Theorem we get
\[ \mathcal{L}[\cos ax] = \frac{1}{1 - e^{-2\pi a}} \int_0^{2\pi} e^{-sx} \cos ax \, dx = \frac{1}{1 - e^{-2\pi a}} \left\{ \frac{e^{-sx} \sin ax}{s^2 + a^2} \right\} \bigg|_0^{2\pi} \]
\[ = \frac{1}{1 - e^{-2\pi a}} \left\{ \frac{e^{-2\pi a}(-s) - (-s)}{s^2 + a^2} \right\} = \frac{s}{s^2 + a^2}. \]

Example 6.23. Find \( \mathcal{L}[f(x)] \) for the function
\[ f(x) = \begin{cases} 1 & 0 < x \leq 1, \\ -1 & 1 < x \leq 2 \end{cases}, \quad f(x + 2n) = f(x) \quad \forall n \in \mathbb{Z}. \]

Solution
The function \( f \) is periodic with period 2, so we have
\[ \mathcal{L}[f(x)] = \frac{1}{1 - e^{-2s}} \int_0^2 e^{-sx} f(x) \, dx = \frac{1}{1 - e^{-2s}} \left\{ \int_0^1 e^{-sx} \, dx - \int_1^2 e^{-sx} \, dx \right\} \]
\[ = \frac{1}{1 - e^{-2s}} \left( \frac{e^{-2s} - 2e^{-s} + 1}{s} \right) = \frac{(1 - e^{-s})^2}{s(1 - e^{-2s})} = \frac{1 - e^{-s}}{s(1 + e^{-s})} = \frac{e^{-s/2} - e^{-s/2}}{s(e^{s/2} + e^{-s/2})} = \frac{1}{s} \tanh\left(\frac{1}{s}\right). \]

6.3 Inverse Laplace Transforms

Recall the solution procedure outlined in Figure 6.1. The final stage in that solution procedure involves calculating inverse Laplace transforms. In this section we look at the problem of finding inverse Laplace transforms. In other words, given \( F(s) \), how do we find \( f(x) \) so that \( F(s) = \mathcal{L}[f(x)] \).

We begin with a simple example which illustrates a small problem on finding inverse Laplace transforms.

Example 6.24. Consider the functions
\[ f(x) = x^2, \quad \text{and} \quad g(x) = \begin{cases} x^2 & x \neq 2, 3, \\ 48 & x = 2, \\ -\pi & x = 3 \end{cases}. \]

Then \( \mathcal{L}[f(x)] = \mathcal{L}[g(x)] = \frac{2}{s^3} \). Since an integral is not affected by the changing of its integrand at a few isolated points, more than one function can have the same Laplace transform. \[ \square \]

Example 6.24 illustrates that inverse Laplace transforms are not unique. However, it can be shown that, if several functions have the same Laplace transform, then at most one of them is continuous. This prompts us to make the following definition.

Definition 6.25. The inverse Laplace transform of \( F(s) \), denoted \( \mathcal{L}^{-1}[F(s)] \), is the function \( f \) defined on \([0, \infty)\) which has the fewest number of discontinuities and satisfies
\[ \mathcal{L}[f(x)] = F(s). \]
1. \( \mathcal{L}^{-1}\left[ \frac{2}{s^3} \right] = x^2 \).
2. \( \mathcal{L}^{-1}\left[ \frac{s}{s^2 + 9} \right] = \cos 3x \).
3. \( \mathcal{L}^{-1}\left[ \frac{s - 1}{s^2 - 2s + 5} \right] = \mathcal{L}^{-1}\left[ \frac{s - 1}{(s - 1)^2 + 4} \right] = e^x \mathcal{L}^{-1}\left[ \frac{s}{s^2 + 4} \right] = e^x \cos 2x \). (using property 1 of Theorem 6.17 in reverse)

The inverse Laplace transform is a linear operator.

Theorem 6.27.
If \( \mathcal{L}^{-1}[F(s)] \) and \( \mathcal{L}^{-1}[G(s)] \) exist, then \( \mathcal{L}^{-1}[\alpha F(s) + \beta G(s)] = \alpha \mathcal{L}^{-1}[F(s)] + \beta \mathcal{L}^{-1}[G(s)] \).

Proof
Starting from the right hand side we have
\[
\mathcal{L}[\alpha \mathcal{L}^{-1}[F(s)] + \beta \mathcal{L}^{-1}[G(s)]] = \alpha \mathcal{L}^{-1}[F(s)] + \beta \mathcal{L}^{-1}[G(s)] = \alpha F(x) + \beta G(x).
\]
The result follows.

Most of the properties of the Laplace transform can be reversed for the inverse Laplace transform.

Theorem 6.28.
If \( \mathcal{L}^{-1}[F(s)] = f(x) \), then the following hold:
1. \( \mathcal{L}^{-1}[F(s + a)] = e^{-ax} f(x) \);
2. \( \mathcal{L}^{-1}[sF(s)] = f'(x) \), if \( f(0) = 0 \);
3. \( \mathcal{L}^{-1}\left[ \frac{1}{s} F(s) \right] = \int_0^x f(t) \, dt \);
4. \( \mathcal{L}^{-1}[e^{-as} F(s)] = u_a(x) f(x-a) \).

Proof
1. \( \mathcal{L}[e^{-ax} f(x)] = F(s+a) \) from Theorem 6.17, property 1. The result follows.
2. \( \mathcal{L}[f'(x)] = -f(0) + sF(s) \) from Theorem 6.17, property 4. The result follows.
3. \( \mathcal{L}\left[ \int_0^x f(t) \, dt \right] = \frac{1}{s} F(s) \), from Theorem 6.17, property 5. The result follows.
4. \( \mathcal{L}[u_a(x) f(x-a)] = e^{-as} \mathcal{L}[f(x)] = e^{-as} F(s) \), from Theorem 6.19. The result follows.

Example 6.29. Find \( \mathcal{L}^{-1}\left[ \frac{1}{s(s^2+1)} \right] \).

Solution
We can write \( \frac{1}{s(s^2+1)} = \frac{1}{s} F(s) \), where \( F(s) = \frac{1}{s^2+1} \). Then \( f(x) = \mathcal{L}^{-1}[F(s)] = \sin x \), so we get
\[
\mathcal{L}^{-1}\left[ \frac{1}{s(s^2+1)} \right] = \mathcal{L}^{-1}\left[ \frac{1}{s} F(s) \right] = \int_0^x f(t) \, dt = \int_0^x \sin t \, dt = 1 - \cos x.
\]
Example 6.30. Find $\mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 2s + 5)}\right]$.

*Solution*

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 2s + 5)}\right] = \mathcal{L}^{-1}\left[\frac{1}{(s + 1)^2 + 4}\right] = e^{-x}\mathcal{L}^{-1}\left[\frac{1}{s^2 + 4}\right] = \frac{1}{2}e^{-x}\sin 2x.$$  

Example 6.31. Find $\mathcal{L}^{-1}\left[\frac{1 + e^{-s}}{s^2}\right]$.

*Solution*

$$\mathcal{L}^{-1}\left[\frac{1 + e^{-s}}{s^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2} + \frac{e^{-s}}{s^2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right] = x + u_1(x)(x - 1).$$

Example 6.32. Find $\mathcal{L}^{-1}\left[\frac{4e^{-2s}}{s^2 + 16}\right]$.

*Solution*

$$\mathcal{L}^{-1}\left[\frac{4e^{-2s}}{s^2 + 16}\right] = \mathcal{L}^{-1}\left[e^{-2s} \cdot \frac{4}{s^2 + 16}\right] = u_2(x)\mathcal{L}^{-1}\left[\frac{4}{s^2 + 16}\right] = u_2(x)\sin 4x.$$  

Many transforms that one encounters are of the form $F(s) = \frac{P(s)}{Q(s)}$, where $P$ and $Q$ are polynomials in $s$ with $\deg\{Q\} > \deg\{P\}$. To evaluate $\mathcal{L}^{-1}[F(s)]$, one writes $\frac{P(s)}{Q(s)}$ in terms of partial fractions.

Example 6.33 (distinct linear factors). Find $\mathcal{L}^{-1}\left[\frac{7s - 1}{(s + 1)(s + 2)(s - 3)}\right]$.

*Solution*

We write the expression in the form

$$\frac{7s - 1}{(s + 1)(s + 2)(s - 3)} = \frac{A}{s + 1} + \frac{B}{s + 2} + \frac{C}{s - 3}.$$  

Solving for the constants yields: $A = 2$, $B = -3$, and $C = 1$. Thus, we get

$$\mathcal{L}^{-1}\left[\frac{7s - 1}{(s + 1)(s + 2)(s - 3)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s + 1}\right] - \mathcal{L}^{-1}\left[\frac{3}{s + 2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s - 3}\right] = 2e^{-x} - 3e^{-2x} + e^3x.$$  

Example 6.34 (repeated linear factors). Find $\mathcal{L}^{-1}\left[\frac{s^2 + 9s + 2}{(s - 1)^2(s + 3)}\right]$.

*Solution*

We write the expression in the form

$$\frac{s^2 + 9s + 2}{(s - 1)^2(s + 3)} = \frac{A}{s - 1} + \frac{B}{(s - 1)^2} + \frac{C}{s + 3}.$$  

Solving for the constants yields: $A = 2$, $B = 3$, and $C = -3$. Thus, we get

$$\mathcal{L}^{-1}\left[\frac{s^2 + 9s + 2}{(s - 1)^2(s + 3)}\right] = \mathcal{L}^{-1}\left[\frac{2}{s - 1}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{(s - 1)^2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s + 3}\right] = 2e^{-x} + 3xe^{x} - e^{-3x}.$$
Example 6.35 (quadratic factors). Find \( \mathcal{L}^{-1}\left[ \frac{2s^2 + 10s}{s^2 - 2s + 5} (s + 1) \right] \).

**Solution**

We write the expression in the form

\[
\mathcal{L}^{-1}\left[ \frac{2s^2 + 10s}{s^2 - 2s + 5} (s + 1) \right] = \frac{A(s-1) + B}{(s-1)^2 + 4} + \frac{C}{s+1}.
\]

Solving for the constants yields: \( A = 3 \), \( B = 8 \), and \( C = -1 \). Thus, we get

\[
\mathcal{L}^{-1}\left[ \frac{2s^2 + 10s}{s^2 - 2s + 5} (s + 1) \right] = 3e^x \cos 2x + 4e^x \sin 2x - e^{-x}.
\]

6.4 Applications to Differential Equations

The easiest way to see how to apply Laplace transforms to differential equations is to work through some examples.

Example 6.36. Solve the following initial value problem:

\[
y'' - y' = 2x, \quad y(0) = 1, \quad y'(0) = -2.
\]

**Solution**

**Method 1** (the old approach)

First solve the homogeneous equation: \( y'' - y = 0. \)

\[
y = e^{rx} \quad \Rightarrow \quad r^2 - r = 0 \quad \Rightarrow \quad r = 0, 1 \quad \Rightarrow \quad y_h(x) = c_1 + c_2 e^x.
\]

Now look for a particular solution: \( y_p(x) = x(Ax + B) = Ax^2 + Bx. \) Plug into the DE to get

\[
y''_p - y'_p = 0 \quad \Rightarrow \quad \begin{cases} 2A - B = 0 \\ -2A = 2 \end{cases} \quad \Rightarrow \quad \begin{cases} A = -1 \\ B = -2 \end{cases} \quad \Rightarrow \quad y_p(x) = -x^2 - 2x.
\]

Thus, we have

\[
y(x) = c_1 + c_2 e^x - x^2 - 2x, \quad y'(x) = c_2 e^x - 2x - 2.
\]

Apply the initial conditions:

\[
\begin{cases} y(0) = 1 \\ y'(0) = -2 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 + c_2 = 1 \\ c_2 - 2 = -2 \end{cases} \quad \Rightarrow \quad \begin{cases} c_1 = 1 \\ c_2 = 0 \end{cases} \quad \Rightarrow \quad y(x) = 1 - x^2 - 2x.
\]

**Method 2** (using Laplace transforms)

Take Laplace transforms of the DE:

\[
\mathcal{L}[y''] - \mathcal{L}[y'] = 2\mathcal{L}[x] \quad \Rightarrow \quad \left[ s^2 Y(s) - sY(0) - y'(0) \right] - \left[ sY(s) - y(0) \right] = \frac{2}{s^2} \]

\[
\Rightarrow \quad s^2 Y(s) - s + 2 - sY(s) + 1 = \frac{2}{s^2} \]

\[
\Rightarrow \quad Y(s) = \frac{s^3 - 3s^2 + 2}{s^2(s-1)} = \frac{(s-1)(s^2 - 2s - 2)}{s^3(s-1)} = \frac{1}{s} - \frac{2}{s^2} - \frac{2}{s^3}
\]
Finally, taking the inverse Laplace transform, we arrive at the final solution:

\[ y(x) = \mathcal{L}^{-1}[Y(s)] = 1 - 2x - x^2. \]

**Example 6.37.** Solve the following initial value problem:

\[ y'' - 2y' + 5y = -8e^{-x}, \quad y(0) = 2, \quad y'(0) = 12. \]

**Solution**

Take Laplace transforms of the DE:

\[
\mathcal{L}[y''] - 2\mathcal{L}[y'] + 5\mathcal{L}[y] = -8\mathcal{L}[e^{-x}]
\]

\[ \Rightarrow [s^2Y(s) - sy(0) - y'(0)] - 2[sY(s) - y(0)] + 5Y(s) = \frac{2}{s^2} \]

\[ \Rightarrow s^2Y(s) - 2s - 12 - 2sY(s) + 4 + 5Y(s) = \frac{-8}{s+1} \]

\[ \Rightarrow Y(s) = \frac{2s^2 + 10s}{(s^2 - 2s + 5)(s + 1)} \]

\[ \Rightarrow Y(s) = \frac{3(s - 1)}{(s - 1)^2 + 4} - \frac{2}{(s - 1)^2 + 4} - \frac{1}{s + 1}. \]

Finally, taking the inverse Laplace transform, we arrive at the final solution:

\[ y(x) = 3e^x \cos 2x + 4e^x \sin 2x - e^{-x}. \]

**Example 6.38.** Solve the following initial value problem:

\[ y'' - 2y' + 5y = -8e^{7-x}, \quad y(7) = 2, \quad y'(7) = 12. \]

**Solution**

It appears that we can not use Laplace transforms since \( \mathcal{L}[y'] = sY(s) - y(0) \), and we don’t know \( y(0) \). But we can get around this by moving the initial point (in this case \( x_0 = 7 \)) to the origin by means of a translation.

Let \( t = x - 7 \) and \( w(t) = y(x) \). Then we get

\[ w'(t) = y'(x), \quad w''(t) = y''(x), \quad w(0) = y(7), \quad w'(0) = y'(7), \]

so, the initial value problem becomes

\[ w'' - 2w' + 5w = -8e^{-t}, \quad w(0) = 2, \quad w'(0) = 12. \]

This is just the initial value problem we had in Example 6.40. The solution is

\[ w(t) = 3e^t \cos 2t + 4e^t \sin 2t - e^{-t}. \]

The solution to the original problem is

\[ y(x) = w(x - 7) = 3e^{x-7} \cos[2(x - 7)] + 4e^{x-7} \sin[2(x - 7)] - e^{-(x-7)}. \]

Next we consider an initial value problem with discontinuous forcing.
Example 6.39. Solve the following initial value problem:

\[ y'' + 4y = g(x), \quad y(0) = 0, \quad y'(0) = 0, \quad \text{where } g(x) = \begin{cases} 
1 & 0 < x < 1 \\
-1 & 1 < x < 2 \\
0 & x > 2 
\end{cases}. \]

**Solution**

We can re-write \( g \) as follows: \( g(x) = 1[u_0(x) - u_1(x)] - 1[u_1(x) - u_2(x)] = u_0(x) - 2u_1(x) + u_2(x) \). Then we have

\[
G(s) = \mathcal{L}[g(x)] = \mathcal{L}[u_0(x)] - 2\mathcal{L}[u_1(x)] + \mathcal{L}[u_2(x)] = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s}.
\]

Take Laplace transforms of the DE:

\[
\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[g(x)] \quad \Rightarrow \quad \left[ s^2Y(s) - sy(0) - y'(0) \right] + 4Y(s) = G(s) \\
\Rightarrow \quad s^2Y(s) + 4Y(s) = G(s) \\
\Rightarrow \quad Y(s) = \frac{G(s)}{s^2 + 4} = \frac{1}{s(s^2 + 4)} - 2\frac{e^{-s}}{s(s^2 + 4)} + \frac{e^{-2s}}{s(s^2 + 4)} \\
\Rightarrow \quad Y(s) = F(s) - 2e^{-s}F(s) + e^{-2s}F(s),
\]

where

\[
F(s) = \frac{1}{s(s^2 + 4)} = \frac{1}{4} \left( \frac{1}{s} - \frac{s}{s^2 + 4} \right).
\]

Thus,

\[
f(x) = \mathcal{L}^{-1}[F(s)] = \frac{1}{4}(1 \cos 2x).
\]

Finally, taking the inverse Laplace transform, we arrive at the final solution:

\[
y(x) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[F(s)] - 2\mathcal{L}^{-1}[e^{-s}F(s)] + \mathcal{L}^{-1}[e^{-2s}F(s)] \\
= f(x) - 2u_1(x)f(x - 1) + u_2(x)f(x - 2) \\
= \frac{1}{4} \left\{ 1 - \cos 2x - 2u_1(x)(1 - \cos[2(x - 1)]) + u_2(x)(1 - \cos[2(x - 2)]) \right\}.
\]

Now we consider an ODE with variable coefficients.

**Example 6.40.** Solve the following initial value problem:

\[ y'' + 2xy' - 4y = 1, \quad y(0) = 0, \quad y'(0) = 0. \]

**Solution**

Take Laplace transforms of the DE:

\[
\mathcal{L}[y''] + 2\mathcal{L}[xy'] - 4\mathcal{L}[y] = \mathcal{L}[1] \\
\Rightarrow \quad \left[ s^2Y(s) - sy(0) - y'(0) \right] - 2 \frac{d}{ds} [sY(s) - y(0)] - 4Y(s) = \frac{1}{s} \\
\Rightarrow \quad s^2Y(s) - 2[sY'(s) + Y(s)] - 4Y(s) = \frac{1}{s} \\
\Rightarrow \quad Y''(s) + \left( \frac{3}{s} - \frac{s}{2} \right) Y(s) = -\frac{1}{2s^2}.
\]
This is a linear ODE in \(Y(s)\). Look for an integrating factor \(\mu\):

\[
\frac{\mu'}{\mu} = \frac{3}{s} - \frac{s}{2} \implies \ln \mu = 3 \ln s - \frac{s^2}{4} \implies \mu = s^3 e^{-s^2/4}.
\]

The ODE for \(Y\) becomes:

\[
\frac{d}{ds}[\mu Y(s)] = \mu \left( Y'(s) + \frac{\mu'}{\mu} Y(s) \right) = -\frac{s}{2} e^{-s^2/4}.
\]

Integrating yields:

\[
\mu Y(s) = -\int \frac{s}{2} e^{-s^2/4} ds = e^{-s^2/4} + C \implies Y(s) = \frac{1}{s^3} + \frac{C}{s^3} e^{s^2/4}.
\]

It remains to determine the value of the constant if integration \(C\). There is no auxiliary condition inherited from the original ODE. To get an appropriate condition to enable us to determine \(C\), we utilize Theorem 6.12 that states that \(Y(s) \to 0\) as \(s \to \infty\). Therefore

\[
\lim_{s \to \infty} Y(s) = 0 \implies C = 0 \implies Y(s) = \frac{1}{s^3}.
\]

Finally, taking the inverse Laplace transform, we arrive at the final solution:

\[
y(x) = \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{x^2}{2}.
\]

We can summarize the application of Laplace transforms to differential equations as follows.

1. For an ODE with constant coefficients, the equation for the Laplace transform is of the form \(AY = B\).
2. For an ODE with polynomial coefficients in \(x\), the equation for the Laplace transform is an ODE with polynomial coefficients in \(s\).
3. If the coefficient functions of the ODE are linear in \(x\), the ODE for \(Y\) is a first order ODE.
4. The auxiliary condition to use when solving a first order ODE in \(Y\) is \(\lim_{s \to \infty} Y(s) = 0\).

### 6.5 Convolution

Consider the following initial value problem:

\[
y'' + y = g(x), \quad y(0) = y'(0) = 0.
\]

Take the Laplace transform of the equation to get:

\[
s^2 Y(s) - sy(0) - y'(0) + Y(s) = G(s) \implies Y(s) = \frac{G(s)}{s^2 + 1} = F(s)G(s), \quad \text{where} \quad F(s) = \frac{1}{s^2 + 1}.
\]

We would like to express the solution \(y(x)\) in terms of \(f(x)\) and \(g(x)\), i.e. we would like to express \(\mathcal{L}^{-1}[F(s)G(s)]\) in terms of \(\mathcal{L}^{-1}[F(s)]\) and \(\mathcal{L}^{-1}[G(s)]\). To do this, we define a special type of product of functions. Let \(f, g \in PC(0, \infty)\).
Definition 6.41. The convolution of $f$ and $g$, denoted $f \ast g$, is defined as:

$$(f \ast g)(x) := \int_0^x f(x-t)g(t) \, dt.$$ 

Example 6.42.

1. $1 \ast x = \int_0^x t \, dt = \frac{x^2}{2}$.
2. $x \ast x^2 = \int_0^x (x-t) \cdot t^2 \, dt = \frac{x^4}{12}$.

Theorem 6.43. The convolution product satisfies the following properties:

1. $f \ast g = g \ast f$; (convolution product is commutative)
2. $f \ast (g+h) = f \ast g + f \ast h$; (convolution product is distributive over addition)
3. $f \ast (g \ast h) = (f \ast g) \ast h$; (convolution product is associative)
4. $f \ast 0 = 0$.

Proof. Exercise.

Remark. While the convolution product has many of the properties of ordinary multiplication of functions, it is different in that it has no multiplicative identity element, i.e. there is no function $g$ with the property that $g \ast f \neq f$ for all functions $f$. □

Theorem 6.44. If

(i) $f, g \in PC(0, \infty)$;
(ii) $F(s) = \mathcal{L}[f(x)]$ and $G(s) = \mathcal{L}[g(x)]$, then

$$\mathcal{L}[f \ast g] = F(s)G(s), \quad \text{or equivalently} \quad \mathcal{L}^{-1}[F(s)G(s)] = (f \ast g)(x).$$

Proof. 

$$\mathcal{L}[f \ast g] = \int_0^\infty e^{-sx}(f \ast g)(x) \, dx = \int_0^\infty e^{-sx} \int_0^x f(x-t)g(t) \, dt \, dx$$

$$= \int_0^\infty \int_t^\infty e^{-sx} f(x-t)g(t) \, dx \, dt = \int_0^\infty \int_0^\xi e^{-s(\xi+t)} f(\xi)g(t) \, d\xi \, dt$$

$$= \left( \int_0^\infty e^{-s\xi} f(\xi) \, d\xi \right) \left( \int_0^\infty e^{-st}g(t) \, dt \right) = F(s)G(s).$$

Example 6.45. Find $\mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 4)}\right]$.

Solution

Method 1 (the old approach)

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 4)}\right] = \mathcal{L}^{-1}\left[\frac{1}{4} \left( \frac{1}{s} - \frac{s}{s^2 + 4} \right) \right] = \frac{1}{4}(1 - \cos 2x).$$

Method 2 (using convolution)
\begin{align*}
\mathcal{L}^{-1}\left[\frac{1}{s(s^2 + 4)}\right] &= \frac{1}{2} \mathcal{L}^{-1}\left[\frac{1}{s}\right] \ast \mathcal{L}^{-1}\left[\frac{2}{s^2 + 4}\right] = \frac{1}{2} \int_0^x \mathcal{L}^{-1}\left[\frac{1}{s}\right](x-t) \mathcal{L}^{-1}\left[\frac{2}{s^2 + 4}\right](t) \, dt \\
&= \frac{1}{2} \int_0^x 1 \cdot \sin 2t \, dt = -\frac{1}{4} \cos 2t \bigg|_0^x = \frac{1}{4} (1 - \cos 2x).
\end{align*}

Now, returning to the origin problem:

\[ y'' + y = g(x), \quad y(0) = y'(0) = 0. \]

The solution satisfies:

\[ y(x) = \mathcal{L}^{-1}\left[\frac{G(s)}{s^2 + 1}\right] = g \ast x = \int_0^x g(t) \sin(x-t) \, dt. \]

**Example 6.46.** Solve the following integro-differential equation:

\[ y'(x) = 1 - e^{-2x} \int_0^x y(t)e^{2t} \, dt, \quad y(0) = 1. \]

**Solution**

The equation may be written as

\[ y' = 1 - \int_0^x j(t)e^{-2(x-t)} \, dt = 1 - y \ast e^{-2x}. \]

Taking Laplace transforms we get

\[ sY(s) - y(0) = \frac{1}{s} - \frac{Y(s)}{s+2} \quad \Rightarrow \quad Y(s) = \frac{s+2}{s(s+1)} = \frac{2}{s} - \frac{1}{s+1} \quad \Rightarrow \quad y(x) = 2 - e^{-x}. \]

### 6.6 The Delta Function

Mechanical systems are often acted upon by external forces that are large in magnitude but are of short duration. Some typical examples are: a hammer hitting a nail; a bat hitting a baseball. These forces are of the form

\[ F(t) = \begin{cases} 
0 & t < t_0 \\
\cdots & t_0 \leq t \leq t_1 \\
0 & t > t_1
\end{cases}. \]

![Figure 6.4: A plot of a localized force y = F(t).](image)
The integral of the force acting over an interval of time is called “impulse:” \( I = \int_{t_0}^{t_1} F(t) \, dt \). From Newton’s Law, \( F = ma \), we get

\[
I = \int_{t_0}^{t_1} F(t) \, dt = \int_{t_0}^{t_1} m \frac{dv}{dt} \, dt = mv(t_1) - mv(t_0) \quad \text{(i.e. impulse = change in momentum)}.
\]

Consider an impulse over shorter and shorter time intervals.

![Diagram showing impulse and force over time intervals](image_url)

Figure 6.5: A plot of a localized force and impulse for decreasing time intervals.

The exact nature of the force in the interval \([t_0, t_1]\) is frequently unknown. What usually is known is the state of the system before and after the application of the force. These duration of the force is often so short, that it is convenient to think of the force as acting instantaneously.
We define the following functions:

\[ \delta_h(t) = \begin{cases} 
\frac{1}{2h} & |t| \leq h \\
0 & |t| > h 
\end{cases} \quad \text{for } h \geq 0. \]

These functions have the following property:

\[ \int_{-\infty}^{\infty} \delta_h(t) \, dt = \int_{-h}^{h} \frac{1}{2h} \, dt = \frac{1}{2h} (h - (-h)) = 1. \]

**Definition 6.47.** The *Dirac Delta Function* is defined implicitly by the following properties:

1. \( \delta(t) = 0 \) for \( t \neq 0 \);
2. \( \int_{-\infty}^{\infty} \delta(t) \, dt = 1. \)

**Remark.** The Dirac delta function is not a function in the usual sense. However, one can “think” of it as follows:

\[ \delta(t) = \lim_{h \to 0} \delta_h(t) = \begin{cases} 
\infty & t = 0 \\
0 & t \neq 0 
\end{cases}. \]
Theorem 6.48. 
If $f$ is continuous on $(-\infty, \infty)$, then for any $c \in \mathbb{R}$,
\[
\int_{-\infty}^{\infty} f(t) \delta(t-c) \, dt = f(c).
\]

Proof 
We have
\[
\delta_h(t-c) = \begin{cases} 
\frac{1}{2h} & |t-c| \leq h \\
0 & |t-c| > h
\end{cases} = \begin{cases} 
\frac{1}{2h} & c-h \leq t \leq c+h \\
0 & |t-c| > h
\end{cases}.
\]

Therefore
\[
\int_{-\infty}^{\infty} f(t) \delta(t-c) \, dt = \lim_{h \to 0} \int_{-\infty}^{\infty} f(t) \delta_h(t-c) \, dt = \lim_{h \to 0} \frac{1}{2h} \int_{c-h}^{c+h} f(t) \, dt
\]
\[
= \lim_{h \to 0} \frac{1}{2h} f(\bar{c})(2h) \quad \text{for some } c-h < \bar{c} < c+h \quad \text{(using the mean value theorem)}
\]
\[
= \lim_{h \to 0} f(\bar{c}) = f(c).
\]

This is sometimes called the “sifting property” of the delta function since, from all possible values of $f$, it “sifts out” the value $f(c)$.

Does the Dirac delta function have a Laplace transform? Yes it does, since
\[
\mathcal{L}[\delta(t-c)] = \int_{0}^{\infty} e^{-st} \delta(t-c) \, dt = e^{-sc}.
\]

Remark. Notice that for $c = 0$ we get $\mathcal{L}[\delta(t)] = 1$. This reinforces what was said earlier, namely that $\delta(t)$ is not a function in the usual sense, since, for a normal function we have
\[
\lim_{s \to \infty} F(s) = 0, \quad \text{but for the delta function we have} \quad \lim_{s \to \infty} \mathcal{L}[\delta(t)] \neq 0.
\]

Example 6.49. Solve the following initial value problem:
\[
y'' + y = 4\delta(t-2\pi), \quad y(0) = y'(0) = 0.
\]

Solution 
Taking Laplace transforms we get,
\[
s^2Y(s) - sy(0) - y'(0) + Y(s) = 4e^{-2\pi s} \quad \implies \quad Y(s) = \frac{4e^{-2\pi s}}{s^2 + 1}.
\]
Taking inverse Laplace transforms we get
\[
y(t) = u_{2\pi}(t) \sin(t - 2\pi) = u_{2\pi}(t) \sin t = \begin{cases} 
0 & 0 \leq t < 2\pi \\
\sin t & t \geq 2\pi
\end{cases}.
\]