THE SUPER $\mathcal{W}_{1+\infty}$ ALGEBRA WITH INTEGRAL CENTRAL CHARGE

THOMAS CREUTZIG AND ANDREW R. LINSHAW

ABSTRACT. The Lie superalgebra $SD$ of regular differential operators on the super circle has a universal central extension $\hat{SD}$. For each $c \in \mathbb{C}$, the vacuum module $\mathcal{M}_c(\hat{SD})$ of central charge $c$ admits a vertex superalgebra structure, and $\mathcal{M}_c(\hat{SD}) \cong \mathcal{M}_{-c}(\hat{SD})$. The irreducible quotient $\mathcal{V}_c(\hat{SD})$ of the vacuum module is known as the super $\mathcal{W}_{1+\infty}$ algebra. We show that for each integer $n > 0$, $\mathcal{V}_n(\hat{SD})$ has a minimal strong generating set consisting of $4n$ fields, and we identify it with a $\mathcal{W}$-algebra associated to the purely odd simple root system of $gl(n|n)$. Finally, we realize $\mathcal{V}_n(\hat{SD})$ as the limit of a family of commutant vertex algebras that generically have the same graded character and possess a minimal strong generating set of the same cardinality.

1. INTRODUCTION

Let $\mathcal{D}$ denote the Lie algebra of regular differential operators on the circle. It has a universal central extension $\hat{\mathcal{D}} = \mathcal{D} \oplus \mathbb{C}\kappa$ which was introduced by Kac-Peterson in [KP]. Although $\hat{\mathcal{D}}$ admits a principal $\mathbb{Z}$-gradation and triangular decomposition, its representation theory is nontrivial because the graded pieces are all infinite-dimensional. The important problem of constructing and classifying the quasifinite irreducible, highest-weight representations (i.e., those with finite-dimensional graded pieces) was solved by Kac-Radul in [KRI]. Explicit constructions of these modules were given in terms of the representation theory of $\hat{gl}(\infty, R_m)$, which is a central extension of the Lie algebra of infinite matrices over $R_m = \mathbb{C}[t]/(t^{m+1})$ having only finitely many nonzero diagonal elements. The authors also classified all such $\mathcal{D}$-modules which are unitary.

In [FKRW], the representation theory of $\hat{\mathcal{D}}$ was developed from the point of view of vertex algebras. For each $c \in \mathbb{C}$, $\hat{\mathcal{D}}$ admits a module $\mathcal{M}_c$ called the vacuum module, which is a vertex algebra freely generated by fields $J^l$ of weight $l+1$, for $l \geq 0$. The highest-weight representations of $\mathcal{D}$ are in one-to-one correspondence with the highest-weight representations of $\mathcal{M}_c$. The irreducible quotient of $\mathcal{M}_c$ by its maximal graded, proper $\mathcal{D}$-submodule $\mathcal{I}_c$ is a simple vertex algebra, and is often denoted by $\mathcal{W}_{1+\infty,c}$. These algebras have been studied extensively in both the physics and mathematics literature (see for example [AFMO][ASV][BS][CTZ][FKRW][KRI]), and they play an important role the theory of integrable systems. The above central extension is normalized so that $\mathcal{M}_c$ is reducible if and only if $c \in \mathbb{Z}$. It was shown in [FKRW] that for every integer $n \geq 1$, $\mathcal{I}_n$ is generated as a vertex algebra ideal by a singular vector of weight $n+1$, and

$$\mathcal{W}_{1+\infty,n} \cong \mathcal{W}(gl_n).$$

Key words and phrases. invariant theory; vertex algebra; deformation; orbifold construction; strong finite generation; $\mathcal{W}$-algebra.
In particular, $\mathcal{W}_{1+n,n}$ has a minimal strong generating set consisting of a field in each weight 1, 2, . . . , $n$. The case of negative integral central charge is more complicated. It was shown in [1] that $L_{-n}$ is generated by a singular vector of weight $(n + 1)\frac{1}{2}$, and $\mathcal{W}_{1+n,-n}$ has a minimal strong generating set consisting of a field in each weight 1, 2, . . . , $n^2 + 2n$. Wang showed in [W] that $\mathcal{W}_{1+\infty,-1}$ is isomorphic to $\mathcal{W}(\mathfrak{gl}_3)$ with central charge $-2$, but for $n > 1$ it is not known if $\mathcal{W}_{1+n,-n}$ can be identified with a standard $\mathcal{W}$-algebra.

The super analogue of $\mathcal{D}$ is the Lie superalgebra $\mathcal{SD}$ of regular differential operators on the super circle $S^{1|1}$. As above, it has a universal central extension $\widehat{SD}$, and for each $c \in \mathbb{C}$ $\widehat{SD}$ admits a vacuum module $\mathcal{M}_c(\widehat{SD})$. This module has a vertex superalgebra structure, and is freely generated by fields
\[
\{ J^{0,k}, J^1,k, J^{+,k}, J^{-,k} | k \geq 0 \}
\]
of weights $k + 1, k + 1, k + 1/2, k + 3/2$, respectively. Unlike the modules $\mathcal{M}_c$ which are all distinct, $\mathcal{M}_c(\widehat{SD}) \cong \mathcal{M}_{-c}(\widehat{SD})$ for all $c$. There are actions of the affine vertex superalgebra $V_c(\mathfrak{gl}(1|1))$ and the $N = 2$ superconformal algebra $A_c$ of central charge $c$ on $\mathcal{M}_c(\widehat{SD})$. Moreover, $\{ J^{0,k} | k \geq 0 \}$ and $\{ J^{1,k} | k \geq 0 \}$ generate copies of $\mathcal{M}_c$ and $\mathcal{M}_{-c}$, respectively, which form a Howe pair (i.e., a pair of mutual commutants) inside $\mathcal{M}_c(\widehat{SD})$.

The super $\mathcal{W}_{1+\infty}$ algebra $V_c(\widehat{SD})$ is the unique irreducible quotient of $\mathcal{M}_c(\widehat{SD})$ by its maximal proper graded $\widehat{SD}$-submodule $\mathcal{SL}_c$. We denote the map $\mathcal{M}_c(\widehat{SD}) \to V_c(\widehat{SD})$ by $\pi_c$, and we denote $\pi_c(J^{a,k})$ by $j^{a,k}$ for $a = 0, 1, \pm$. There are induced actions of $V_c(\mathfrak{gl}(1|1))$ and $A_c$ on $V_c(\widehat{SD})$, as well as copies of $\mathcal{W}_{1+\infty,c}$ and $\mathcal{W}_{1+\infty,-c}$ which form a Howe pair inside $V_c(\widehat{SD})$.

For $n \in \mathbb{Z}$, $\mathcal{M}_n(\widehat{SD})$ is reducible and $V_n(\widehat{SD})$ has a nontrivial structure. Our main goal in this paper is to elucidate this structure. Since $V_n(\widehat{SD}) \cong V_{-n}(\widehat{SD})$ and $V_0(\widehat{SD}) \cong \mathbb{C}$, it suffices to consider the case $n \geq 1$. This problem was posed by Cheng-Wang; see Problem 3 at the end of [CW]. Our starting point is a free field realization of $V_n(\widehat{SD})$ due to Awata-Fukuma-Matsuo-Odake as the $GL_n$-invariant subalgebra of the $bc\beta\gamma$-system $\mathcal{F}$ of rank $n$ [AFMO]. We will show that $\mathcal{SL}_n$ is generated as a vertex algebra ideal by a singular vector of weight $n + 1/2$, and that $V_n(\widehat{SD})$ has a minimal strong generating set consisting of the following $4n$ fields:
\[
\{ j^{0,k}, j^{1,k}, j^{+,k}, j^{-,k} | k = 0, \ldots, n - 1 \}.
\]

Next we show that $V_n(\widehat{SD})$ admits a deformation as the limit of a family of commutant algebras. The rank $n$ $bc\beta\gamma$-system $\mathcal{F}$ has a natural action of $V_0(\mathfrak{gl}_n)$, and we obtain a diagonal homomorphism $V_k(\mathfrak{gl}_n) \to V_k(\mathfrak{gl}_n) \otimes \mathcal{F}$ for all $k$. We define
\[
B_{n,k} = \text{Com}(V_k(\mathfrak{gl}_n), V_k(\mathfrak{gl}_n) \otimes \mathcal{F}).
\]
We have $\lim_{k \to \infty} B_{n,k} = V_n(\widehat{SD})$, and for generic values of $k$, $B_{n,k}$ has a minimal strong generating set consisting of $4n$ generators and has the same graded character as $V_n(\widehat{SD})$.

Next, we consider a family of $\mathcal{W}$-algebras $\mathcal{W}_{n,k}$ associated naturally to the Lie superalgebra $\mathfrak{gl}(n|n)$. It is defined as a certain subalgebra of the joint kernel of screening operators corresponding to the purely odd simple root system of $\mathfrak{gl}(n|n)$. We expect that $\mathcal{W}_{n,k}$ coincides with the joint kernel of the screening charges, although we are unable to prove this at present. The $\mathcal{W}$-algebras of simple affine Lie (super) algebras $\hat{\mathfrak{g}}$ [FF1, FF2, KRW]
are defined via the quantum Hamiltonian reduction, which is a certain semi-infinite cohomology. These \( \mathcal{W} \)-algebras are associated to the principal embedding of \( \mathfrak{sl}_2 \) in \( \mathfrak{g} \) and they usually can also be realized as the intersection of kernels of screening charges corresponding to a simple root system of \( \mathfrak{g} \). A simple root system of a Lie superalgebra is not unique, and in our case it turns out that a purely odd simple root system is most suitable. We will show that
\[
(1.2) \quad \mathcal{W}_n(\mathcal{SD}) \cong \lim_{k \to \infty} \mathcal{W}_{n,k},
\]
and we regard this as an analogue of the isomorphism \( \mathcal{W}_{1+\infty,n} \cong \mathcal{W}(\mathfrak{gl}_n) \). In the case \( n = 2 \), we find a minimal strong generating set for \( \mathcal{W}_{2,k} \) consisting of eight fields, and show by explicit computation that \( \mathcal{W}_{2,k+2} \) has the same operator product algebra as \( \mathcal{B}_{2,k} \). More generally, we conjecture that \( \mathcal{W}_{n,k+n} \) is isomorphic to \( \mathcal{B}_{n,k} \) for all \( k \) and \( n \).

There is also a commutant realization of the deformable family of \( \mathcal{W} \)-algebras of \( \mathfrak{sl}_n \), namely \( \text{Com}(\mathcal{W}_{k+1}(\mathfrak{sl}_n), \mathcal{V}_k(\mathfrak{sl}_n) \otimes \mathcal{V}_1(\mathfrak{sl}_n)) \) \([BS]\). In physics the corresponding conformal field theories are called \( \mathcal{W}_n \) minimal models and they have received much attention recently as tentative dual theories to three dimensional higher spin gravity \([GG]\). The supersymmetric analogue \([CHR]\) has the \( \mathcal{W} \)-superalgebra of \( \mathfrak{sl}(n+1|n) \) as coset algebra whose twisted algebra in turn is argued to be related to the \( \mathcal{W} \)-superalgebra of \( \mathfrak{gl}(n|n) \) \([I]\).

2. Vertex Algebras

In this section, we define vertex algebras, which have been discussed from various different points of view in the literature \([B][FBZ][FLH][FLM][K][LiI][LZ]\). We will follow the formalism developed in \([LZ]\) and partly in \([Li]\). Let \( V = V_0 \oplus V_1 \) be a super vector space over \( \mathbb{C} \), and let \( z, w \) be formal variables. By \( \text{QO}(V) \), we mean the space of all linear maps
\[
V \to V((z)) := \{ \sum_{n \in \mathbb{Z}} v(n) z^{-n-1} | v(n) \in V, v(n) = 0 \text{ for } n > 0 \}.
\]
Each element \( a \in \text{QO}(V) \) can be uniquely represented as a power series
\[
a = a(z) := \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \in \text{End}(V)[[z, z^{-1}]].
\]
We refer to \( a(n) \) as the \( n \)th Fourier mode of \( a(z) \). Each \( a \in \text{QO}(V) \) is of the shape \( a = a_0 + a_1 \) where \( a_i : V_j \to V_{i+j}((z)) \) for \( i, j \in \mathbb{Z}/2\mathbb{Z} \), and we write \( |a_i| = i \).

On \( \text{QO}(V) \) there is a set of nonassociative bilinear operations \( \circ_n \), indexed by \( n \in \mathbb{Z} \), which we call the \( n \)th circle products. For homogeneous \( a, b \in \text{QO}(V) \), they are defined by
\[
a(w) \circ_n b(w) = \text{Res}_z a(z) b(w) \mathcal{i}_{|z|>|w|}(z-w)^n - (-1)^{|a||b|} \text{Res}_z b(w) a(z) \mathcal{i}_{|w|>|z|}(z-w)^n.
\]
Here \( \mathcal{i}_{|z|>|w|} f(z, w) \in \mathbb{C}[[z, z^{-1}, w, w^{-1}]] \) denotes the power series expansion of a rational function \( f \) in the region \( |z| > |w| \). We usually omit the symbol \( \mathcal{i}_{|z|>|w|} \) and just write \( (z-w)^{-1} \) to mean the expansion in the region \( |z| > |w| \), and write \( -(w-z)^{-1} \) to mean the expansion in \( |w| > |z| \). It is easy to check that \( a(w) \circ_n b(w) \) above is a well-defined element of \( \text{QO}(V) \).
The nonnegative circle products are connected through the operator product expansion (OPE) formula. For \( a, b \in \text{QO}(V) \), we have
\[
(2.1) \quad a(z)b(w) = \sum_{n \geq 0} a(w) \circ_n b(w) (z-w)^{-n-1} + :a(z)b(w):,
\]
which is often written as \( a(z)b(w) \sim \sum_{n \geq 0} a(w) \circ_n b(w) (z-w)^{-n-1} \), where \( \sim \) means equal modulo the term
\[
:a(z)b(w) := a(z)_- b(w) + (-1)^{|a||b|} b(w) a(z)_+.
\]
Here \( a(z)_- = \sum_{n < 0} a(n) z^{-n-1} \) and \( a(z)_+ = \sum_{n \geq 0} a(n) z^{-n-1} \). Note that \( :a(w)b(w): \) is a well-defined element of \( \text{QO}(V) \). It is called the Wick product of \( a \) and \( b \), and it coincides with \( a \circ_{-1} b \). The other negative circle products are related to this by\[ n! a(z) \circ_{-n-1} b(z) = (\partial^n a(z)) b(z) :,\]
where \( \partial \) denotes the formal differentiation operator \( \frac{d}{dz} \). For \( a_1(z), \ldots, a_k(z) \in \text{QO}(V) \), the \( k \)-fold iterated Wick product is defined to be
\[
(2.2) \quad :a_1(z)a_2(z) \cdots a_k(z): := a_1(z)b(z) :,
\]
where \( b(z) = :a_2(z) \cdots a_k(z) :. \) We often omit the formal variable \( z \) when no confusion can arise.

The set \( \text{QO}(V) \) is a nonassociative algebra with the operations \( \circ_n \), which satisfy \( 1 \circ_n a = \delta_{n,-1} a \) for all \( n \), and \( a \circ_n 1 = \delta_{n,-1} a \) for \( n \geq -1 \). In particular, \( 1 \) behaves as a unit with respect to \( \circ_{-1} \). A linear subspace \( A \subset \text{QO}(V) \) containing \( 1 \) which is closed under the circle products will be called a quantum operator algebra (QOA). Note that \( A \) is closed under \( \partial \) since \( \partial a = a \circ_{-2} 1 \). Many formal algebraic notions are immediately clear: a homomorphism is just a linear map that sends \( 1 \) to \( 1 \) and preserves all circle products; a module over \( A \) is a vector space \( M \) equipped with a homomorphism \( A \to \text{QO}(M) \), etc. A subset \( S = \{a_i \mid i \in I\} \) of \( A \) is said to generate \( A \) if every element \( a \in A \) can be written as a linear combination of nonassociative words in the letters \( a_i, \circ_n \), for \( i \in I \) and \( n \in \mathbb{Z} \). We say that \( S \) strongly generates \( A \) if every \( a \in A \) can be written as a linear combination of words in the letters \( a_i, \circ_n \) for \( n < 0 \). Equivalently, \( A \) is spanned by the collection \( \{ :\partial^{k_1} a_{i_1}(z) \cdots \partial^{k_m} a_{i_m}(z) : \mid i_1, \ldots, i_m \in I, k_1, \ldots, k_m \geq 0\} \).

We say that \( a, b \in \text{QO}(V) \) quantum commute if \( (z-w)^N [a(z), b(w)] = 0 \) for some \( N \geq 0 \). Here \([,]\) denotes the super bracket. This condition implies that \( a \circ_n b = 0 \) for \( n \geq N \), so \((2.1)\) becomes a finite sum. A commutative quantum operator algebra (CQOA) is a QOA whose elements pairwise quantum commute. Finally, the notion of a CQOA is equivalent to the notion of a vertex algebra. Every CQOA \( A \) is itself a faithful \( A \)-module, called the left regular module. Define \( \rho : A \to \text{QO}(A), \quad a \mapsto \hat{a}, \quad \hat{a}(\zeta)b = \sum_{n \in \mathbb{Z}} (a \circ_n b) \zeta^{-n-1} \).

Then \( \rho \) is an injective QOA homomorphism, and the quadruple of structures \((A, \rho, 1, \partial)\) is a vertex algebra in the sense of [FLM]. Conversely, if \((V, Y, 1, D)\) is a vertex algebra, the collection \( Y(V) \subset \text{QO}(V) \) is a CQOA. We will refer to a CQOA simply as a vertex algebra throughout the rest of this paper.

Let \( \mathcal{R} \) be the category of vertex algebras \( A \) equipped with a \( \mathbb{Z}_{\geq 0} \) filtration
\[
(2.3) \quad A_{(0)} \subset A_{(1)} \subset A_{(2)} \subset \cdots, \quad A = \bigcup_{k \geq 0} A_{(k)}
\]
such that $\mathcal{A}_{(0)} = \mathbb{C}$, and for all $a \in \mathcal{A}_{(k)}, b \in \mathcal{A}_{(l)}$, we have
\begin{align}
(2.4) & \quad a \circ_n b \in \mathcal{A}_{(k+l)}, \quad \text{for } n < 0, \\
(2.5) & \quad a \circ_n b \in \mathcal{A}_{(k+l-1)}, \quad \text{for } n \geq 0.
\end{align}
Elements $a(z) \in \mathcal{A}_{(d)} \setminus \mathcal{A}_{(d-1)}$ are said to have degree $d$.

Filtrations on vertex algebras satisfying (2.4)-(2.5) were introduced in \cite{LiII}, and are known as good increasing filtrations. Setting $\mathcal{A}_{(-1)} = \{0\}$, the associated graded object $gr(\mathcal{A}) = \bigoplus_{k \geq 0} \mathcal{A}_{(k)}/\mathcal{A}_{(k-1)}$ is a $\mathbb{Z}_{\geq 0}$-graded associative, (super)commutative algebra with a unit 1 under a product induced by the Wick product on $\mathcal{A}$. For each $r \geq 1$ we have the projection
\begin{equation}
(2.6) \quad \phi_r : \mathcal{A}_{(r)} \to \mathcal{A}_{(r)}/\mathcal{A}_{(r-1)} \subset gr(\mathcal{A}).
\end{equation}
Moreover, $gr(\mathcal{A})$ has a derivation $\partial$ of degree zero (induced by the operator $\partial = \frac{d}{dz}$ on $\mathcal{A}$), and for each $a \in \mathcal{A}_{(d)}$ and $n \geq 0$, the operator $a\circ_n$ on $\mathcal{A}$ induces a derivation of degree $d - k$ on $gr(\mathcal{A})$, which we denote by $a(n)$. Here
\begin{equation}
k = \sup\{j \geq 1 \mid \mathcal{A}_{(r)} \circ_n \mathcal{A}_{(s)} \subset \mathcal{A}_{(r+s-j)}, \forall r, s, n \geq 0\},
\end{equation}
as in \cite{LL}. Finally, these derivations give $gr(\mathcal{A})$ the structure of a vertex Poisson algebra.

The assignment $\mathcal{A} \mapsto gr(\mathcal{A})$ is a functor from $\mathcal{R}$ to the category of $\mathbb{Z}_{\geq 0}$-graded (super)commutative rings with a differential $\partial$ of degree zero, which we will call $\partial$-rings. A $\partial$-ring is just an abelian vertex algebra, that is, a vertex algebra $\mathcal{V}$ in which $[a(z), b(w)] = 0$ for all $a, b \in \mathcal{V}$. A $\partial$-ring $A$ is said to be generated by a subset $\{a_i \mid i \in I\}$ if $\{\partial^k a_i \mid i \in I, k \geq 0\}$ generates $A$ as a graded ring. The key feature of $\mathcal{R}$ is the following reconstruction property \cite{LL}:

**Lemma 2.1.** Let $\mathcal{A}$ be a vertex algebra in $\mathcal{R}$ and let $\{a_i \mid i \in I\}$ be a set of generators for $gr(\mathcal{A})$ as a $\partial$-ring, where $a_i$ is homogeneous of degree $d_i$. If $a_i(z) \in \mathcal{A}_{(d_i)}$ are vertex operators such that $\phi_{d_i}(a_i(z)) = a_i$, then $\mathcal{A}$ is strongly generated as a vertex algebra by $\{a_i(z) \mid i \in I\}$.

As shown in \cite{LL}, there is a similar reconstruction property for kernels of surjective morphisms in $\mathcal{R}$. Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism in $\mathcal{R}$ with kernel $\mathcal{J}$, such that $f$ maps $\mathcal{A}_{(k)}$ onto $\mathcal{B}_{(k)}$ for all $k \geq 0$. The kernel $J$ of the induced map $gr(f) : gr(\mathcal{A}) \to gr(\mathcal{B})$ is a homogeneous $\partial$-ideal (i.e., $\partial J \subset J$). A set $\{a_i \mid i \in I\}$ such that $a_i$ is homogeneous of degree $d_i$ is said to generate $J$ as a $\partial$-ideal if $\{\partial^k a_i \mid i \in I, k \geq 0\}$ generates $J$ as an ideal.

**Lemma 2.2.** Let $\{a_i \mid i \in I\}$ be a generating set for $J$ as a $\partial$-ideal, where $a_i$ is homogeneous of degree $d_i$. Then there exist vertex operators $a_i(z) \in \mathcal{A}_{(d_i)}$ with $\phi_{d_i}(a_i(z)) = a_i$, such that $\{a_i(z) \mid i \in I\}$ generates $\mathcal{J}$ as a vertex algebra ideal.

3. The $\mathcal{W}_{1+\infty}$ Algebra

Let $\mathcal{D}$ be the Lie algebra of regular differential operators on $\mathbb{C} \setminus \{0\}$, with coordinate $t$. A standard basis for $\mathcal{D}$ is
\begin{equation}
J^l_k = -t^{l+k}(\partial_t)^l, \quad k \in \mathbb{Z}, \quad l \in \mathbb{Z}_{\geq 0},
\end{equation}
where $\partial_t = \frac{d}{dt}$. An alternative basis is $\{t^k D^l \mid k \in \mathbb{Z}, \ l \in \mathbb{Z}_{\geq 0}\}$, where $D = t \partial_t$. There is a 2-cocycle on $\mathcal{D}$ given by
\begin{equation}
(3.1) \quad \Psi(f(t)(\partial_t)^m, g(t)(\partial_t)^n) = \frac{m!n!}{(m + n + 1)!} \text{Res}_{t=0} f^{(n+1)}(t)g^{(m)}(t) dt,
\end{equation}
and a corresponding central extension \( \hat{D} = D \oplus \mathbb{C} \kappa \), which was first studied by Kac-Peterson in [KP]. \( \hat{D} \) has a \( \mathbb{Z} \)-grading \( \hat{D} = \bigoplus_{j \in \mathbb{Z}} \hat{D}_j \) by weight, given by
\[
\text{wt}(J_k^i) = k, \quad \text{wt}(\kappa) = 0,
\]
and a triangular decomposition \( \hat{D} = \hat{D}_+ \oplus \hat{D}_0 \oplus \hat{D}_- \), where \( \hat{D}_\pm = \bigoplus_{j \in \pm \mathbb{N}} \hat{D}_j \) and \( \hat{D}_0 = D_0 \oplus \mathbb{C} \kappa \).

Let \( P \) be the parabolic subalgebra of \( D \) consisting of differential operators which extend to all of \( \mathbb{C} \), which has a basis \( \{ J^i_k \mid l \geq 0, \ l+k \geq 0 \} \). The cocycle \( \Psi \) vanishes on \( P \), so \( P \) may be regarded as a subalgebra of \( \hat{D} \), and \( \hat{D}_0 \oplus \hat{D}_+ \subset \hat{P} \), where \( \hat{P} = P \oplus \mathbb{C} \kappa \). Given \( c \in \mathbb{C} \), let \( C_c \) denote the one-dimensional \( \hat{P} \)-module on which \( \kappa \) acts by \( c \cdot \text{id} \) and \( J^i_k \) acts by zero. The induced \( \hat{D} \)-module
\[
M_c = U(\hat{D}) \otimes_{U(\hat{P})} C_c
\]
is known as the vacuum \( \hat{D} \)-module of central charge \( c \). \( M_c \) has a vertex algebra structure and is generated by fields
\[
J^i(z) = \sum_{k \in \mathbb{Z}} J^i_k z^{-k-1}, \quad l \geq 0
\]
of weight \( l + 1 \). The modes \( J^i_k \) represent \( \hat{D} \) on \( M_c \), and as in [LI], we rewrite these fields in the form
\[
J^i(z) = \sum_{k \in \mathbb{Z}} J^i(k) z^{-k-1},
\]
where \( J^i(k) = J^i_{k-l} \). In fact, \( M_c \) is freely generated by \( \{ J^i(z) \mid l \geq 0 \} \); the set of iterated Wick products
\[
: \partial^{i_1} J^{i_1}(z) \cdots \partial^{i_r} J^{i_r}(z) :,
\]
such that \( l_1 \leq \cdots \leq l_r \) and \( i_a \leq i_b \) if \( l_a = l_b \), forms a basis for \( M_c \).

A weight-homogeneous element \( \omega \in M_c \) is called a singular vector if \( J^i \circ_k \omega = 0 \) for all \( k > l \geq 0 \). The maximal proper \( D \)-submodule \( L_c \) is the vertex algebra ideal generated by all singular vectors \( \omega \neq 0 \), and the unique irreducible quotient \( M_c/L_c \) is denoted by \( \mathcal{W}_{1+\infty,c} \). The cocycle (3.1) is normalized so that \( M_c \) is reducible if and only if \( c \in \mathbb{Z} \). For each integer \( n \geq 1 \), \( L_n \) is generated by a singular vector of weight \( n+1 \), and \( \mathcal{W}_{1+\infty,n} \) is isomorphic to \( \mathcal{W}(\mathfrak{gl}_n) \) with central charge \( n \) [FKRW]. In [LI] it was shown that \( L_{-n} \) is generated by a singular vector of weight \( (n+1)^2 \), and \( \mathcal{W}_{1+\infty,-n} \) has a minimal strong generating set consisting of a field in each weight \( 1, 2, \ldots, n^2+2n \). It is known that \( \mathcal{W}_{1+\infty,-1} \) is isomorphic to \( \mathcal{W}(\mathfrak{gl}_3) \) of central charge \( -2 \) [W], but no identification of \( \mathcal{W}_{1+\infty,-n} \) with a standard \( \mathcal{W} \)-algebra is known for \( n > 1 \).

### 4. The Super \( \mathcal{W}_{1+\infty} \) Algebra

Following the notation in [CW], we denote by \( SD \) the Lie superalgebra of regular differential operators on the super circle \( S^{1|1} \). There is a standard basis for \( SD \) given by
\[
t^{k+1}(\partial_t)^l \theta \partial_{\theta}, \quad t^{k+1}(\partial_t)^l \partial_{\theta} \theta, \quad t^{k+1}(\partial_t)^l \theta, \quad t^{k+1}(\partial_t)^l \partial_{\theta}, \quad l \in \mathbb{Z}_{\geq 0}, \quad k \in \mathbb{Z}.
\]
Here \( \theta \) is an odd indeterminate which commutes with \( t \). The odd elements \( \theta \) and \( \partial_{\theta} \) generate a four-dimensional Clifford algebra \( Cl \) with relation \( \theta \partial_{\theta} + \partial_{\theta} \theta = 1 \), and \( SD = D \otimes Cl \).
Let $M(1,1)$ be the set of $2 \times 2$ matrices of the form
\[
\begin{pmatrix}
\alpha^0 & \alpha^+ \\
\alpha^- & \alpha^1
\end{pmatrix},
\]
where $\alpha^a \in \mathbb{C}$ for $a = 0, 1, \pm$. There is a natural $\mathbb{Z}_2$-gradation on $M(1,1)$ where we define $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ to be even, and $M_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $M_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to be odd. The supertrace $\text{Str}$ of the above matrix is $\alpha^0 - \alpha^1$. We have an isomorphism $\text{Cl} \cong M(1,1)$ of associative superalgebras given by
\[
M_0 \mapsto \partial_\theta \theta, \quad M_1 \mapsto \theta \partial_\theta, \quad M_+ \mapsto \partial_\theta, \quad M_- \mapsto \theta.
\]
Therefore we can regard $\hat{SD}$ as the superalgebra of $2 \times 2$ matrices with coefficients in $\mathcal{D}$. Let $F(D)$ denote the matrix
\[
\begin{pmatrix}
f_0(D) & f_+(D) \\
f_-(D) & f_1(D)
\end{pmatrix}, \quad D = t \partial_t, \quad f_a(x) \in \mathbb{C}[x],
\]
which we regard as an element of $\hat{SD}$. Define a 2-cocycle $\Psi$ on $\hat{SD}$ by
\[
\Psi(t^r F(D), t^s G(D)) = \begin{cases}
\sum_{-r \leq j \leq -1} \text{Str}(F(j)G(j + r)) & r = -s \geq 0 \\
0 & r + s \neq 0.
\end{cases}
\]
We obtain the corresponding one-dimensional central extension $\hat{SD} = SD \oplus \mathbb{C}C$, with bracket
\[
[t^r F(D), t^s G(D)] = t^{r+s} (F(D + s)G(D) - (-1)^{|F||G|} F(D)G(D + r)) + \Psi(t^r F(D), t^s G(D)) C.
\]
Here $| \cdot |$ denotes the $\mathbb{Z}_2$-gradation. We introduce the principal $\mathbb{Z}$-gradation on $\hat{SD}$ by
$\text{wt}(C) = 0, \quad \text{wt}(t^n f(D) \partial_\theta) = \text{wt}(t^n f(D) \theta \partial_\theta) = n, \quad \text{wt}(t^{n+1} f(D) \partial_\theta) = \text{wt}(t^n f(D) \theta) = n + \frac{1}{2}.$

This defines the triangular decomposition
\[
\hat{SD} = \hat{SD}_- \oplus \hat{SD}_0 \oplus \hat{SD}_+, \quad \hat{SD}_\pm = \bigoplus_{j \in \mathbb{N}/2} \hat{SD}_j.
\]
Define $J_n^{a,k} = J_n^{k} M_a$ for $a = 0, 1, \pm$, and define the parabolic subalgebra $\mathcal{SP} \subset \hat{SD}$ to be the Lie algebra spanned by
\[
\{ J_n^{a,k} | k + n \geq 0, n \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}, a = 0, 1, \pm \}.
\]
The cocyle (4.1) vanishes on $\mathcal{SP}$, so $\mathcal{SP}$ is a subalgebra of $\hat{SD}$, and $\hat{SD}_0 \oplus \hat{SD}_+ \subset \hat{SP}$, where $\hat{SP} = SP \oplus \mathbb{C}C$.

Given $c \in \mathbb{C}$, let $C_c$ denote the one-dimensional $\hat{SP}$-module on which $C$ acts by $c \cdot \text{id}$ and $J_n^{a,k}$ acts by zero. The induced $\hat{SD}$-module
\[
\mathcal{M}_c(\hat{SD}) = U(\hat{SD}) \otimes_{U(\hat{SP})} C_c,
\]
is known as the vacuum $\hat{SD}$-module of central charge $c$.

**Proposition 4.1.** The vacuum modules at central charge $c$ and $-c$ are isomorphic
\[
\mathcal{M}_c(\hat{SD}) \cong \mathcal{M}_{-c}(\hat{SD}).
\]
Proof. We will construct an automorphism of $\hat{SD}$ that maps $C$ to $-C$ and hence $M_c(\hat{SD})$ carries also an action of $\hat{SD}$ at central charge $-c$, establishing the isomorphism.

Define the map $\Pi$ on $SD$ by

$$\Pi(F(D)) = \begin{pmatrix} f_1(D) & f_-(D) \\ f_+(D) & f_0(D) \end{pmatrix},$$

this map respects the graded commutator of $2 \times 2$ matrices and $\Pi \circ \Pi$ acts as the identity, hence $\Pi$ defines an automorphism on $SD$. $\Pi$ does not respect the supertrace but changes sign, hence also the cocycle satisfies

$$\Psi(t^r \Pi(F(D)), t^s \Pi(G(D))) = -\Psi(t^r F(D), t^s G(D)).$$

Defining $\Pi(C) = -C$ extends $\Pi$ to an automorphism of $\hat{SD}$. \hfill \Box

The module $M_c(\hat{SD})$ possesses a vertex superalgebra structure, and is freely generated by fields

$$J_{0,k}^0(z) = \sum_{n \in \mathbb{Z}} J_{n,k}^0 z^{-n-k-1}, \quad J_{1,k}^1(z) = \sum_{n \in \mathbb{Z}} J_{n,k}^1 z^{-n-k-1},$$

$$J_{+,-}^{+,-}(z) = \sum_{n \in \mathbb{Z}} J_{n,k}^{+,-} z^{-n-k-1/2}, \quad J_{-,-}^{-,-}(z) = \sum_{n \in \mathbb{Z}} J_{n,k}^{-,-} z^{-n-k-3/2},$$

for $k \geq 0$. Here $J_{0,0}^0, J_{1,0}^1, J_{+,-}^{+,-}, J_{-,-}^{-,-}$ have weights $k+1, k+1, k+1/2, k+3/2$, respectively. The modes $J_{n,k}^a$ represent $\hat{SD}$ on $M_c(\hat{SD})$, and we rewrite these fields in the form

$$J_{a,k}^a(z) = \sum_{k \in \mathbb{Z}} J_{a,k}^a(n) z^{-n-1}, \quad J_{a,k}^a(n) = J_{n-k}^a.$$

Define a filtration

$$(M_c(\hat{SD}))_{(0)} \subset (M_c(\hat{SD}))_{(1)} \subset \cdots$$

on $M_c(\hat{SD})$ as follows: for $k \geq 0$, $(M_c(\hat{SD}))_{(2k)}$ is the span of iterated Wick products in the generators $J_{a,k}$ and their derivatives of length at most $k$, and $(M_c(\hat{SD}))_{(2k+1)} = (M_c(\hat{SD}))_{(2k)}$. In particular, each $J_{a,k}^a$ and its derivatives have degree 2. Equipped with this filtration, $M_c(\hat{SD})$ lies in the category $R$, and $\text{gr}(M_c(\hat{SD}))$ is the polynomial superalgebra $\mathbb{C}[\partial^l J_{a,k}^a | l, k \geq 0]$. Each element $J_{a,k}^a(m) \in SP$ for $k, m \geq 0$ gives rise to a derivation of degree zero on $\text{gr}(M_c(\hat{SD}))$, and this action of $\hat{SP}$ on $\text{gr}(M_c(\hat{SD}))$ is independent of $c$.

There are some substructures of $M_c(\hat{SD})$ that will be important to us. First, the fields $J_{0,0}^0, J_{1,0}^1, J_{+,-}^{+,-}, J_{-,-}^{-,-}$ satisfy

$$J_{0,0}^0(z) J_{0,0}^0(w) \sim c(z-w)^{-2}, \quad J_{1,0}^1(z) J_{1,0}^1(w) \sim -c(z-w)^{-2},$$

$$J_{0,0}^0(z) J_{+,-}^{+,-}(w) \sim J_{+,-}^{+,-}(w)(z-w)^{-1}, \quad J_{0,0}^0(z) J_{-,-}^{-,-}(w) \sim -J_{+,-}^{+,-}(w)(z-w)^{-1},$$

$$J_{1,0}^1(z) J_{+,-}^{+,-}(w) \sim -J_{+,-}^{+,-}(w)(z-w)^{-1}, \quad J_{1,0}^1(z) J_{-,-}^{-,-}(w) \sim J_{+,-}^{+,-}(w)(z-w)^{-1},$$

$$J_{+,-}^{+,-}(z) J_{+,-}^{+,-}(w) \sim c(z-w)^{-2} - (J_{0,0}^0(w) + J_{1,0}^1(w))(z-w)^{-1},$$

so they generate a copy of the affine vertex superalgebra associated to $\mathfrak{gl}(1|1)$ at level $c$. Next, recall that the $N = 2$ superconformal vertex algebra $A_c$ of central charge $c$ is
generated by fields $F, L, G^\pm$, where $L$ is a Virasoro element of central charge $c$, $F$ is an even primary of weight one, and $G^\pm$ are odd primaries of weight $\frac{3}{2}$. These fields satisfy

$$F(z)F(w) \sim \frac{c}{3}(z-w)^{-1}, \quad G^\pm(z)G^\pm(w) \sim 0,$$

(4.3) $$F(z)G^\pm(w) \sim \pm G^\pm(w)(z-w)^{-1},$$

$$G^+(z)G^-(w) \sim \frac{c}{3}(z-w)^{-3} + F(w)(z-w)^{-2} + (L(w) + \frac{1}{2}\partial F(w))(z-w)^{-1}.$$

We have a vertex algebra homomorphism $A_c \to M_c(\mathcal{SD})$ given by

$$F \mapsto \frac{2}{3}J^{0,0} - \frac{1}{3}J^{1,0}, \quad L \mapsto J^{0,1} + J^{1,1} - \frac{2}{3}\partial J^{0,0} - \frac{1}{6}\partial J^{1,1}, \quad G^+ \mapsto J^{-0}, \quad G^- \mapsto -J^{+,1} + \frac{1}{3}\partial J^{+,0}.$$ 

(4.4)

Finally, \{\(J^{0,k}\mid k \geq 0\}\} and \{\(J^{1,k}\mid k \geq 0\}\} generate copies of $M_c$ and $M_{-c}$ inside $M_c(\mathcal{SD})$, respectively.

**Lemma 4.2.** $M_c$ and $M_{-c}$ form a Howe pair, i.e., a pair of mutual commutants, inside $M_c(\mathcal{SD})$.

**Proof.** We show that $\text{Com}(M_c, M_c(\mathcal{SD})) = M_{-c}$; the proof that $\text{Com}(M_{-c}, M_c(\mathcal{SD})) = M_c$ is the same. Let $\omega(\in \text{Com}(M_c, M_c(\mathcal{SD})))$, and write $\omega$ as a sum of monomials

$$\partial^{a_1}J^{0,i_1} \ldots \partial^{a_r}J^{0,i_r}\partial^{b_1}J^{1,j_1}\ldots \partial^{b_s}J^{1,j_s}\partial^{c_1}J^{+,k_1}\ldots \partial^{c_t}J^{+,k_t}\partial^{d_1}J^{-,l_1}\ldots \partial^{d_u}J^{-,l_u}.$$ 

(4.5)

Since $\omega$ commutes with $J^{0,0}$, we have $t = u$ for each such term. Suppose that $u > 0$ for some such monomial, and let $l$ be the maximal integer such that $J^{-,l}$ appears. Since $J^{0,1} \circ J^{-,l} = J^{-,l+1}$, we would have $J^{0,1} \circ \omega \neq 0$, so we conclude that $u = 0$. Therefore $\omega \in M_c \otimes M_{-c}$, and since the center of $M_c$ is trivial, we conclude that $\omega \in M_{-c}$.

**Lemma 4.3.** For each $c \in \mathbb{C}$, the sets

$$S = \{J^{0,0}, J^{1,0}, J^{+,0}, J^{-,0}, J^{0,1}\}, \quad T = \{J^{0,0}, J^{1,0}, J^{+,0}, J^{-,0}, J^{+,1}, J^{-,1}\}$$

both generate $M(\mathcal{SD})_c$ as a vertex algebra.

**Proof.** Let $\langle S \rangle$ and $\langle T \rangle$ denote the vertex subalgebras of $M(\mathcal{SD})_c$ generated by $S$ and $T$, respectively. Note that $J^{+,0} \circ J^{0,1} = J^{+,1}$ and $J^{-,0} \circ J^{0,1} = -J^{-,1}$, so $J^{+,1}$ and $J^{-,1}$ both lie in $\langle S \rangle$. Next, $J^{+,0} \circ J^{-,1} = -J^{0,1} - J^{1,1}$, which shows that $J^{1,1} \in \langle S \rangle$. So far, $J^{a,k} \in \langle S \rangle$ for $a = 0, 1, \pm$ and $k = 0, 1$. Next, we have

$$J^{0,1} \circ J^{-,1} = J^{-2}, \quad J^{1,1} \circ J^{+,1} = J^{+,2},$$

$$J^{-2} \circ J^{+,2} - (J^{-,1} \circ J^{+,2}) = -3J^{1,2}, \quad J^{1,2} \circ J^{+,2} + 2(J^{-,1} \circ J^{+,2}) = -6J^{0,2}. $$

This shows that $J^{a,k} \in \langle S \rangle$ for $a = 0, 1, \pm$ and $k = 2$.

For $k \geq 1$, we have

$$J^{0,2} \circ J^{0,k-1} = (k+1)J^{0,k} - 2\partial J^{0,k-1}, \quad J^{0,1} \circ J^{0,k} = \partial J^{0,k}.$$ 

It follows that $J^{0,k-1} = (k+1)J^{0,k}$, where $\alpha = J^{0,2} - 2\partial J^{0,1}$. Since $\alpha \in \langle S \rangle$, it follows by induction that $J^{0,k} \in \langle S \rangle$ for all $k$. Next, we have

$$J^{+,0} \circ J^{0,k} = J^{+,k}, \quad J^{-,0} \circ J^{0,k} = -J^{-,k},$$

so $J^{+,k}$ and $J^{-,k}$ lie in $\langle S \rangle$ for all $k$. Finally, $J^{+,0} \circ J^{-,k} = -J^{0,k} - J^{1,k}$, which shows that $J^{1,k}$ lies in $\langle S \rangle$ for all $k$. This shows that $M(\mathcal{SD})_c = \langle S \rangle$.\]
To prove that $\mathcal{M}(\widehat{SD})_c = \langle T \rangle$, it is enough to show that $J^{0,1} \in \langle T \rangle$. First, we have
\[
(J^{-1} \circ_0 J^{+1}) \circ_1 J^{+1} = -4J^{+2} + 2\partial J^{+1},
\]
which implies that $J^{+2} \in \langle T \rangle$. Finally, we have
\[
J^{-0} \circ_1 J^{+2} = -2J^{0,1},
\]
which shows that $J^{0,1} \in \langle T \rangle$. □

Lemma 4.3 shows that $\mathcal{M}_c(\widehat{SD})$ is a finitely generated vertex algebra. However, $\mathcal{M}_c(\widehat{SD})$ is not strongly generated by any finite set of vertex operators. This follows from the fact that gr($\mathcal{M}_c(\widehat{SD})$) is the polynomial superalgebra with generators $\partial^i J^{a,k}$ for $k, l \geq 0$ and $a = 0, 1, \pm$, which implies that there are no nontrivial normally ordered polynomial relations in $\mathcal{M}_c(\widehat{SD})$. A weight-homogeneous element $\omega \in \mathcal{M}_c(\widehat{SD})$ is called a singular vector if $\omega$ is annihilated by the operators
\[
J^{0,k}_\circ_m, \quad J^{1,k}_\circ_m, \quad J^{-k}_\circ_m, \quad J^{+,k}_\circ_r, \quad m > k, \quad r \geq k.
\]
The maximal proper $\widehat{SD}$-submodule $SL_c$ is the ideal generated by all singular vectors $\omega \neq 1$, and the super $\mathcal{W}_{1+\infty}$ algebra $V_c(\widehat{SD})$ is the unique irreducible quotient $\mathcal{M}_c(\widehat{SD})/SL_c$. We denote the projection $\mathcal{M}(\widehat{SD})_c \to V_c(\widehat{SD})$ by $\pi_c$, and we write
\[
(4.6) \quad j^{a,k} = \pi_c(J^{a,k}), \quad k \geq 0.
\]
Clearly $V_c(\widehat{SD})$ is generated as a vertex algebra by the corresponding sets
\[
\{j^{0,0}, j^{1,0}, j^{+,0}, j^{-,0}, j^{0,1}\}, \quad \{j^{0,0}, j^{1,0}, j^{+,0}, j^{-,0}, j^{+,1}, j^{-,1}\},
\]
but there may now be normally ordered polynomial relations among $\{j^{a,k}| k \geq 0\}$ and their derivatives. The actions of $V_c(\mathfrak{gl}(1|1))$ and the $N = 2$ superconformal algebra $A_c$ on $\mathcal{M}_c(\widehat{SD})$ descend to actions on $V_c(\widehat{SD})$ given by the same formulas, where $J^{a,k}$ is replaced by $j^{a,k}$. Likewise, $\{j^{0,k}| k \geq 0\}$ and $\{j^{+,k}| k \geq 0\}$ generate copies of $\mathcal{W}_{1+\infty,c}$ and $\mathcal{W}_{1+\infty,-c}$ inside $V_c(\widehat{SD})$, respectively, and these subalgebras form a Howe pair.

5. The Case of Positive Integral Central Charge

For $n \in \mathbb{Z}$, $\mathcal{M}_n(\widehat{SD})$ is reducible and $V_n(\widehat{SD})$ has a nontrivial structure. For $n \neq 0$, $J^{+,0}$ is a singular vector so $V_0(\widehat{SD}) \cong \mathbb{C}$, and since $V_n(\widehat{SD}) \cong V_{-n}(\widehat{SD})$ it suffices to consider the case $n \geq 1$. The starting point of our study is a free field realization of $V_n(\widehat{SD})$ as the $GL_n$-invariant subalgebra of the $bc\beta\gamma$-system $\mathcal{F}$ of rank $n$ [AFMO]. This indicates that the structure of $V_n(\widehat{SD})$ is deeply connected to classical invariant theory.

Given a vector space $V$ of dimension $n$, the $\beta\gamma$-system $\mathcal{S} = S(V)$, or algebra of chiral differential operators on $V$, was introduced in [FMS]. It is the unique even vertex algebra with generators $\beta^x, \gamma^x$ for $x \in V, x' \in V^*$, which satisfy the OPE relations
\[
(5.1) \quad \beta^x(z)\beta^y(w) \sim 0, \quad \gamma^x(z)\gamma^y(w) \sim 0.
\]
There is a one-parameter family of conformal structures

\[(5.2) \quad L_\lambda^S = \lambda \sum_{i=1}^n : \beta^x_i \partial \gamma^x_i : + (\lambda - 1) \sum_{i=1}^n : \partial \beta^x_i \gamma^x_i : \]

of central charge \( n(12\lambda^2 - 12\lambda + 2) \), under which \( \beta^x \) and \( \gamma^x \) are primary of conformal weights \( \lambda \) and \( 1 - \lambda \), respectively. Here \( \{x_1, \ldots, x_n\} \) is a basis for \( V \) and \( \{x'_1, \ldots, x'_n\} \) is the dual basis for \( V^* \).

Similarly, the \( bc \)-system \( \mathcal{E} = \mathcal{E}(V) \) is the unique vertex superalgebra with odd generators \( b^x, c^x \) for \( x \in V, x' \in V^* \), which satisfy the OPE relations

\[(5.3) \quad b^x(z)c^x(w) \sim \langle x', x \rangle (z - w)^{-1}, \quad c^x(z)b^x(w) \sim \langle x', x \rangle (z - w)^{-1}, \]

There is a similar family of conformal structures

\[(5.4) \quad L_\lambda^E = (1 - \lambda) \sum_{i=1}^n : \partial b^x_i c^x_i : - \lambda \sum_{i=1}^n : b^x_i \partial c^x_i : \]

of central charge \( n(-12\lambda^2 + 12\lambda - 2) \), under which \( b^x \) and \( c^x \) are primary of conformal weights \( \lambda \) and \( 1 - \lambda \), respectively. The \( bc \beta \gamma \) system \( \mathcal{F} \) is just \( \mathcal{E} \otimes \mathcal{S} \), and we will assign \( \mathcal{F} \) the conformal structure

\[ L^F = L^S_{5/6} + L^E_{1/3}, \]

under which \( \beta^x, \gamma^x, b^x, c^x \) have weights \( 5/6, 1/6, 1/3, 2/3 \), respectively.

\( \mathcal{F} \) admits a good increasing filtration

\[(5.5) \quad \mathcal{F}(0) \subset \mathcal{F}(1) \subset \cdots, \quad \mathcal{F} = \bigcup_{k \geq 0} \mathcal{F}(k), \]

where \( \mathcal{F}(k) \) is spanned by iterated Wick products of the generators \( b^x, c^x, \beta^x, \gamma^x \) and their derivatives, of length at most \( k \). This filtration is \( GL_n \)-invariant, and we have an isomorphism of supercommutative rings

\[(5.6) \quad \text{gr}(\mathcal{F}) \cong \text{Sym}(\bigoplus_{k \geq 0} (V_k \oplus V_k^*)) \otimes (\bigoplus_{k \geq 0} (U_k \oplus U_k^*)). \]

Here \( V_k, U_k \) are copies of \( V \), and \( V_k^*, U_k^* \) are copies of \( V^* \). The generators of \( \text{gr}(\mathcal{F}) \) are \( \beta_k^x, \gamma_k^x, b_k^x, c_k^x \), which correspond to the vertex operators \( \partial^k \beta^x, \partial^k \gamma^x, \partial^k b^x, \) and \( \partial^k c^x \), respectively for \( k \geq 0 \).

**Theorem 5.1.** *(Awata-Fukuma-Matsuo-Odake)* There is an isomorphism \( \mathcal{V}_n(\hat{SD}) \to \mathcal{F}^{GL_n} \) given by

\[(5.7) \quad j^{0,k} \mapsto - \sum_{i=1}^n : b^x_i \partial^k c^{x_i} :, \quad j^{1,k} \mapsto \sum_{i=1}^n : \beta^x_i \partial^k \gamma^{x_i} :, \]

\[ j^{+,k} \mapsto - \sum_{i=1}^n : b^x_i \partial^k \gamma^{x_i} :, \quad j^{-k} \mapsto \sum_{i=1}^n : \beta^x_i \partial^k c^{x_i} :, \]
The Virasoro element $L$ of $\mathcal{V}_\nu(\mathcal{S}D)$ is given by

$$L = j^{0,1} + j^{1,1} - \frac{2}{3} \partial j^{0,0} - \frac{1}{6} \partial j^{1,0}.$$  

Clearly $L$ maps to $L^F$, and the above map is a morphism in the category $\mathcal{R}$. The identification $\mathcal{V}_\nu(\mathcal{S}D) \cong \mathcal{F}^{GL_n}$ suggests an alternative strong generating set for $\mathcal{V}_\nu(\mathcal{S}D)$ coming from classical invariant theory. Since $GL_n$ preserves the filtration on $\mathcal{F}$, we have

$$\text{gr}(\mathcal{V}_\nu(\mathcal{S}D)) \cong \text{gr}(\mathcal{F}^{GL_n}) \cong \text{gr}(\mathcal{F})^{GL_n}.$$  

The generators and relations for $\text{gr}(\mathcal{F})^{GL_n}$ are given by Weyl’s first and second fundamental theorems of invariant theory for the standard representation of $GL_n$ [We]. This theorem was originally stated for the $GL_n$-invariants in the symmetric algebra, but the following is an easy generalization to the case of odd as well as even variables.

**Theorem 5.2.** (Weyl) For $k \geq 0$, let $V_k$ and $U_k$ be the copies of the standard $GL_n$-module $\mathbb{C}^n$ with basis $x_{i,k}$ and $y_{i,k}$, for $i = 1, \ldots, n$, respectively. Let $V_k^*$ and $U_k^*$ be the copies of $V^*$ with basis $x_{i,k}'$ and $y_{i,k}'$, respectively. The invariant ring

$$R = \left(\left(\text{Sym} \bigoplus_{k=0}^{\infty} (V_k \oplus V_k^*)\right) \otimes \bigwedge \bigoplus_{k=0}^{\infty} (U_k \oplus U_k^*)\right)^{GL_n}$$

is generated by the quadratics

$$q_{a,b}^0 = \sum_{i=1}^{n} y_{i,a} y_{i,b}, \quad q_{a,b}^1 = \sum_{i=1}^{n} x_{i,a} x_{i,b},$$

$$q_{a,b}^+ = \sum_{i=1}^{n} y_{i,a} x_{i,b}', \quad q_{a,b}^- = \sum_{i=1}^{n} x_{i,a} y_{i,b}'.$$

Let $Q_{k,l}^0, Q_{k,l}^1$ be even indeterminates and let $Q_{k,l}^+, Q_{k,l}^-$ be odd indeterminates for $k, l \geq 0$. The kernel $I_n$ of the homomorphism

$$\mathbb{C}[Q_{k,l}^a] \to R, \quad Q_{k,l}^a \mapsto q_{k,l}^a,$$

is generated by homogeneous polynomials $d_{I,J}$ of degree $n + 1$ in the variables $Q_{k,l}^a$. Here $I = (i_0, \ldots, i_n)$ and $J = (j_0, \ldots, j_n)$ are lists of nonnegative integers, where $i_r$ corresponds to either $V_{i_r}$ or $U_{i_r}$, and $j_s$ corresponds to either $V_{j_s}^*$ or $U_{j_s}^*$. We call indices $i_r$ and $j_s$ bosonic if they correspond to $V_{i_r}$ and $V_{j_s}^*$, and fermionic if they correspond to $U_{i_r}$ and $U_{j_s}^*$, respectively. Bosonic indices appearing in either $I$ or $J$ must be distinct, but fermionic indices can be repeated. Finally, $d_{I,J}$ is uniquely characterized by the condition that it changes sign if bosonic indices in either $I$ or $J$ are permuted, and remains unchanged if fermionic indices are permuted. If all indices are bosonic,

$$d_{I,J} = \det \begin{bmatrix} Q_{i_0,j_0}^1 & \cdots & Q_{i_0,j_n}^1 \\ \vdots & & \vdots \\ Q_{i_n,j_0}^1 & \cdots & Q_{i_n,j_n}^1 \end{bmatrix}.$$
Under the identification (5.9), the generators \( q_{k,l}^n \) correspond to strong generators

\[
\omega_{k,l}^0 = \sum_{i=1}^n: \partial^k b^x_i \partial^l c^{x_i} :, \quad \omega_{k,l}^1 = \sum_{i=1}^n: \partial^k \beta b^x_i \partial^l \gamma^{x_i} :,
\]

(5.13)

\[
\omega_{k,l}^+ = \sum_{i=1}^n: \partial^k b^x_i \partial^l \gamma^{x_i} :, \quad \omega_{k,l}^- = \sum_{i=1}^n: \partial^k \beta b^x_i \partial^l c^{x_i} :.
\]

of \( \mathcal{V}_{-n} \), satisfying \( \phi_2(\omega_{a,b}) = q_{a,b} \). In this notation, we have

\[
j^{0,k} = -\omega_{0,k}^0, \quad j^{1,k} = \omega_{0,k}^1, \quad j^{+k} = -\omega_{0,k}^+, \quad j^{-k} = \omega_{0,k}^-, \quad k \geq 0.
\]

(5.14)

For each \( m \geq 0 \), let \( A_m^a \) denote the vector space with basis \( \{ \omega_{k,l}^n \mid k + l = m \} \). We have \( \dim(A_m^a) = m + 1 \), and \( \dim(A_m^a/\partial(A_m^a-1)) = 1 \). Hence \( A_m^a \) has a decomposition

\[
A_m^a = \partial(A_m^a-1) \oplus \langle j^{a,m} \rangle,
\]

(5.15)

where \( \langle j^{a,m} \rangle \) is the linear span of \( j^{a,m} \). Clearly \( \{ \partial^l j^{0,m} \mid 0 \leq l \leq m \} \) is a basis of \( A_m^a \), so for \( k + l = m \), \( \omega_{k,l}^a \in A_m^a \) can be expressed uniquely in the form

\[
\omega_{k,l}^a = \sum_{i=0}^m \lambda_i \partial^i j^{a,m-i},
\]

(5.16)

for constants \( \lambda_i \). Hence \( \{ \partial^k j^{a,m} \mid k, m \geq 0 \} \) and \( \{ \omega_{k,m}^a \mid k, m \geq 0 \} \) are related by a linear change of variables. Using (5.16), we can define an alternative strong generating set \( \{ \Omega_{k,l}^a \mid k, l \geq 0 \} \) for \( \mathcal{M}_n(\hat{SD}) \) by the same formula: for \( k + l = m \),

\[
\Omega_{k,l}^a = \sum_{i=0}^m \lambda_i \partial^i j^{a,m-i}.
\]

Clearly \( \pi_n(\Omega_{k,l}^a) = \omega_{k,l}^a \).

6. THE STRUCTURE OF THE IDEAL \( SI_n \)

Recall that the projection \( \pi_n : \mathcal{M}_n(\hat{SD}) \rightarrow \mathcal{V}_n(\hat{SD}) \) with kernel \( SI_n \) is a morphism in the category \( \mathcal{R} \). Under the identifications

\[
\text{gr}(\mathcal{M}_n(\hat{SD})) \cong \mathbb{C}[Q_{k,l}], \quad \text{gr}(\mathcal{V}_n(\hat{SD})) \cong \mathbb{C}[q_{k,l}]/I_n,
\]

\( \text{gr}(\pi_n) \) is just the quotient map (5.11).

**Lemma 6.1.** For each classical relation \( d_{I,J} \) there exists a unique vertex operator

\[
D_{I,J} \in (\mathcal{M}_n(\hat{SD}))(2n+2) \cap SI_n
\]

satisfying

\[
\phi_{2n+2}(D_{I,J}) = d_{I,J}.
\]

These elements generate \( SI_n \) as a vertex algebra ideal.

**Proof.** Clearly \( \pi_n \) maps each filtered piece \( (\mathcal{M}_n(\hat{SD}))(k) \) onto \( (\mathcal{V}_n(\hat{SD}))(k) \), so the hypotheses of Lemma 2.2 are satisfied. Since \( I_n = \text{Ker}(\text{gr}(\pi_n)) \) is generated by \( \{ d_{I,J} \} \), we can find \( D_{I,J} \in (\mathcal{M}_n(\hat{SD}))(2n+2) \cap SI_n \) satisfying \( \phi_{2n+2}(D_{I,J}) = d_{I,J} \), such that \( \{ D_{I,J} \} \) generates
Let $\mathcal{D}'_{I,J}$ also satisfies (6.2), we would have $D_{I,J} - D'_{I,J} \in (\mathcal{M}_n(\mathcal{L}))_{2n+2} \cap \mathcal{L}_n$. Since there are no relations in $\mathcal{V}_n(\mathcal{D})$ of degree less than $2n + 2$, we have $D_{I,J} - D'_{I,J} = 0$. □

Recall the generators $b_{j,i}^c, c_{j,i}^x, z_{j,i}^x, z_{j,i}^y$ of $\text{gr}(\mathcal{F})$ corresponding to $\partial b_{j,i}^c, \partial c_{j,i}^x, \partial z_{j,i}^x, \partial z_{j,i}^y$. Let $W \subset \text{gr}(\mathcal{F})$ be the vector space with basis $\{b_{j,i}^c, c_{j,i}^x, z_{j,i}^x, z_{j,i}^y \mid j \geq 0\}$, and for each $m \geq 0$, let $W_m$ be the subspace with basis $\{b_{j,i}^c, c_{j,i}^x, z_{j,i}^x, z_{j,i}^y \mid 0 \leq j \leq m\}$. Let $\phi : W \to W$ be a linear map of weight $w \geq 1$, such that

\[
\phi(b_{j,i}^c) = \lambda_j b_{j+w}^c, \quad \phi(c_{j,i}^x) = \mu_j c_{j+w}^x, \quad \phi(z_{j,i}^x) = \lambda_j^\beta z_{j+w}^x, \quad \phi(z_{j,i}^y) = \mu_j^\gamma z_{j+w}^y
\]

for constants $\lambda_j, \mu_j, \lambda_j^\beta, \mu_j^\gamma \in \mathbb{C}$ which are independent of $i$. For example, the restrictions of $j^{0,k}(k-w)$ and $j^{1,k}(k-w)$ to $W$ is such a map for $k > w$.

**Lemma 6.2.** Fix $w \geq 1$ and $m \geq 0$, and let $\phi$ be a linear map satisfying (6.3). Then the restriction $\phi|_{W_m}$ can be expressed uniquely as a linear combination of the operators

\[
\{j^{0,k}(k-w)|_{W_m}, \ j^{1,k}(k-w)|_{W_m} \mid 0 \leq k \leq 2m + 1\}.
\]

**Proof.** The argument is the same as the proof of Lemma 6 of [111]. □

**Lemma 6.3.** Fix $w \geq 1$ and $m \geq 0$, and let $\phi$ be a linear map satisfying

\[
\phi(b_{j,i}^c) = \lambda_j z_{j+w}^c, \quad \phi(c_{j,i}^x) = 0, \quad \phi(z_{j,i}^x) = 0, \quad \phi(z_{j,i}^y) = 0
\]

for constants $\lambda_j \in \mathbb{C}$ which are independent of $i$. Then the restriction $\phi|_{W_m}$ can be expressed as a linear combination of $j^{+,k}(k-w)$ for $0 \leq k \leq 2m + 1$.

Similarly, let $\psi$ be a linear map satisfying

\[
\phi(b_{j,i}^c) = 0, \quad \phi(c_{j,i}^x) = \mu_j z_{j+w}^x, \quad \phi(z_{j,i}^x) = 0, \quad \phi(z_{j,i}^y) = 0
\]

for constants $\mu_j \in \mathbb{C}$. Then the restriction $\psi|_{W_m}$ can be expressed as a linear combination of the operators $j^{-,k}(k-w)|_{W_m}$ for $0 \leq k \leq 2m + 1$.

**Proof.** This is easy to extract from Lemma 6.2 using the $\text{gl}(1|1)$ structure. □

Let $\langle D_{I,J} \rangle$ denote the vector space with basis $\{D_{I,J}\}$ where $I, J$ are as in Theorem 5.2. We have $\langle D_{I,J} \rangle = (\mathcal{M}_n(\mathcal{L}))_{2n+2} \cap \mathcal{L}_n$, and clearly $\langle D_{I,J} \rangle$ is a module over the Lie algebra $\mathcal{S}D' \subset \mathcal{L}$ generated by $\{J^a(m) \mid m, k \geq 0\}$, since $\mathcal{S}D'$ preserves both the filtration on $\mathcal{M}_n(\mathcal{L})$ and the ideal $\mathcal{L}_n$. The action of $\mathcal{S}D'$ on $\langle D_{I,J} \rangle$ is by “weighted derivation” in the following sense. Given $I = (i_0, \ldots, i_n)$, $J = (j_0, \ldots, j_n)$ and given $\phi \in \mathcal{S}D'$ satisfying (6.3), we have

\[
\phi(D_{I,J}) = \sum_{r=0}^n \lambda_{i_r} D_{I',J} + \mu_{j_r} D_{I,J'},
\]

for lists $I' = (i_0, \ldots, i_r + w, \ldots, i_n)$ and $J' = (j_0, \ldots, j_r + w, \ldots, j_n)$. Here $\lambda_{i_r} = \lambda_{i_r}^c$ if $i_r$ is fermionic, and $\lambda_{i_r} = \lambda_{i_r}^\beta$ if bosonic. Moreover, $i_r + w$ has the same parity as $i_r$, i.e., it is bosonic (respectively fermionic) if and only if $i_r$ is. Similarly, $\mu_{j_r} = \mu_{j_r}^c$ if $j_r$ is fermionic, and $\mu_{j_r} = \mu_{j_r}^\gamma$ if $j_r$ is bosonic, and the parity of $\mu_{j_r}$ is preserved. The odd operators $\phi \in \mathcal{S}D'$
given by Lemma 6.3 have a similar derivation property except that they reverse the parity of the entries $i_r$ and $j_r$.

For each $n \geq 1$, there are four distinguished element in $\langle D_{I,J} \rangle$, which correspond to $I = (0,\ldots,0) = J$. Define $D_+$ to be the element where all entries of $I$ are fermionic, and one entry $J$ is bosonic. Similarly, define $D_-$ to be the element where one entry of $I$ is bosonic and all entries of $J$ are fermionic. Finally, define $D_0$ to be the element where all entries in both $I$ and $J$ are fermionic, and define $D_1$ to be the element where one entry of $I$ and one entry of $J$ are bosonic. Clearly $D_0, D_1, D_+, D_-$ have weights $n + 1, n + 1, n + 1/2$, and $n + 3/2$, respectively. It is clear that $D_+$ is the unique element of $SI_n$ of minimal weight $n + 1/2$, and hence is a singular vector in $\mathcal{M}_n(\tilde{S}D)$.

**Theorem 6.4.** $D_+$ generates $SI_n$ as a vertex algebra ideal.

**Proof.** Since $SI_n$ is generated by $\langle D_{I,J} \rangle$ as a vertex algebra ideal, it suffices to show that $\langle D_{I,J} \rangle$ is generated by $D_+$ as a module over $\tilde{S}\mathcal{P}$. Let $SI_n'$ denote the ideal in $\mathcal{M}_n(\tilde{S}D)$ generated by $D_+$, and let $\langle D_{I,J} \rangle^{(m)}$ denote the subspace spanned by elements $D_{I,J}$ with $|I| + |J| = m$. We will prove by induction on $m$ that $\langle D_{I,J} \rangle^{(m)} \subseteq SI_n'$.

First we need to show that $\langle D_{I,J} \rangle^{(0)} \subseteq SI_n'$, i.e., $D_0, D_1$, and $D_-$ lie in $SI_n'$. Note that $J^{-0} \circ_0 D_+ = D_0 + (n + 1)D_1$, so $D_0 + (n + 1)D_1$ lies in $SI_n'$. By Lemma 6.2 we can find $\phi \in \tilde{S}\mathcal{P}$ such that

\[
\phi(\beta_0^{x_i}) = \beta_0^{x_i}, \quad \phi(b_0^{x_i}) = 0, \quad \text{for } r > 0, \quad \phi(\gamma_0^{x_i}) = 0, \quad \phi(c_0^{x_i}) = 0, \quad \phi(b_s^{x_i}) = 0, \quad s \geq 0.
\]

We have $\phi(D_0) = 0$ and $\phi(D_1) = D_{I,J}$ where $I = (1,0,\ldots,0)$ and $J = (0,\ldots,0)$. Moreover, the entry 1 in $I$ is bosonic, and all other entries of $I$ are fermionic, and $J$ contains one bosonic entry and $n - 1$ fermionic entries. It follows that $\phi(D_0 + (n + 1)D_1) = (n + 1)D_{I,J}$, so $D_{I,J} \in SI_n'$. Next, note that $J^{-1}(1)(\beta_0^{x_i}) = 2\beta_0^{x_i}$, and $J^{-1} \circ_2 (D_{I,J}) = 2D_{I,J}$. This shows that $D_1 \in SI_n'$, so $D_0 \in SI_n'$ as well. Finally, $J^{-0} \circ_0 (D_0) = D_-$, so $D_-$ also lies in $SI_n'$.

For $m > 0$, we assume inductively that $\langle D_{I,J} \rangle^{(r)}$ lies in $SI_n'$ for $0 \leq r < m$. Fix $I = (i_0,\ldots,i_n)$ and $J = (j_0,\ldots,j_n)$ with $|I| + |J| = m$.

**Case 1:** $I = (0,\ldots,0)$, and $j_0,\ldots,j_n$ are all fermionic. Since $m > 0$, at least one of the $j_k$’s is nonzero. Let $J'$ be obtained from $J$ by replacing $j_k$ with 0. By Lemma 6.2 we can find $\phi \in \tilde{S}\mathcal{P}$ with the property that

\[
\phi(c_0^{x_i}) = c_0^{x_i}, \quad \phi(c_r^{x_i}) = 0 \quad \text{for } r > 0, \quad \phi(\gamma_0^{x_i}) = 0, \quad \phi(c_0^{x_i}) = 0, \quad \phi(b_s^{x_i}) = 0, \quad s \geq 0.
\]

Then $\phi(D_{I,J'}) = \lambda D_{I,J}$ where $\lambda$ is a nonzero constant depending on the number of indices appearing in $J$ which are zero. Since $D_{I,J'} \in SI_n'[m - j_k]$, we have $D_{I,J} \in SI_n'$.

**Case 2:** $I = (0,\ldots,0)$, and for some $0 \leq r < n$, $j_0,\ldots,j_r$ are fermionic and $j_{r+1},\ldots,j_n$ are bosonic. If one of the fermionic entries $j_k \neq 0$ for $0 \leq k \leq r$, we proceed as in Case 1. If $j_0 = \cdots = j_r$, there exists $j_k > 0$ for some $k = r + 1,\ldots,n$. Let $J'$ be obtained from $J$ by replacing the bosonic entry $j_k$ with the fermionic entry 0. Then $D_{I,J'} \in \langle D_{I,J} \rangle^{(m-j_k)} \subseteq SI_n'$. Using Lemma 6.3 we can find $\phi \in \tilde{S}\mathcal{P}$ such that

\[
\phi(c_0^{x_i}) = \gamma_0^{x_i}, \quad \phi(c_r^{x_i}) = 0 \quad \text{for } r > 0, \quad \phi(\gamma_0^{x_i}) = 0, \quad \phi(b_s^{x_i}) = 0, \quad s \geq 0.
\]

It follows that, up to a nonzero constant, $\phi(D_{I,J'}) = D_{I,J}$, so $D_{I,J} \in SI_n'$. 


Case 3: $I \neq (0, \ldots, 0)$. This is the same as Cases 1 and 2 with the roles of $I$ and $J$ reversed.

7. A MINIMAL STRONG FINITE GENERATING SET FOR $V_\eta(\widehat{SD})$

Recall that $\{j^{0,k} \mid k \geq 0\}$ generate a copy of $\mathcal{W}_{1+\infty,n}$ inside $V_\eta(\widehat{SD})$. It is well known [FKRW] that the relation $D_0$ above is a singular vector for the action of the Lie subalgebra $\mathcal{P} \subset \mathcal{SP}$, and is of the form

$$j^{0,n} = P(j^{0,0}, \ldots, j^{0,n-1}),$$

where $P$ is a normally ordered polynomial in $j^{0,0}, \ldots, j^{0,n-1}$ and their derivatives. Applying the projection $\pi_n : \mathcal{M}(\widehat{SD}) \to V_\eta(\widehat{SD})$ yields a decoupling relation

$$j^{0,n} = P(j^{0,0}, \ldots, j^{0,n-1}).$$

This relation is responsible for the isomorphism $\mathcal{W}_{1+\infty,n} \cong \mathcal{W}(\mathfrak{gl}_n)$. In fact, by applying the operators $j^{0,2}\circ_1$ repeatedly, it is easy to construct higher decoupling relations

$$j^{0,m} = Q_m(j^{0,0}, j^{1,1}, \ldots, j^{n-1}),$$

for all $m > n$. In particular, $\{j^{0,k} \mid 0 \leq k < n\}$ strongly generate $\mathcal{W}_{1+\infty,n}$. There are no nontrivial normally ordered polynomial relations among these generators and their derivatives, so they freely generate $\mathcal{W}_{1+\infty,n}$.

**Theorem 7.1.** The set $\{j^{0,k}, j^{1,k}, j^{+,k}, j^{-,k} \mid k = 0, 1, \ldots, n-1\}$ is a minimal strong generating set for $V_\eta(\widehat{SD})$ as a vertex algebra.

**Proof.** Via the inclusion $\mathcal{W}_{1+\infty,n} \to V_\eta(\widehat{SD})$ we obtain decoupling relation

$$(7.1) \quad j^{0,m} = P_m(j^{0,0}, j^{1,1}, \ldots, j^{n-1}), \quad m \geq n.$$  

We shall find all the decoupling relations by acting on this set by the copy of $\mathfrak{gl}(1|1)$ spanned by $j^{a,0}(0)$ for $a = 0, 1, \pm$.

First, acting by $j^{+,0}(0)$ on the relations (7.1) and using the fact that

$$J^{+,0}(0)(\partial^m j^{0,k}) = \partial^m J^{+,k},$$

we get relations

$$(7.2) \quad j^{+,m} = Q_m(j^{0,0}, j^{+,0}, j^{0,1}, j^{+,1}, \ldots, j^{0,n-1}, j^{+,n-1}).$$

Similarly, by acting on (7.1) by $j^{-,0}(0)$ and using

$$J^{-,0}(0)(\partial^m j^{0,k}) = -\partial^m J^{-,k},$$

we obtain decoupling relations

$$(7.3) \quad j^{-,m} = R_m(j^{0,0}, j^{-,0}, j^{0,1}, j^{-,1}, \ldots, j^{0,n-1}, j^{-,n-1}).$$

Finally, acting by $j^{+,0}(0)$ on (7.3) and using

$$J^{+,0}(0)(\partial^m j^{-,k}) = -\partial^m J^{0,k} - \partial^m j^{1,k},$$

we obtain relations

$$(7.4) \quad j^{0,m} + j^{1,m} = S(j^{0,0}, j^{1,0}, j^{+,0}, j^{-,0}, j^{0,1}, j^{1,1}, j^{+,1}, j^{-,1}, \ldots, j^{0,n-1}, j^{1,n-1}, j^{+,n-1}, j^{-,n-1}).$$

We can subtract from this the relation (7.1), obtaining

$$(7.5) \quad j^{1,m} = T(j^{0,0}, j^{1,0}, j^{+,0}, j^{-,0}, j^{0,1}, j^{1,1}, j^{+,1}, j^{-,1}, \ldots, j^{0,n-1}, j^{1,n-1}, j^{+,n-1}, j^{-,n-1}).$$
The relations (7.1)-(7.3) and (7.5) imply that \( \{j^{0,k}, j^{1,k}, j^{+,k}, j^{-k} | k = 0, 1, \ldots, n - 1 \} \) strongly generates \( \mathcal{V}_n(\mathcal{S} \mathcal{D}) \). The fact that this set is \textit{minimal} is a consequence of Weyl's second fundamental theorem of invariant theory for \( GL_n \); there are no relations of weight less than \( n + 1/2 \). \( \square \)

The cases \( n = 1 \) and \( n = 2 \). It is immediate from Theorem 7.1 that \( \mathcal{V}_1(\mathcal{S} \mathcal{D}) \cong \mathcal{V}_1(\mathfrak{gl}(1|1)) \). In the case \( n = 2 \), the decoupling relations for \( j^{a,2} \) for \( a = 0, 1, \pm \) are as follows:

\[
j^{0,2} = -\frac{1}{6} : j^{0,0} j^{0,0} \hat{j}^{0,0} : - \frac{1}{2} : j^{0,0} \partial j^{0,0} : + : j^{0,0} j^{0,1} : + \partial j^{0,1} - \frac{1}{6} \partial^2 j^{0,0}.
\]

\[
j^{+,2} = -\frac{1}{2} : j^{+,0} j^{0,0} \hat{j}^{0,0} : - \frac{1}{2} : j^{+,0} \partial j^{0,0} : + : j^{+,1} j^{0,0} : + : j^{+,0} j^{0,1} : .
\]

\[
j^{-2} = -\frac{1}{2} : j^{-,0} j^{0,0} \hat{j}^{0,0} : - \frac{1}{2} : j^{-,0} \partial j^{0,0} : - : \partial j^{-,0} j^{0,0} : + : j^{-,1} j^{0,0} : + : j^{-,0} j^{0,1} : - \partial^2 j^{-,0} + 2 \partial j^{-1}.
\]

(7.6)

\[
j^{1,2} = - : j^{-,0} j^{+,0} \hat{j}^{0,0} : - \frac{1}{2} : j^{1,0} j^{0,0} \hat{j}^{0,0} : - \frac{1}{3} : j^{0,0} j^{0,0} j^{0,0} : - : \partial j^{-,0} j^{+,0} : + : j^{-,1} j^{0,0} : + : j^{-1} j^{0,1} : \\
+ : j^{-,0} \partial j^{0,0} : + : j^{0,1} j^{0,0} : - \frac{5}{6} \partial^2 j^{0,0} + \partial j^{0,1} - \partial^2 j^{1,0} + 2 \partial j^{1,1}.
\]

Since the original generating set \( \{j^{a,k} | k \geq 0 \} \) closes linearly under OPE, these decoupling relations allow us to write down all nonlinear OPE relations in \( \mathcal{V}_2(\mathcal{S} \mathcal{D}) \) among the strong generating set \( \{j^{a,k} | k = 0, 1 \} \). For example, \( j^{-1}(z) j^{+,1}(w) \sim 2(z-w)^{-4} + (j^{1,1} j^{0,1})(w)(z-w)^{-2} + (\partial j^{1,1} j^{-,1} j^{0,2})(w)(z-w)^{-1} \), which yields

\[
j^{-1}(z) j^{+,1}(w) \sim 2(z-w)^{-4} + (j^{1,1} j^{0,1})(w)(z-w)^{-2} + \partial j^{1,1}(w)(z-w)^{-1} \\
- \left( -\frac{1}{6} : j^{0,0} j^{0,0} \hat{j}^{0,0} : - \frac{1}{2} : j^{0,0} \partial j^{0,0} : + : j^{0,0} j^{0,1} : + \partial j^{0,1} - \frac{1}{6} \partial^2 j^{0,0} \right) \\
- : j^{-,0} j^{+,0} \hat{j}^{0,0} : - \frac{1}{2} : j^{1,0} j^{0,0} \hat{j}^{0,0} : - \frac{1}{3} : j^{0,0} j^{0,0} j^{0,0} : - : \partial j^{-,0} j^{+,0} : + : j^{-,1} j^{0,0} : \\
+ : j^{-,0} j^{+,1} : - : j^{1,0} j^{0,0} : - \frac{1}{2} : j^{1,0} \partial j^{0,0} : + : j^{1,1} j^{0,0} : + : j^{1,0} j^{0,1} : \\
- : j^{0,0} \partial j^{0,0} : + : j^{0,1} j^{0,0} : - \frac{5}{6} \partial^2 j^{0,0} + \partial j^{0,1} - \partial^2 j^{1,0} + 2 \partial j^{1,1} \right)(w)(z-w)^{-1}.
\]

(7.8)

Similarly, we have the following additional nonlinear OPEs:

\[
j^{0,1}(z) j^{-1}(w)(z-w)^{-2} + \left( \frac{1}{2} : j^{-,0} j^{0,0} \hat{j}^{0,0} : - \frac{1}{2} : j^{-,0} \partial j^{0,0} : - : \partial j^{-,0} j^{0,0} : + : j^{-,1} j^{0,0} : - \partial^2 j^{-,0} + 2 \partial j^{-1} \right)(w)(z-w)^{-1},
\]

(7.9)
\[ j^{1,1}(z)j^{-1}(w) \sim j^{-1}(w)(z-w)^{-2} + \left( \partial j^{-1} - \frac{1}{2} : j^{-0}j^{0,0}j^{0,0} : + \frac{1}{2} : j^{-0}\partial j^{0,0} : \right. \]
\[ + : \partial j^{-0}j^{0,0} : - : j^{-1}j^{0,0} : - : j^{-0}j^{0,1} : + \partial^2 j^{-0} - 2\partial j^{-1} \left) (w) (z-w)^{-1}. \right. \]

(7.10)

\[ j^{0,1}(z)j^{+1}(w) \sim j^{+1}(w)(z-w)^{-2} + \partial j^{+1}(z-w)^{-1} - \left( - \frac{1}{2} : j^{+0}j^{0,0}j^{0,0} : \right. \]
\[ - \frac{1}{2} : j^{+0}\partial j^{0,0} : + : j^{+1}j^{0,0} : + : j^{+0}j^{0,1} : \left) (w) (z-w)^{-1}. \right. \]

(7.11)

\[ j^{1,1}(z)j^{+1}(w) \sim j^{+1}(w)(z-w)^{-2} + \left( - \frac{1}{2} : j^{+0}j^{0,0}j^{0,0} : - \frac{1}{2} : j^{+0}\partial j^{0,0} : \right. \]
\[ + : j^{+1}j^{0,0} : + : j^{+0}j^{0,1} : \left) (w) (z-w)^{-1}. \right. \]

(7.12)

The remaining nontrivial OPEs in \( \mathcal{V}_{\infty}(\hat{S}\hat{D}) \) are linear in the generators, and are omitted.

8. A deformable family with limit \( \mathcal{V}_n(\hat{S}\hat{D}) \)

We will construct a deformable family of vertex algebras \( \mathcal{B}_{n,k} \) with the property that \( \mathcal{B}_{n,\infty} = \lim_{k \to \infty} \mathcal{B}_{n,k} = \mathcal{V}_n(\hat{S}\hat{D}) \). The key property will be that for generic values of \( k \), \( \mathcal{B}_{n,k} \) has a minimal strong generating set consisting of \( 4n \) fields, and has the same graded character as \( \mathcal{V}_n(\hat{S}\hat{D}) \).

First, we need to formalize what we mean by a deformable family. Let \( K \subset \mathbb{C} \) be a subset which is at most countable, and let \( F_K \) denote the \( \mathbb{C} \)-algebra of rational functions in a formal variable \( \kappa \) of the form \( \frac{p(\kappa)}{q(\kappa)} \) where \( \deg(p) \leq \deg(q) \) and the roots of \( q \) lie in \( K \). A deformable family will be a free \( F_K \)-module \( \mathcal{B} \) with the structure of a vertex algebra with coefficients in \( F_K \). Vertex algebras over \( F_K \) are defined in the same way as ordinary vertex algebras over \( \mathbb{C} \). We assume that \( \mathcal{B} \) possesses a \( \mathbb{Z}_{\geq 0} \)-grading \( \mathcal{B} = \bigoplus_{m \geq 0} \mathcal{B}[m] \) by conformal weight where each \( \mathcal{B}[m] \) is free \( F_K \)-module of finite rank. For \( k \notin K \), we have a vertex algebra

\[ \mathcal{B}_k = \mathcal{B}/(\kappa - k), \]

where \( (\kappa - k) \) is the ideal generated by \( \kappa - k \). Clearly \( \dim_{\mathbb{C}}(\mathcal{B}_k[m]) = \text{rank}_{F_K}(\mathcal{B}[m]) \) for all \( k \notin K \) and \( m \geq 0 \). We have a vertex algebra \( \mathcal{B}_\infty = \lim_{\kappa \to \infty} \mathcal{B} \) with basis \( \{ \alpha_i | i \in I \} \), where \( \{ \alpha_i | i \in I \} \) is any basis of \( \mathcal{B} \) over \( F_K \), and \( \alpha_i = \lim_{\kappa \to \infty} \alpha_i \). By construction, \( \dim_{\mathbb{C}}(\mathcal{B}_\infty[m]) = \text{rank}_{F_K}(\mathcal{B}[m]) \) for all \( m \geq 0 \). The vertex algebra structure on \( \mathcal{B}_\infty \) is defined by

(8.1) \[ \alpha_i \circ_n \alpha_j = \lim_{\kappa \to \infty} \alpha_i \circ_n \alpha_j, \quad i, j \in I, \quad n \in \mathbb{Z}. \]

It is immediate that the \( F_K \)-linear map \( \phi : \mathcal{B} \to \mathcal{B}_\infty \) sending \( \alpha_i \mapsto \alpha_i \) satisfies

(8.2) \[ \phi(\omega \circ_n \nu) = \phi(\omega) \circ_n \phi(\nu), \quad \omega, \nu \in \mathcal{B}, \quad n \in \mathbb{Z}. \]

Moreover, all normally ordered polynomial relations \( P(\alpha_i) \) among the generators \( \alpha_i \) and their derivatives are of the form

\[ \lim_{\kappa \to \infty} \bar{P}(\alpha_i), \]

18
where $\tilde{P}(a_i)$ is a normally ordered polynomial relation among the $a_i$’s and their derivatives, which converges termwise to $P(\alpha_i)$. In other words, if

$$P(\alpha_i) = \sum_j c_j m_j(\alpha_i)$$

where $m_j(\alpha_i)$ is a normally ordered monomial and $c_j \in \mathbb{C}$, there exists a relation

$$\tilde{P}(a_i) = c_j(\kappa) m_j(\alpha_i)$$

where $\lim_{\kappa \to \infty} c_j(\kappa) = c_j$ and $m_j(\alpha_i)$ is obtained from $m_j(\alpha_i)$ by replacing $\alpha_i$ with $a_i$.

We are interested in the relationship between strong generating sets for $B_{\infty}$ and $B$.

**Lemma 8.1.** Let $B$ be a vertex algebra over $F_K$ as above. Let $U = \{\alpha_i | i \in I\}$ be a strong generating set for $B_{\infty}$, and let $T = \{a_i | i \in I\}$ be the corresponding subset of $B$, so that $\phi(a_i) = \alpha_i$. There exists a subset $S \subset \mathbb{C}$ which is at most countable, such that $F_S \otimes_{\mathbb{C}} B$ is strongly generated by $T$. Here we have identified $T$ with the set $\{1 \otimes a_i | i \in I\} \subset F_S \otimes_{\mathbb{C}} B$.

**Proof.** Without loss of generality, we may assume that $U$ is linearly independent. Complete $U$ to a basis $U'$ for $B_{\infty}$ containing finitely many element in each weight, and let $T'$ be the corresponding basis of $B$ over $F_{K'}$. Let $d$ be the first weight such that $U'$ contains elements which do not lie in $U$, and let $\alpha_{1,d}, \ldots, \alpha_{r,d}$ be the set of elements of $U' \setminus U$ of weight $d$. Since $U$ strongly generates $B_{\infty}$, we have decoupling relations in $B_{\infty}$ of the form

$$\alpha_{j,d} = P_j(\alpha_i), \quad j = 1, \ldots, r.$$ 

Here $P$ is a normally ordered polynomial in the generators $\{\alpha_i | i \in I\}$ and their derivatives. Let $a_{j,d}$ be the corresponding elements of $T'$. There exist relations

$$a_{j,d} = \tilde{P}_j(a_i, a_{1,d}, \ldots, a_{j,d}, \ldots, a_{r,d}), \quad j = 1, \ldots, r,$$

which converge termwise to $P_j(\alpha_i)$. Here $\tilde{P}_j$ does not depend on $a_{j,d}$ but may depend on $a_{k,d}$ for $k \neq j$. Since each $a_{k,d}$ has weight $d$ and $\tilde{P}_j$ is homogeneous of weight $d$, $\tilde{P}_j$ depends linearly on $a_{k,d}$. We can therefore rewrite these relations in the form

$$\sum_{k=1}^{r} b_{jk} a_{k,d} = Q_j(a_i), \quad b_{jk} \in F_K,$$

where $b_{jj} = 1$, $\lim_{\kappa \to \infty} b_{jk} = 0$ for $j \neq k$, and

$$Q_j(a_i) = \tilde{P}_j(a_i, a_{1,d}, \ldots, a_{j,d}, \ldots, a_{r,d}) + \sum_{k=1}^{j-1} b_{jk} a_{k,d} + \sum_{k=j+1}^{r} b_{jk} a_{k,d}.$$ 

Clearly $\lim_{\kappa \to \infty} \det[b_{jk}] = 1$, so this matrix is invertible over the field of rational functions in $\kappa$. Let $S_d$ denote the set of distinct roots of the numerator of $\det[b_{jk}]$ regarded as a rational function of $\kappa$. We can solve this linear system over the ring $F_{S_d}$, so in $F_{S_d} \otimes_{\mathbb{C}} B$ we obtain decoupling relations

$$t_{a,d} = \tilde{Q}_j(a_i), \quad j = 1, \ldots, r.$$ 

For each weight $d + 1, d + 2 \ldots$ we repeat this procedure, obtaining finite sets

$$S_d \subset S_{d+1} \subset S_{d+2} \subset \cdots$$

and decoupling relations

$$a = P(a_i)$$
in $F_{d+i} \otimes \mathbb{C} B$, for each $a \in T' \setminus T$ of weight $d + i$. Letting $S = \bigcup_{i > 0} S_{d+i}$, we obtain decoupling relations in $F_S \otimes \mathbb{C} B$ expressing each $a \in T' \setminus T$ as a normally ordered polynomial in $a_1, \ldots, a_s$ and their derivatives. 

\[ \text{Corollary 8.2.} \quad \text{For } k \neq K \cup S, \text{the vertex algebra } B_k = B/(\kappa - k) \text{ is strongly generated by the image of } T \text{ under the map } B \to B_k. \]

Next we consider a special class of deformable families that are well known in the physics literature. Let $V$ be a vertex algebra equipped a conformal weight grading $V = \bigoplus_{m \geq 0} V[m]$ with each $V[m]$ finite-dimensional. Let $g$ be a simple, finite-dimensional Lie algebra. Fix an orthonormal basis $\xi_1, \ldots, \xi_n$ for $g$ relative to the normalized Killing form, so that the generators $X^{\xi_i}$ of $V_1(g)$ satisfy

\[ X^{\xi_i}(z)X^{\xi_j}(w) \sim 2\delta_{ij}(z - w)^{-2} + X^{[\xi_i, \xi_j]}(w)(z - w)^{-1}. \]

Let $V_1(g) \to V$ be a vertex algebra homomorphism and assume that the action of $g$ on $V$ integrates to an action of a connected, reductive group $G$ on $V$ with $g = \text{Lie}(G)$, so that $V^G$ coincides with the joint kernel of the zero modes $X^{\xi}(0)$.

It is well-known (see [BFH]) that $V^G$ admits a deformation as follows. We have the diagonal homomorphism $V_{k+1}(g) \to V_k(g) \otimes V$ sending $X^\xi \mapsto X^\xi \otimes 1 + 1 \otimes X^\xi$. Here $\tilde{X}^{\xi_i}$ and $\tilde{X}^{\xi}$ are the generators of $V_k(g)$ and $V_{k+1}(g)$, respectively. Define

\[ B_k = \text{Com}(V_{k+1}(g), V_k(g) \otimes V). \]

There is a linear map $B_k \to V^G$ defined as follows. Each element $\omega \in B_k$ of weight $d$ can be written uniquely in the form $\omega = \sum_{r=0}^{d} \omega_r$ where $\omega_r$ lies in the space

(8.3)

\[ (V_k(g) \otimes V)^{(r)} \]

spanned by terms of the form $\alpha \otimes \nu$ where $\alpha \in V_k(g)$ has weight $r$. Clearly $\omega_0 \in V^G$ so we have a well-defined linear map

(8.4)

\[ \phi_k : B_k \to V^G, \quad \omega \mapsto \omega_0. \]

Note that $\phi_k$ is not a vertex algebra homomorphism for any $k$.

\[ \text{Lemma 8.3.} \quad \phi_k \text{ is surjective for } k \neq 0. \]

\[ \text{Proof.} \quad \text{Fix } \nu \in V^G \text{ of weight } d. \text{ For } k \neq 0, \text{ we will construct } \omega \in B_k \text{ such that } \phi_k(\omega) = \nu. \text{ First, let } \omega_0 = \nu \text{ and} \]

(8.5)

\[ \omega_1 = -\frac{1}{k} \sum_{i=1}^{n} \tilde{X}^{\xi_i} \otimes (X^{\xi_i} \circ_1 \nu). \]

Clearly $\omega_0 + \omega_1$ is $G$-invariant (equivalently, it is annihilated by $\tilde{X}^{\xi_i} \circ_0$ for $i = 1, \ldots, n$), and has the property that $\tilde{X}^{\xi_i} \circ_1 (\omega_0 + \omega_1)$ lies in $(V_k(g) \otimes V)^{(1)}$. Inductively, suppose that $\omega_{r-1}$ has been defined so that $\omega_{r-1}$ is $G$-invariant and $\sum_{s=0}^{r-1} (\tilde{X}^{\xi_i} \circ_1 \omega_s)$ lies in $(V_k(g) \otimes V)^{(r-1)}$. Define

(8.6)

\[ \omega_r = -\frac{1}{k} \sum_{i=1}^{n} \left( \tilde{X}^{\xi_i} \otimes \sum_{s=0}^{r-1} (\tilde{X}^{\xi_i} \circ_1 \omega_s) \right). \]

Clearly $\omega_r$ is $G$-invariant and $\sum_{s=0}^{r} (\tilde{X}^{\xi_i} \circ_1 \omega_s)$ lies in $(V_k(g) \otimes V)^{(r)}$. This process terminates after at most $d$ steps, and $\omega = \sum_{r=0}^{d} \omega_r$ lies in $B_k$ since $\omega$ is $G$-invariant and is annihilated by $\tilde{X}^{\xi_i} \circ_1$ for $i = 1, \ldots, n$. By definition, $\phi_k(\omega) = \nu$. \[ \square \]
Lemma 8.4. \(\phi_k\) is injective whenever \(V_k(g)\) is a simple vertex algebra.

Proof. Assume that \(V_k(g)\) is simple. Fix \(\omega \in B_k\), and suppose that \(\phi_k(\omega) = 0\). If \(\omega \neq 0\), there is a minimal integer \(r > 0\) such that \(\omega_r \neq 0\). We may express \(\omega_r\) as a linear combination of terms of the form \(\alpha \otimes \nu\) for which the \(\nu\)'s are linearly independent. Since \(\omega\) lies in the commutant \(B_k\), it follows that each of the above \(\alpha\)'s must be annihilated by \(\hat{X}^\xi(m)\) for \(i = 1, \ldots, n\) and all \(m > 0\). Since \(\text{wt}(\alpha) = r > 0\), this implies that \(\alpha\) generates a nontrivial ideal in \(V_k(g)\), which is a contradiction. \(\square\)

Let \(K \subset \mathbb{C}\) be the set of values of \(k\) such that \(V_k(g)\) is not simple. This set is countable and is described explicitly by Theorem 0.2.1 in the paper \([\text{GK}]\) by Kac-Gorelik. As above, there exists a vertex algebra \(B\) with coefficients in \(F_K\) with the property that \(B/(\kappa - k) = B_k\) for all \(k \notin K\). The generators of \(B\) are the same as the generators of \(B_k\), where \(k\) has been replaced by the formal variable \(\kappa\), and the OPE relations are the same as well. The maps \(\phi_k\) above give rise to a linear isomorphism \(\phi_n : B \to F_K \otimes_C \mathcal{V}^G\), which is not a vertex algebra homomorphism.

Corollary 8.5. The induced map \(\phi = \lim_{k \to \infty} \phi_k\) is a vertex algebra isomorphism from \(B_\infty \to \mathcal{V}^G\).

Proof. It is clear from (8.5) and (8.6) that \(\phi\) is a vertex algebra homomorphism. Since \(\dim(B_\infty[m]) = \dim(B_k[m]) = \dim(\mathcal{V}^G[m])\) for all \(k \notin K\) and all \(m \geq 0\), \(\phi\) must be an isomorphism. \(\square\)

Corollary 8.6. Let \(\{\nu_i | i \in I\}\) be a strong generating set for \(\mathcal{V}^G\), and let \(\{\omega_i | i \in I\}\) be the corresponding subset of \(B_k\), where \(\phi_k(\omega_i) = \nu_i\). Then \(\{\omega_i | i \in I\}\) strongly generates \(B_k\) for generic values of \(k\).

Proof. This is immediate from Lemma 8.1 and Corollary 8.2. \(\square\)

Corollary 8.7. Suppose that \(\{\nu_i | i \in I\}\) generates \(\mathcal{V}^G\), not necessarily strongly. Then the corresponding subset \(\{\omega_i | i \in I\}\) generates \(B_k\) for generic values of \(k\).

Proof. This is immediate from the fact that if \(\{\nu_i | i \in I\}\) generates \(\mathcal{V}^G\), the set \(\{\nu_{i_1} \circ_{j_1} \cdots \nu_{i_r} \circ_{j_r-1} \nu_{i_r}\} | i_1, \ldots, i_r \in I, j_1, \ldots, j_r-1 \geq 0\}\) strongly generates \(\mathcal{V}^G\). \(\square\)

Now we specialize this construction to the example where \(\mathcal{V}\) is the rank \(n\) bc\(\beta\gamma\)-system \(\mathcal{F}\), which carries an action of \(V_0(\mathfrak{gl}_n)\) which is just the sum of the action of \(V_1(\mathfrak{gl}_n)\) on the bc-system \(\mathcal{E}\) and \(V_{-1}(\mathfrak{gl}_n)\) on the \(\beta\gamma\)-system \(\mathcal{S}\). Even though \(\mathfrak{gl}_n\) is not semisimple, the proof of Lemma 8.3 is easily modified to handle this case. First, any element \(\nu \in \mathcal{F}_{\mathfrak{gl}_{n}}\) can be corrected to an element \(\omega \in \text{Com}(V_k(\mathfrak{gl}_n), V_k(\mathfrak{gl}_n) \otimes \mathcal{F})\) such that \(\phi_k(\omega) = \nu\). We can further correct \(\omega\) to make it invariant under the Heisenberg algebra corresponding to the central term in \(\mathfrak{gl}_n\) without destroying the property \(\phi_k(\omega) = \nu\).

We have \(\mathcal{F}_{\mathfrak{gl}_{n}} \simeq \mathcal{V}_n(\hat{\mathcal{S}}\hat{\mathcal{D}})\), and we obtain a deformable family of vertex algebras

\[ B_{n,k} = \text{Com}(V_k(\mathfrak{gl}_n), V_k(\mathfrak{gl}_n) \otimes \mathcal{F}), \]

such that \(B_{n,\infty} \simeq \mathcal{V}_n(\hat{\mathcal{S}}\hat{\mathcal{D}})\) and \(B_{n,k}\) has the same graded character as \(\mathcal{V}_n(\hat{\mathcal{S}}\hat{\mathcal{D}})\) for generic values of \(k\).
Theorem 8.8. Let $U$ be the strong generating set $\{j^{0,l}, j^{1,l}, j^{+,l}, j^{-,l} | l=0,1, \ldots, n-1\}$ for $\mathcal{V}_n(\hat{SD})$ given by Theorem 7.1, and let $T = \{t^{0,l}, t^{1,l}, t^{+,l}, t^{-,l} | l=0,1, \ldots, n-1\}$ be the corresponding subset of $\mathcal{B}_{n,k}$ with $\phi_k(t^{a,l}) = j^{a,l}$. For generic values of $k$, $T$ is a minimal strong generating set for $\mathcal{B}_{n,k}$.

Proof. By Lemma 8.1 and Corollary 8.2, $T$ strongly generates $\mathcal{B}_{n,k}$ for generic $k$. If $T$ were not minimal, we would have a decoupling relation expressing $t^{a,l}$ as a normally ordered polynomial in the remaining elements of $T$ and their derivatives, for some $l \leq n-1$. This relation has weight at most $n$, and taking the limit as $k \to \infty$ would give us a nontrivial relation in $\mathcal{V}_n(\hat{SD})$ of the same weight. But this is impossible by Weyl’s theorem, which implies that there are no relations in $\mathfrak{F}^{GL}$ of weight less than $n+1/2$. □

9. $\mathcal{W}$-algebras of $\hat{gl}(n|n)$

As mentioned in the introduction, $\mathcal{W}$-algebras can often be realized in various ways. In this section, we find a family of $\mathcal{W}$-algebras $\mathcal{W}_{n,k}$ associated to a certain simple and purely odd root system of $\hat{gl}(n|n)$. We will see that $\mathcal{V}_n(\hat{SD}) = \lim_{k \to \infty} \mathcal{W}_{n,k}$. In the case $n = 2$ we write down all nontrivial OPE relations in $\mathcal{W}_{2,k}$ explicitly.

Definition 9.1. Let $X = \{E_{ij} | 1 \leq i, j \leq 2n\}$ be the basis of a $\mathbb{Z}_2$-graded $\mathbb{C}$-vector space with gradation given by

\[
|E_{ij}| = \begin{cases} 
0 & \text{for } 1 \leq i, j \leq n \text{ or } n+1 \leq i, j \leq 2n \\
1 & \text{for } 1 \leq i \leq n, \ n+1 \leq j \leq 2n \text{ or } 1 \leq j \leq n, \ n+1 \leq i \leq 2n 
\end{cases}
\] (9.1)

Then

\[
[E_{ij}, E_{kl}] = \delta_{j,k}E_{il} - (-1)^{|E_{ij}||E_{kl}|}\delta_{i,l}E_{kj}
\] (9.2)

provides the $\mathbb{C}$-span of $X$ with the structure of a Lie superalgebra, namely $\hat{gl}(n|n)$. A consistent, graded symmetric, invariant and non-degenerate bilinear form is given by

\[
B(E_{ij}, E_{kl}) = \delta_{j,k}\delta_{i,l} \times \begin{cases} 
1 & \text{for } 1 \leq i \leq n \\
-1 & \text{for } n+1 \leq i \leq 2n
\end{cases}
\] (9.3)

A Cartan subalgebra has a basis given by the $E_{ii}$. A root system is given by $\alpha_{ij}$ for $1 \leq i \neq j \leq 2n$ with $\alpha_{ij}(E_{kk}) = \delta_{ik} - \delta_{jk}$. The parity of a root is defined as $|\alpha_{ij}| = |E_{ij}|$. The distinguished system of positive simple roots is given by $\alpha_{ii+1}$, where only $\alpha_{nn+1}$ is an odd root. We are interested in a system of positive simple and purely odd roots. Such a system is given by $\{\alpha_i = \alpha_{i,n+i}, \beta_i = \alpha_{n+i,i+1}\}$. Define $2n$ bosonic fields $\phi_i^\pm$, $1 \leq i \leq n$, and $2n$ fermionic fields $\psi_i^\pm$, with operator products

\[
\phi_i^\pm(z)\phi_j^\pm(w) \sim \pm k\delta_{ij}\ln(z-w), \quad \psi_i^\pm(z)\psi_j^\pm(w) \sim \pm \frac{k\delta_{ij}}{(z-w)}.
\] (9.4)

and all others are regular. The complex number $k$ will be called level. We define the even and odd Cartan subalgebra valued fields

\[
\phi = \sum_{i=1}^{n} \phi_i^+ E_{ii} + \phi_i^- E_{n+nn+i}, \quad \psi = \sum_{i=1}^{n} \psi_i^+ E_{ii} + \psi_i^- E_{n+nn+i}.
\] (9.5)
Finally, we are ready to define the set of screening associated to our purely odd simple root system

\[ Q_{\alpha_i} = \text{Res}_z \left( : \alpha_i(\psi(z)) e^{\alpha_i(\phi(z))} : \right) \]

\[ Q_{\beta_i} = \frac{1}{k} \text{Res}_z \left( : \beta_i(\psi(z)) e^{\beta_i(\phi(z))} : \right) \]

(9.6)

It is convenient to change basis as follows.

\[ Y_i(z) = \phi_i^+(z) - \phi_i^-(z) \]

\[ X_i(z) = \frac{1}{2k} \left( \phi_i^+ + \phi_i^- + \sum_{j=1}^{i-1} Y_j(z) - \sum_{j=i+1}^{n} Y_j(z) \right) \]

\[ b_i(z) = \psi_i^+(z) - \psi_i^-(z) \]

\[ c_i(z) = \frac{1}{2k} \left( \psi_i^+ + \psi_i^- + \sum_{j=1}^{i-1} b_j(z) - \sum_{j=i+1}^{n} b_j(z) \right) \]

(9.7)

in this new basis, the non-regular OPEs are

\[ Y_i(z) X_j(w) \sim \delta_{i,j} \ln(z-w), \quad b_i(z) c_j(w) \sim \frac{\delta_{i,j}}{(z-w)} \]

(9.8)

The screening charges read in this basis

\[ Q_{\alpha_i} = \text{Res}_z \left( : b_i(z) e^{Y_i(z)} : \right) \]

\[ Q_{\beta_i} = \text{Res}_z \left( : (c_i(z) - c_{i+1}(z)) e^{k(X_i(z)-X_{i+1}(z))} : \right) \]

(9.9)

Let \( M_i \) be the vertex algebra generated by \( \partial Y_i(z), \partial X_i(z), b_i(z), c_i(z) \) and let \( M = \bigoplus_i M_i \). We have

**Lemma 9.2.** Let

\[ N_i(z) = \partial X_i(z) - : b_i(z) c_i(z) :, \quad E_i(z) = \partial Y_i(z), \]

\[ \Psi_i^+(z) = b_i(z), \quad \Psi_i^-(z) = \partial c_i(z) - : c_i(z) \partial Y_i(z) : \]

then the vertex algebra generated by \( N_i, E_i, \Psi_i^\pm \) is a homomorphic image of \( V_1(\mathfrak{gl}(1|1)) \), moreover this algebra is contained in the kernel of \( Q_{\alpha_i} \).

**Proof.** The non-regular operator products of \( N_i, E_i, \Psi_i^\pm \) are

\[ N_i(z) E_i(w) \sim \frac{1}{(z-w)^2} \]

\[ N_i(z) N_i(w) \sim \frac{1}{(z-w)^2} \]

\[ N_i(z) \Psi_i^+(w) \sim \mp \Psi_i^+(w) \]

\[ \Psi_i^+(z) \Psi_i^-(w) \sim -\frac{1}{(z-w)^2} - \frac{E_i(w)}{(z-w)} \]

(9.10)
Lemma 9.3. We have

\[ E_i + E_{i+1}, N_i + N_{i+1} - \frac{1}{k} E_i, \Psi_i^\pm + \Psi_{i+1}^\pm \in \text{Ker}_M(Q_{\beta_i}) \]

\[ :\Psi_i^+N_i: + :\Psi_i^{i+1}N_{i+1} - \frac{1}{k} \partial \Psi_i^+ \in \text{Ker}_M(Q_{\beta_i}) \]

\[ :N_i\Psi_i^- + :N_{i+1}\Psi_{i+1}^- + \frac{1}{k} (E_{i+1}\Psi_i^- - E_i\Psi_{i+1}^-) - \frac{1}{k} \partial \Psi_i^- \in \text{Ker}_M(Q_{\beta_i}) \]

Proof. The first two lines are obvious, while the last one is a lengthy OPE computation. One needs

\[ :N_i(z)\Psi_i^-(z) : = : b_i(z) : \partial c_i(z) c_i(z) : = : c_i(z) : \partial X_i(z) \partial Y_i(z) : + \]

\[ + : \partial c_i(z) \partial X_i(z) : - \partial c_i(z) \partial Y_i(z) : + \frac{1}{2} \partial^2 c_i(z). \]

We define some fields in \( M \)

\[ E(z) = - \sum_{i=1}^n E_i(z), \quad N(z) = - \sum_{i=1}^n N_i(z) + \frac{1}{k} \sum_{i=1}^n (n - i) E_i(z) \]

\[ \Psi^+(z) = \sum_{i=1}^n \Psi_i^+(z), \quad \Psi^-(z) = \sum_{i=1}^n \Psi_i^-(z) \]

(9.11)

\[ F^+(z) = \sum_{i=1}^n : \Psi_i^+(z) N_i(z) : - \frac{1}{k} \sum_{i=1}^n (n - i) \partial \Psi_i^+(z) \]

\[ F^-(z) = \sum_{i=1}^n : N_i(z) \Psi_i^-(z) : - \frac{1}{k} \sum_{i=1}^n (n - i) \partial \Psi_i^-(z) + \]

\[ + \frac{1}{2k} \left( \sum_{1 \leq i < j \leq n} : E_j(z) \Psi_i^+(z) : - \sum_{1 \leq j < i \leq n} : E_j(z) \Psi_i^-(z) : \right) \]

Theorem 9.4.

\[ E, N, \Psi^\pm, F^\pm \in \bigcap_{i=1}^n \text{Ker}_M(Q_{\alpha_i}) \cap \bigcap_{i=1}^{n-1} \text{Ker}_M(Q_{\beta_i}) \]

Proof. By Lemma 9.2, we have \( E, N, \Psi^\pm, F^\pm \in \bigcap_{i=1}^n \text{Ker}_M(Q_{\alpha_i}) \) and Lemma 9.3 implies \( E, N, \Psi^\pm, F^\pm \in \bigcap_{i=1}^{n-1} \text{Ker}_M(Q_{\beta_i}) \)

Definition 9.5. We call the vertex algebra generated by \( E, N, \Psi^\pm, F^\pm \) by \( W_{n,k} \). We define \( W_{n,\infty} \) as the limit \( k \to \infty \).

Theorem 9.6. \( \mathcal{V}_n(S\mathcal{D}) \cong W_{n,\infty} \)
Proof. We will construct the isomorphism explicitly. Let \( \tilde{\beta}_i, \tilde{\gamma}_i, \tilde{b}_i, \tilde{c}_i \) be the generators of a rank \( n \) bc\( \beta \gamma \)-ghost vertex algebra. Then let \( \tilde{\phi}^\pm_i, \tilde{\phi}_i \) be bosonic fields with OPE
\[
\tilde{\phi}^\pm_i(z) \tilde{\phi}^\pm_j(w) \sim \pm \delta_{ij} \ln(z-w), \quad \tilde{\phi}_i(z) \tilde{\phi}_j(w) \sim \delta_{ij} \ln(z-w).
\]
Using the well-known bosonization isomorphism, one obtains
\[
\tilde{b}_i(z) = e^{-\tilde{\phi}_i(z)} :, \quad \tilde{c}_i(z) = e^{\tilde{\phi}_i(z)} :, \quad \tilde{b}_i(z) \tilde{c}_i(z) := -\partial \tilde{\phi}_i(z)
\]
and
\[
\tilde{\beta}_i(z) = e^{-\tilde{\phi}^{-}_i(z)+\tilde{\phi}^{+}_i(z)} \partial \tilde{\phi}_i^+(z) :, \quad \tilde{\gamma}_i(z) = e^{\tilde{\phi}^{-}_i(z)-\tilde{\phi}^{+}_i(z)} :, \quad \tilde{\beta}_i(z) \tilde{\gamma}_i(z) := \partial \tilde{\phi}_i^-(z)
\]
We define
\[
Y_i := \tilde{\phi}_i - \tilde{\phi}^+_i, \quad X_i = \tilde{\phi}_i - \tilde{\phi}^+_i, \quad \phi = \tilde{\phi}_- - \tilde{\phi}^+_i + \tilde{\phi}^+_i
\]
then the non-zero OPEs of these fields are
\[
Y_i(z) X_j(w) \sim \delta_{ij} \ln(z-w), \quad \phi_i(z) \phi_j(w) \sim \delta_{ij} \ln(z-w).
\]
Finally we use bosonization again to obtain
\[
b_i(z) = e^{-\phi_i(z)} :, \quad c_i(z) = e^{\phi_i(z)} :, \quad b_i(z)c_i(z) := -\partial \phi_i(z).
\]
Under this isomorphism, we get the following identifications
\[
E_i(z) = \partial Y_i(z) = \partial \tilde{\phi}_i(z) - \partial \tilde{\phi}^+_i(z) = - \tilde{b}_i(z) \tilde{c}_i(z) := - \tilde{\beta}_i(z) \tilde{\gamma}_i(z) :
\]
\[
N_i(z) = \partial Y_i(z) - b_i(z) c_i(z) := \partial \tilde{\phi}_i(z) = - \tilde{b}_i(z) \tilde{c}_i(z) :
\]
\[
\Psi^+(z) = b_i(z) = e^{-\phi_i(z)} := e^{-\tilde{\phi}_i(z)+\tilde{\phi}^+_i(z)-\tilde{\phi}^+_i(z)} := \tilde{b}_i(z) \tilde{\gamma}_i(z) :
\]
\[
\Psi^-(z) = \partial c_i(z) - c_i(z) \partial Y_i(z) := e^{\phi_i(z)} (\partial \phi_i(z) - \partial Y_i(z))
\]
\[
= e^{\tilde{\phi}_i^{-}(z)-\tilde{\phi}^+_i(z)+\tilde{\phi}^+_i(z)} \partial \tilde{\phi}_i^+(z) := \tilde{c}_i(z) \tilde{\beta}_i(z) :
\]
and hence
\[
E(z) = \sum_{i=1}^{n} \tilde{b}_i(z) \tilde{c}_i(z) : + \tilde{\beta}_i(z) \tilde{\gamma}_i(z) :, \quad N(z) = \sum_{i=1}^{n} \tilde{b}_i(z) \tilde{c}_i(z) :
\]
\[
\Psi^+(z) = \sum_{i=1}^{n} \tilde{b}_i(z) \tilde{\gamma}_i(z) :, \quad \Psi^-(z) = \sum_{i=1}^{n} \tilde{c}_i(z) \tilde{\beta}_i(z) :
\]
\[
F^+(z) = \sum_{i=1}^{n} \Psi^+_i(z) N_i(z) := \sum_{i=1}^{n} \tilde{b}_i(z) \tilde{\gamma}_i(z) :: \tilde{b}_i(z) \tilde{c}_i(z) :: = - \sum_{i=1}^{n} \tilde{b}_i(z) \partial \tilde{\gamma}_i(z) :
\]
\[
F^-(z) = \sum_{i=1}^{n} N_i(z) \Psi^+_i(z) := \sum_{i=1}^{n} \tilde{b}_i(z) \tilde{c}_i(z) :: \tilde{c}_i(z) \tilde{\beta}_i(z) :: = - \sum_{i=1}^{n} \tilde{\beta}_i(z) \partial \tilde{c}_i(z) :
\]
But these fields are by Lemma 4.3 a generating set of \( \mathcal{V}_n(SD) \). \( \square \)

The operator product algebra of \( \mathcal{W}_{2,k} \) can be computed explicitly. For this, we choose a slightly different basis from (9.11). Let \( n = 2 \), and define
\[
G^\pm = F^\pm \pm \left( \frac{1}{2k} - \frac{1}{2} \right) \partial \Psi^\pm.
\]
There will be two additional fields, a Virasoro field of central charge zero,

\[ T = :E_1(z)N_1(z): + :E_2(z)N_2(z): - :\Psi^+_1(z)\Psi^-_1(z): - :\Psi^+_2(z)\Psi^-_2(z): + \]

\[ - \frac{1+k}{2k} \partial E_1(z) + \frac{1-k}{2k} \partial E_2(z) \]

and another dimension two field

\[ H = - \frac{1}{2} ( :N_1(z)N_1(z): + :N_2(z)N_2(z): - :E_1(z)N_1(z): - :E_2(z)N_2(z): + \]

\[ + :\Psi^+_1(z)\Psi^-_1(z): - :\Psi^+_2(z)\Psi^-_2(z): ) + \frac{1}{2k} ( :E_1(z)N_2(z): - :E_2(z)N_1(z): + \]

\[ + :\Psi^+_1(z)\Psi^-_2(z): + :\Psi^+_2(z)\Psi^-_1(z): + \frac{1}{k} :E_1(z)E_2(z): + \partial N_1(z) - \partial N_2(z)) + \]

\[ - \frac{1}{8k^2} ((2k^2 + 2k + 1) \partial E_1(z) + (2k^2 - 2k + 1) \partial E_2(z)) \]

Then \( E, N, \Psi^\pm \) have the operator product algebra of \( \mathfrak{gl}(1|1) \) at level two, and \( G^\pm \) are Virasoro primaries of dimension two, while \( H \) is the partner of \( T \),

\[ T(z)H(w) \sim \left( \frac{3}{k^2} - 1 \right) \left( \frac{1}{(z-w)^4} + \frac{3}{4^2 (z-w)^3} + \frac{2H(w)}{(z-w)^2} + \frac{\partial H(w)}{(z-w)} \right). \]

In addition the operator products of the dimension two fields with the currents are

\[ N(z)H(w) \sim \frac{3}{2k^2} \frac{1}{(z-w)^3} - \left( \frac{1}{4} - \frac{3}{4^2 k^2} \right) \frac{E(w)}{(z-w)^2}, \quad N(z)G^\pm(w) \sim \frac{\pm G^\pm(w)}{(z-w)} \]

\[ \Psi^\pm(z)H(w) \sim - \frac{G^\pm(w)}{(z-w)}, \quad \Psi^\pm(z)G^\mp(w) \sim \frac{N(w)}{(z-w)^2} \pm \frac{T(w)}{(z-w)} \]

\[ E(z)H(w) \sim - \frac{N(w)}{(z-w)^2}, \quad E(z)G^\pm(w) \sim - \frac{\pm \psi^\pm(w)}{(z-w)^2}. \]

Introduce the following normally ordered polynomials in the currents and their derivatives

\[ X_0 = \frac{1}{2} \left( 2 :HE : -2 :TE : -2 :TN : -2 :G^+\Psi^- : -2 :G^-\Psi^+ : + :\partial \Psi^- \Psi^+ : + :\partial \Psi^+ \Psi^- : \right. \]

\[ + :\partial EN : -2 :N : \Psi^+ \Psi^- : + :N :NE : - :E : \Psi^+ \Psi^- : + :N :EE : \left.) \right. \]

\[ - \frac{1}{8k^2} \left( (1 - 2k^2) \partial^2 E + (3 - 2k^2) :\partial EE : + (1 - k^2) :E :EE : \right) \]

\[ X^+ = \frac{1}{2} \left( :N \partial \Psi^+ : -2 :H \Psi^+ : -2NG^+ : + :T \Psi^+ : - :EG^+ : - :N :N \Psi^+ : - :N :E \Psi^+ : \right) \]

\[ - \frac{1}{8k^2} \left( (2 + 2k^2) \partial^2 \Psi^+ : -2 :E \partial \Psi^+ : - (2 + 2k^2) :E E \partial \Psi^+ : - (1 - k^2) :E :E \Psi^+ : \right) \]

\[ X^- = \frac{1}{2} \left( 2NG^- : - :N \partial \Psi^- : -2 :H \Psi^- : + :T \Psi^- : - :EG^- : - :N :N \Psi^- : - :N :E \Psi^- : \right) \]

\[ + \frac{1}{8k^2} \left( (2 + 2k^2) \partial^2 \Psi^- : 5 :E \partial \Psi^- : - (2 + 2k^2) :E E \partial \Psi^- : + (1 - k^2) :E :E \Psi^- : \right) \]

\[ X_2 = 3 \partial^2 N + (2 + 2k^2) \partial^2 E + 4 :\partial \psi^- \psi^+ : -4 :\partial \psi^+ \psi^- : + 4 :\partial NE : + 4 :\partial EN : + 2 :\partial EE : \]
Then the operator products of the dimension two fields with themselves are
\[
H(z)H(w) \sim -\frac{1}{4k^2} \frac{1}{(z-w)^2} \left( 2\partial E(w) + 3\partial N(w) - (2k^2 + 2)T(w) + 4 : N(w)E(w) : + E(w)E(w) : -4 : \psi^+(w)\psi^-(w) : \right) - \frac{1}{8k^2} \frac{X_2(w)}{(z-w)}
\]
\[
H(z)G^+(w) \sim \left( \frac{1}{2} - \frac{3}{4k^2} \right) \frac{\Psi^+(w)}{(z-w)^3} + \frac{1}{4} - \frac{3}{4k^2} \frac{\partial \Psi^+(w)}{(z-w)^2} + \frac{G^+(w) + X^+(w)}{(z-w)}
\]
\[
H(z)G^-(w) \sim \left( \frac{1}{2} - \frac{9}{4k^2} \right) \frac{\Psi^-(w)}{(z-w)^3} + \frac{1}{4} - \frac{3}{4k^2} \frac{\partial \Psi^-(w)}{(z-w)^2} + \frac{G^-(w) + X^-(w)}{(z-w)}
\]
\[
G^+(z)G^-(w) \sim -\left( 1 - \frac{3}{k^2} \right) \frac{1}{(z-w)^4} - \frac{1}{2} - \frac{3}{2k^2} \frac{E(w)}{(z-w)^3} - \frac{1}{4} \frac{\partial E(w) - 8H(w)}{(z-w)^2} + (H(w) + X_0(w))(z-w)^{-1}.
\]

We see

**Proposition 9.7.** \( W_{2,k} \) is strongly generated by \( E, N, \Psi^\pm, T, H, G^\pm \).

**10. The Relationship Between \( B_{n,k} \) and \( W_{n,k} \)**

Recall the algebra \( B_{n,k} \) that we constructed in Section 8, which has a minimal strong generating set consisting of \( 4n \) fields, for generic values of \( k \). In this section we show that for \( n = 2, W_{2,k+2} \) and \( B_{2,k} \) have the same generators and OPE relations. More generally, we conjecture that \( W_{n,k+n} \) is isomorphic to \( B_{n,k} \) for all \( n \) and \( k \).

Recall that \( V_k(\mathfrak{gl}_n) \) has a strong generating set \( \{X^{ij} | 1 \leq i, j \leq n\} \) satisfying

\[
X^{ij}(z)X^{lm}(w) \sim \frac{k\delta_{i,l}\delta_{j,m}}{(z-w)^2} + \frac{\delta_{i,l}X^{im}(w) - \delta_{i,m}X^{lj}(w)}{(z-w)}.
\]

Recall the bcβγ system \( \mathcal{F} = \mathcal{E} \otimes \mathcal{S} \) of rank \( n \). There is a map \( V_1(\mathfrak{gl}_n) \rightarrow \mathcal{E} \) sending \( X^{ij} \mapsto c_i b_j : \) and a map \( V_1(\mathfrak{gl}_n) \rightarrow \mathcal{S} \) sending \( X^{ij} \mapsto - : \gamma_i \beta_j : \). These combine to give us a map \( V_0(\mathfrak{gl}_n) \rightarrow \mathcal{F} \) sending \( X^{ij} \mapsto c_i b_j : - : \gamma_i \beta_j : \).

A straightforward computation shows that \( B_{n,k} = \text{Com}(V_k(\mathfrak{gl}_n), V_k(\mathfrak{gl}_n) \otimes \mathcal{F}) \) contains the following elements:

\[
\Psi^- = -\sum_{l=1}^{n} : c_i \beta_l : , \quad \Psi^+ = -\sum_{l=1}^{n} : \gamma_i b_l : ,
\]

\[
\tilde{E} = -\sum_{l=1}^{n} : c_i b_l : + : \gamma_i \beta_l : , \quad \tilde{N} = -\sum_{l=1}^{n} \frac{2}{k} X^{ii} - : c_i b_l : + : \gamma_i \beta_l : ,
\]

\[
\tilde{F}^- = \frac{1}{k} \sum_{1 \leq k, l \leq n} : X^{kl} c_i \beta_k : + \frac{n}{k} : c_i \partial \beta_l : , \quad \tilde{F}^+ = \frac{1}{k} \sum_{1 \leq k, l \leq n} : X^{kl} \gamma_i b_k : + \frac{n}{k} : \gamma_i \partial b_l : .
\]

By Lemma 4.3, the elements of \( B_{n,\infty} = \mathcal{V}_n(\hat{S}D) \) corresponding to these six elements under \( \phi_k : B_{n,k} \rightarrow \mathcal{V}_n(\hat{S}D) \), are a generating set for \( \mathcal{V}_n(\hat{S}D) \). By Corollary 8.7, these six fields generate \( B_{n,k} \) for generic values of \( k \).

**Theorem 10.1.** For generic values of \( k \), \( W_{2,k+2} \) and \( B_{2,k} \) have the same OPE algebra.
Proof. This is a computer computation, where the field identification is given by, with 
\[ s = \frac{(1 + k)}{(4 + 2k)}, \]
\[
E \rightarrow \tilde{E}, \quad N \rightarrow \tilde{N} - \frac{\tilde{E}}{k}, \quad \Psi^\pm \rightarrow \tilde{\Psi}^\pm
\]
(10.2)
\[
G^+ \rightarrow (4s - 1)\tilde{F}^+ + s\tilde{\Psi}^+ + (2s - 1) : \tilde{N}\tilde{\Psi}^+ : ,
\]
\[
G^- \rightarrow (4s - 1)\tilde{F}^- - (3s - 1)\partial\tilde{\Psi}^- - (2s - 1) : \tilde{N}\tilde{\Psi}^- : .
\]

□

Remark 10.2. There exist other realizations of \( W_{2,k} \). Let \( E_{ij} \) for \( 1 \leq i, j \leq 4 \) be the basis of \( \mathfrak{gl}(2|2) \) of Definition 9.1, and we denote the corresponding fields of \( V_k(\mathfrak{gl}(2|2)) \) by \( E_{ij}(z) \). Then, we find that for level \( k = -2 \) the fields
\[
E' = -\frac{1}{2} \left( \sum_{i=1}^{4} E_{ii} \right), \quad N' = \frac{1}{2} (E_{11} + E_{22} - E_{33} - E_{44}),
\]
\[
\Psi'^+ = \frac{1}{\sqrt{-2}} (E_{13} + E_{24}), \quad \Psi'^- = \frac{1}{\sqrt{-2}} (E_{31} + E_{42})',
\]
\[
G'^+ = \frac{1}{\sqrt{-2}} \left( : E_{12}E_{23} : + : E_{21}E_{14} : + : (E_{11} - E_{22})(E_{13} - E_{24}) : + \right.
\]
\[- \frac{1}{2} \partial\Psi'^+ - \frac{1}{2} : N'\Psi'^+ : + \frac{1}{4} : E'\Psi'^+ : ,
\]
\[
G'^- = \frac{1}{\sqrt{-2}} \left( : E_{12}E_{41} : + : E_{21}E_{32} : + : (E_{11} - E_{22})(E_{31} - E_{42}) : + \right.
\]
\[- \frac{1}{2} \partial\Psi'^- + \frac{1}{2} : N'\Psi'^- : - \frac{1}{4} : E'\Psi'^- : ,
\]
are elements of \( \text{Com}(V_0(\mathfrak{sl}(2)), V_{-2}(\mathfrak{gl}(2|2))) \) and satisfy the operator product algebra of \( W_{2,-1} \) where the field identification is given by \( X \rightarrow X' \), for \( X = E, N, \Psi^\pm, G^\pm \).

REFERENCES


**Fachbereich Mathematik, Technische Universität Darmstadt and Hausdorff Research Institute for Mathematics in Bonn**

*E-mail address: tcreutzig@mathematik.tu-darmstadt.de*

**Department of Mathematics, Brandeis University**

*E-mail address: linshaw@brandeis.edu*