algebra. As a Lie algebra, this is equal to the vector space of polynomial maps \( f : \mathbb{C} \to \mathfrak{g} \), with Lie structure given pointwise. Equivalently,

\[
\mathfrak{g}[v] = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[v], \quad [X v^p, Y v^q] = [X, Y] v^{p+q} \quad \forall X, Y \in \mathfrak{g}, \; p, q \in \mathbb{N}.
\]

Here \( X v^p = X \otimes v^p \), and \( \mathbb{N} \) includes zero. This is an \( \mathbb{N} \)-graded Lie algebra:

\[
\mathfrak{g}[v] = \bigoplus_{p \in \mathbb{N}} \mathfrak{g}[v]_p, \quad [\mathfrak{g}[v]_p, \mathfrak{g}[v]_q] \subset \mathfrak{g}[v]_{p+q}, \quad \text{where} \; \mathfrak{g}[v]_p = \mathfrak{g} \otimes \mathbb{C} v^p.
\]

To describe the cocommutator \( \delta \) of \( \mathfrak{g}[v] \), let \( \Omega \in \mathfrak{g} \otimes \mathfrak{g} \) be the Casimir 2-tensor with respect to the Killing form \((\ , \) ) or any other fixed non-degenerate, symmetric and \( \mathfrak{g} \)-invariant bilinear form. If \( \{x_b\}_{b \in \mathcal{B}} \) is an orthonormal basis of \( \mathfrak{g} \) with respect to this form, then

\[
\Omega = \sum_{b \in \mathcal{B}} x_b \otimes x_b \in \mathfrak{g} \otimes \mathfrak{g}.
\]

Then \( \delta : \mathfrak{g}[v] \to \mathfrak{g}[v] \otimes \mathfrak{g}[u] \cong (\mathfrak{g} \otimes \mathfrak{g})[v, u] \) is defined by

\[
\delta(X v^p) = \left( \frac{v^p - u^p}{v - u} \right) [X \otimes 1, \Omega] = \sum_{n=0}^{p-1} [X, x_b] v^n \otimes x_b u^{p-n-1} \quad \forall X \in \mathfrak{g} \; \text{and} \; p \in \mathbb{N}.
\]

Since \([\Delta(X), \Omega] = 0\), this is equivalent to

\[
\delta(f(v)) = \left[ f(v) \otimes 1 + 1 \otimes f(u), \frac{\Omega}{v - u} \right] \quad \forall f(v) \in \mathfrak{g}[v],
\]

where for now \( \Omega/(v - u) \) should be just viewed as a purely formal rational function. We will soon see a better way of interpreting it.

**Remark 1.3** (On the Casimir \( \Omega \)). A better way of defining \( \Omega \), which is “coordinate free”, is as follows. Since the form \((\ , \) ) is non-degenerate, it induces an isomorphism

\[
\nu : \mathfrak{g} \to \mathfrak{g}^*, \quad \nu(X)(Y) = [X, Y] \quad \forall X, Y \in \mathfrak{g}.
\]

On the other hand, we have a canonical isomorphism \( \kappa : \mathfrak{g} \otimes \mathfrak{g}^* \to \text{End}_\mathbb{C}(\mathfrak{g}) \) given by

\[
\kappa(X \otimes \varphi) = \varphi_X, \quad \text{where} \; \varphi_X(Y) = \varphi(Y)X \quad \forall X, Y \in \mathfrak{g}, \; \varphi \in \mathfrak{g}^*.
\]

Composing \( \kappa \) with \( \text{id}_\mathbb{C} \otimes \nu \), we obtain an isomorphism of vector spaces

\[
\kappa \circ (\text{id}_\mathbb{C} \otimes \nu) : \mathfrak{g} \otimes \mathfrak{g} \to \text{End}_\mathbb{C}(\mathfrak{g}).
\]

The Casimir 2-tensor \( \Omega \) is precisely the preimage of \( \text{id}_\mathbb{C} \) under this isomorphism. It is an easy exercise to check that the previous definition is equivalent to this one.

**Definition 1.4.** Let \((\mathfrak{g}, \delta)\) be a Lie bialgebra.

1. \((\mathfrak{g}, \delta)\) is *coboundary* if there is \( r \in \mathfrak{g} \otimes \mathfrak{g} \) such that

\[
\delta(X) = [\Delta(X), r] = [X \otimes 1 + 1 \otimes X, r] \quad \forall X \in \mathfrak{g}.
\]
(2) \((g, \delta)\) is triangular if is coboundary with \(r \in g \wedge g\) and
\[
[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \quad \text{in} \quad g \otimes g \otimes g.
\]
(3) \((g, \delta)\) is quasitriangular if is coboundary with \(r + r_{21}\) a \(g\)-invariant element of \(g \otimes g\) satisfying the relation (1.2).

**Remark 1.5.**

1. The relation (1.2) is called the classical Yang-Baxter equation, and can be expressed as \(\text{CYB}^c(r) = 0\), where \(\text{CYB} : g \otimes g \rightarrow g \otimes g \otimes g\) is the map which sends \(r\) to the left-hand side of (1.2).
2. It is a theorem of Drinfeld that, given \(r \in g \otimes g\), the assignment \(\delta(X) = [\Delta(X), r]\) for all \(X \in g\) determines a Lie bialgebra structure on \(g\) if and only if a) \(r + r_{21}\) is \(g\)-invariant and b) \(\text{CYB}(r)\) is \(g\)-invariant.

The easiest way for these to be satisfied is by demanding \(r + r_{21} = 0\) and \(\text{CYB}(r) = 0\).

This is what it means to be a triangular Lie bialgebra. Often, this is too strong a requirement. A weaker requirement is that \(r + r_{21}\) \(g\)-invariant and \(\text{CYB}(r) = 0\). This is precisely what it means to be quasitriangular.

The problems of finding solutions of the classical Yang-Baxter equation and constructing quasitriangular Lie bialgebras are thus closely linked. The latter is actually more approachable as Drinfeld’s double construction gives a systematic way of producing quasitriangular Lie bialgebras.

**Theorem 1.6.** Let \((g, \delta_g)\) be a finite-dimensional Lie bialgebra.

1. The vector space \(D(g) = g \oplus g^*\) can be given a Lie algebra structure with bracket
\[
[X + \varphi, Y + \phi] = [X, Y]_g + ([X, \varphi] - [Y, \varphi]) + [\varphi, \phi]_{g^*} \quad \forall \; X, Y \in g \quad \text{and} \quad \varphi, \phi \in g^*,
\]
where \([\varphi, \phi]_{g^*} = \delta_g^*(\varphi \otimes \phi)\) and the term \(([X, \varphi] - [Y, \varphi])\) is determined by
\[
[X, \phi] = \text{ad}^*(\phi)(X) - \text{ad}^*(X)(\phi),
\]
with \(\text{ad}^*(\phi)(X)\) (resp. \(\text{ad}^*(X)(\phi)\)) the coadjoint action of \(\phi\) on \(X\) (resp. \(X\) on \(\phi\)).
2. Let \(\{x_\lambda\}_{\lambda \in \Lambda} \subset g\) and \(\{x^\lambda\}_{\lambda \in \Lambda} \subset g^*\) be dual bases. Then
\[
r = \sum_{\lambda \in \Lambda} x_\lambda \otimes x^\lambda \in g \otimes g^* \subset D(g) \otimes D(g)
\]
defines a quasitriangular Lie bialgebra structure on \(D(g)\).
3. \(D(g)\) naturally contains \((g, \delta_g)\) and \((g^*, \delta_{g^*}^\text{op})\), where \(\delta_{g^*}^\text{op} = -\delta\) as Lie sub-bialgebras.

The quasitriangular Lie bialgebra \(D(g)\) defined by this theorem is called the Drinfeld double of \((g, \delta_g)\).

**Remark 1.7.**

1. The construction of \(D(g)\) is closely related to the notion of Manin triples. Matt discussed this, but I won’t say more about this here.
(2) The construction of the Drinfeld double relies on the fact that the dual of a finite-dimensional Lie bialgebra is itself a Lie bialgebra. This symmetry breaks in the infinite-dimensional setting. Nonetheless, there are still suitable modifications of the procedure used to construct the double in the infinite-dimensional setting, but one must be more careful.

1.2. Quick review of October 10th. Let us now fix $\mathfrak{g}$ to be finite-dimensional simple Lie algebra over the complex numbers. Since our focus is the infinite-dimensional Lie algebra $\mathfrak{g}[v]$, we should think about how the definitions and constructions of the previous lectures fit $\mathfrak{g}[v]$.

**Question:** Is $(\mathfrak{g}[v], \delta)$ (defined in Example 1.2) a quasitriangular Lie bialgebra?

The answer is no. In particular, there is no way to interpret $\Omega_{v-u}$ as an element of $\mathfrak{g}[v] \otimes \mathfrak{g}[u]$.

To get a quasitriangular Lie bialgebra, we could take the Drinfeld double of $\mathfrak{g}[v]$. Since it is infinite-dimensional, this must be done carefully. Consider the Lie algebra $\mathfrak{g}((v^{-1})) = \mathfrak{g} \otimes \mathbb{C}((v^{-1}))$. Recall\(^1\) that this Lie algebra comes equipped with a non-degenerate, symmetric, invariant bilinear form $\langle , \rangle : \mathfrak{g}((v^{-1})) \otimes \mathfrak{g}((v^{-1})) \rightarrow \mathbb{C}$, $\langle f(v), g(u) \rangle = -\text{Res}_v(f(v), g(v))$ where

- $\text{Res}_v : \mathbb{C}((v^{-1})) \rightarrow \mathbb{C}$, $\sum_{p \in \mathbb{Z}} f_p v^{-p-1} \mapsto f_0$ is the formal residue map,
- $( , )$ is the killing form of $\mathfrak{g}$ naturally extended to $\mathfrak{g}((v^{-1})) \otimes \mathfrak{g}((v^{-1})) \rightarrow \mathbb{C}((v^{-1}))$.

The form $( , )$ induces an isomorphism $\mathfrak{g}[v]^* \cong v^{-1} \mathfrak{g}[[v^{-1}]]$ and we have $\mathfrak{g}((v^{-1})) \cong \mathfrak{g}[v] \oplus v^{-1} \mathfrak{g}[[v^{-1}]]$.

Moreover, the element $\frac{\Omega}{v-u}$ can naturally be viewed as an element of $\mathfrak{g}((v^{-1})) \hat{\otimes} \mathfrak{g}((u^{-1}))$\(^2\) by expanding $(v-u)^{-1}$ as a geometric series in $u^{-1}$:

$$\frac{\Omega}{v-u} = -\sum_{b \in \mathcal{B}} \sum_{p \geq 0} x_b v^p \otimes x_b u^{-p-1} \in \mathfrak{g}[v] \hat{\otimes} u^{-1} \mathfrak{g}[u^{-1}].$$

As the set $\{-x_b u^{-p-1} \}_{b \in \mathcal{B}, p \in \mathbb{N}}$ is dual to $\{x_b v^p \}_{b \in \mathcal{B}, p \in \mathbb{N}}$ under $( , )$, it is tempting to declare that $\mathfrak{g}((v^{-1}))$ should be called the Drinfeld double of $\mathfrak{g}[v]$.

However, the full dual of $\mathfrak{g}[v]$ is too big and this would lead to several problems. In particular, $\mathfrak{g}((v^{-1}))$ is not even itself a Lie bialgebra. If it were a Lie bialgebra with coboundary structure given by $\Omega/(v-u)$, then we would have

$$\delta(\sum_{q \geq 0} X v^{-q-1}) \subset \mathfrak{g}((v^{-1})) \otimes \mathfrak{g}((u^{-1})) \quad \forall X \in \mathfrak{g},$$

but we have

$$\delta(\sum_{q \geq 0} X v^{-q-1}) = \sum_{q \geq 0} \left(\frac{v^{-q-1} - u^{-q-1}}{v-u}\right) [X \otimes 1, \Omega] = \sum_{q \geq 0} \sum_{p=0}^q v^{-p-1} u^{p-q-1} [X \otimes 1, \Omega].$$

\(^1\)This has not been typed, but was discussed in one of our first meetings when we constructed $\delta$ using the Manin triple formalism.

\(^2\)Here $\mathfrak{g}((v^{-1})) \hat{\otimes} \mathfrak{g}((u^{-1}))$ denotes a completion of $\mathfrak{g}((v^{-1})) \hat{\otimes} \mathfrak{g}((u^{-1}))$. It’s precise definition will not be needed (see Remark 1.9).
Since $\sum_{q \geq 0} \sum_{p=0}^{q} v^{-p-1} \otimes u^{-q-1}$ does not belong to $\mathbb{C}[[v^{-1}]] \otimes \mathbb{C}[[u^{-1}]]$, the above element does not belong to $\mathfrak{g}[[v^{-1}]] \otimes \mathfrak{g}[[u^{-1}]]$.

A much better option is to exploit the fact that $\mathfrak{g}[v]$ is graded and replace $\ast$ by the graded dual $\circ$. We have

$$\mathfrak{g}[v]^{\circ} = \bigoplus_{p \in \mathbb{N}} (\mathfrak{g}[v]_{p})^{\ast} \cong \bigoplus_{p \in \mathbb{N}} \mathfrak{g} \otimes \mathbb{C} v^{-p-1} \cong v^{-1} \mathfrak{g}[v^{-1}].$$

**Definition 1.8.** The graded Drinfeld double of $\mathfrak{g}[v]$ is equal to $D(\mathfrak{g}[v]) = \mathfrak{g}[v] \oplus v^{-1} \mathfrak{g}[v^{-1}] = \mathfrak{g}[v^{\pm 1}]$ as a Lie algebra, and has quasitriangular Lie bialgebra structure given by

$$\delta(f(v)) = [f(v) \otimes 1 + 1 \otimes f(u), r_{g}] \quad \forall f(v) \in D(\mathfrak{g}[v]),$$

where

$$r_{g} = - \sum_{b \in B} \sum_{p \geq 0} x_{b} v^{p} \otimes x_{b} u^{-p-1} \in \mathfrak{g}[v] \hat{\otimes} u^{-1} \mathfrak{g}[u^{-1}] \subset D(\mathfrak{g}[v]) \hat{\otimes} D(\mathfrak{g}[u]).$$

**Remark 1.9.** $D(\mathfrak{g}[v]) \hat{\otimes} D(\mathfrak{g}[u])$ can be defined as follows. Set $D(\mathfrak{g}[v])_{k} = \mathfrak{g} \otimes \mathbb{C} v^{k}$. Then

$$D(\mathfrak{g}[v]) \hat{\otimes} D(\mathfrak{g}[u]) = \left\{ \sum_{k \in \mathbb{Z}} A_{k} \in \prod_{k \in \mathbb{Z}} (D(\mathfrak{g}[v])_{k} \otimes D(\mathfrak{g}[u])_{k}) : A_{m} = 0 \quad \forall \ m \gg 0 \right\}.$$

In conclusion, we have seen that $\mathfrak{g}[v]$ is not itself quasitriangular, but it embeds in the topologically quasitriangular Lie bialgebra $D(\mathfrak{g}[v]) = \mathfrak{g}[v^{\pm 1}]$, which is equal to its graded Drinfeld double. Here the word “topologically” just indicates that $r_{g}$ belongs to a completion of $D(\mathfrak{g}[v]) \otimes D(\mathfrak{g}[u])$ and not the genuine tensor square of $D(\mathfrak{g}[v])$.

In the next section, we will see that $\mathfrak{g}[v]$ has what is called a pseudo-quasitriangular Lie bialgebra structure, which in many ways is much more interesting than an honest quasitriangular structure.

October 17th 2018

2. PSEUDO QUASITRIANGULAR STRUCTURE ON $\mathfrak{g}[v]$

2.1. Shift automorphisms and their formal analogues.

**Lemma 2.1.** Let $c \in \mathbb{C}$. Then

$$\tau_{c} : f(v) \mapsto f(v + c) \quad \forall f(v) \in \mathfrak{g}[v]$$

defines a Lie algebra automorphism of $\mathfrak{g}[v]$.

These are called *shift automorphisms* and they play a critical role in the representation theory of $\mathfrak{g}[v]$ (and the Yangian!). We will give one interesting application in Proposition 2.3 below, but first we will need some terminology.

For each $c \in \mathbb{C}$, set

$$V(c) = \tau_{c}^{\ast}(V).$$

Consider the *evaluation homomorphism* $ev_{\mathfrak{g}}$ defined by

$$ev_{\mathfrak{g}} : \mathfrak{g}[v] \to \mathfrak{g}, \quad f(v) \mapsto f(0) \quad \forall f(v) \in \mathfrak{g}[v].$$
Of course, ev\_g uniquely extends to U(g[v]) \rightarrow U(g). We will not distinguish between these two epimorphisms.

**Definition 2.2.** Given a U(g)-module V, we define V\_ev to be the U(g[z])-module

\[ V\_ev = ev\_g^*(V). \]

Any U(g[z])-module of the form V\_ev(c) is called an *evaluation module*.

**Proposition 2.3.** Let V, W be finite-dimensional irreducible U(g)-modules. The U(g[v])-module

\[ V\_ev(c) \otimes W\_ev(b) \]

is irreducible whenever \( c - b \in \mathbb{C}^\times \).

**Proof.** We begin by reducing the proof to the case where \( b = 0 \).

**Claim:** It suffices to prove the assertion of the proposition for \( b = 0 \).

To see why this is true, note that \( \tau_c \) is a coalgebra morphism (or more precisely, its extension to U(g[v]) is). This means that

\[ \tau_c \otimes \tau_c \circ \Delta = \Delta \circ \tau_c, \]

where \( \Delta \) is the standard coproduct on U(g[v]). Moreover, since \( \tau_{-b} \) is an automorphism, \( V\_ev(c) \otimes W\_ev(b) \) is irreducible if and only if \( \tau^{*}\_b(V\_ev(c) \otimes W\_ev(b)) \) is. By (2.1) and the fact that \( \tau_\lambda \circ \tau_\gamma = \tau_{\lambda + \gamma} \), we have

\[ \tau^{*}\_b(V\_ev(c) \otimes W\_ev(b)) = V\_ev(c - b) \otimes W\_ev, \]

which proves the claim.

Hence, we can assume that \( b = 0 \) and \( c \in \mathbb{C}^\times \). Let \( \pi_V : U(g) \rightarrow \text{End}_\mathbb{C}(V) \) be the homomorphism equipping \( V \) with a U(g)-module structure and define \( \pi_W \) similarly. Let \( \pi : U(g[v]) \rightarrow \text{End}(V \otimes W) \) be the algebra homomorphism associated to \( V\_ev(c) \otimes W\_ev \), so

\[ \pi = (\pi_V \circ \tau_c) \otimes \pi_W. \]

To prove the proposition it suffices to show that \( \pi \) is surjective. By Jacobson’s Density Theorem, both \( \pi_V \) and \( \pi_W \) are themselves surjective. Thus, \( \text{End}_\mathbb{C}(V \otimes W) \cong \text{End}_\mathbb{C}(V) \otimes \text{End}_\mathbb{C}(W) \) is generated by elements of the form

\[ \pi_V(X) \otimes \pi_W(Y) = (\pi_V(X) \otimes \text{id}_W)(\text{id}_V \otimes \pi_W(Y)) \quad \forall X, Y \in g. \]

By definition,

\[ \pi(c^{-1}Xv) = \pi_V(X) \otimes \text{id}_W \quad \text{and} \quad \pi(X - c^{-1}Xv) = \text{id}_V \otimes \pi_W(X) \quad \forall X \in g. \]

**Definition 2.4.** The formal shift homomorphism is the Lie algebra injection

\[ \tau_z : g[v] \hookrightarrow (g[v])[z], f(v) \rightarrow f(v + z) \quad \forall f(v) \in g[v]. \]

Note that \( \tau_z \) is a \( \mathbb{N} \)-graded homomorphism, provided we view \( (g[v])[z] \cong g[v, z] \) as a graded Lie algebra with \( \text{deg} \ z = \text{deg} \ v = 1 \). It satisfies

\[ \tau_c = ev\_c \circ \tau_z \quad \forall c \in \mathbb{C}, \]

where \( ev\_c : (g[v])[z] \rightarrow g[v] \) is given by \( z \mapsto c \).

**Goal A.** Show \( \tau_z \) extends to \( \Phi_z : D(g[v]) \hookrightarrow (g[v])((z^{-1})) \) which has image in a \( \mathbb{Z} \)-graded subalgebra of \( (g[v])((z^{-1})) \), and is itself a \( \mathbb{Z} \)-graded homomorphism.
The hidden subgoal is that we want to use $\Phi_z$ to “project” the quasitriangular structure of $D(\mathfrak{g}[v])$ onto $\mathfrak{g}[v]$. Realizing the above goal is in fact not at all difficult, but let us first consider a simpler version of this story.

Consider the shift homomorphism

$$\gamma_z : \mathbb{C}[w] \to \mathbb{C}[v][z], \quad f(w) \to f(v + z) \quad \forall f(w) \in \mathbb{C}[w].$$

Note that $\tau_z = \text{id}_\mathbb{C} \otimes \gamma_z$.

**Lemma 2.5.**

1. $\gamma_z$ extends uniquely to $\Gamma_z : \mathbb{C}[w^{\pm 1}] \hookrightarrow \mathbb{C}[v](z^{-1})$.
2. $\Gamma_z$ has image in $\bigoplus_{n \in \mathbb{Z}} z^n \mathbb{C}[v]$ with $\mathbb{C}[v] = \prod_{k \in \mathbb{N}} \mathbb{C}v^k z^{-k}$.
3. $\Gamma_z$ is $\mathbb{Z}$-graded.

**Proof.** To prove (1), note that for each $p \geq 0$,

$$\gamma_z(w^p) = (v + z)^p \in \mathbb{C}[v](z^{-1})^x.$$ 

Indeed, $\sum_{k \geq 0} (-v)^k z^{-k-1}$ is the inverse of $\gamma_z(w)$, and its $p$-th power is the inverse of $\gamma_z(w^p)$. Since

$$\mathbb{C}[w^{\pm 1}] = S^{-1}(\mathbb{C}[w]),$$

where $S = \{w^p : p \geq 0\}$, the universal property of localization implies that $\gamma_z$ extends uniquely to $\Gamma_z$, as desired.

Parts 2 and 3 follow from the fact that

$$\Gamma_z(w^p) = z^p \sum_{k=0}^p \binom{p}{k} w^k z^{-k}, \quad \Gamma_z(w^{-p}) = z^{-p} \sum_{k \geq 0} \binom{p+k}{k} (-1)^k w^k z^{-k}. \quad \square$$

As a corollary, we get Goal A for free by setting $\Phi_z = \text{id}_\mathbb{C} \otimes \Gamma_z$.

**Proposition 2.6.** $\Phi_z = \text{id}_\mathbb{C} \otimes \Gamma_z : \mathfrak{g}[w^{\pm 1}] \to \mathfrak{g} \otimes \mathbb{C}[v](z^{-1}) \subset (\mathfrak{g}[v])(z^{-1})$ is a Lie algebra injection satisfying

1. $\Phi_z$ is $\mathbb{Z}$-graded with image in $\bigoplus_{n \in \mathbb{Z}} z^n \mathfrak{g}^\circ[v]$, where $\mathfrak{g}_z[v] = \prod_{p \in \mathbb{N}} \mathfrak{g}[v]_k z^{-k}$.
2. $\Phi_z|_{\mathfrak{g}[w]} = \tau_z$ and $\Phi_z(w^{-1} \mathfrak{g}[w^{-1}]) \subset (\mathfrak{g}[v])[z^{-1}]$.

There are a few nice ways of writing down $\Phi_z$ explicitly. One such way is to use (2.2).

Another is as follows: for each $X \in \mathfrak{g}$, set

$$X(u) = \sum_{p \in \mathbb{Z}} X w^p u^{-p-1} \in (\mathfrak{g}[w^{\pm 1}])[u, u^{-1}].$$

Then we have

$$\Gamma_z(X(u)) = \sum_{n \in \mathbb{N}} \frac{X}{n!} \partial_z^n (u^{-1} \delta(z/u))$$

$$= \exp(v \partial_z) (u^{-1} \delta(z/u) X) = u^{-1} \delta(z+v) X,$$

Where $\delta(x) = \sum_{p \in \mathbb{Z}} x^p$ and the above equality is understood to mean that the $u^{-p-1}$ coefficient of $X(u)$ gets sent to the $u^{-p-1}$ coefficient of the right-hand side for each $p \in \mathbb{Z}$.

The following exercise is not difficult, but it is important and I unfortunately don’t have time to discuss it.

**Exercise.** Let $\lambda \in \mathbb{C}^\times$. 
(1) Show that there is a Lie algebra homomorphism

\[ \Phi_\lambda : g[w^\pm 1] \to g[[v]] = g \otimes C[v] \]

given by composing \( \Phi_z \) with the evaluation map at \( z = \lambda \).

(2) Show that \( \Phi_\lambda \) is injective.

(3) Let \( \widehat{g[w^\pm 1]} = \varprojlim_n (g[w^\pm 1]/J_n) \), where \( J_n = (z - 1)^n g[z] \). Show that \( \Phi = \Phi_{\lambda=1} \) induces an isomorphism

\[ \widehat{\Phi} : \widehat{g[w^\pm 1]} \xrightarrow{\sim} g[v]. \]

As a hint for the last part, I would recommend rephrasing the problem just in terms of \( C[w^\pm 1] \) and \( C[[v]] \) and using that \( C[[v]] = \varprojlim_n (C[v]/v^n C[v]) \).
2.2. The classical universal r-matrix. We can now define
\[ r(z) = (\text{id} \otimes \Phi_z)(r_g) = \frac{-\Omega}{z + u - v} = \sum_{n \in \mathbb{N}} \left( \sum_{b \in B} \sum_{s=0}^{n} \binom{n}{s} (-1)^{n-s+1} x_b v^s \otimes x_b u^{n-s} \right) z^{-n-1} \in (\mathfrak{g}[v] \otimes \mathfrak{g}[u])[z^{-1}]. \]

By definition, \( r(z) \) satisfies the coboundary type relation
\[ (1 \otimes \tau_z) \delta(f(v)) = [f(v) \otimes 1 + 1 \otimes f(u + z), r(z)] \quad \forall f(v) \in \mathfrak{g}[v]. \]

It also has the following remarkable properties.

**Theorem 2.7.** \( r(z) \) satisfies the following properties:

1. \( r(z) = -r_{21}(-z) \),
2. \( (\tau_c \otimes \tau_d)r(z) = r(z - c + d) \) for all \( c, d \in \mathbb{C} \),
3. \( r(z) \) satisfies the CYBE with additive spectral parameter:
   \[ [r_{12}(z), r_{13}(w)] + [r_{12}(z), r_{23}(w - z)] + [r_{13}(w), r_{23}(w - z)] = 0. \]

**Proof.** Part (1) and (2) follow easily from the explicit form of \( r(z) \) and the fact that it is the expansion of \(-\Omega/(z + u - v)\). To prove (3), apply \( 1 \otimes \Phi_z \otimes \Phi_w \) to
\[ [(r_g)_{12}, (r_g)_{13}] + [(r_g)_{12}, (r_g)_{23}] + [(r_g)_{13}, (r_g)_{23}] = 0 \]
(recalling that \( r_g \) gives a quasitriangular structure on \( \mathfrak{g}[v^{\pm 1}] \)) and then use (2), which implies
\[ (\Phi_z \otimes \Phi_w)r_g = (\Phi_z \otimes \text{id})r(w) = (\tau_z \otimes \text{id})r(w) = r(w - z). \]
Alternatively, use the fact that \( r(z) \) is the expansion at \( z = \infty \) of \(-\Omega/(z + u - v)\) and the relation \([\Delta(X), \Omega] = 0\), which implies
\[ -[\Omega_{12}, \Omega_{23}] = [\Omega_{13}, \Omega_{23}] = [\Omega_{12}, \Omega_{13}]. \]

**Definition 2.8.** We call \((\mathfrak{g}[v], r(z))\) a pseudo-quasitriangular Lie bialgebra with universal \( r \)-matrix \( r(z) \).

**Remark 2.9.** Perhaps a more descriptive name would encode the fact that the definition of \( r(z) \) involves twisting by an action of the additive group \( \mathbb{C} \). In addition, it would take into account the fact that \( r(z) \) evaluates to a rational function of \( z \) on finite-dimensional representations (to be proven in the next section).

3. Rationality properties of the current algebra

In this final section on the current algebra, we will prove that \( r(z) \) evaluates to a rational function of \( z \) on finite-dimensional representations. In this way, one can understand \( \mathfrak{g}[z] \) as a vehicle for producing rational solutions to the classical Yang-Baxter equation with spectral parameter.
3.1. Rationality of generating series. Recall that, for each \( X \in \mathfrak{g} \), \( X(u) \) denotes the series
\[
X(u) = \sum_{p \in \mathbb{Z}} (Xv^p)u^{-p-1} \in (\mathfrak{g}[v])[[u^{\pm 1}]] = (D(\mathfrak{g}[v])[[u^{\pm 1}]].
\]
We decompose \( X(u) = X(u)_+ - X(u)_- \), where
\[
X(u)_+ = \sum_{p \geq 0} (Xv^p)u^{-p-1} \in (\mathfrak{g}[v])[u^{-1}], \quad X(u)_- = -\sum_{p < 0} (Xv^p)u^{-p-1} \in (v^{-1}\mathfrak{g}[v^{-1}])[u].
\]

\textbf{Remark 3.1.} Note that \( \tau_z(X(u)_+) = X(u-z)_+ \). That is, for each \( p \geq 0 \), the \( u^{-p-1} \) coefficient of \( X(u)_+ \) is mapped to the \( u^{-p-1} \) coefficient of \( X(u-z)_+ \) under \( \tau_z \). Similarly, \( \Phi_z(X(u)_-) \) is equal to the expansion of \( X(u-z)_+ \) in powers of \( z^{-1} \).

\textbf{Proposition 3.2.} Let \((\rho, V)\) be a finite-dimensional \( U(\mathfrak{g}[v])\)-module. Then, for each \( X \in \mathfrak{g} \),
\[
\rho(X(u)_+) \in \text{End}_C(V)[[u^{-1}]]
\]
is the expansion at \( u = \infty \) of a \( \text{End}_C(V)\)-valued rational function of \( u \).

\textbf{Remark 3.3.} Let’s dissect this algebraically. Since \( \mathbb{C}(u) \) is the field of fractions of \( \mathbb{C}[u^{-1}] \), the natural embedding
\[
\mathbb{C}[u^{-1}] \hookrightarrow \mathbb{C}(u^{-1})
\]
extends uniquely to \( \iota : \mathbb{C}(u) \hookrightarrow \mathbb{C}(u^{-1}) \). Then \( \text{id} \otimes \iota \) gives an injection
\[
\text{End}_C(V) \otimes \mathbb{C}(u) \hookrightarrow \text{End}_C(V) \otimes \mathbb{C}(u^{-1}) = \text{End}_C(V)[(u^{-1})].
\]
The statement of the proposition is that each series \( X(u)_+ \) belongs to the image of this map.

\textit{Proof of Proposition 3.2.} It suffices to prove the assertion for \( \{y(u)_+\}_{y \in \mathcal{G}} \) where \( \mathcal{G} \) is some Lie algebra generating set for \( \mathfrak{g} \). Let us take \( \mathcal{G} \) to be \( \{x^{\pm}_{\alpha}\}_{\alpha \in \Delta_+} \), where \( x^\pm_{\alpha} \in \mathfrak{g}_{\pm \alpha} \) are nonzero root vectors (taking only simple roots \( \alpha = \alpha_i \) would suffice, but this won’t make things easier). By Remark 3.3, we need to show
\[
\rho(x^{\pm}_{\alpha}(u)_+) \in \text{End}_C(V) \otimes \mathbb{C}(u) \quad \forall \alpha \in \Delta_+.
\]
For brevity, we drop the \( \rho \). For each \( h \) in the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \), we have
\[
[hv, x^{\pm}_{\alpha}(u)] = \pm \alpha(h) x^{\pm}_{\alpha} v^{p+1} u^{-p-1} = \pm \alpha(h) x^{\pm}_{\alpha}(u)_+ - x^{\pm}_{\alpha}.
\]
Choose \( h \in \mathfrak{h} \) such that \( \alpha(h) \neq 0 \). Then we can rewrite the above as
\[
\left( \frac{\text{ad}(hv)}{u} \mp \alpha(h) \text{id} \right) (x^{\pm}_{\alpha}(u)_+) = -u^{-1} x^{\pm}_{\alpha}
\]
Let \( A(u) = \frac{\text{ad}(hv)}{u} \mp \alpha(h) \text{id} \) and set \( W = \text{End}_C(V) \), a finite-dimensional vector space. Then
\[
A(u) \in \text{End}(W) \otimes \mathbb{C}(u),
\]
a finite-matrix with entries in \( \mathbb{C}(u) \). It is also invertible; this follows from the fact that it is a unit in \( \text{End}(W)[[u^{-1}]] \), or that \( \lim_{u \to \infty} \det(A(u)) = \mp \alpha(h) \neq 0 \). Therefore we get
\[
x^{\pm}_{\alpha}(u)_+ = -u^{-1} A(u)^{-1}(x^{\pm}_{\alpha}).
\]
Necassarily, \( A(u)^{-1} \) is also an element of \( \text{End}(W) \otimes \mathbb{C}(u) \cong M_{(\dim W)^2}(\mathbb{C}(u)) \) (the inverse of an invertible matrix over a field has entries in the same field). Hence, the right-hand side of the above belongs to \( \text{End}(V) \otimes \mathbb{C}(u) \), as desired. \( \square \)
As a consequence of the proposition, we can think of a finite-dimensional \( \mathfrak{g}[v] \)-module as a collection of operator valued rational functions of \( u \) satisfying certain relations.

**Definition 3.4.** Let \((\rho, V)\) be as above. We define the set of poles of \( V \) to be

\[
\sigma(V) = \bigcup_{X \in \mathfrak{g}} \text{Poles}(\rho(X(u)_+)) ,
\]

where \( \text{Poles}(X(u)_+) \) is the set of poles of the rational function \( \rho(X(u)_+) \).

**Remark 3.5.** It suffices to take the above union over our generating set \( \{x_{\alpha}^{\pm}\}_{\alpha \in \Delta_+} \), or even a set of simple root vectors. In particular, \( \sigma(V) \) is a finite set.

**Example 3.6.** Let \((\pi_V, V)\) be a finite-dimensional \( \mathfrak{g} \)-module and consider the \( U(\mathfrak{g}[v]) \)-module \( V_{ev}(c) \). For each \( X \in \mathfrak{g} \), \( X(u)_+ \) operates as

\[
\sum_{p \geq 0} \pi_V(X)c^pu^{-p-1} = \frac{\pi_V(X)}{u-c} ,
\]

it follows that \( \sigma(V_{ev}(c)) = \{c\} \) unless \( V \) has no non-trivial components (in which case \( \sigma(V_{ev}(c)) = \emptyset \)).

### 3.2. Rationality of the universal \( r \)-matrix.

**Theorem 3.7.** Let \((\rho_U, U)\) and \((\rho_V, V)\) be finite-dimensional \( U(\mathfrak{g}[v]) \)-modules. Then

1. The evaluation \( r_{U,V}(z) = (\rho_U \otimes \rho_V)r(z) \in \text{End}(V \otimes W)[z^{-1}] \) is the expansion at \( z = \infty \) of a rational \( \text{End}_\mathbb{C}(U \otimes V) \)-valued rational function of \( z \).
2. The set of poles of \( r_{U,V}(z) \) is contained in \( \sigma(U) - \sigma(V) \).
3. For each \( c \in \mathbb{C} \setminus (\sigma(U) - \sigma(V)) \), \( r_{U,V}(c) = r_{U,V}(z)|_{z=c} \) is given by

\[
(3.1) \quad r_{U,V}(c) = \sum_{b \in B} \oint_C x_b(u)_+ \otimes x_b(u - c)_+ du ,
\]

where

- \( x_b(u)_+ \otimes x_b(u - c)_+ \) is identified with the rational function \( \rho_U(x_b(u)_+) \otimes \rho_V(x_b(u - c)_+) \).
- \( C \) is a Jordan curve in \( \mathbb{C} \) enclosing all of \( \sigma(V) + c \) but none of \( \sigma(U) \).

**Remark 3.8.**

1. The statement of (1) is that \( r_{U,V}(z) \in \text{End}(V \otimes W) \otimes \mathbb{C}(z) \), where this space is embedded in \( \text{End}(V \otimes W)[(z^{-1})] \) as in Remark 3.3.
2. In part (3), it is a priori clear that each integral in (3.1) is a rational function of \( z \). This is a consequence of Cauchy’s Integral Formula. This will be made clear in the proof below.
3. It is tempting to say \( r_{U,V}(z) \) is rational because \( r(z) = -\frac{\Omega}{z+u-v} \). This is not correct because \( u \) and \( v \) are not scalars or even parameters.

**Proof of Theorem 3.7.** First note that

\[
r(z) = \sum_{b \in B} \text{Res}_u (x_b(u)_+ \otimes \Phi(x_b(u)_-)) = \sum_{b \in B} \text{Res}_u (x_b(u)_+ \otimes \Phi(x_b(u))) .
\]
It thus suffices to prove the assertion of the theorem for
\[ r_b(u) = \text{Res}_u (x_b(u)_+ \otimes \Phi_z(x_b(u))) \]
for each fixed \( b \in B \).

Fix \( b \in B \). By Proposition 3.2, \( \rho_U(x_b(u)_+) \) and \( \rho_V(x_b(u)_+) \) are rational functions of \( u \) which converge to 0 as \( u \to \infty \). They therefore admit partial fraction decompositions of the form
\[
(3.2) \quad \rho_U(x_b(u)_+) = \sum_{\gamma \in \mathbb{C}, k \in \mathbb{N}} \frac{X^U_{\gamma,k}}{(u - \gamma)^{k+1}}, \quad \rho_V(x_b(u)_+) = \sum_{\lambda \in \mathbb{C}, n \in \mathbb{N}} \frac{X^V_{\lambda,n}}{(u - \lambda)^{n+1}},
\]
where, for \( W = U \) or \( V \), \( \{X^W_{\gamma,k}\}_{\gamma \in \mathbb{C}, k \in \mathbb{N}} \in \text{End}_\mathbb{C}(W) \) with \( X^W_{\gamma,k} = 0 \) for all but finitely many pairs \( (\gamma, k) \).

**Claim:** Set \( \delta_z^{(n)} = \frac{1}{n!} \delta_z^n \). Then, we have
\[
\rho_V(\Phi_z(x_b(u))) = \sum_{\lambda,n} X^V_{\lambda,n} \delta_z^{(n)} \left( u^{-1} \delta \left( \frac{z + w}{u} \right) \right)_{w=\lambda}.
\]

To prove the claim, first note that by (3.2) we have
\[
\rho_V(x_bv^p) = \sum_{\lambda,n} X^V_{\lambda,n} \text{Res}_w \left( \frac{w^p}{(w - \lambda)^{n+1}} \right) = \sum_{\lambda,n} X^V_{\lambda,n} \delta_z^{(n)} (w^p)_{w=\lambda}.
\]

Since \( \Phi_z(X(u)) = \sum_{m \in \mathbb{N}} X^m \partial_z^m (u^{-1} \delta(z/w)) \), we thus have
\[
\rho_V(\Phi_z(x_b(u))) = \sum_{m \in \mathbb{N}} \sum_{\lambda,n} X^V_{\lambda,n} \delta_z^{(n)} (w^m)_{w=\lambda} \partial_z^m (u^{-1} \delta(z/u)) = \sum_{\lambda,n} X^V_{\lambda,n} \delta_z^{(n)} (\exp(w \partial_z) (u^{-1} \delta(z/u)))_{w=\lambda}.
\]

Using the claim, we deduce that
\[
(\rho_U \otimes \rho_V) r_b(z) = \sum_{\gamma,k} \sum_{\lambda,n} X^U_{\gamma,k} \otimes X^V_{\lambda,n} \partial_z^{(n)} \left( \frac{1}{(z + w - \gamma)^{n+1}} \right)_{w=\lambda}.
\]

This is a rational function of \( z \) with poles in \( \sigma(U) - \sigma(V) \). Hence, we have proven Parts (1) and (2).

Moreover, to prove Part (3) it suffices to show that
\[
\oint_C \rho_U(x_b(u)_+) \otimes \rho_U(x_b(u-c)_+) du = \sum_{\gamma,k} \sum_{\lambda,n} X^U_{\gamma,k} \otimes X^V_{\lambda,n} \partial_z^{(n)} \left( \frac{1}{(c + w - \gamma)^{n+1}} \right)_{w=\lambda}.
\]

The left-hand side is
\[
\sum_{\gamma,k} \sum_{\lambda,n} X^U_{\gamma,k} \otimes X^V_{\lambda,n} \oint_C \frac{1}{(u - \gamma)^{k+1}(u - c - \lambda)^{n+1}} du.
\]
We can assume that the summations only go over those pairs \( (\gamma, k) \) and \( (\lambda, n) \) such that \( X^U_{\gamma,k} \neq 0 \) and \( X^V_{\lambda,n} \neq 0 \), respectively. By assumption \( \frac{1}{(u-\gamma)^{k+1}} \) is holomorphic inside \( C \), and
each pole of \( \frac{1}{(u-c-\lambda)^{n+1}} \) is contained in the interior of \( C \). Cauchy’s Integral formula therefore yields
\[
\oint_C \rho_U(x_b(u)_+ \otimes \rho_U(x_b(u-c)_+) du = \sum_{\gamma,k} \sum_{\lambda,n} X^{U}_{\gamma,k} \otimes X^{V}_{\lambda,n} \partial^{(n)}_{w} \left( \frac{1}{(w-\gamma)^{k+1}} \right)_{w=c+\lambda}.
\]

\[\square\]

\textbf{Part II. The Yangian} \( Y_h(\mathfrak{g}) \)

\textbf{November 7th 2018}

4. Quantization

Roughly speaking, quantization is the process of passing from a Lie bialgebra \((\mathfrak{g},\delta)\) to a Hopf algebra \(U_h(\mathfrak{g})\) over \(\mathbb{C}[h]\) which equal \(U(\mathfrak{g})\) modulo \(h\) and whose coproduct encodes all the information about \(\delta\). If \(\delta \neq 0\), this process sends \((\mathfrak{g},\delta)\) to a non-cocommutative non-commutative Hopf algebra.

Our focus will be on studying the quantization of the pseudo-quasitriangular Lie bialgebra \((\mathfrak{g}[v],\delta,r(z))\). In this special case, we can (and will) work over \(\mathbb{C}[h]\) instead of \(\mathbb{C}[h]\). As a disclaimer, I emphasis that many of the definitions related to quantization given below are usually given over \(\mathbb{C}[h]\) with (Hopf) algebras replaced by topological (Hopf) algebras.

4.1. Quantizations over \(\mathbb{C}[h]\). Let \(\mathfrak{g}\) be an arbitrary complex Lie algebra.

\textbf{Definition 4.1.} A quantized universal enveloping algebra (over \(\mathbb{C}[h]\)) is a Hopf algebra \(U_h(\mathfrak{g})\) over \(\mathbb{C}[h]\) such that

1. There is a Hopf algebra isomorphism
   \[
   U_h(\mathfrak{g})/hU_h(\mathfrak{g}) \cong U(\mathfrak{g}).
   \]
2. \(U_h(\mathfrak{g})\) is flat as a \(\mathbb{C}[h]\)-module. Equivalently, \(U_h(\mathfrak{g})\) is a torsion free \(\mathbb{C}[h]\)-module:
   \[
   hx = 0 \implies x = 0 \text{ for all } x \in U_h(\mathfrak{g}).
   \]

Given a quantized universal enveloping (QUE) algebra \(U_h(\mathfrak{g})\), we will denote the natural map \(U_h(\mathfrak{g}) \to U(\mathfrak{g})\) by \(ev_h\).

\textbf{Proposition 4.2} (Props. 6.2.3 & 6.2.7, [CP]). Let \(U_h(\mathfrak{g})\) be a QUE algebra with coproduct \(\Delta_h\). Then \(\mathfrak{g}\) is a Lie bialgebra with cocommutator \(\delta\) given by
\[
(4.1) \quad \delta(x) = ev_h \left( \frac{\Delta_h(x)}{h} - \Delta_{h}^{\text{op}}(\hat{x}) \right),
\]
where \(\hat{x}\) is such that \(ev_h(\hat{x}) = x\) and \(\Delta_{h}^{\text{op}} = \text{flip} \circ \Delta_h\).

\textbf{Remark 4.3.} Let \(\Delta\) be the coproduct for \(U(\mathfrak{g})\). Since \(\Delta \equiv \Delta_h\) mod \(h\) and \(\Delta = \Delta_{h}^{\text{op}}\),
\[
\Delta_h(X) - \Delta_{h}^{\text{op}}(X) \in hU_h(\mathfrak{g}) \otimes_{\mathbb{C}[h]} U_h(\mathfrak{g}) \quad \forall \ X \in U_h(\mathfrak{g}).
\]

Hence
\[
\Delta_h(x) - \Delta_{h}^{\text{op}}(\hat{x}) \in U_h(\mathfrak{g}) \otimes_{\mathbb{C}[h]} U_h(\mathfrak{g}).
\]
This implies that \(\delta(x) \in U(\mathfrak{g}) \wedge U(\mathfrak{g})\). Additional work is required to show \(\delta(x) \in \mathfrak{g} \wedge \mathfrak{g}\); the idea is to show that each factor of \(\delta(x)\) is primitive.
Definition 4.4. A quantization of a Lie bialgebra \((\mathfrak{g}, \delta)\) is a quantized enveloping algebra \(U_\hbar(\mathfrak{g})\) with \(\delta\) given by (4.1).

Example 4.5 (The trivial quantization). Fix \(\mathfrak{g}\) to be any complex Lie algebra and consider the algebra \(U_\hbar(\mathfrak{g}) = U(\mathfrak{g})[\hbar]\). We have
\[
U_\hbar(\mathfrak{g}) \otimes_{\mathbb{C}[\hbar]} U_\hbar(\mathfrak{g}) \cong (U(\mathfrak{g}) \otimes_{\mathbb{C}} U(\mathfrak{g}))[\hbar],
\]
and the Hopf structure of \(U(\mathfrak{g})\) trivially extends to \(U_\hbar(\mathfrak{g})\). This is a cocommutative Hopf structure, and hence \(U_\hbar(\mathfrak{g})\) quantizes the trivial Lie bialgebra structure on \(\mathfrak{g}\), i.e. where \(\delta = 0\).

Remark 4.6. The problem of whether or not every Lie bialgebra \((\mathfrak{g}, \delta)\) admits a quantization was posed by Vladimir Drinfeld and solved by David Kazhdan and Pavel Etingof in [EK1]. The short answer is that every Lie bialgebra \((\mathfrak{g}, \delta)\) admits a quantization.

4.2. Homogeneous quantizations. We have seen that the fact that \(\mathfrak{g}[v]\) is graded plays an important role. It is thus reasonable to look at quantizations which somehow preserve gradings. Our situation is a special case of the following setup:

Let \(\mathfrak{g} = \bigoplus_{k \in \mathbb{N}} \mathfrak{g}_k\) be a \(\mathbb{N}\)-graded Lie algebra (so \([\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}\)), and suppose that \(\mathfrak{g}\) is in addition a Lie bialgebra with cocommutator \(\delta\) of degree \(-1\). That is,
\[
\delta(\mathfrak{g}_k) \subseteq \bigoplus_{n=0}^{k-1} \mathfrak{g}_n \otimes \mathfrak{g}_{k-n-1}.
\]
The grading on \(\mathfrak{g}\) induces a graded Hopf algebra structure on \(U(\mathfrak{g})\).

Definition 4.7. A homogeneous quantization of \((\mathfrak{g}, \delta)\) is a quantization \(U_\hbar(\mathfrak{g})\) of \((\mathfrak{g}, \delta)\) such that \(U_\hbar(\mathfrak{g})\) is an \(\mathbb{N}\)-graded Hopf algebra (with \(\deg \hbar = 1\)) and the isomorphism
\[
U_\hbar(\mathfrak{g})/\hbar U_\hbar(\mathfrak{g}) \cong U(\mathfrak{g})
\]
is an isomorphism of graded Hopf algebras.

The condition that \(\deg \hbar = 1\) ensures that \(\delta\) given by (4.1) will be of degree \(-1\). Indeed, if \(x \in \mathfrak{g}_k\) and \(\hat{x}\) is an element of degree \(k\) in \(U_\hbar(\mathfrak{g})\) such that \(ev_\hbar(\hat{x}) = x\), then we can write
\[
\Delta_\hbar(\hat{x}) = \sum_{n \in \mathbb{N}} \hbar^n \Delta_{\hbar,k-n}(\hat{x}) \quad \text{with} \quad \deg \Delta_{\hbar,k-n}(\hat{x}) = ev_\hbar(\Delta_{\hbar,k-n}(\hat{x})) = k - n,
\]
(This is because \(\Delta_\hbar\) is graded, by assumption.) Hence,
\[
\delta(x) = ev_\hbar(\Delta_{\hbar,k-1}(\hat{x}) - \Delta_{\hbar,k-1}(\hat{x})).
\]

5. The Yangian defined

Now let us turn to explicitly constructing a homogeneous quantization of \((\mathfrak{g}[v], \delta)\).

5.1. Chevalley-Serre presentation for \(\mathfrak{g}[v]\). Let \(\mathfrak{g}\) be a finite-dimensional simple Lie algebra over \(\mathbb{C}\) with invariant form ( , ). Let \(\mathfrak{h} \subset \mathfrak{g}\) be a Cartan subalgebra, \(\{\alpha_i\}_{i \in \mathbf{I}} \subset \mathfrak{h}^*\) a basis of simple positive roots of \(\mathfrak{g}\), and \(\mathbf{A} = (a_{ij})_{i,j \in \mathbf{I}}\) the Cartan matrix, so that
\[
a_{ij} = \frac{2(\alpha_i, \alpha_j)}{\langle \alpha_i, \alpha_i \rangle} \quad \forall \ i, j \in \mathbf{I}.
\]
For each $i, j \in I$, define
\[
d_{ij} = \frac{(\alpha_i, \alpha_j)}{2} \quad \text{and} \quad d_i = d_{ii}.
\]
The Lie algebra $g$ is isomorphic to the Lie algebra generated by $\{x_i^+, h_i\}_{i \in I}$ subject to the defining relations
\[
[h_i, h_j] = 0,
\]
\[
[h_i, x_j^+] = \pm (\alpha_i, \alpha_j) x_j^+ = \pm 2d_{ij} x_j^+,
\]
\[
[x_i^+, x_j^-] = \delta_{ij} h_i,
\]
\[
\text{ad}(x_i^+) = d_{ij} a_{ij} (x_j^+) = 0 \quad \forall i \neq j.
\]
Indeed, this is just the usual Chevalley–Serre presentation of $g$ with the Chevalley generators renormalized appropriately. To arrive at it, let $\nu : h \rightarrow h^*$ be the isomorphism given by
\[
\nu(h)(h') = (h, h') \quad \forall h, h' \in h,
\]
and set $h_i = \nu^{-1}(\alpha_i)$. One then chooses $x_i^+$ to be root vectors in $g_{\pm \alpha_i}$ such that $[x_i^+, x_i^-] = h_i$. Using this presentation of $g$, we can build a Chevalley–Serre type presentation for the current algebra $g[v]$ as follows.

**Lemma 5.1.** The current algebra $g[v]$ is isomorphic to the complex Lie algebra generated by $\{x_i^+, h_{ir}\}_{i \in I, r \in \mathbb{N}}$ subject to the defining relations
\[
[h_{ir}, h_{js}] = 0,
\]
\[
[h_{i0}, x_{js}^+] = \pm 2d_{ij} x_{js}^+,
\]
\[
[h_{i,r+1}, x_{js}^+] = [h_{ir}, x_{js+1}^+]
\]
\[
[x_{i,r+1}^+, x_{js}^+] = [x_{ir}^+, x_{js+1}^+]
\]
\[
[x_{ir}^+, x_{js}^-] = \delta_{ij} h_{i,r+s},
\]
\[
\text{ad}(x_{i0}^+) = d_{ij} a_{ij} (x_{js}^+) = 0 \quad \forall i \neq j.
\]

**Proof (reference only).** Let us denote the Lie algebra with the above generators and defining relations by $a$. It is clear that there is a surjective Lie algebra homomorphism
\[
\psi : a \rightarrow g[v], \quad h_{ir} \mapsto h_i \otimes v^r, \quad x_{ir}^+ \mapsto x_i^+ \otimes v^r \quad \forall i \in I \text{ and } r \in \mathbb{N}.
\]
It would be wrong to say that it is obvious that this homomorphism is injective; for instance, if $g$ is replaced by an affine Kac-Moody algebra then this completely false. One way this can be proved is by using the same argument as used to prove Proposition 3.2 of [MRY]. That proposition is for the case where $g$ is an affine Kac-Moody algebra, but the argument goes through after making only very small modifications. \hfill \Box

5.2. **Definition of $Y_h(g)$.** We are now prepared to define the Yangian $Y_h(g)$ as an algebra. For a fixed $m > 0$, let $S_m$ denote the symmetric group on the set $\{1, \ldots, m\}$.

**Definition 5.2.** The Yangian $Y_h(g)$ is the unital associative $\mathbb{C}[h]$-algebra generated by $\{x_{ir}^+, h_{ir}\}_{i \in I, r \in \mathbb{N}}$, subject to the following relations for $i, j \in I$ and $r, s \in \mathbb{N}$:
\[
[h_{ir}, h_{js}] = 0,
\]
\[
[h_{i0}, x_{js}^+] = \pm 2d_{ij} x_{js}^+,
\]
\[
[h_{i0}, x_{js}^+] = \pm 2d_{ij} x_{js}^+.
\]

(5.2)
To complete the proof, it suffices to show this is an isomorphism. Let $\bar{h}_{i,r+1}^{\pm} = \delta_{ij} h_{i,r+s}$,

$$[h_{i,r+1}^+, x_{js}^+] - [h_{i,r}, x_{js+1}^+] = \pm h d_{ij} (h_{i,r} x_{js}^+ + x_{js+1}^+ h_{i,r}),$$

$$[x_{i,r+1}^+, x_{js}^+] - [x_{i,r}^+, x_{js+1}^+] = \pm h d_{ij} (x_{i,r}^+ x_{js}^+ + x_{js+1}^+ x_{i,r}^+),$$

$$\sum_{\pi \in S_m} \left[ x_{i,r_n(1)}, x_{i,r_n(2)}, \ldots, x_{i,r_n(m)}, x_{j,s} \right] = 0,$$

where in the last relation $i \neq j$, $m = 1 - a_{ij}$ and $r_1, \ldots, r_m \in \mathbb{N}$.

The Yangian $Y_h(\mathfrak{g})$ is an $\mathbb{N}$-graded algebra with

$$\deg h = 1 \quad \text{and} \quad \deg x_{ir}^\pm = \deg h_{ir} = r \quad \forall i \in I \text{ and } r \in \mathbb{N}.$$

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We may equip $U(\mathfrak{g}[v])$ (with presentation given by Lemma 5.1) with the structure of a $\mathbb{C}[h]$-algebra via the the evaluation morphism $\mathbb{C}[h] \to \mathbb{C}$, $h \mapsto 0$. In order to distinguish between the generators of $\mathfrak{g}[v]$ and of $Y_h(\mathfrak{g})$, let us denote the former by $\{H_{ir}, X_{ir}^\pm\}_{i \in I, r \in \mathbb{N}}$.

**Lemma 5.3.** The assignment

$$\text{ev}_h : h_{ir} \mapsto H_{ir}, \quad x_{ir}^\pm \mapsto X_{ir}^\pm \quad \forall i \in I \text{ and } r \in \mathbb{N}$$

extends to an epimorphism of graded $\mathbb{C}[h]$-algebras $\text{ev}_h : Y_h(\mathfrak{g}) \to U(\mathfrak{g}[v])$ with kernel $hY_h(\mathfrak{g})$.

In particular,

$$Y_h(\mathfrak{g})/hY_h(\mathfrak{g}) \cong U(\mathfrak{g}[v])$$

as $\mathbb{N}$-graded $\mathbb{C}[h]$-algebras.

**Proof.** This lemma is quite simple, but for completeness I will give all details. Since $h$ acts as zero in $U(\mathfrak{g}[v])$, it is clear from Lemma 5.1 that the relations (5.1)–(5.5) are preserved by the assignment $\text{ev}_h$. To show (5.6) is preserved, we must prove that

$$\sum_{\pi \in S_m} \left[ x_{i,r_n(1)}^\pm, x_{i,r_n(2)}^\pm, \ldots, x_{i,r_n(m)}^\pm, x_{j,s}^\pm \right] = 0 \quad \forall i \neq j, m = 1 - a_{ij}$$

holds in $\mathfrak{g}[v]$. We know that, by definition of the bracket in $\mathfrak{g}[v]$, the left-hand side of the above equality is equal to

$$\sum_{\pi \in S_m} \text{ad}(X_{i0}^\pm)^{1-a_{ij}} (X_{j,r_n(1)+\ldots+r_n(m)+s})^\pm,$$

and each term of this sum is zero by the Serre relations of $\mathfrak{g}[v]$ (i.e. the last relation of Lemma 5.1). This proves $\text{ev}_h$ extends to an epimorphism $\text{ev}_h : Y_h(\mathfrak{g}) \to U(\mathfrak{g}[v])$ with $hY_h(\mathfrak{g}) \subset \text{Ker}(\text{ev}_h)$. In particular, we have an induced epimorphism

$$\overline{\text{ev}}_h : Y_h(\mathfrak{g})/hY_h(\mathfrak{g}) \to U(\mathfrak{g}[v]).$$

To complete the proof, it suffices to show this is an isomorphism. Let $\bar{h}_{ir}$ and $\bar{x}_{ir}^\pm$ denote the images of $h_{ir}$ and $x_{ir}^\pm$ in $Y_h(\mathfrak{g})/hY_h(\mathfrak{g})$. It is clear that these elements satisfy the relations of Lemma 5.1, and hence we obtain a homomorphism

$$U(\mathfrak{g}[v]) \to Y_h(\mathfrak{g})/hY_h(\mathfrak{g}), \quad H_{ir} \mapsto \bar{h}_{ir}, \quad X_{ir}^\pm \mapsto \bar{x}_{ir}^\pm \quad \forall i \in I \text{ and } r \in \mathbb{N}.$$

This is clearly $(\overline{\text{ev}}_h)^{-1}$. \qed
5.3. **Hopf structure and quantization.** The previous lemma allows us to think of \( Y_h(\mathfrak{g}) \) as an algebra deformation of \( U(\mathfrak{g}[v]) \), but says nothing about the flatness of this deformation and nothing about a Hopf algebra structure related to the Lie bialgebra structure on \( \mathfrak{g}[v] \).
In fact, Both of these statements will depend on the Hopf structure of \( Y_h(\mathfrak{g}) \). To tell you how it is defined, we will first need to specify a more minimal generating set.

**Lemma 5.4.** \( Y_h(\mathfrak{g}) \) is generated by \( \{x_{i0}^\pm, h_{i1}\}_{i \in \mathbb{I}} \). Explicitly, for \( s > 0 \), \( x_{is}^\pm \) and \( h_{is+1} \) are determined by

\[
x_{is}^\pm = \pm \frac{1}{2d_i} [t_{i1}, x_{i,s-1}^\pm], \quad \text{where} \quad t_{i1} = h_{i1} - \frac{h}{2} h_{i0}^2,
\]

\[
h_{is} = [x_{is}^+, x_{i0}^-].
\]

**Proof.** By (5.4) with \( r = 0 \) and \( i = j \), we have

\[
[h_{i1}, x_{is}^\pm] = [h_{i0}, x_{is+1}^\pm] \pm h d_i \{h_{i0}, x_{is}^\pm\} = \pm 2d_i x_{is+1}^\pm \pm h d_i \{h_{i0}, x_{is}^\pm\},
\]

where the second equality is due to (5.2). Using again (5.2), we obtain

\[
\frac{1}{2} [h_i^2, x_{is}^\pm] = \pm d_i \{h_{i0}, x_{is}^\pm\}.
\]

Hence,

\[
[t_{i1}, x_{is}^\pm] = \pm 2d_i x_{is+1}^\pm.
\]

This proves the first equality in the statement of the Lemma. The second is immediate from (5.3).

Of course, we can take \( \{x_{i0}^\pm, t_{i1}\}_{i \in \mathbb{I}} \) as an alternative generating set, and this is in fact what we will normally do.

Next, let us denote the set of positive roots of \( \mathfrak{g} \) by \( \Delta_+ \). For each \( \alpha \in \Delta_+ \), choose root vectors \( x_\alpha^\pm \in \mathfrak{g}_\pm \) such that \( (x_\alpha^+, x_\alpha^-) = 1 \) and \( x_\alpha^+ = x_\alpha^- \). There is a natural homomorphism of \( \mathbb{C} \)-algebras

\[
\iota : U(\mathfrak{g}) \to Y_h(\mathfrak{g}), \quad h_i \mapsto h_{i0}, \quad x_i^\pm \mapsto x_{i0}^\pm \quad \forall \ i \in \mathbb{I}.
\]

It is injective because \( \text{ev}_h \circ \iota = \text{id}_{U(\mathfrak{g})} \). Henceforth, we will make use of this fact and not distinguish between \( x \in \mathfrak{g} \) and \( \iota(x) \in Y_h(\mathfrak{g}) \).

**Theorem 5.5.** \( Y_h(\mathfrak{g}) \) is an \( \mathbb{N} \)-graded Hopf algebra over \( \mathbb{C}[h] \) with

1. **Coproduct** \( \Delta \) given by

\[
\Delta(x_{i0}^\pm) = x_{i0}^\pm \otimes 1 + 1 \otimes x_{i0}^\pm,
\]

\[
\Delta(t_{i1}) = t_{i1} \otimes 1 + 1 \otimes t_{i1} - h \sum_{\alpha \in \Delta_+} (\alpha_i, \alpha) x_\alpha^- \otimes x_\alpha^+
\]

2. **Antipode** \( S \) given by

\[
S(x_{i0}^\pm) = -x_{i0}^\pm, \quad S(t_{i1}) = -t_{i1} - h \sum_{\alpha \in \Delta_+} (\alpha_i, \alpha) x_\alpha^- x_\alpha^+
\]

3. **Counit** \( \varepsilon \) given by

\[
\varepsilon(x_{i0}^\pm) = 0 = \varepsilon(t_{i1}).
\]
Proof (reference only). The proof of this theorem is actually quite (technically) difficult. I will give an indication of what the hard parts are and provide a proper reference to a full proof.

First, it should be clear that the assignment \( \varepsilon \) extends to a \( \mathbb{C}[h] \)-algebra homomorphism
\[
\varepsilon : Y_h(\mathfrak{g}) \to \mathbb{C}[h], \quad x^+_i \mapsto 0, \quad h_i \mapsto 0 \quad \forall \ i \in I \text{ and } r \in \mathbb{N}.
\]
This is easily verified; \( \varepsilon \) gives the trivial representation for \( Y_h(\mathfrak{g}) \). Let us now assume the following:

(A1). The assignment \( \Delta \) extends to a \( \mathbb{C}[h] \)-algebra homomorphism
\[
\Delta : Y_h(\mathfrak{g}) \to Y_h(\mathfrak{g}) \otimes_{\mathbb{C}[h]} Y_h(\mathfrak{g}).
\]

(A2). The assignment \( S \) extends to an algebra homomorphism
\[
S : Y_h(\mathfrak{g}) \to Y_h(\mathfrak{g})^{\text{op}},
\]
where \( Y_h(\mathfrak{g})^{\text{op}} \) is the opposite algebra of \( Y_h(\mathfrak{g}) \) (equivalently, we assume \( S \) extends uniquely to an anti-endomorphism of \( Y_h(\mathfrak{g}) \)).

Given these highly non-trivial assumptions, it is not hard to complete the proof of the theorem. What is left to show is

1. \((Y_h(\mathfrak{g}), \Delta, \mathbb{C}[h], \varepsilon)\) is a coalgebra. Since \( \Delta \) and \( \varepsilon \) are algebra homomorphisms this will imply \( Y_h(\mathfrak{g}) \) has a (graded) bialgebra structure.
2. \( S \) is an antipode on the bialgebra \( Y_h(\mathfrak{g}) \), i.e.
\[
(5.7) \quad \eta \circ \varepsilon = m \circ S \otimes \text{id} \circ \Delta = m \circ \text{id} \otimes S \circ \Delta,
\]
where \( m \) is the multiplication on \( Y_h(\mathfrak{g}) \) and \( \eta : \mathbb{C}[h] \to Y_h(\mathfrak{g}) \) is the unit map \( \eta(f(h)) = f(h)1_{Y_h(\mathfrak{g})} \).

To check (1), we need to check that the identities
\[
(5.8) \quad (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta \quad \text{and} \quad (\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta,
\]
where all tensor products are over \( \mathbb{C}[h] \) and in the second equality we have made use of the natural identifications
\[
\mathbb{C}[h] \otimes_{\mathbb{C}[h]} Y_h(\mathfrak{g}) \cong Y_h(\mathfrak{g}) \cong Y_h(\mathfrak{g}) \otimes_{\mathbb{C}[h]} \mathbb{C}[h].
\]
As \( \Delta \) and \( \varepsilon \) are algebra homomorphisms, \( (5.8) \) need only be checked on \( \{x^\pm_{i0}, t_i\}_{i \in I} \). That they are true on \( \{x^\pm_{i0}\}_{i \in I} \) is because \( \Delta \) and \( \varepsilon \) restrict to the usual coproduct and counit of \( U(\mathfrak{g}) \). On \( t_{i1} \), we have
\[
(\varepsilon \otimes \text{id})\Delta(t_{i1}) = \varepsilon(t_{i1}) \otimes 1 + 1 \otimes t_{i1} - h \sum_{\alpha \in \Delta^+_+} (\alpha_i, \alpha)\varepsilon(x^-_{\alpha}) \otimes x^+_{\alpha} = 1 \otimes t_{i1},
\]
as desired. One similarly checks \( (\text{id} \otimes \varepsilon)\Delta(t_{i1}) = t_{i1} \otimes 1 \). Hence the second equality in \( (5.8) \) holds. To check the first identity in \( (5.8) \) (the coassociativity of \( \Delta \)), let us set
\[
(X)_a = 1^\otimes(a-1) \otimes X \otimes 1^\otimes(3-a) \quad \forall \ 1 \leq a \leq 3.
\]
Then we have
\[
(\Delta \otimes \text{id})\Delta(t_{i1}) = \Delta(t_{i1}) \otimes 1 + 1 \otimes t_{i1} - h \sum_{\alpha \in \Delta^+_+} (\alpha_i, \alpha)\Delta(x^-_{\alpha}) \otimes x^+_{\alpha}
\]
Consider now (2). By Lemma III.3.6 of [K], it is enough to show that (5.7) holds on \{x_i^\pm, t_i\}_{i \in \mathbf{I}}. On \{x_i^\pm\} this is a \(U(g)\)-statement. On \(t_i\), we have
\[
m(S \otimes \text{id})\Delta(t_i) = S(t_i) + t_i - \hbar \sum_{\alpha \in \Delta_+} (\alpha,\alpha) S(x^-_\alpha)x^+_\alpha = 0 = m(\text{id} \otimes S)\Delta(t_i).
\]
Since \(\varepsilon(t_i) = 0\), we have proven the assertion.

To complete the proof of theorem, we would need to prove that the assumptions (A1) and (A2) do in fact hold. The proof of (A2) is easier in the so-called \(J\)-presentation of the Yangian originally given by Drinfeld [Dr1]; in that presentation it is easy to identify \(S\) with the composition of an automorphism and an anti-automorphism. The isomorphism between the \(J\)-presentation of the Yangian and the one considered here was also established by Drinfeld [Dr2]. For a complete proof, see [GRW1, Thm. 2.6].

**Corollary 5.6.** If \(Y_\hbar(g)\) is a flat \(\mathbb{C}[\hbar]\)-module, then it is a homogeneous quantization of \(g[v]\).

**Proof.** The definition of the \(h\)-Hopf structure on \(Y_\hbar(g)\) implies that the epimorphism of Lemma 5.3 is a homomorphism of graded Hopf algebras. It follows that the induced isomorphism \(Y_\hbar(g)/hY_\hbar(g) \cong U(g[v])\) is an isomorphism of \(N\)-graded Hopf algebras.

We are left to verify that the Lie bialgebra cocommutator \(\delta\) on \(g[v]\) induced by \(Y_\hbar(g)\) (see Prop. 4.2) coincides with the standard cocommutator on \(g[v]\) we studied in Example 1.2. Since these cocommutators are determined by their values on any set of Lie algebra generators (by the cocycle equation), it is enough to prove that
\[
\delta(x^+_i) = 0 \quad \text{and} \quad \delta(H_{i1}) = [h_i \otimes 1, \Omega] \quad \forall \, i \in \mathbf{I}.
\]
Here I kindly remind the reader that we are using the uppercase letters to denote the generators of \(g[v]\) in the presentation of Lemma 5.1, so \(H_{i1} = h_i \otimes v\), with \(x^+_i, h_i\) as in §5.1.

Since
\[
\Delta(x^\pm_i) = \Delta^{\text{op}}(x^\pm_i),
\]
the equality \(\delta(x^+_i) = 0\) does hold. To prove \(\delta(H_{i1}) = [h_i \otimes 1, \Omega]\), recall that if we set \(g_0 = \mathfrak{h}\) we have
\[
g = g_0 \oplus \bigoplus_{\alpha \in \Delta_+} (g_\alpha \oplus g_{-\alpha}) \quad \text{with} \quad (g_\alpha, g_\beta) = \delta_{\alpha,-\beta}\mathbb{C} \quad \forall \, \alpha, \beta \in \Delta_+ \cup \{0\}.
\]
It follows that we may write the Casimir tensor \(\Omega\) as
\[
\Omega = \Omega_\mathfrak{h} + \sum_{\alpha \in \Delta_+} (x^\alpha_+ \otimes x^-_\alpha + x^-_\alpha \otimes x^+_\alpha)
\]
with \(\Omega_\mathfrak{h}\) the Casimir two-tensor of \(\mathfrak{h}\) in \(\mathfrak{h} \otimes \mathfrak{h}\). The definition of \(\delta\) then yields the sequence of equalities
\[
\delta(H_{i1}) = \text{ev}_\mathfrak{h} \left( \frac{\Delta(h_{i1}) - \Delta^{\text{op}}(h_{i1})}{\hbar} \right)
\]
\[= \sum_{\alpha \in \Delta_\pm} (\alpha_i, \alpha)(x^+_{\alpha} \otimes x^-_{\alpha} - x^-_{\alpha} \otimes x^+_{\alpha}) \]
\[= \left[ h_i \otimes 1, \Omega h + \sum_{\alpha \in \Delta_\pm} (x^+_{\alpha} \otimes x^-_{\alpha} + x^+_{\alpha} \otimes x^-_{\alpha}) \right] = [h_i \otimes 1, \Omega]. \]

The flatness condition of \( Y_h(\mathfrak{g}) \) is the most non-trivial property left for us to establish. It is also the most interesting property, as our proof of this result will tell us a lot about the structure of the Yangian. This will be delayed a few sections.

5.4. Generating series and automorphisms. We saw in §3.1 that it was convenient to introduce the notion of generating series for \( \mathfrak{g}[v] \) (and for \( D(\mathfrak{g}[v]) = \mathfrak{g}[v^\pm 1] \)). We can also organize the generators of \( Y_h(\mathfrak{g}) \) into such series. Set
\[ x^\pm_i(u) = \sum_{r \geq 0} x^\pm_{ir} u^{-r-1} \in Y_h(\mathfrak{g})[u^{-1}] \quad \text{and} \quad h_i(u) = \sum_{r \geq 0} h_{ir} u^{-r-1} \in Y_h(\mathfrak{g})[u^{-1}] \quad \forall \ i \in I. \]

**Proposition 5.7.** The defining relations (5.1)–(5.6) of \( Y_h(\mathfrak{g}) \) are equivalent to the following relations for \( i, j \in I \):

(5.9) \[ [h_i(u), h_j(v)] = 0, \]
(5.10) \[ [h_{i0}, x^\pm_j(u)] = \pm 2d_{ij} x^\pm_j(u), \]
(5.11) \[ (u - v \mp hd_{ij}) h_i(u) x^\pm_j(v) \]
\[ = (u - v \pm hd_{ij}) x^\pm_j(v) h_i(u) \pm 2d_{ij} x^\pm_j(v) - [h_i(u), x^\pm_j(u)], \]
(5.12) \[ (u - v \mp hd_{ij}) x^\pm_i(u) x^\pm_j(v) \]
\[ = (u - v \pm hd_{ij}) x^\pm_j(v) x^\pm_i(u) + [x^\pm_i(u), x^\pm_j(v)] - [x^\pm_i(u), x^\pm_j(u)], \]
(5.13) \[ (u - v)[x^\pm_i(u), x^\pm_j(v)] = \delta_{ij}(h_i(v) - h_i(u)), \]
(5.14) \[ \sum_{\pi \in S_m} \left[ x^\pm_i(u_{\pi(1)}), \ldots, x^\pm_i(u_{\pi(m)}), x^\pm_j(v) \right] = 0, \]

where in the last relation \( i \neq j \) and \( m = 1 - \alpha_{ij} \).

**Remark 5.8.** The relations (5.9)–(5.13) can (and will) be viewed as identities in the algebra \( Y_h(\mathfrak{g})[u^{-1}, v^{-1}] \) (one must absorb the polynomial coefficients). Similarly, the Serre relations (5.14) should be understood as identities in \( Y_h(\mathfrak{g})[u^{-1}_1, \ldots, u^{-1}_m, v^{-1}] \).

**Proof of Prop. 5.7.** This is essentially [GTL, Prop. 2.3], but let us give a few details just to provide an indication of how this works. The relations (5.9) and (5.10) are easily seen to be equivalent to (5.1) and (5.2), respectively. Indeed:

\[ [h_i(u), h_j(v)] = 0 \iff \sum_{r,s \geq 0} [h_{ir}, h_{js}] u^{-r-1} v^{-s-1} = 0 \iff [h_{ir}, h_{js}] = 0 \forall r, s \in \mathbb{N}. \]

\[ [h_{i0}, x^\pm_j(u)] = \pm 2d_{ij} x^\pm_j(u) \iff \sum_{r \geq 0} \left[ [h_{i0}, x^\pm_{jr}] \pm 2d_{ij} x^\pm_{jr} \right] u^{-r-1} = 0 \]
\[ \iff [h_{i0}, x^\pm_{jr}] = \pm 2d_{ij} x^\pm_{jr} \forall r \in \mathbb{N}. \]
Next, let us show that, given (5.10), (5.4) is equivalent to (5.11). Note that given a series
\[ a(u) = \sum_{r \geq 0} a_r u^{-r-1}, \]
we have
\[ \sum_{r \geq 0} a_{r+1} u^{-r-1} = u \sum_{r \geq 0} a_{r+1} u^{-(r+1)-1} = ua(u) - a_0. \]
Multiplying (5.4) by \( u^{-r-1} v^{-s-1} \) and summing over \( r, s \in \mathbb{N} \), we find it is equivalent to
\[ \sum_{r \geq 0} [h_i, v_j] u^{-r-1} - \sum_{s \geq 0} [h_i(u), x_{j,s+1}^\pm] v^{-s-1} = \pm h_{ij} \{h_i(u), x_j^\pm(v)\}, \]
where \( \{x, y\} = xy + yx \). Using (5.15) and (5.10), we find that the above is equivalent to
\[ u[h_i(u), x_j^\pm(v)] \mp 2d_{ij} x_j^\pm(v) - v[h_i(u), x_j^\pm(v)] + [h_i(u), x_j^\pm(v)] = \pm h_{ij} \{h_i(u), x_j^\pm(v)\}. \]
Rearranging terms, one easily deduces this is exactly (5.11). The equivalence of the remaining relations is checked similarly.

**Remark 5.9.** The relation (5.10) is in fact redundant and follows from (5.11). Indeed, taking the \( u^0 \) coefficient of (5.11) returns (5.10).

This presentation of \( Y_h(\mathfrak{g}) \) is particularly convenient for defining homomorphisms and the study of representations of \( Y_h(\mathfrak{g}) \).

**Corollary 5.10.** Let \( c \in \mathbb{C} \). The the assignment
\[ \tau_c : x_i^\pm(u) \mapsto x_i^\pm(u - c), \quad h_i(u) \mapsto h_i(u - c) \quad \forall i \in I, \]
extends to an automorphism of \( Y_h(\mathfrak{g}) \) with inverse \( \tau_{-c} \). Moreover, \( \tau_c \) specializes (modulo \( h \)) to the shift automorphism of \( U(\mathfrak{g}[v]) \) given by Lemma 2.1.

**Remark 5.11.** The above means \( \tau_c \) sends \( x_i^\pm \) (resp. \( h_i \)) to the coefficient of \( u^{-r-1} \) of the series \( x_i^\pm(u - c) \) (resp. \( h_i(u - c) \)), which is to be expanded in \( Y_h(\mathfrak{g})[u^{-1}] \). Since
\[ (u - c)^{-r-1} = \sum_{p \geq 0} \binom{r + p}{p} c^p u^{-p-r-1}, \]
this means that
\[ x_i^\pm \mapsto \sum_{k=0}^r \binom{r}{k} c^{-k} x_{ik}, \quad h_i \mapsto \sum_{k=0}^r \binom{r}{k} c^{-k} h_{ik} \quad \forall i \in I, r \in \mathbb{N}. \]

**Proof of Cor. 5.10.** Replacing \( u \) and \( v \) by \( u - c \) and \( v - c \) in the relations of Proposition 5.7 does not change any of the polynomial coefficients, which depend only on \( u - v \). It also sends \( x_i^\pm(w) \) and \( h_i(w) \) to \( \tau_c(x_i^\pm(w)) = x_i^\pm(w - c) \) and \( \tau_c(h_i(w)) = h_i(w - c) \) \((w = u \) or \( v \)). As \( \tau_c \) fixes \( x_i^\pm \), it follows that \( \tau_c(x_i^\pm(u)) \) and \( \tau_c(x_i^\pm(u)) \) satisfy the relations of Proposition 5.7. This proves \( \tau_c \) extends to an algebra endomorphism of \( Y_h(\mathfrak{g}) \). It is clear that \( \tau_c \circ \tau_{-c} = \text{id} = \tau_{-c} \circ \tau_c \).

The assertion that \( \tau_c \) specializes modulo \( h \) to the automorphism of \( U(\mathfrak{g}[v]) \) given by Lemma 2.1 is also easily verified.

The Yangian \( Y_h(\mathfrak{g}) \) also admits an analogue of the Chevalley involution of \( \mathfrak{g} \):
Corollary 5.12. The assignment
\[ \omega : x_i^\pm(u) \mapsto -x_i^\mp(-u), \quad h_i(u) \mapsto h_i(-u) \quad \forall \ i \in I \]
extends to an automorphism of \( Y_h(g) \) with \( \omega^2 = \text{id} \).

The proof of this result is left as an exercise to the reader.

November 21st 2018

6. Filtered algebras and Rees algebras

Although we have thus far worked with algebras over the polynomial ring \( \mathbb{C}[h] \), the Yangian is often studied over \( \mathbb{C} \) with \( h \) evaluated to a complex number \( \lambda \). This setting is ideal for the study of representation theory, for instance. When one passes from a formal variable \( h \) to a complex number \( \lambda \), the grading on \( Y_h(g) \) is destroyed as \( h \), which had degree one, is being specialized to an element of degree zero. The resulting algebra, denoted \( Y_\lambda(g) \), is a \( \mathbb{N} \)-filtered (and not \( \mathbb{N} \)-graded) algebra. This turns out to be a very good thing. In this section we review the basics of filtered algebras and then explain how to go back from a filtered algebra to a deformation algebra over \( \mathbb{C}[h] \).

The ideas we will discuss are not difficult, but are incredibly useful. In the next section we will use them to prove the flatness of \( Y_h(g) \).

6.1. Filtered algebras.

Definition 6.1. Let \( \mathcal{B} \) be a \( \mathbb{C} \)-algebra and \( \mathbf{F} = \{ \mathbf{F}_k \}_{k \in \mathbb{N}} \) a collection of subspaces of \( \mathcal{B} \). Then \( \mathbf{F} \) is called an ascending algebra filtration on \( \mathcal{B} \) if
\[ 0 \subseteq \mathbf{F}_0 \subseteq \mathbf{F}_1 \subseteq \cdots \subseteq \mathbf{F}_n \subseteq \cdots \quad \text{and} \quad \mathbf{F}_n \cdot \mathbf{F}_m \subset \mathbf{F}_{n+m} \quad \text{and} \quad \bigcup_{k \in \mathbb{N}} \mathbf{F}_k = \mathcal{B}. \]
Whenever this is the case, we call \( \mathcal{B} \) a \( \mathbb{N} \)-filtered (or just filtered) algebra with filtration \( \mathbf{F} \). By convention we usually set \( \mathbf{F}_{-1} = \{0\} \).

Given a filtered algebra \( (\mathcal{B}, \mathbf{F}) \), we define
\[ \text{gr}(\mathcal{B}) = \bigoplus_{n \in \mathbb{N}} \mathbf{F}_n/\mathbf{F}_{n-1}. \]

The algebra structure on \( \mathcal{B} \) induces an algebra structure on \( \text{gr}(\mathcal{B}) \), which is naturally \( \mathbb{N} \)-graded. The algebra \( \text{gr}(\mathcal{B}) \) is called the associated graded algebra of \( (\mathcal{B}, \mathbf{F}) \).

Thus to any filtered algebra we can naturally attach a graded algebra. We can do the same thing for homomorphisms.

Definition 6.2. Let \( (\mathcal{B}, \mathbf{F}) \) and \( (\mathcal{C}, \mathbf{H}) \) be filtered algebras. Then an algebra homomorphism \( \varphi : \mathcal{B} \to \mathcal{C} \) is said to be filtered if
\[ \varphi(\mathbf{F}_k) \subset \mathbf{H}_k \quad \forall \ k \in \mathbb{N}. \]

Any filtered homomorphism \( \varphi \) induces a family of linear maps
\[ \varphi_k : \mathbf{F}_k/\mathbf{F}_{k-1} \to \mathbf{H}_k/\mathbf{H}_{k-1}. \]
We then set
\[ \text{gr}(\varphi) = \bigoplus_{k \in \mathbb{N}} \varphi_k : \text{gr}(\mathcal{B}) \to \text{gr}(\mathcal{C}). \]
It is easily seen this is an algebra homomorphism which is, by definition, \(\mathbb{N}\)-graded. We call \(\text{gr}(\varphi)\) the associated graded morphism to \(\varphi\).

**Remark 6.3.** What we have really just observed is that there is a functor \(\text{gr}\) from the category of \(\mathbb{N}\)-filtered algebras to the category of \(\mathbb{N}\)-graded algebras. It turns out that this functor encodes a lot of important information. The idea is that some filtered algebras may be very difficult to deal with directly, but in some special cases their associated graded algebras are extremely nice and easy to deal with graded algebras (of course, this does not always happen).

One of the most important classical examples is given by enveloping algebras.

**Example 6.4.** Let \(\mathfrak{a}\) be a complex Lie algebra. Then the enveloping algebra \(U(\mathfrak{a})\) of \(\mathfrak{a}\) is a \(\mathbb{N}\)-filtered algebra with filtration given by the length of a monomial in \(U(\mathfrak{a})\). That is, one defines \(F = \{F_k\}_{k \in \mathbb{N}}\) by setting

\[
F_k = \text{Span}\{x_1 \ldots x_\ell : x_i \in \mathfrak{a}, \ell \leq k\}.
\]

In \(U(\mathfrak{a})\) we have \(xy - yx = [x,y] \in \mathfrak{a}\) for all \(x,y \in \mathfrak{a}\). Since the left-hand side is in \(F_2\) and the right-hand side is in \(F_1\), we obtain

\[
\bar{x}y = \bar{y}\bar{x} \in F_2/F_1 \subset \text{gr}(U(\mathfrak{a})) \quad \forall \, x,y \in \mathfrak{a},
\]

where \(\bar{x}\) is the image of \(x\) in \(F_1/F_0\). This implies that \(\text{gr}(U(\mathfrak{a}))\) is a commutative algebra.

Now let \(S(\mathfrak{a})\) be the symmetric algebra of \(\mathfrak{a}\). By its univeral property, the linear map

\[
a \mapsto \text{gr}(U(\mathfrak{a})), \quad x \mapsto \bar{x} \quad \forall \, x \in \mathfrak{a}
\]

uniquely extends to an algebra homomorphism

\[
\varphi_\mathfrak{a} : S(\mathfrak{a}) \to \text{gr}(U(\mathfrak{a}))
\]

This homomorphism is surjective because \(\text{gr}(U(\mathfrak{a}))\) is generated as an algebra by the image of \(\mathfrak{a}\). It is also injective, and this is nothing but the Poincaré-Birkhoff-Witt Theorem for \(U(\mathfrak{a})\):

**Theorem 6.5 (PBW Theorem).** The algebra homomorphism \(\varphi_\mathfrak{a}\) is an isomorphism. Consequently

\[
S(\mathfrak{a}) \cong \text{gr}(U(\mathfrak{a})).
\]

This is a particularly nice way of stating the PBW theorem since it does not depend on specifying a basis; it is coordinate free. We will see why it implies the regular version of the PBW theorem momentarily.

One of the reasons the associated graded functor is so powerful is given by the following proposition.

**Proposition 6.6.** Let \((\mathfrak{B}, \mathcal{F})\) and \((\mathcal{D}, \mathcal{H})\) be filtered and let \(\varphi : \mathfrak{B} \to \mathcal{D}\) be a filtered homomorphism. Then

(1) If \(\text{gr}(\varphi)\) is injective, then \(\varphi\) is injective.
(2) If \(\text{gr}(\varphi)\) is surjective, then \(\varphi\) is surjective.
(3) If \(\text{gr}(\varphi)\) is an isomorphism, then \(\varphi\) is an isomorphism.
Proof. It suffices to prove (1) and (2). Suppose that $\text{gr}(\varphi)$ is injective, and assume $x \in \mathfrak{B}$ is a nonzero. Since $\mathfrak{B} = \bigcup_{k \in \mathbb{N}} F_k$, $x \in F_k$ for some $k \in \mathbb{N}$. We may assume $k$ is minimal with this property. Then $\bar{x} \in F_k/F_{k-1}$ is nonzero. As $\text{gr}(\varphi)$ is injective, $\text{gr}(\varphi)(\bar{x}) \neq 0$. But

$$\text{gr}(\varphi)(\bar{x}) = \bar{\varphi(x)} \in H_k/H_{k-1}.$$ 

Therefore $\varphi(x)$ cannot be zero. This proves (1).

To prove (2), it suffices to show $\varphi|_{F_k}$ is surjective for each $k$. This is proven by induction on $k$. By assumption, $\varphi_k = \text{gr}(\varphi)|_{F_k/F_{k-1}}$ is surjective for each $k$. The base case of the induction amounts to the fact that $\varphi_0 = \varphi_{F_0}$. Assume inductively that $\varphi|_{F_k}$ is injective for some fixed $k \geq 0$.

Let $y_k \in H_k$ be arbitrary. The surjectivity of $\varphi_k$ implies that there is $x_k \in F_k$ with

$$y_{k-1} := \varphi(x_k) - y_k \in F_{k-1}.$$ 

By assumption, we can thus find $x_{k-1} \in F_{k-1}$ with $\varphi(x_{k-1}) = y_{k-1}$. Therefore

$$\varphi(x_k - x_{k-1}) = y_k + y_{k-1} - y_{k-1} = x_k. \quad \square$$

There is a generalization of this result which will be particularly useful to us. Let $\mathcal{J}$ be (a possible infinite) set, and suppose that $(\mathfrak{O}_j, \{H^j_k\}_{k \in \mathbb{N}})_{j \in \mathcal{J}}$ is a family of filtered algebras over $\mathbb{C}$. Suppose that $(\mathfrak{B}, F)$ is another filtered algebra, and assume we are given a family of filtered algebra homomorphisms

$$\psi_j : \mathfrak{B} \to \mathfrak{O}_j \quad \forall j \in \mathcal{J}.$$ 

By the universal property of direct product, we obtain an algebra homomorphism

$$\Psi : \mathfrak{B} \to \prod_{j \in \mathcal{J}} \mathfrak{O}_j, \quad P_j \circ \Psi = \psi_j \quad \forall j \in \mathcal{J},$$

where $P_j : \prod_{i \in \mathcal{J}} \mathfrak{O}_i \to \mathfrak{O}_j$ is the natural projection.

Since $\{\psi_j\}_{j \in \mathcal{J}}$ is a filtered family, we obtain the family of associated graded maps

$$\text{gr}(\psi_j) : \text{gr}(\mathfrak{B}) \to \text{gr}(\mathfrak{O}_j).$$

Taking the direct, product we obtain

$$\bar{\Psi} : \text{gr}(\mathfrak{B}) \to \prod_{j \in \mathcal{J}} \text{gr}(\mathfrak{B}_j), \quad P_j \circ \bar{\Psi} = \text{gr}(\psi_j) \quad \forall j \in \mathcal{J}.$$ 

Proposition 6.7. Suppose that $\bar{\Psi}$ is injective. Then $\Psi$ is injective.

Proof. The argument is the same as the proof of Part (1) of Prop. 6.6. \quad \square

Another particularly important result is given by the following proposition.

Proposition 6.8. Let $(\mathfrak{B}, F)$ be a filtered algebra. Suppose that $B_k = \bigcup_{k \in \mathbb{N}} B_k$ is a graded basis of $\text{gr}(\mathfrak{B})$, i.e. that $B_k$ is a basis of $\text{gr}(\mathfrak{B})_k = F_k/F_{k-1}$. For each $k$, let $D_k \subset F_k$ be any lift of $B_k$. Then $D = \bigcup_{k \in \mathbb{N}} D_k$ is a basis of $\mathfrak{B}$.

Remark 6.9. This proposition says that any graded basis of $\text{gr}(\mathfrak{B})$ lifts to a basis of $\mathfrak{B}$. This is precisely why Theorem 6.5 implies the perhaps more standard basis-dependent version of the PBW theorem. Indeed, a graded basis of $S(\mathfrak{a})$ is given by ordered monomials in any basis of $\mathfrak{a}$ (the grading being given by the length of each monomial), and thus the same is true of $\text{gr}(U(\mathfrak{a}))$. 

Proof of Prop. 6.8. Let us first show $\mathcal{D}$ is a linearly independent set. Suppose a relation of the form

$$0 = \sum_{Y \in \mathcal{D}} \lambda_Y Y$$

holds, where $\lambda_Y$ are zero except for finitely many $Y$. Assume towards a contradiction not all $\lambda_Y$ are zero. Since this is a finite sum there is $k$ sufficiently large such that $\lambda_Y = 0$ whenever $Y \in \mathcal{D}_n$ for $n > k$. We may choose $k$ minimal with this property, i.e. $\lambda_Y \neq 0$ for some $Y \in \mathcal{D}_k$.

Applying the quotient map $F_k \to F_k/F_{k-1}$, we get

$$0 = \sum_{Y \in \mathcal{D}_k} \lambda_Y \bar{Y} = \sum_{Y \in B_k} \lambda_Y \bar{Y}.$$ 

The linear independence of $B_k$ then gives $\lambda_Y = 0$ for all $Y \in \mathcal{D}_k$, which contradicts our minimality assumption on $k$. Therefore $\mathcal{D}$ is linearly independent.

To show $\mathcal{D}$ spans, we argue that $F_k \subset \text{span}(\mathcal{D})$ by induction on $k$. Since $F_0 \subset \text{gr}(\mathfrak{B})$, the base case is trivial. Assume the assertion holds for a fixed $k \geq 0$ and let $x \in F_{k+1}$. By the assumption of the proposition,

$$\bar{x} = \sum_{Y \in \mathcal{B}_k} \lambda_Y \bar{Y} \in F_{k+1}/F_k$$

for some $\lambda_Y \in \mathbb{C}$.

Therefore $x - \sum_{Y \in \mathcal{D}_k} \lambda_Y Y \in F_k$, and applying the induction hypothesis we are done. □

6.2. Rees algebras.

Definition 6.10. Let $\Omega$ be a $\mathbb{C}$-algebra.

(1) A $\mathbb{C}[h]$-algebra $\Omega_h$ is said to be a flat deformation of $\Omega$ if $\Omega_h$ is flat as a $\mathbb{C}[h]$-module and

$$\Omega_h/h\Omega_h \cong \Omega.$$ 

When the flatness condition is omitted, we will just say that $\Omega_h$ is a deformation of $\Omega$.

(2) Now assume in addition that $\Omega$ is graded. We say that a $\mathbb{C}$-algebra $\mathfrak{A}$ is a filtered deformation of $\Omega$ if

$$\Omega \cong \text{gr}(\mathfrak{A}).$$ 

In this subsection we will see how to pass from filtered deformations to graded flat deformations.

Definition 6.11. Let $(\mathfrak{B}, F)$ be a filtered algebra. The Rees algebra of $\mathfrak{B}$ (with respect to $F$) is the $\mathbb{C}[h]$-subalgebra of $\mathfrak{B}[h]$ given by

$$R_h[\mathfrak{B}] = \bigoplus_{n \in \mathbb{N}} h^n F_n \subset \mathfrak{B}[h].$$ 

Proposition 6.12. Let $(\mathfrak{B}, F)$ be a filtered algebra. Then

(1) $R_h[\mathfrak{B}]$ is a flat deformation of $\text{gr}(\mathfrak{B})$.

(2) $R_h[\mathfrak{B}]$ satisfies

$$R_h[\mathfrak{B}]/(h - \lambda)R_h[\mathfrak{B}] \cong \mathfrak{B} \quad \forall \lambda \in \mathbb{C}^\times.$$
**Proof.** We have

\[
R_h[\mathcal{B}] \big/ hR_h[\mathcal{B}] \cong \bigoplus_{n \in \mathbb{N}} h^nF_n \big/ \bigoplus_{n \in \mathbb{N}\cup\{-1\}} h^nF_{n-1} \cong \bigoplus_{n \in \mathbb{N}} F_n / F_{n-1} = \text{gr}(\mathcal{B}),
\]

and \(R_h[\mathcal{B}]\) is flat because it is torsion free (being a subalgebra of \(\mathcal{B}[h]\)). Thus it is a flat deformation of \(\text{gr}(\mathcal{B})\).

Consider now (2). There is a natural epimorphism of algebras \(\text{ev}_\lambda : \mathcal{B}[h] \to \mathcal{B}\) given by sending \(h\) to \(\lambda\). It has kernel \((h - \lambda)\mathcal{B}[h]\). Restricting \(\text{ev}_\lambda\) to \(R_h[\mathcal{B}]\), we obtain

\[\text{ev}_\lambda^{\mathfrak{g}} : R_h[\mathcal{B}] \to \mathcal{B}\text{.}\]

It is surjective because \(\mathcal{B} = \bigcup_k F_k\) and the image contains \(F_k\) for each \(k\). We also have

\[\text{Ker}(\text{ev}_\lambda^{\mathfrak{g}}) = R_h[\mathcal{B}] \cap (h - \lambda)\mathcal{B}[h] = (h - \lambda)R_h[\mathcal{B}].\]

□

Thus, the Rees algebra construction allows us to pass from filtered algebras to graded flat deformations. In fact, we can also go the other way, but this is left as an exercise:

**Exercise.** Let \(\mathfrak{U}_h\) be a graded \(\mathbb{C}[h]\)-algebra with \(\text{deg} h = 1\). Assume that \(\mathfrak{U}_h\) is a flat deformation of \(\mathfrak{U}\). Show that, for any fixed \(\lambda \in \mathbb{C}^\times\), \(A = \mathfrak{U}_h / (h - \lambda)\mathfrak{U}_h\) is a filtered deformation of \(\mathfrak{U}\).

**References**


Department of Mathematical and Statistical Sciences, University of Alberta.

E-mail address: cwendlan@ualberta.ca