False theta functions and the Verlinde formula

Thomas Creutzig and Antun Milas

May 22, 2014

Abstract

We discover new analytic properties of classical partial and false theta functions and their potential applications to representation theory of $\mathcal{W}$-algebras and vertex algebras in general. More precisely, motivated by clues from conformal field theory, first, we are able to determine modular-like transformation properties of *regularized* partial and false theta functions. Then, after suitable identification of regularized partial/false theta functions with the characters of atypical modules for the singlet vertex algebra $\mathcal{W}(2,2p-1)$, we formulate a Verlinde-type formula for the fusion rules of irreducible $\mathcal{W}(2,2p-1)$-modules.

1 Introduction: partial and false theta functions

In this paper, we are primarily concerned with modular-like transformation properties of functions

$$P_{a,b}(u,\tau)=\sum_{n=0}^{\infty} z^{n+\frac{b}{2a}} q^{a(n+\frac{b}{2a})^2}, \quad q=e(\tau), \quad z=e(u), \tag{1.1}$$

where $a,b\in\mathbb{N}$, $\tau\in\mathbb{H}$, $u\in\mathbb{C}$, called *partial theta functions* [AB][1]. If z is specialized to be q^c, we call them partial theta series. The names explain themselves as the usual Jacobi theta functions/series are also given by (1.1), but with the summation over all integers. We will also be interested in closely related series called *false theta series*, where the summation is over $\mathbb{Z}$, but the sign choice does not correspond to any specialization of the theta function. A typical example is

$$\sum_{n\in\mathbb{Z}} \operatorname{sgn}(n) q^{a(n+\frac{b}{2a})^2}, \tag{1.2}$$

which plays a prominent role in our work. Actually, the only false theta series appearing in this paper are simply differences of two partial thetas.

While theta functions of course enjoy modular transformation properties, and are related to numerous concepts in mathematics and theoretical physics, partial/false theta functions do not seem to have any modular properties and their relevance in mathematics is somewhat obscure and random. Most prominently, false/partial thetas appear in various identities involving q-hypergeometric series and even some partition identities (for a thorough selection of results see [AB] and references therein). They also seem to arise in computation of topological invariants. For example, in [LZ],

[1] In [AB] a slightly different definition was used.

the (rescaled) Witten-Reshetikhin-Turaev (WRT) invariant associated to the homology spheres was studied as radial limiting value of certain partial/false theta series. Also, it was recently shown in [GL] (see Section 14) that the generating series of colored Jones polynomials for alternating knots are given by (1.2). But as far as we know, there were no serious attempts to relate false theta functions to concepts in infinite-dimensional representation theory and to ideas in conformal field theory (eg. Verlinde formula). We also point out that partial/false theta functions are *not* mock theta functions as studied in [Z], although there is a connection (see [Za] for instance).

In this paper we take a radical different point of view to (1.1) and eventually (1.2). Our starting observation is that some of the series discussed earlier are essentially (i.e. up to Dedekind η-function factor) graded dimensions of modules for the vertex operator algebra $\mathcal{W}(2, 2p-1)$, also called the singlet algebra (this was also noticed by Flohr in [Fl]). This vertex algebra did not attract so much attention, primarily because it is not C_2-cofinite, although it is instrumental for studying more interesting triplet vertex algebra [FHST], [FGST1], [FGST2], [NT], [TW], [AdM1], [CF], [Fl], etc. Characters of modules for the triplet are well-understood; they can be organized so that they form a vector-valued (logarithmic) modular form. Using this approach a Verlinde-type formula can be also obtained [FHST] (in the rational setup see [Ve], and the proof of the Verlinde conjecture by Huang [Hu]). Thus, it is very natural to ask: (i) Do irreducible characters of the singlet algebra also obey modular-like transformations properties, and (ii) is there a Verlinde-type formula for irreps based on these properties? Of course, because we have infinitely many irreps, in modular transformation formulas we also allow integral part as in other works on "continuous" Verlinde-type formulas [AC], [BR], [CR1], [CR2], [CR3]. In order to describe the family of singlet algebras $\mathcal{W}(2, 2p-1)$ parameterized by integers $p \geq 2$ we introduce the numbers $\alpha_+ = \sqrt{2p}, \alpha_- = -\sqrt{2/p}$ and $\alpha_0 = \alpha_+ + \alpha_-$. For our purposes we first note that the singlet algebra admits two types of $\mathbb{Z}_{\geq 0}$-graded representations:

(1) generic (or typical). Fock space representations F_λ, with the character

$$\mathrm{ch}[F_\lambda](\tau) = \frac{q^{\frac{1}{2}(\lambda - \alpha_0/2)^2}}{\eta(\tau)}; \quad \lambda \in \mathbb{C}$$

and $\eta(\tau) = q^{\frac{1}{24}} \prod_{i=1}^{\infty}(1 - q^i)$ is the usual Dedekind eta-function.

(2) Non-generic (or atypical). Certain subquotients or reducible Fock spaces, denoted by $M_{r,s}$, with the character

$$\mathrm{ch}[M_{r,s}](\tau) = \frac{P_{p,pr-s}(0,\tau) - P_{p,pr+s}(0,\tau)}{\eta(\tau)},$$

where $P_{a,b}$ is as in (1.1). The range is $r \in \mathbb{Z}$ and $1 \leq s \leq p$ [2].

Modular transformation properties of (1) are easily computed via Gauss' integral, but those of (2) are much more delicate. As a remedy, we introduce the ϵ-regularized characters $\mathrm{ch}[X^\epsilon]$ (see Section 2 for details; see also [Fl]) such that $\lim_{\epsilon \to 0} \mathrm{ch}[X^\epsilon] = \mathrm{ch}[X]$. These regularized atypical characters transform much nicer as illustrated in our first main results of the paper.

Theorem 1 *The modular S-transformation of atypical characters is*

$$ch[M^\epsilon_{r,s}]\Big(-\frac{1}{\tau}\Big) = \int_{\mathbb{R}} S^\epsilon_{(r,s),\mu+\alpha_0/2} \mathrm{ch}[F^\epsilon_{\mu+\alpha_0/2}](\tau) d\mu + X^\epsilon_{r,s}(\tau)$$

[2]Strictly speaking $M_{r,p}$ modules can be also viewed as typical representations. For a precise definition of typical/atypical see Definition 18.

with

$$S^{\epsilon}_{(r,s),\mu+\alpha_0/2} = -e^{-2\pi\epsilon((r-1)\alpha_+/2+\mu)} e^{\pi i(r-1)\alpha_+\mu} \frac{\sin\big(\pi s\alpha_-(\mu+i\epsilon)\big)}{\sin\big(\pi\alpha_+(\mu+i\epsilon)\big)}$$

and

$$X^{\epsilon}_{r,s}(\tau) = \frac{1}{4\eta(\tau)}(\mathrm{sgn}(\mathrm{Re}(\epsilon))+1)\sum_{n\in\mathbb{Z}}(-1)^{rn} e^{\pi i\frac{s}{p}n} q^{\frac{1}{2}(\frac{n^2}{\alpha_+^2}-\epsilon^2)}\big(q^{-i\epsilon\frac{n}{\alpha_+}} - q^{i\epsilon\frac{n}{\alpha_+}}\big).$$

Although this result concerns certain characters of modules it relies on another key result for the partial theta function (see Theorem 4). Interestingly, the previous theorem does not provide us with the usual modular transformation properties one would expect in non-rational theories due to the theta-like term $X^{\epsilon}_{r,s}(\tau)$ that has no obvious interpretation as a regularized character. Surprisingly, the term disappears for $\mathrm{Re}(\epsilon) < 0$, which we assume to hold. By using the above S-transformation formulas we define a suitable product on the space of characters by mimicking the Verlinde formula for the fusion rules. We obtain the following result

Theorem 2 *(Verlinde-type formula) With parametrization of irreps as in Section 4, and with multiplication of regularized characters as in (4.5), we have*

$$\begin{aligned}
\mathrm{ch}[F^{\epsilon}_{\lambda}] \times \mathrm{ch}[F^{\epsilon}_{\mu}] &= \sum_{\ell=0}^{p-1} \mathrm{ch}[F^{\epsilon}_{\lambda+\mu+\ell\alpha_-}] \\
\mathrm{ch}[M^{\epsilon}_{r,s}] \times \mathrm{ch}[F^{\epsilon}_{\mu}] &= \sum_{\substack{\ell=-s+2\\ \ell+s=0 \bmod 2}}^{s} \mathrm{ch}[F^{\epsilon}_{\mu+\alpha_{r,\ell}}] \\
\mathrm{ch}[M^{\epsilon}_{r,s}] \times \mathrm{ch}[M^{\epsilon}_{r',s'}] &= \sum_{\substack{\ell=|s-s'|+1\\ \ell+s+s'=1 \bmod 2}}^{min\{s+s'-1,p\}} \mathrm{ch}[M^{\epsilon}_{r+r'-1,\ell}] \\
&+ \sum_{\substack{\ell=p+1\\ \ell+s+s'=1 \bmod 2}}^{s+s'-1} \Big(\mathrm{ch}[M^{\epsilon}_{r+r'-2,\ell-p}] + \mathrm{ch}[M^{\epsilon}_{r+r'-1,2p-\ell}] + \mathrm{ch}[M^{\epsilon}_{r+r',\ell-p}]\Big).
\end{aligned}$$

The algebra structure on the integer span of characters given by this Verlinde-type formula we also call Verlinde algebra (of characters). Notice that the product does not depend on the regularization parameter ϵ. Independence of the choice of this parameter for the final answer is exactly the requirement for a good regularization in mathematical physics.

At last, in parallel with the triplet algebra [FGST1, FGST2], and partially motivated by [KL], we expect the category of (ordinary) $\mathcal{W}(2,2p-1)$-modules to be equivalent to the category of finite-dimensional representations for a certain infinite-dimensional quantum group. This conjecture is amplified with the computation of regularized quantum dimensions in Section 4.3 (see Theorem 28). We hope to return to the problem of identifying the relevant quantum group in our future publications.

David Ridout and Simon Wood have informed us that they are preparing a manuscript on the Verlinde formula of the (p,p')-singlet algebra [RW]. Instead of an analytic approach they follow the strategy of the previous works on the Verlinde formula, see e.g. [CR4] for an introduction and the example of $\mathcal{W}(2,3)$, of resolving atypical modules in terms of typical ones. They find agreement with our Verlinde formula when restricting to the case of $\mathcal{W}(2, 2p-1)$. Very recently, in [CMW], we extended methods of this paper to obtain rigorous derivation of results in [RW].

Acknowledgements: We thank K. Bringmann on some discussions related to this paper. We also thank D. Ridout and S. Wood for very useful discussion, suggestions and (S.W.) for pointing out some inconsistencies in a previous version of the paper. T.C. appreciates that D.R. shared some computations on the singlet algebra.

2 Modularity of regularized partial theta functions

In this part we study modular-like transformation properties of a partial theta function, which can be used to express more complicated partial and eventually false theta functions. Let

$$P(u,\tau) = \sum_{k \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} z^k q^{k^2/2}.$$

where $q = e(\tau)$, $\tau \in \mathbb{H}$, the upper half-plane, and $z = e(u)$, where $u \in \mathbb{C}$. This function is obviously holomorphic. If one tries to compute modular transformation properties under $(u,\tau) \mapsto (\frac{u}{\tau}, -\frac{1}{\tau})$, some divergent integrals quickly appear. To fix this inconvenience we use a method sometimes used in the physics literature, and in particular in [Fl], called *regularization*. The idea is to deform the "charge" variable u by introducing an additional parameter denoted by ϵ. On one hand ϵ can be viewed as the contour deformation parameter, but also as a quantum group parameter (the two seem to be connected). In this paper we do not try to make this connection precise, leaving it for future considerations.

Definition 3 Let $\epsilon \in \mathbb{C} \setminus i\mathbb{R}$, then the partial regularized theta function is

$$P_\epsilon(u,\tau) = \sum_{k \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} z^k e^{2\pi\epsilon k} q^{k^2/2}.$$

The main result of this section will be

Theorem 4 *Let $\epsilon \in \mathbb{C} \setminus i\mathbb{R}$, then the modular properties of the regularized partial theta function are:*

$$P_\epsilon(u,\tau+1) = e^{\pi i/4} P_\epsilon(u,\tau),$$
$$P_\epsilon((u/\tau, -1/\tau) = \frac{e^{\pi i u^2/\tau}\sqrt{-i\tau}}{2}\Big(-i\int_{\mathbb{R}} \frac{q^{x^2/2} z^x}{\sin(\pi(x+i\epsilon))}\, dx + \frac{1}{2}(\mathrm{sgn}(\mathrm{Re}(\epsilon))+1)\vartheta_{4,\epsilon}(u,\tau)\Big),$$

with the regularized ordinary theta function

$$\vartheta_{4,\epsilon}(u,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n z^{n-i\epsilon} q^{(n-i\epsilon)^2/2}.$$

The modular T-transformation ($\tau \mapsto \tau+1$) is obvious, and the rest of this section will be devoted to proving the modular S-transformation of the theorem.

Let us start with its elliptic transformation properties.

Proposition 5 *The elliptic transformations of* P_ϵ *are*

$$P_\epsilon(u+1,\tau) = -P_\epsilon(u,\tau),$$
$$P_\epsilon(u+\tau,\tau) = z^{-1}q^{-1/2}e^{-2\pi\epsilon}P_\epsilon(u,\tau) - z^{-1/2}q^{-3/8}e^{-\pi\epsilon}\,.$$

Proof A straightforward rewriting of the sums. □

As a corollary, we get

Corollary 6 *Let* $\tilde{u} = u/\tau$ *and* $\tilde{\tau} = -1/\tau$*, then*

$$P_\epsilon((u+1)/\tau, -1/\tau) = \tilde{z}\tilde{q}^{-1/2}e^{2\pi\epsilon}P_\epsilon(\tilde{u},\tilde{\tau}) + \tilde{z}^{1/2}\tilde{q}^{-3/8}e^{\pi\epsilon}$$
$$P_\epsilon((u+\tau)/\tau, -1/\tau) = -P_\epsilon(\tilde{u},\tilde{\tau})\,.$$

Proof This follows from Proposition 5 as follows. Let $\tilde{u} = u/\tau$, $\tilde{\tau} = -1/\tau$ and $u' = \tilde{u} - \tilde{\tau}$ and correspondingly $\tilde{z} = e^{2\pi i\tilde{u}}, \tilde{q} = e^{2\pi i\tilde{\tau}}, z' = e^{2\pi i u'}$, then

$$P_\epsilon((u+1)/\tau, -1/\tau) = P_\epsilon(\tilde{u}-\tilde{\tau},\tilde{\tau}) = P_\epsilon(u',\tilde{\tau})$$

and hence with Proposition 5

$$P_\epsilon(u',\tilde{\tau}) = z'\tilde{q}^{1/2}e^{2\pi\epsilon}P_\epsilon(u'+\tilde{\tau},\tilde{\tau}) + z'^{1/2}\tilde{q}^{1/8}e^{\pi\epsilon} = \tilde{z}\tilde{q}^{-1/2}e^{2\pi\epsilon}P_\epsilon(\tilde{u},\tilde{\tau}) + \tilde{z}^{1/2}\tilde{q}^{-3/8}e^{\pi\epsilon}.$$

The second equation is obvious. □

Proposition 7 *Let* $\gamma(u,\tau) = e^{-\pi i u^2/\tau}$*, then* $f_\epsilon(u,\tau) = \gamma(u,\tau)P_\epsilon(\tilde{u},\tilde{\tau})$ *satisfies*

$$f_\epsilon(u,\tau) - e^{-2\pi\epsilon}f_\epsilon(u+1,\tau) \;=\; -e^{-\pi\epsilon}\gamma(u+1/2,\tau) \qquad (2.1)$$
$$f_\epsilon(u,\tau) + zq^{1/2}f_\epsilon(u+\tau,\tau) \;=\; 0\,. \qquad (2.2)$$

Proof This follows from corollary 6. □

Lemma 8 *Let* $\tilde{z} = e(u/\tau)$*,* $\tilde{q} = e(-1/\tau)$*, then we have*

$$\tilde{z}^k\tilde{q}^{\frac{k^2}{2}} \;=\; \sqrt{-i\tau}e^{\frac{\pi i u^2}{\tau}}\int_{-\infty}^{\infty} q^{\frac{w^2}{2}}z^w e^{-2\pi i w k}\,dw\,.$$

Proof This is a Gauss integral. □

Proposition 9 *Let*

$$h_\epsilon(u,\tau) \;=\; -\frac{i\sqrt{-i\tau}}{2}\int_{\mathbb{R}}\frac{q^{x^2/2}z^x}{\sin(\pi(x+i\epsilon))}\,dx,$$

then h_ϵ *satisfies*

$$h_\epsilon(u,\tau) - e^{-2\pi\epsilon}h_\epsilon(u+1,\tau) \;=\; -e^{-\pi\epsilon}\gamma(u+1/2,\tau) \qquad (2.3)$$
$$h_\epsilon(u,\tau) + zq^{1/2}h_\epsilon(u+\tau,\tau) \;=\; 0\,. \qquad (2.4)$$

Also note, that since $\epsilon \notin i\mathbb{R}$*, the integral is absolutely convergent for all* $u \in \mathbb{C}, \tau \in \mathbb{H}$*.*

Proof The left-hand side of the first equation is a Gauss integral, so the equation follows with Lemma 8. The second equation follows with the substitution $x \to x+1$ in the second integral. $\square$

We define the correction

$$R_\epsilon = f_\epsilon - h_\epsilon.$$

Proposition 10 *The correction term is holomorphic and satisfies the following functional equation*

$$R_\epsilon(u,\tau) - e^{-2\pi\epsilon} R_\epsilon(u+1,\tau) \;=\; 0 \tag{2.5}$$

$$R_\epsilon(u,\tau) + z q^{1/2} R_\epsilon(u+\tau,\tau) \;=\; 0\,. \tag{2.6}$$

Proof Follows from Proposition 7 and 9. $\square$

Proposition 11 *The Jacobi-theta-like function (for fixed $\epsilon \in \mathbb{C}$)*

$$\vartheta_{4,\epsilon}(u,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n z^{n-i\epsilon} q^{(n-i\epsilon)^2/2}$$

satisfies

$$\vartheta_{4,\epsilon}(u,\tau) - e^{-2\pi\epsilon} \vartheta_{4,\epsilon}(u+1,\tau) \;=\; 0 \tag{2.7}$$

$$\vartheta_{4,\epsilon}(u,\tau) + z q^{1/2} \vartheta_{4,\epsilon}(u+\tau,\tau) \;=\; 0 \tag{2.8}$$

and up to a scalar multiple it is the unique holomorphic function in u with this property.

Proof Well-known fact for standard theta-functions, but $q^{-\epsilon^2/2} z^{-i\epsilon} \vartheta_{4,0}(u - i\epsilon\tau, \tau) = \vartheta_{4,\epsilon}(u,\tau)$ can be expressed in terms of the standard Jacobi-theta function $\vartheta_{4,0}(u,\tau)$. $\square$

We define $\alpha_\epsilon(\tau)$ by $R_\epsilon(u,\tau) = \alpha_\epsilon(\tau)\vartheta_{4,\epsilon}(u,\tau)$.

Proposition 12 $\alpha_\epsilon(\tau) = \alpha_\epsilon \sqrt{-i\tau}$ *where α_ϵ is independent of τ.*

Proof Define

$$\Delta = \frac{1}{\pi i}\partial_\tau - \frac{1}{(2\pi i)^2}\partial_u^2,$$

then we compute that

$$\Delta \frac{f_\epsilon(u,\tau)}{\sqrt{-i\tau}} \;=\; \Delta \frac{h_\epsilon(u,\tau)}{\sqrt{-i\tau}} \;=\; \Delta \vartheta_{4,\epsilon}(u,\tau) \;=\; 0$$

now, let $\sqrt{-i\tau}\beta_\epsilon(\tau) = \alpha_\epsilon(\tau)$, then the statement implies $\partial_\tau \beta_\epsilon(\tau) = 0$. $\square$

Proposition 13 *There exists $\alpha \in \mathbb{C}$, such that*

$$\alpha_\epsilon \;=\; \begin{cases} \alpha & \text{if } \mathrm{Re}(\epsilon) > 0 \\ \alpha - 1 & \text{if } \mathrm{Re}(\epsilon) < 0 \end{cases}.$$

Proof Let $\mu \in \mathbb{C}$ with $\mathrm{Re}(\epsilon) \neq -\mathrm{Im}(\mu)$, then rewriting the appropriate sums yields

$$z^\mu q^{\mu^2/2} \vartheta 4, \epsilon - i\mu(u + \mu\tau, \tau) = \vartheta_{4,\epsilon}(u, \tau) \qquad , \qquad z^\mu q^{\mu^2/2} f_{\epsilon - i\mu}(u + \mu\tau, \tau) = f_\epsilon(u, \tau).$$

With the substitution $y = x + \mu$, we get

$$z^\mu q^{\mu^2/2} h_{\epsilon-i\mu}(u + \mu\tau, \tau) = -\frac{i\sqrt{-i\tau}}{2} \int_{\mathbb{R}} \frac{q^{x^2/2+\mu x+\mu^2/2} z^{x+\mu}}{\sin(\pi(x + \mu + i\epsilon))} dx = -\frac{i\sqrt{-i\tau}}{2} \int_{\mathbb{R}+\mu} \frac{q^{y^2/2} z^y}{\sin(\pi(y + i\epsilon))} dy.$$

Since τ is in the upper half plane, we can thus write the difference $h_\epsilon(u, \tau) - z^\mu q^{\mu^2/2} h_{\epsilon-i\mu}(u+\mu\tau, \tau)$ as a contour integral over the contour $\mathcal{C}_\mu$ which connects the two areas of integration of the previous equation,

$$h_\epsilon(u, \tau) - z^\mu q^{\mu^2/2} h_{\epsilon-i\mu}(u + \mu\tau, \tau) \ = \ -\frac{i\sqrt{-i\tau}}{2} \int_{\mathcal{C}_\mu} \frac{q^{x^2/2} z^x}{\sin(\pi(x + i\epsilon))}\, dx.$$

Let $\epsilon' = \epsilon - i\mu$, we can then evaluate the integral by the residuum theorem and get

$$\begin{aligned} -\frac{i\sqrt{-i\tau}}{2} \int_{\mathcal{C}_\mu} \frac{q^{x^2/2} z^x}{\sin(\pi(x + i\epsilon))}\, dx \ &= \ \sqrt{-i\tau} \sum_{n\in\mathbb{Z}} (-1)^n z^{n-i\epsilon} q^{(n-i\epsilon)^2/2} \begin{cases} 0 & \text{if } \mathrm{Re}(\epsilon) > 0\ ,\ \mathrm{Re}(\epsilon') > 0 \\ 1 & \text{if } \mathrm{Re}(\epsilon) > 0\ ,\ \mathrm{Re}(\epsilon') < 0 \\ -1 & \text{if } \mathrm{Re}(\epsilon) < 0\ ,\ \mathrm{Re}(\epsilon') > 0 \\ 0 & \text{if } \mathrm{Re}(\epsilon) < 0\ ,\ \mathrm{Re}(\epsilon') < 0 \end{cases} \\ &= \ \sqrt{-i\tau}\vartheta_{4,\epsilon}(u, \tau) \begin{cases} 0 & \text{if } \mathrm{Re}(\epsilon) > 0\ ,\ \mathrm{Re}(\epsilon') > 0 \\ 1 & \text{if } \mathrm{Re}(\epsilon) > 0\ ,\ \mathrm{Re}(\epsilon') < 0 \\ -1 & \text{if } \mathrm{Re}(\epsilon) < 0\ ,\ \mathrm{Re}(\epsilon') > 0 \\ 0 & \text{if } \mathrm{Re}(\epsilon) < 0\ ,\ \mathrm{Re}(\epsilon') < 0 \end{cases} \end{aligned}$$

but this implies

$$\alpha_\epsilon \ = \ \begin{cases} \alpha & \text{if } \mathrm{Re}(\epsilon) > 0 \\ \alpha - 1 & \text{if } \mathrm{Re}(\epsilon) < 0 \end{cases}$$

for some $\alpha \in \mathbb{C}$. □

It remains to compute α.

Definition 14 We define the regularized false theta function

$$F\vartheta_\epsilon(u, \tau) = P_\epsilon(u, \tau) - P_{-\epsilon}(-u, \tau).$$

Proposition 15 *The modular S-transformation of the regularized false theta function is*

$$F\vartheta_\epsilon(u/\tau, -1/\tau) = e^{\pi i u^2/\tau} \sqrt{-i\tau} \Big(-i \int_{\mathbb{R}} \frac{q^{x^2/2} z^x}{\sin(\pi(x + i\epsilon))}\, dx + \mathrm{sgn}(\mathrm{Re}(\epsilon)) \vartheta_{4,\epsilon}(u, \tau) \Big)$$

Proof This follows directly from $h_{-\epsilon}(-u, \tau) = -h_\epsilon(u, \tau)$, $\vartheta_{4,-\epsilon}(-u, \tau) = \vartheta_{4,\epsilon}(u, \tau)$, $\alpha_\epsilon - \alpha_{-\epsilon} = \mathrm{sgn}(\mathrm{Re}(\epsilon))$ and the expression for $\alpha_\epsilon(\tau)$ that is Proposition 12 and 13. □

Proposition 16 $\alpha = 1$

Proof Define $\vartheta_{2,\epsilon}(u,\tau) = \sum_{k\in\mathbb{Z}+\frac{1}{2}} z^k e^{2\pi\epsilon k} q^{k^2/2}$. Then a short computation gives

$$\vartheta_{2,\epsilon}(u/\tau, -1/\tau) = e^{\pi i u^2/\tau}\sqrt{-i\tau}\vartheta_{4,\epsilon}(u,\tau).$$

The partital regularized theta function satisfies $2P_\epsilon(u,\tau) = F\vartheta_\epsilon(u,\tau) + \vartheta_{2,\epsilon}(u,\tau)$ and hence with Proposition 15 and above equation the claim follows. □

This proposition completes the proof of Theorem 4.

Remark 17 Clearly,

$$P_{a,b}(u,\tau) = \sum_{n=0}^{\infty} z^{n+\frac{b}{2a}} q^{a(n+\frac{b}{2a})^2} = z^{\frac{b}{2a}-\frac{1}{2}} q^{a(\frac{b}{2a}-\frac{1}{2})^2} P(u + (b-a)\tau, 2a\tau)$$

so the previous theorem solves the problem of finding the modular-like transformations of the non-generic family of regularized singlet algebra characters.

3 The singlet vertex algebra $\mathcal{W}(2, 2p-1)$

Let

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}k.$$

denote the rank one Heisenberg Lie algebra. We choose its generators to be $\varphi(n)$, $n \in \mathbb{Z}$, such that $[\varphi(n), \varphi(m)] = \delta_{m+n,0}k$. Denote by $F_\lambda \cong U(\hat{\mathfrak{h}}_-)$, $\lambda \in \mathbb{C}$, the usual Fock space of charge λ, so that $\varphi(0) \cdot e^{\lambda\varphi} = \lambda e^{\lambda\varphi}$, where $e^{\lambda\varphi}$ is a lowest weight vector for F_λ. We fix the level to be one, i.e. the central element k will act by multiplication with one on all Fock spaces. For every $p \geq 2$, we choose the conformal vector to be

$$\omega = \frac{1}{2}\varphi(-1)^2\mathbf{1} + \frac{p-1}{\sqrt{2p}}\varphi(-2)\mathbf{1} \in F_0.$$

This equips F_0 with a VOA structure of central charge $c_{p,1} = 1 - 6\frac{(p-1)^2}{p}$. We shall be using standard parametrization of $(1,p)$ lowest weight logarithmic minimal models following

$$h_{m,n} = \frac{(np-m)^2 - (p-1)^2}{4p}.$$

In this parametrization we may assume $1 \leq m \leq p$ and $n \in \mathbb{Z}$.

Next, we introduce the singlet vertex algebra after Kausch [Ka], where we prefer to follow the approach from [A], [AdM1] and [AdM2] (see also [FHST], [FGST1], [FGST2]).

3.1 Definition of $\mathcal{W}(2, 2p-1)$

Again, here $p \in \mathbb{N}_{\geq 2}$. Denote by

$$V_L = \bigoplus_{\lambda\in\sqrt{2p}\mathbb{Z}} F_\lambda$$

the lattice vertex algebra associated to rank one even lattice $\sqrt{2p}\mathbb{Z}$ [AdM2] (see also [LL]). Consider its dual lattice $\widetilde{L} = \mathbb{Z}(\frac{1}{\sqrt{2p}})$. Then we have a generalized vertex algebra structure on

$$V_{\widetilde{L}} = \bigoplus_{\lambda \in \frac{1}{\sqrt{2p}}\mathbb{Z}} F_\lambda = \bigoplus_{i=0}^{2p-1} V_{L+\frac{i}{\sqrt{2p}}}.$$

The vertex algebra V_L is then a vertex subalgebra of $V_{\widetilde{L}}$.

The element $L(0)$ of the Virasoro algebra defines a $\mathbb{N}$–gradation on V_L. As in [Ka] define the following *long* and *short* screening operators

$$Q = e_0^{\sqrt{2p}\varphi}, \qquad \widetilde{Q} = e_0^{-\sqrt{\frac{2}{p}}\varphi},$$

respectively, where we use

$$e^{\gamma}(x) = \sum_{n\in\mathbb{Z}} e_n^{\gamma} x^{-n-1},$$

the Fourier expansion of e^{γ}. Then we have

$$[Q, \widetilde{Q}] = 0, \quad [L(n), Q] = [L(n), \widetilde{Q}] = 0 \quad (n \in \mathbb{Z}).$$

Thus, the operators Q and $\widetilde{Q}$ are intertwinners among Virasoro algebra modules. In fact, the Virasoro vertex operator algebra $L(c_{p,1}, 0) \subset \mathcal{F}_0$ is the kernel of the screening operator Q. Define

$$\mathcal{W}(2, 2p-1) = \mathrm{Ker}_{F_0} \widetilde{Q},$$

called the *singlet vertex algebra*. Since $\widetilde{Q}$ commutes with the action of the Virasoro algebra, we have

$$L(c_{p,1}, 0) \subset \mathcal{W}(2, 2p-1).$$

The vertex operator algebra $\mathcal{W}(2, 2p-1)$ is completely reducible as a Virasoro algebra module and the following decomposition holds:

$$\mathcal{W}(2, 2p-1) = \bigoplus_{n=0}^{\infty} U(Vir).\, u^{(n)} = \bigoplus_{n=0}^{\infty} L(c_{p,1}, n^2p + np - n),$$

where

$$u^{(n)} = Q^n e^{-n\sqrt{2p}\varphi}. \tag{3.1}$$

In addition, $\mathcal{W}(2, 2p-1)$ is strongly generated by ω and the primary vector

$$H = Q e^{-\sqrt{2p}\varphi} \tag{3.2}$$

of conformal weight $2p-1$.

3.2 Irreducible $\mathcal{W}(2, 2p-1)$-modules

Complete classification of all (weak) irreducible $\mathcal{W}(2, 2p-1)$-modules is presently unknown. On the other hand, $\mathbb{Z}_{\geq 0}$-gradable irreducible modules were classified in [A] (see also [AdM1] for some additional details). From now on we consider finitely generated $\mathbb{Z}_{\geq 0}$-gradable ordinary modules whose characters are well-defined. We do not consider logarithmic modules in this work. First we distinguish between typical and atypical modules.

Definition 18 An irreducible ($\mathbb{Z}_{\geq 0}$-graded) $\mathcal{W}(2, 2p-1)$-module is called *typical* if it remains irreducible as a Virasoro module, and *atypical* otherwise.

We denote by $\mathrm{ch}[X](\tau)$ the usual character of X, the trace of $q^{L(0)-c/24}$. Generic characters are easily computed. Denote by F_λ the Fock space of charge λ as in the previous section. Then clearly,

$$\mathrm{ch}[F_\lambda](\tau) = \frac{q^{h_\lambda - c_{p,1}/24}}{(q;q)_\infty} = \frac{q^{(\lambda-\alpha_0/2)^2/2}}{\eta(\tau)}.$$

We also observe the symmetry

$$\mathrm{ch}[F_\lambda](\tau) = \mathrm{ch}[F_{\alpha_0-\lambda}](\tau).$$

By using results from [A] and [AdM1], we easily infer that all atypical irreducible $\mathcal{W}(2, 2p-1)$-modules can be constructed as subquotients of F_λ, where $\lambda \in \sqrt{2p}\mathbb{Z} + \frac{i}{\sqrt{2p}}$, $0 \leq i \leq 2p-1$ (the dual lattice $\tilde{L}$). Every such Fock space yields a unique irreducible $\mathcal{W}(2, 2p-1)$-module. To see this, we shall first slightly adjust the parametrization of $\tilde{L}$. Let $\alpha_+ = \sqrt{2p}$, $\alpha_- = -\sqrt{2/p}$ and let also

$$\alpha_0 = \alpha_+ + \alpha_-.$$

Further let

$$\alpha_{r,s} = -\frac{1}{2}(r\alpha_+ + s\alpha_- - \alpha_0) = -\frac{r-1}{2}\sqrt{2p} + \frac{s-1}{\sqrt{2p}} \in \tilde{L}.$$

The range for r is the set of integers and $1 \leq s \leq p$. Now to each $F_{\alpha_{r,s}}$ we associate an irreducible module $M_{r,s}$.

It is known that for $s = p$, the singlet module $F_{\alpha_{r,p}}$ is irreducible [AdM1]. So for $r \in \mathbb{Z}$, we let

$$M_{r,p} := F_{\alpha_{r,p}}.$$

From now on we may assume the range to be $1 \leq s \leq p-1$. The module $F_{\alpha_{r,s}}$ has a composition series of length 2 with respect to $\mathcal{W}(2, 2p-1)$. We denote by

$$M_{r,s} := \mathrm{soc}(F_{\alpha_{r,s}}).$$

The socle can be also taken with respect to the Virasoro algebra though. We first consider the case $r \geq 1$. Then $M_{r,s}$ is of the same highest weight as $F_{\alpha_{r,s}}$. We clearly have a short exact sequence

$$0 \to M_{r,s} \to F_{\alpha_{r,s}} \to N_{r,s} \to 0,$$

where $N_{r,s}$ is another irreducible module. By using well-known formulas for decomposition of $M_{r,s}$ into irreducible Virasoro modules [CRW], [AdM1] we easily obtain

$$\mathrm{ch}[M_{r,s}](\tau) = \frac{1}{\eta(\tau)} \left(\sum_{n=0}^{\infty} q^{p(\frac{r}{2}+n-\frac{s}{2p})^2} - q^{p(\frac{r}{2}+n+\frac{s}{2p})^2} \right). \tag{3.3}$$

Now we focus on $F_{\alpha_{r,s}}$, $r \leq 0$. Similarly, we get

$$\mathrm{ch}[M_{r,s}] = \frac{1}{\eta(\tau)} \left(\sum_{n=0}^{\infty} q^{p(-\frac{r}{2}+\frac{1}{2}+n+\frac{p-s}{2p})^2} - q^{p(-\frac{r}{2}+\frac{1}{2}+n+\frac{p+s}{2p})^2} \right).$$

Observe now the relation (for $r \leq 0$)

$$\sum_{n=0}^{\infty} q^{p(-\frac{r}{2}+\frac{1}{2}+n+\frac{p-s}{2p})^2} - q^{p(-\frac{r}{2}+\frac{1}{2}+n+\frac{p+s}{2p})^2} = \sum_{n=0}^{\infty} q^{p(\frac{r}{2}+n-\frac{s}{2p})^2} - q^{p(\frac{r}{2}+n+\frac{s}{2p})^2},$$

due to cancellations in the second sum. To summarize, for $r \in \mathbb{Z}$ and $1 \leq s \leq p$ we have

$$\mathrm{ch}[M_{r,s}](\tau) = \frac{P_{p,pr-s}(0,\tau) - P_{p,pr+s}(0,\tau)}{\eta(\tau)},$$

where $P_{a,b}(\tau)$ is as in the introduction. In particular, for $M_{1,1} = \mathcal{W}(2, 2p-1)$, we get

$$\mathrm{ch}[\mathcal{W}(2,2p-1)](\tau) = \frac{\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{p(n+\frac{p-1}{2p})^2}}{\eta(\tau)}.$$

Remark 19 *In addition to considerations coming from decomposition of Fock spaces into Virasoro algebra modules it is also useful to apply Felder's resolution in the category of $\mathcal{W}(2, 2p-1)$-modules (see formula (2.26) [CRW], for instance):*

$$\cdots \to F_{\alpha_{r,s}} \xrightarrow{\tilde{Q}^{[s]}} F_{\alpha_{r+1,p-s}} \xrightarrow{\tilde{Q}^{[p-s]}} F_{\alpha_{r+2,s}} \xrightarrow{\tilde{Q}^{[s]}} F_{\alpha_{r+3,p-s}} \to \cdots,$$

where $\tilde{Q}^{[s]}$ are suitable "powers" of the short screening operator $\tilde{Q}$. It can be shown $M_{r,s} =$ $\mathrm{Ker}\, \tilde{Q}^{[s]} \subset F_{\alpha_{r,s}}$ [CRW], so by the Euler-Poincaré principle we easily get

$$ch[M_{r,s}](\tau) = \sum_{n=0}^{\infty} ch[F_{\alpha_{r-2n-1,p-s}}](\tau) - ch[F_{\alpha_{r-2n-2,s}}](\tau).$$

This formula will be useful in the next section

3.3 Regularized characters of $\mathcal{W}(2, 2p-1)$-modules

Now, we define the regularized characters by introducing a parameter ϵ. We let

$$\begin{aligned} \mathrm{ch}[F^{\epsilon}_{\lambda}](\tau) &= e^{2\pi\epsilon(\lambda-\alpha_0/2)} \frac{q^{(\lambda-\alpha_0/2)^2/2}}{\eta(\tau)} \\ \mathrm{ch}[M^{\epsilon}_{r,s}](\tau) &= \sum_{n=0}^{\infty} \mathrm{ch}[F^{\epsilon}_{\alpha_{r-2n-1,p-s}}](\tau) - \mathrm{ch}[F^{\epsilon}_{\alpha_{r-2n-2,s}}](\tau) \end{aligned} \tag{3.4}$$

Observe that typical ϵ-regularized characters are simply $\mathrm{tr}_{F_\lambda} e^{2\pi\epsilon(\varphi(0)-\alpha_0/2)} q^{L(0)-c/24}$. But atypical regularization is more subtle although very natural in view of Remark 19.

Note, that the characters of the atypical modules are parameterized by $r, s \in \mathbb{Z}$ with $1 \leq s \leq p$. Also, $\mathrm{ch}[M^{\epsilon}_{r,0}](\tau) = 0$ and in the case $s > p$, $\mathrm{ch}[M^{\epsilon}_{r,s}](\tau)$ is actually an integral combination of atypical module characters (virtual character):

Proposition 20 *The regularized typical and atypical characters satisfy the following relations*

$$\mathrm{ch}[F^{\epsilon}_{\alpha_{r,s}}](\tau) = \mathrm{ch}[M^{\epsilon}_{r,s}](\tau) + \mathrm{ch}[M^{\epsilon}_{r+1,p-s}](\tau)$$

while $\mathrm{ch}[M^\epsilon_{r,s}](\tau)$ *for* $p < s \le 2p-1$ [3] *is the following linear combination of atypical module characters*

$$\mathrm{ch}[M^\epsilon_{r,s}](\tau) = \mathrm{ch}[M^\epsilon_{r-1,s-p}](\tau) + \mathrm{ch}[M^\epsilon_{r,2p-s}](\tau) + \mathrm{ch}[M^\epsilon_{r+1,s-p}](\tau).$$

Proof The first equality follows directly from (3.4), while for the second one, we use that $\alpha_{r,s} = \alpha_{r+1,s+p}$ for all $r, s \in \mathbb{Z}$ to compute

$$\begin{aligned}
\mathrm{ch}[M^\epsilon_{r,s}](\tau) &= \sum_{n=0}^{\infty} \mathrm{ch}[F^\epsilon_{\alpha_{r-2n-1,p-s}}](\tau) - \mathrm{ch}[F^\epsilon_{\alpha_{r-2n-2,s}}](\tau) \\
&= \sum_{n=0}^{\infty} \mathrm{ch}[F^\epsilon_{\alpha_{r-2n,2p-s}}](\tau) - \mathrm{ch}[F^\epsilon_{\alpha_{r-2n-3,s-p}}](\tau) \\
&= \mathrm{ch}[F^\epsilon_{\alpha_{r,2p-s}}](\tau) + \mathrm{ch}[M^\epsilon_{r-1,s-p}](\tau) \\
&= \mathrm{ch}[M^\epsilon_{r-1,s-p}](\tau) + \mathrm{ch}[M^\epsilon_{r,2p-s}](\tau) + \mathrm{ch}[M^\epsilon_{r+1,s-p}](\tau).
\end{aligned}$$

□

Proposition 21 *Let* $\beta^\pm_{r,s} = ((r-1)\alpha_+ \pm s\alpha_-)/2$*, then the atypical characters are*

$$\mathrm{ch}[M^\epsilon_{r,s}](\tau) = \mathrm{ch}[F^\epsilon_{\alpha_0/2-\beta^-_{r,s}}](\tau) P_{\alpha_+\epsilon}(-\alpha_+\beta^-_{r,s}\tau; \alpha_+^2\tau) - \mathrm{ch}[F^\epsilon_{\alpha_0/2-\beta^+_{r,s}}](\tau) P_{\alpha_+\epsilon}(-\alpha_+\beta^+_{r,s}\tau; \alpha_+^2\tau)$$

Proof This is a straightforward rewriting. □

3.4 Modular properties of characters

Proposition 22 *The modular S-transformation of typical characters is*

$$\mathrm{ch}[F^\epsilon_{\lambda+\alpha_0/2}]\left(\frac{-1}{\tau}\right) = \int_{\mathbb{R}} S^\epsilon_{\lambda+\alpha_0/2,\mu+\alpha_0/2} \mathrm{ch}[F^\epsilon_{\mu+\alpha_0/2}](\tau) d\mu,$$

with $S^\epsilon_{\lambda+\alpha_0/2,\mu+\alpha_0/2} = e^{2\pi\epsilon(\lambda-\mu)} e^{-2\pi i\lambda\mu}$.

Proof This follows from the Gauss integral of Lemma 8. □

Proposition 23 *The modular S-transformation of atypical characters is*

$$ch[M^\epsilon_{r,s}]\Big(-\frac{1}{\tau}\Big) = \int_{\mathbb{R}} S^\epsilon_{(r,s),\mu+\alpha_0/2} \mathrm{ch}[F^\epsilon_{\mu+\alpha_0/2}](\tau) d\mu + X^\epsilon_{r,s}(\tau)$$

with

$$S^\epsilon_{(r,s),\mu+\alpha_0/2} = -e^{-2\pi\epsilon((r-1)\alpha_+/2+\mu)} e^{\pi i(r-1)\alpha_+\mu} \frac{\sin\big(\pi s\alpha_-(\mu+i\epsilon)\big)}{\sin\big(\pi\alpha_+(\mu+i\epsilon)\big)}$$

and

$$X^\epsilon_{r,s}(\tau) = \frac{1}{4\eta(\tau)} (\mathrm{sgn}(\mathrm{Re}(\epsilon)) + 1) \sum_{n\in\mathbb{Z}} (-1)^{rn} e^{\pi i\frac{s}{p}n} q^{\frac{1}{2}(\frac{n^2}{\alpha_+^2}-\epsilon^2)} \big(q^{-i\epsilon\frac{n}{\alpha_+}} - q^{i\epsilon\frac{n}{\alpha_+}}\big).$$

Note, that in the limit $\epsilon \to 0$, $X^\epsilon_{r,s}$ vanishes.

[3]Similar formulas can be obtained for higher s but we do not need them right now.

Proof With Theorem 4, we get

$$\mathrm{ch}[F^{\epsilon}_{\alpha_0/2-\beta^{\pm}_{r,s}}]\Big(\frac{-1}{\tau}\Big)P_{\alpha_+\epsilon}\Big(\frac{\alpha_+\beta^{\pm}_{r,s}}{\tau};-\frac{\alpha_+^2}{\tau}\Big)=\frac{1}{2i}\int_{\mathbb{R}}\frac{e^{-2\pi\epsilon\mu}e^{2\pi i\beta^{\pm}_{r,s}(\mu+i\epsilon)}\mathrm{ch}[F^{\epsilon}_{\mu+\alpha_0/2}](\tau)}{\sin\big(\pi\alpha_+(\mu+i\epsilon)\big)}d\mu+$$
$$\frac{1}{4\eta(\tau)}(\mathrm{sgn}(\mathrm{Re}(\epsilon))+1)\vartheta_{4,\alpha_+\epsilon}\Big(\frac{\beta^{\pm}_{r,s}}{\alpha_+};\frac{\tau}{\alpha_+^2}\Big)$$

and hence with Proposition 21 the statement follows. □

4 A Verlinde-type formula

Let us consider the case $\mathrm{Re}(\epsilon)<0$ so there is no correction term present. We are interested in applying the Verlinde formula. This requires a unitary S-matrix (actually, S-kernel), which is spoiled by the regularization and instead we have

$$\text{"}\int_{\mathbb{R}}\text{"}S^{\epsilon}_{\lambda\mu}\overline{S^{-\bar\epsilon}_{\mu\nu}}d\mu=\text{"}\int_{\mathbb{R}}\text{"}e^{-2\pi i\mu(\lambda-\nu)}e^{2\pi\epsilon(\lambda-\nu)}d\mu=e^{2\pi\epsilon(\lambda-\nu)}\delta(\lambda-\nu)=\delta(\lambda-\nu),\tag{4.1}$$

where $\delta(x-y)$ is the Dirac delta-function supported at $x=y$. We thus define the regularized fusion coefficients

$${N^{\epsilon}_{ab}}^{c}=\text{"}\int_{\mathbb{R}}\text{"}\frac{S^{\epsilon}_{a\rho}S^{\epsilon}_{b\rho}\overline{S^{-\bar\epsilon}_{c\rho}}}{S^{\epsilon}_{(1,1)\rho}}d\rho\tag{4.2}$$

where $(1,1)$ refers to the vacuum module $M_{1,1}$. Consider the vector space $\mathcal{V}_{ch}$ generated by $\mathrm{ch}[V^{\epsilon}]$ where $V=M_{r,s}$ or $V=F_{\lambda}$.

We want to turn $\mathcal{V}_{ch}$ into a commutative associative algebra (called the *Verlinde algebra of characters*) by defining the product to be

$$\mathrm{ch}[V^{\epsilon}_a]\times\mathrm{ch}[V^{\epsilon}_b]:=\text{"}\int_{\mathbb{R}}\text{"}{N^{\epsilon}_{ab}}^{c}\mathrm{ch}[V^{\epsilon}_c]dc\tag{4.3}$$

and extending this multiplication by linearity. We expect the right hand side to be a finite sum with non-negative integer multiplicities (as in the Verlinde formula).

4.1 Making the Verlinde algebra of characters rigorous

The integration in (4.1) and (4.2) over $\mathbb{R}$ should not be taken literally. As we shall see shortly, the function that we would like to integrate is clearly non-integrable. Yet, as we are primarily interested in (4.3), and not so much (4.2), we explain first how to make (4.3) rigorous and then how to view (4.2) not as a numerical quantity but rather as a distribution. Thus instead of working with

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{S^{\epsilon}_{a\rho}S^{\epsilon}_{b\rho}\overline{S^{-\bar\epsilon}_{\rho\mu}}}{S^{\epsilon}_{(1,1)\rho}}d\rho\right)\mathrm{ch}[F^{\epsilon}_{\mu}]d\mu,\tag{4.4}$$

we redefine the fusion product in the Verlinde algebra of characters as

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{S^{\epsilon}_{a\rho}S^{\epsilon}_{b\rho}\overline{S^{-\bar\epsilon}_{\rho\mu}}}{S^{\epsilon}_{(1,1)\rho}}\mathrm{ch}[F^{\epsilon}_{\mu}]d\mu\right)d\rho\tag{4.5}$$

This double integral turns out to be well-defined in our examples.

To see this let us start from the classical Fourier inversion formula. Suppose that $f(x)$ and its Fourier transform $\hat{f}(x)$ lie in an appropriate L^1-space. Then we have

$$f(x) = \int_{\mathbb{R}} (\int_{\mathbb{R}} e^{2\pi i(x-y)z} f(y)dy)dz.$$

Going back to (4.5), in this setup the test functions are essentially

$$f(\mu) = e^{2\pi\epsilon(\mu+c)} q^{(\mu-\alpha_0/2)^2/2} e^{2\pi\epsilon(\mu-\alpha_0/2)},$$

where we ignore the Dedekind η denominator and c does not depend on μ. We are integrating

$$\int_{\mathbb{R}} (\int_{\mathbb{R}} e^{-2\pi i\rho(\mu+c)} f(\mu)d\mu)d\rho = f(-c) = q^{(-c-\alpha_0/2)^2/2} e^{2\pi\epsilon(-c-\alpha_0/2)}. \tag{4.6}$$

Notice that the same result can be inferred by working with (4.4) and by using (heuristic) δ-function

$$\delta(x-y) = \int_{\mathbb{R}} e^{2\pi i\rho(x-y)} d\rho,$$

as in (4.1). Then second integration, against the delta function (now viewed as distribution) is simply evaluation so we obtain the same result as in (4.6). To handle infinite sums (see below) we only have to notice that for every $Re(\epsilon) < 0$ and $f(\mu)$ as before, we have

$$\int_{\mathbb{R}} \frac{f(\mu)}{\sin(\mu+\epsilon i)} d\mu = 2i \sum_{m=0}^{\infty} \int_{\mathbb{R}} f(\mu) e^{\epsilon(2m+1)-i\mu(2m+1)} d\mu.$$

We first fix $\epsilon = \epsilon_1 + i\epsilon_2$, where $\epsilon_1 < 0$ and $\epsilon_2 \in \mathbb{R}$. To prove the last formula we first observe that $|q^{(\lambda-\alpha_0/2)^2}| = |q|^{(\lambda-\alpha_0/2)^2}$ with $|q| < 1$. Now, Gauss' integral formula shows that $\sum_{m=0}^{\infty} \int_{\mathbb{R}} |f(\mu) e^{(\epsilon_1+\epsilon_2 i)(2m+1)-i\mu(2m+1)}| d\mu = \sum_{m=0}^{\infty} \int_{\mathbb{R}} |f(\mu) e^{\epsilon_1(2m+1)}| d\mu$ is convergent. Finally, by Fubini's theorem we can interchange the sum and integration. This section clarifies all future computations involving the formal delta function.

4.2 Verlinde algebra of characters

In the next theorem we explicitly determine this algebra.

Theorem 24 *Let $1 \leq s, s' \leq p$, then the Verlinde algebra of characters is associative and commu-*

tative and is given by

$$\mathrm{ch}[F^\epsilon_\lambda] \times \mathrm{ch}[F^\epsilon_\mu] = \sum_{\ell=0}^{p-1} \mathrm{ch}[F^\epsilon_{\lambda+\mu+\ell\alpha_-}]$$

$$\mathrm{ch}[M^\epsilon_{r,s}] \times \mathrm{ch}[F^\epsilon_\mu] = \sum_{\substack{\ell=-s+2\\ \ell+s=0 \bmod 2}}^{s} \mathrm{ch}[F^\epsilon_{\mu+\alpha_{r,\ell}}]$$

$$\begin{aligned}\mathrm{ch}[M^\epsilon_{r,s}] \times \mathrm{ch}[M^\epsilon_{r',s'}] = &\sum_{\substack{\ell=|s-s'|+1\\ \ell+s+s'=1 \bmod 2}}^{min\{s+s'-1,p\}} \mathrm{ch}[M^\epsilon_{r+r'-1,\ell}]\\ &+ \sum_{\substack{\ell=p+1\\ \ell+s+s'=1 \bmod 2}}^{s+s'-1} \Big(\mathrm{ch}[M^\epsilon_{r+r'-2,\ell-p}] + \mathrm{ch}[M^\epsilon_{r+r'-1,2p-\ell}] + \mathrm{ch}[M^\epsilon_{r+r',\ell-p}]\Big),\end{aligned}$$

where $\sum_{l=i}^{j}(\cdot) = 0$ *for* $i > j$.

Remark 25 In mathematical physics a regularization is introduced to avoid divergent quantities. It is then required that the final result is independent of the regularization scheme. In our case the final result is the Verlinde algebra, which indeed is independent of the choice of our regularization parameter ϵ.

Proof We first note the following identity

$$\frac{\sin(sx)}{\sin(x)} = \sum_{\substack{\ell=-s+1\\ \ell+s=1 \bmod 2}}^{s-1} e^{ix\ell}$$

which is verified by multiplying both sides with $\sin(x)$. It follows the Verlinde fusion of typicals with themselves. Note that $\alpha_+ = -p\alpha_-$.

$$\begin{aligned}N^\epsilon_{\lambda+\alpha_0/2,\mu+\alpha_0/2}{}^{\nu+\alpha_0/2} &= -\int_{\mathbb{R}} e^{-2\pi i(\rho+i\epsilon)(\lambda+\mu-\nu)} \frac{\sin(\pi\alpha_+(\rho+i\epsilon))}{\sin(\pi\alpha_-(\rho+i\epsilon))} d\rho\\ &= \int_{\mathbb{R}} \sum_{\substack{\ell=-p+1\\ \ell+p=1 \bmod 2}}^{p-1} e^{-2\pi i(\rho+i\epsilon)(\lambda+\mu-\nu+\ell\alpha_-/2)} d\rho\\ &= \sum_{\substack{\ell=-p+1\\ \ell+p=1 \bmod 2}}^{p-1} \delta(\lambda+\mu-\nu+\ell\alpha_-/2) e^{2\pi\epsilon(\lambda+\mu-\nu+\ell\alpha_-/2)}\\ &= \sum_{\substack{\ell=-p+1\\ \ell+p=1 \bmod 2}}^{p-1} \delta(\lambda+\mu-\nu+\ell\alpha_-/2)\end{aligned}$$

and hence

$$\mathrm{ch}[F^\epsilon_{\lambda+\alpha_0/2}]\times\mathrm{ch}[F^\epsilon_{\mu+\alpha_0/2}] = \sum_{\substack{\ell=-p+1\\ \ell+p=1 \bmod 2}}^{p-1} \mathrm{ch}[F^\epsilon_{\lambda+\mu+\ell\alpha_-/2+\alpha_0/2}]$$

so that

$$\mathrm{ch}[F^\epsilon_{\lambda}]\times\mathrm{ch}[F^\epsilon_{\mu}] = \sum_{\substack{\ell=-p+1\\ \ell+p=1 \bmod 2}}^{p-1} \mathrm{ch}[F^\epsilon_{\lambda+\mu+\ell\alpha_-/2-\alpha_0/2}] = \sum_{\ell=0}^{p-1}\mathrm{ch}[F^\epsilon_{\lambda+\mu+\ell\alpha_-}]$$

since $-\alpha_0=(p-1)\alpha_-$. The Verlinde fusion of atypicals with typicals is proven in the same manner as the previous case.

$$\begin{aligned}
{N^\epsilon_{(r,s),\mu+\alpha_0/2}}^{\nu+\alpha_0/2} &= \int_{\mathbb{R}} e^{-2\pi i(\rho+i\epsilon)(\mu-(r-1)\alpha_+/2-\nu)}\frac{\sin(\pi s\alpha_-(\rho+i\epsilon))}{\sin(\pi\alpha_-(\rho+i\epsilon))}d\rho\\
&= \int_{\mathbb{R}} \sum_{\substack{\ell=-s+1\\ \ell+s=1 \bmod 2}}^{s-1} e^{-2\pi i(\rho+i\epsilon)(\mu-(r-1)\alpha_+/2-\nu+\ell\alpha_-/2)}d\rho\\
&= \sum_{\substack{\ell=-s+1\\ \ell+s=1 \bmod 2}}^{s-1} \delta(\mu-(r-1)\alpha_+/2-\nu+\ell\alpha_-/2)\\
&= \sum_{\substack{\ell=-s+1\\ \ell+s=1 \bmod 2}}^{s-1} \delta(\alpha_{r,\ell+1}+\mu-\nu)
\end{aligned}$$

Recall that $\alpha_{r,s}=-\frac{1}{2}(r\alpha_++s\alpha_--\alpha_0)$. It follows that

$$\mathrm{ch}[M^\epsilon_{r,s}]\times\mathrm{ch}[F^\epsilon_{\mu+\alpha_0/2}] = \sum_{\substack{\ell=-s+2\\ \ell+s=0 \bmod 2}}^{s} \mathrm{ch}[F^\epsilon_{\mu+\alpha_0/2+\alpha_{r,\ell}}].$$

Finally, the case of atypicals uses the following identity for $Im(x)<0$, and positive integers s,s',p

$$-\frac{\sin(sx)\sin(s'x)}{\sin(x)\sin(px)} = \sum_{\ell'=0}^{\infty} e^{-ipx(2\ell'+1)} \sum_{\substack{\ell=|s'-s|+1\\ s+s'+\ell=1 \bmod 2}}^{s'+s-1} \left(e^{-ix\ell}-e^{ix\ell}\right),$$

$Im(x)<0$ ensures convergence and the identity is verified by multiplying both sides with the denominator of the left-hand side. We thus get

$$\begin{aligned}
{N^\epsilon_{(r,s)(r',s')}}^{\nu+\alpha_0/2} &= -\int_{\mathbb{R}} e^{-\pi i(\rho+i\epsilon)(-(r+r'-2)\alpha_+-2\nu)}\frac{\sin(\pi s\alpha_-(\rho+i\epsilon))\sin(\pi s'\alpha_-(\rho+i\epsilon))}{\sin(\pi\alpha_-(\rho+i\epsilon))\sin(\pi\alpha_+(\rho+i\epsilon))}d\rho\\
&= \int_{\mathbb{R}}\sum_{\ell'=0}^{\infty} \sum_{\substack{\ell=|s'-s|+1\\ s+s'+\ell=1 \bmod 2}}^{s'+s-1} e^{-\pi i(\rho+i\epsilon)((2\ell'+3-r-r')\alpha_+-2\nu)}\left(e^{-\pi i\alpha_-\ell(\rho+i\epsilon)}-e^{\pi i\alpha_-\ell(\rho+i\epsilon)}\right)d\rho\\
&= \sum_{\ell'=0}^{\infty} \sum_{\substack{\ell=|s'-s|+1\\ s+s'+\ell=1 \bmod 2}}^{s'+s-1} \Big(\delta(\alpha_{r+r'-2\ell'-2,-\ell+1}-\nu)-\delta(\alpha_{r+r'-2\ell'-2,\ell+1}-\nu)\Big)
\end{aligned}$$

Now using $\alpha_{r,s}+\alpha_0/2=\alpha_{r-1,s-1}$ and $\alpha_{r,s}=\alpha_{r+1,p+s}$ and the expression of atypical characters in terms of typicals, we get

$$\begin{aligned}\mathrm{ch}[M^{\epsilon}_{r,s}]\times\mathrm{ch}[M^{\epsilon}_{r',s'}] &= \sum_{\ell'=0}^{\infty}\sum_{\substack{\ell=|s'-s|+1\\ s+s'+\ell=1 \bmod 2}}^{s'+s-1}\Big(\mathrm{ch}[F^{\epsilon}_{\alpha_{r+r'-1-2\ell'-1,p-\ell}}]-\mathrm{ch}[F^{\epsilon}_{\alpha_{r+r'-1-2\ell'-2,\ell}}\Big)\\ &= \sum_{\substack{\ell=|s-s'|+1\\ \ell+s+s'=1 \bmod 2}}^{s+s'-1}\mathrm{ch}[M^{\epsilon}_{r+r'-1,\ell}]\end{aligned}$$

In the case of $p<s+s'-1\leq 2p-1$ we have to use Proposition 20 to obtain

$$\begin{aligned}\mathrm{ch}[M^{\epsilon}_{r,s}]\times\mathrm{ch}[M^{\epsilon}_{r',s'}] = &\sum_{\substack{\ell=p+1\\ \ell+s+s'=1 \bmod 2}}^{s+s'-1}\Big(\mathrm{ch}[M^{\epsilon}_{r+r'-2,\ell-p}]+\mathrm{ch}[M^{\epsilon}_{r+r'-1,2p-\ell}]+\mathrm{ch}[M^{\epsilon}_{r+r',\ell-p}]\Big)+\\ &\sum_{\substack{\ell=|s-s'|+1\\ \ell+s+s'=1 \bmod 2}}^{p}\mathrm{ch}[M^{\epsilon}_{r+r'-1,\ell}].\end{aligned}$$

We also have to check that the following relation inside the Verlinde algebra of characters

$$\mathrm{ch}[F^{\epsilon}_{\alpha_{r-1,p-s}}]=\mathrm{ch}[M^{\epsilon}_{r,s}]+\mathrm{ch}[M^{\epsilon}_{r-1,p-s}]$$

is consistent with the proposed multiplication. This follows immediately from the relation

$$\begin{aligned}\mathrm{ch}[F^{\epsilon}_{\alpha_{r-1,p-s}}]\times\mathrm{ch}[F^{\epsilon}_{\nu}] &= (\mathrm{ch}[M^{\epsilon}_{r,s}]+\mathrm{ch}[M^{\epsilon}_{r-1,p-s}])\times\mathrm{ch}[F^{\epsilon}_{\mu}]\\ &= \sum_{l=-s+2;2}^{s}\mathrm{ch}[F^{\epsilon}_{\mu+\alpha_{r,l}}]+\sum_{l=-(p-s)+2;2}^{p-s}\mathrm{ch}[F^{\epsilon}_{\mu+\alpha_{r-1,l}}]\\ &= \sum_{l=0}^{p-1}\mathrm{ch}[F^{\epsilon}_{\alpha_{r-1,p-s-2l}+\mu}].\end{aligned}$$

Commutativity is clear from the definition, while associativity can be easily checked directly (It also follows from Theorem 28 below). This completes the proof of the theorem. □

The previous computations with the characters is very useful because it gives us lots of hints about the fusion rules between triples of modules for the singlet. Although we do not have a proof that the category of $\mathcal{W}(2,2p-1)$-Mod is a braided tensor category, we believe this to be the case (or at least a suitable sub-category). Thus we can talk about its Grothendieck ring.

Conjecture 26 *The relations in Theorem 24 also hold inside the Grothendieck ring.*

A vertex operator algebra approach to this conjecture will be subject of [AdM4].

4.3 Regularized quantum dimensions

We introduce the regularized quantum dimension of a module for $\mathrm{Re}(\epsilon) < 0$ as

$$\mathrm{qdim}[V^\epsilon] = \lim_{\tau\to 0+} \frac{\mathrm{ch}[V^\epsilon(\tau)]}{\mathrm{ch}[M^\epsilon_{1,1}](\tau)}. \tag{4.7}$$

They are

Proposition 27 *The regularized quantum dimension of typical characters are*

$$\mathrm{qdim}[F^\epsilon_\lambda] = q_\epsilon^{2\lambda-\alpha_0}\frac{\sin(-\pi\alpha_+\epsilon i)}{\sin(\pi\alpha_-\epsilon i)} = q_\epsilon^{2\lambda-\alpha_0}\sum_{\substack{\ell=-p+1\\ \ell+p=1 \bmod 2}}^{p-1} q_\epsilon^{\alpha_-\ell}$$

and of atypicals they are

$$\mathrm{qdim}[M^\epsilon_{r,s}] = q_\epsilon^{-(r-1)\alpha_+}\frac{\sin(\pi s\alpha_-\epsilon i)}{\sin(\pi\alpha_-\epsilon i)} = q_\epsilon^{-(r-1)\alpha_+}\sum_{\substack{\ell=-s+1\\ \ell+s=1 \bmod 2}}^{s-1} q_\epsilon^{\alpha_-\ell}$$

for $q_\epsilon = e^{\pi\epsilon}$.

Proof We have the limits

$$\lim_{\tau\to 0}\eta(q)\mathrm{ch}[F^\epsilon_\lambda] = e^{2\pi\epsilon(\lambda-\alpha_0/2)} = q_\epsilon^{2\lambda-\alpha_0}$$

and for positive a

$$\lim_{\tau\to 0} P_{a\epsilon}(b\tau; c\tau) = \sum_{n=0}^{\infty} e^{2\pi\epsilon a(n+1/2)} = (e^{-a\pi\epsilon} - e^{a\pi\epsilon})^{-1}$$

Observe that here it was essential that $\mathrm{Re}(\epsilon) < 0$.

$$\lim_{\tau\to 0}\eta(q)\mathrm{ch}[M^\epsilon_{r,s}] = \frac{e^{-\beta^-_{r,s}2\pi\epsilon} - e^{-\beta^+_{r,s}2\pi\epsilon}}{e^{-a\pi\epsilon} - e^{a\pi\epsilon}} = e^{-(r-1)\alpha_+\epsilon}\frac{\sin(\pi s\alpha_-\epsilon i)}{\sin(-\pi\alpha_+\epsilon i)}.$$

The quantum dimensions follow. The sum expansion of the quotients of $\sin(x)$ are as in the proof of Theorem 24. □

The regularized quantum dimensions should be regarded as functions of the regularization parameter ϵ with $\mathrm{Re}(\epsilon) < 0$. Consider the vector space $\mathcal{Q}$ spanned by (regularized) quantum dimensions of atypical and typical modules. Pointwise multiplication of quantum dimensions defines a commutative product on $\mathcal{Q}$, which compares nicely to the Verlinde algebra.

Theorem 28 *The algebra of regularized quantum dimensions $\mathcal{Q}$ is isomorphic to the Verlinde algebra $\mathcal{V}_{ch}$.*

Proof The products of regularized quantum dimensions are

$$\mathrm{qdim}[F^\epsilon_\lambda]\times\mathrm{qdim}[F^\epsilon_\mu] = q_\epsilon^{2\lambda+2\mu-\alpha_0}\frac{\sin(-\pi\alpha_+\epsilon i)}{\sin(\pi\alpha_-\epsilon i)}\sum_{\substack{\ell=-p+1\\ \ell+p=1 \bmod 2}}^{p-1} q_\epsilon^{\alpha_-\ell-\alpha_0}$$
$$= \sum_{\ell=0}^{p-1}\frac{\sin(-\pi\alpha_+\epsilon i)}{\sin(\pi\alpha_-\epsilon i)}q_\epsilon^{2\lambda+2\mu+2\alpha_-\ell-\alpha_0} = \sum_{\ell=0}^{p-1}\mathrm{qdim}[F^\epsilon_{\lambda+\mu+\ell\alpha_-}]$$

$$\mathrm{qdim}[M^\epsilon_{r,s}]\times\mathrm{qdim}[F^\epsilon_\mu] = q_\epsilon^{2\mu-\alpha_0-(r-1)\alpha_+}\frac{\sin(-\pi\alpha_+\epsilon i)}{\sin(\pi\alpha_-\epsilon i)}\sum_{\substack{\ell=-s+1\\ \ell+s=1 \bmod 2}}^{s-1} q_\epsilon^{\alpha_-\ell-\alpha_0}$$
$$= \sum_{\substack{\ell=-s+2\\ \ell+s=0 \bmod 2}}^{s}\frac{\sin(-\pi\alpha_+\epsilon i)}{\sin(\pi\alpha_-\epsilon i)}q_\epsilon^{2\mu-(r\alpha_++\ell\alpha_--\alpha_0)-\alpha_0} = \sum_{\substack{\ell=-s+2\\ \ell+s=0 \bmod 2}}^{s}\mathrm{qdim}[F^\epsilon_{\mu+\alpha_{r,\ell}}]$$

$$\mathrm{qdim}[M^\epsilon_{r,s}]\times\mathrm{qdim}[M^\epsilon_{r',s'}] = \frac{q_\epsilon^{-(r+r'-2)\alpha_+}}{\sin(\pi\alpha_-\epsilon i)}\frac{q_\epsilon^{-s\alpha_-}-q_\epsilon^{s\alpha_-}}{2i}\sum_{\substack{\ell=-s'+1\\ \ell+s'=1 \bmod 2}}^{s'-1} q_\epsilon^{\alpha_-\ell}$$
$$= \sum_{\substack{\ell=-s'+1\\ \ell+s'=1 \bmod 2}}^{s'-1} q_\epsilon^{-(r+r'-2)\alpha_+}\frac{\sin(\pi\alpha_-(\ell+s)\epsilon i)}{\sin(\pi\alpha_-\epsilon i)} = \sum_{\substack{\ell=|s-s'|+1\\ \ell+s+s'=1 \bmod 2}}^{s+s'-1}\mathrm{qdim}[M^\epsilon_{r+r'-1,\ell}].$$

Here $\mathrm{qdim}[M^\epsilon_{r,s}]$ for $s>p$ is defined as

$$\lim_{\tau\to 0+}\frac{\mathrm{ch}[M^\epsilon_{r,s}(\tau)]}{\mathrm{ch}[M^\epsilon_{1,1}](\tau)}$$

so that by Proposition 20

$$\mathrm{qdim}[M^\epsilon_{r,s}] = \mathrm{qdim}[M^\epsilon_{r-1,s-p}] + \mathrm{qdim}[M^\epsilon_{r,2p-s}] + \mathrm{qdim}[M^\epsilon_{r+1,s-p}]$$

since all limits involved exist. It follows that

$$\mathrm{qdim}[M^\epsilon_{r,s}]\times\mathrm{qdim}[M^\epsilon_{r',s'}] = \sum_{\substack{\ell=|s-s'|+1\\ \ell+s+s'=1 \bmod 2}}^{s+s'-1}\mathrm{qdim}[M^\epsilon_{r+r'-1,\ell}]$$

for $s+s'-1\le p$ and

$$\mathrm{qdim}[M^\epsilon_{r,s}]\times\mathrm{qdim}[M^\epsilon_{r',s'}] = \sum_{\substack{\ell=|s-s'|+1\\ \ell+s+s'=1 \bmod 2}}^{s+s'-1}\mathrm{qdim}[M^\epsilon_{r+r'-1,\ell}]+$$
$$\sum_{\substack{\ell=p+1\\ \ell+s+s'=1 \bmod 2}}^{s+s'-1}\Big(\mathrm{qdim}[M^\epsilon_{r+r'-2,\ell-p}] + \mathrm{qdim}[M^\epsilon_{r+r'-1,2p-\ell}] + \mathrm{qdim}[M^\epsilon_{r+r',\ell-p}]\Big).$$

for $s+s'-1>p$.

Observe also the relation

$$\mathrm{qdim}[F^{\epsilon}_{\alpha_{r-1,p-s}}]=\mathrm{qdim}[M^{\epsilon}_{r,s}]+\mathrm{qdim}[M^{\epsilon}_{r-1,p-s}],$$

which is true as all three limits involved exist. The rest of the proof is the observation that the linear map from $\mathcal{V}_{ch}$ to $\mathcal{Q}$ induced by

$$\mathrm{ch}(X^{\epsilon})\mapsto \mathrm{qdim}[X^{\epsilon}]$$

is one-to-one, which follows easily by using the linear independence of power functions q_{ϵ}^{ν}. □

Remark 29 In rational conformal field theory, the map from the Verlinde algebra to quantum dimensions is an algebra homomorphism with non-trivial kernel. The information about fusion rules obtained from quantum dimension is then very limited. In our case the quantum dimension in the limit $\epsilon \to 0$ of typical modules F_{λ} is p and of atypicals $M_{r,s}$ is s. On the other hand the regularized quantum dimension as a function of ϵ contains the same information as the Verlinde algebra.

Remark 30 Recall the fusion rules for atypical representations of the Virasoro vertex algebra $L(c_{p,1},0)$ (cf. [L], [Fl], and [M]):

$$L(c_{p,1},h_{r,s})\times L(c_{p,1},h_{r',s'})=\sum_{r''\in A(r,r'),s''\in A(s,s')} L(c_{p,1},h_{r'',s''}),$$

where we assume that all indices are positive and $A_{i,j}=\{i+j-1,i+j-3,\cdots,|i-j|+1\}$. This formula merely indicates the fusion rules among triples of irreducibles, defined as dimensions of the space of intertwining operators [HLZ], and should not be viewed as a relation in the (hypothetical) Grothendieck ring. If we view $L(c_{p,1},h_{r'',s''})$ as the top (summand) component of an atypical singlet module, the above fusion rules can be used to give an upper bound for the singlet fusion rules. But this "upper bound" is of course different compared to the proposed fusion rules in Conjecture 26.

Remark 31 As shown in [BM], partial and false theta functions admit higher rank generalizations coming from higher rank ADE-type Lie algebras in a way that $\mathfrak{g}=sl_2$ recovers the functions studied in this paper. In the same paper, various properties of characters of modules and their quantum dimensions are studied. In particular, we expect their q-dimensions to be positive integers. In [CMW] we also study the (p_+,p_-) singlet algebra and the supertriplet introduced in [AdM3].

References

[A] D. Adamović, Classification of irreducible modules of certain subalgebras of free boson vertex algebra, J. Algebra 270 (2003) 115-132.

[AdM1] D. Adamović and A. Milas, Logarithmic intertwining operators and $W(2,2p-1)$-algebras, J. Math. Phys., **48** (2007), p. 073503.

[AdM2] D. Adamović and A. Milas, On the triplet vertex algebra $W(p)$, Advances in Math. **217** (2008), 2664–2699.

[AdM3] D. Adamović and A. Milas, The $N=1$ triplet vertex operator superalgebras, Comm. Math. Phys. **288** (2009), 225-270; arXiv:0712.0379.

[AdM4] D. Adamović and A. Milas, in progress.

[AB] G. Andrews and B. Berndt, Ramanujan's Lost Notebook: Part II, Springer, 2009.

[AC] C. Alfes and T. Creutzig, The Mock Modular Data of a Family of Superalgebras, Proc. Amer. Math. Soc. 142 (2014), 2265-2280.

[BM] K. Bringmann and A.Milas, preprints.

[BR] A. Babichenko and D. Ridout, Takiff superalgebras and Conformal Field Theory, J. Phys. A 46 (2013) 125204.

[CF] N. Carqueville and M. Flohr, Nonmeromorphic operator product expansion and C_2-cofiniteness for a family of W-algebras, Journal of Physics A: Mathematical and General **39** (2006): 951.

[CMW] T. Creutzig, A.Milas and S. Wood, in preparation.

[CR1] T. Creutzig and D. Ridout, Modular Data and Verlinde Formulae for Fractional Level WZW Models I, Nucl. Phys. B 865 (2012) 83.

[CR2] T. Creutzig and D. Ridout, Modular Data and Verlinde Formulae for Fractional Level WZW Models II, Nucl. Phys. B 875 (2013) 423.

[CR3] T. Creutzig and D. Ridout, Relating the Archetypes of Logarithmic Conformal Field Theory, Nucl. Phys. B 872 (2013) 348.

[CR4] T. Creutzig and D. Ridout, Logarithmic Conformal Field Theory: Beyond an Introduction, J. Phys. A 46 (2013), no. 49, 494006, 72pp.

[CRW] T. Creutzig and D. Ridout, and S. Wood, Coset construction of logarithmic $(1, p)$-models, Lett. Math. Phys. 104, 5 (2014) 553-583.

[FGST1] B.L. Feigin, A.M. Gaĭnutdinov, A. M. Semikhatov, and I. Yu Tipunin, I, The Kazhdan-Lusztig correspondence for the representation category of the triplet W-algebra in logorithmic conformal field theories. (Russian) Teoret. Mat. Fiz. **148** (2006), no. 3, 398–427.

[FGST2] B.L. Feigin, A.M. Gaĭnutdinov, A. M. Semikhatov, and I. Yu Tipunin, Modular group representations and fusion in logarithmic conformal field theories and in the quantum group center. Comm. Math. Phys. **265** (2006), 47–93.

[FHST] J. Fuchs, S. Hwang, A.M. Semikhatov and I. Yu. Tipunin, Nonsemisimple Fusion Algebras and the Verlinde Formula, Comm. Math. Phys. **247** (2004), no. 3, 713–742.

[Fl] M. Flohr, On modular invariant partition functions of conformal field theories with logarithmic operators, International Journal of Modern Physics A **11** 22 (1996): 4147–4172.

[GL] S. Garoufalidis and T.Q. Le, Nahm sums, stability and the colored Jones polynomial, `arXiv:1112.3905`.

[Hu] Y.-Z. Huang, Vertex operator algebra and the Verlinde conjecture, Commun. Contemp. Math., **10** (2008), 108.

[HLZ] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor product theory for generalized modules for a conformal vertex algebra, `arXiv:0710.2687`. (also Parts I-VIII: arXiv:1012.4193, arXiv:1012.4196, arXiv:1012.4197, arXiv:1012.4198, arXiv:1012.4199, arXiv:1012.4202, arXiv:1110.1929, arXiv:1110.1931).

[Ka] H.G. Kausch, Extended conformal algebras generated by a multiplet of primary fields, Phys. Lett. B 259 (1991) 448.

[KL] D. Kazhdan and G. Lusztig, Affine Lie algebras and quantum groups, *International Mathematics Research Notices*, 1991:21-29.

[LZ] R. Lawrence and D. Zagier, Modular Forms and Quantum Invariants of 3-manifolds, Asian J. Math. **3** (1999)

[LL] J. Lepowsky and H. Li, *Introduction to Vertex Operator Algebras and Their Representations*, Birkhäuser, Boston, 2003.

[L] X. Lin, Fusion rules of Virasoro vertex operator algebras, `arXiv:1204.4855`.

[M] A. Milas, Fusion rings for degenerate minimal models, J. of Algebra **254** (2002), 300–335.

[NT] K. Nagatomo and A. Tsuchiya, The Triplet Vertex Operator Algebra $W(p)$ and the Restricted Quantum Group at Root of Unity, Exploring new structures and natural constructions in mathematical physics, 1-49, Adv. Stud. Pure Math., 61, Math. Soc. Japan, Tokyo, 2011.

[RW] D. Ridout and S. Wood, Modular Transformations and Verlinde Formulae for Logarithmic (p_+, p_-)-Models, *Nucl Phys* B **880** (2014), 175-202.

[TW] A. Tsuchiya and S. Wood, The tensor structure on the representation category of the W_p triplet algebra, J. Phys. A 46 (2013), no. 44, 445203, 40pp.

[Ve] E. Verlinde, Fusion rules and modular transformations in 2D conformal field theory, Nuclear Physics B **300**, (1988), 360-376.

[Za] D. Zagier, Quantum modular forms, In Quanta of Maths: Conference in honor of Alain Connes, Clay Mathematics Proceedings 11, AMS and Clay Mathematics Institute 2010, 659-675.

[Z] S. Zwegers, Mock Theta Functions, Utrecht PhD thesis, 2002.

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada. *email: creutzig@ualberta.ca*

Department of Mathematics and Statistics, SUNY-Albany, 1400 Washington Avenue, Albany, NY 12222, USA. *email: amilas@albany.edu*