Logarithmic Hopf Link Invariants for the Unrolled Restricted Quantum Group of $sl(2)$

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Science

in

Mathematics

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University of Alberta

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Abstract

Little is known about Vertex Operator Algebras (VOAs) which are neither semi-simple nor rational, and most of the work on such VOAs has been focused around specific examples such as the Singlet VOA \( \mathcal{W}(2, 2r - 1) \). In this thesis, the relationship between subcategories of the module categories of the Singlet VOA and the unrolled restricted quantum group associated to \( \mathfrak{sl}(2) \), \( \mathcal{U}_q^{H}(\mathfrak{sl}(2)) \) at \( 2r \)-th root of unity \( q = e^{\pi i / r} \) with \( r \geq 2 \) is studied. A family of deformable modules \( X_\epsilon \) is used to efficiently compute open Hopf Links for \( \mathcal{U}_q^{H}(\mathfrak{sl}(2)) \) and particular \( (1, 1) \)-tangle invariants colored with projective modules of \( \mathcal{U}_q^{H}(\mathfrak{sl}(2)) \). These tangle invariants are extensions of the Alexander invariants defined by Murakami. It is also shown that normalized modified traces of open Hopf links for \( \mathcal{U}_q^{H}(\mathfrak{sl}(2)) \)-modules correspond exactly with the asymptotic quantum dimensions for certain \( \mathcal{W}(2, 2r - 1) \)-modules.
Acknowledgements

I would like to thank my supervisor professor Thomas Creutzig for all of the time he has spent answering my questions and the insight he has provided throughout the course of my graduate studies, this work wouldn’t have been possible without his guidance and advice. I would like to thank my wife, Prachi Loliencar, for all of her support, encouragement, and understanding. I would also like to thank the University of Alberta for providing an excellent learning environment, and I thank my supervisory committee.
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1 Introduction

1.1 Motivation

A conformal field theory (CFT) is a quantum field theory invariant under conformal transformations, essentially, a quantum field theory with added symmetry. This added symmetry makes many problems in CFT exactly solvable in two dimensions unlike most quantum field theories. Conformal field theory plays a significant role in string theory as well as statistical mechanics in the study of higher order phase transitions. Vertex operator algebras (VOAs) were first introduced by Richard Borcherds [34] in an attempt to formalize CFT. They provide a mathematical formulation of the symmetry algebras for conformal field theory, called chiral algebras, and provide a rigorous mathematical approach to string theory and two-dimensional conformal field theory. VOAs have found applications to many other areas of mathematics such as Lie theory, algebraic geometry, topology, and modular forms, and have played an important role in connecting seemingly distant areas of mathematics. One such example is given by monstrous moonshine, a term used by John Conway and Simon Norton [35] in 1979 to describe the unexpected connections between modular forms and the monster group. The conjectures of Conway and Norton were proven by Richard Borcherds [40] in 1992 using the monster VOA constructed by Igor Frenkel, James Lepowsky, and Arne Meurman [36].

Much of the study of vertex operator algebras has been focused on VOAs which are $C_2$-cofinite or rational. It is known that the representation category of a rational and $C_2$-cofinite (regular) VOA is semi-simple, and also modular and ribbon given some weak assumptions [11,12]. In this case, the VOA also carries three coinciding actions of the modular group: one on the linear span of torus one point functions, a categorical action given by twists and Hopf links, and an action diagonalizing fusion rules (this is only a $S$-matrix). $C_2$-cofinite VOAs have representation categories with only finitely many isomorphism classes of simple objects. Clearly, it is desirable to extend our understanding to those VOAs which have infinitely many simple objects. However, once we drop the $C_2$-cofiniteness assumption, our VOAs become much more difficult to understand because their representation categories are so large. One approach to this problem has been to restrict ourselves to a smaller, more manageable subcategory. It was shown in [16] by Huang, Lepowsky, and Zhang that if all objects in an abelian subcategory $C$ of generalized $V$-modules for a VOA $V$ are $C_1$-cofinito with some additional requirements, then $C$ is braided and monoidal. One of the fundamental results for modularity of rational VOAs was given by Zhu in [23], where it was shown that characters are closed under modular transformations. This property and the action of the modular group on the space of one-point functions on the torus can both be preserved for irrational $C_2$-cofinite VOAs given some additional assumptions [24]. Furthermore, if the category of modules for a $C_2$-cofinite VOA is ribbon, then there is also an action of the modular group in the category [15,25].
VOAs which are rational and $C_2$-cofinite (regular) are fairly well understood through modularity of characters and the automorphicity of chiral blocks, as well as the categorical interpretation of VOA modules. These concepts are connected through the Verlinde formula which relates fusion product coefficients in the category of modules to the dimensions of chiral blocks. We noted above that all modules of a regular VOA are completely reducible and that there are only finitely many inequivalent simple modules. Let $M_0 = V, M_1, ..., M_n$ denote the inequivalent simple modules for a regular VOA $V$. Then, there exists a tensor product satisfying

$$M_i \otimes M_j \simeq \bigoplus_{k=0}^{n} N_{ij}^k M_k.$$

For any $\tau \in \mathbb{H}$ (the upper half of the complex plane) and $v \in V$, the 1-point function $F_{M_i}$ is defined as

$$F_{M_i}(\tau, v) := \text{tr}_{M_i}(o(v)q^{L_0 - \frac{c}{24}}),$$

where $o(v)$ is the zero mode of the field associated to $v$. When $v = 1$, $F_{M_i}$ is called the character of $M_i$ and is denoted $\text{ch}[M_i](\tau)$. The modular $S$-transformation $\tau \mapsto -1/\tau$ defines a matrix $S^\chi$ by

$$F_{M_i}(-1/\tau, v) = \tau^{wt(v)} \sum_{j=0}^{n} S^\chi_{ij} F_{M_j}(\tau, v).$$

This matrix and the tensor coefficients are related through the Verlinde formula [11,41,42]:

$$N_{ij}^k = \sum_{\ell=0}^{n} S^\chi_{i\ell} S^\chi_{j\ell} (S^\chi_{\ell k})^{-1}.$$  

The Verlinde formula is one of the deepest results for regular VOAs and has been studied by many authors. It is natural to wonder if one could prove results for irrational non $C_2$-cofinite VOAs analogous to existing results for rational and $C_2$-cofinite VOAs, and in particular, if there is a generalization of the Verlinde formula. For $C_2$-cofinite VOAs some progress has been made by T. Creutzig and T. Gannon on the case of the triplet VOA in [39]. Logarithmic Hopf link invariants of $U_q(\mathfrak{sl}(2))$ were shown to give a Verlinde formula for certain structure coefficients for the tensor ring, which together with some additional information completely determines the tensor ring. The modular $S$-matrix of the triplet vertex algebra is compared to the logarithmic Hopf link invariants and are found to be in agreement, and a comparison of Jordan blocks with open Hopf link operators of the triplet VOA is drawn. We would like to perform a similar analysis for VOAs which are not $C_2$-cofinite. Little is known about the general theory of irrational non $C_2$-cofinite VOAs, and most of the work on such VOAs has been on developing and understanding specific examples, with a particularly nice choice of example being the singlet VOA $W(2,2r-1)$. The singlet is a desirable choice for the study of irrational non $C_2$-cofinite VOAs as all of its irreducible modules are $C_1$-cofinite [38], and has been studied by many authors [2,4,5,31,32,33].
The characters of irrational non $C_2$-cofinite VOAs are not closed under the modular $S$-transformation, so we cannot construct the matrix $S^X$ as we did for regular VOAs. Instead, we will use quantities called regularized quantum dimensions. The quantum dimension of a regular VOA is defined to be

$$\text{qdim}[X] := \lim_{\tau \to 0^+} \frac{\text{ch}[X](\tau)}{\text{ch}[M_0](\tau)}.$$ 

Applying this to $X = M_i$, we see that

$$\text{qdim}[M_i] = \lim_{\tau \to 0^+} \frac{\text{ch}[M_i](\tau)}{\text{ch}[M_0](\tau)} = \lim_{\tau \to i\infty} \frac{\text{ch}[M_i](-1/\tau)}{\text{ch}[M_0](-1/\tau)} = \lim_{\tau \to i\infty} \frac{\sum_{j=0}^n S^X_{ij} \text{ch}[M_j](\tau)}{\sum_{j=0}^n S^X_{0j} \text{ch}[M_j](\tau)} = \lim_{\tau \to i\infty} \frac{\sum_{j=0}^n S^X_{ij} \text{tr}_{M_j}(q^{L_0 - \frac{c}{24}})}{\sum_{j=0}^n S^X_{0j} \text{tr}_{M_j}(q^{L_0 - \frac{c}{24}})}$$

where $q = e^{2\pi i \tau}$ tends to zero as tau tends to $i\infty$. It is conjectured that there exists an $M_r$ with dominating conformal dimension among the $M_i$, which gives

$$\text{qdim}[M_i] = \frac{S^X_{ir}}{S^X_{0r}}.$$ 

So the quantum dimensions are closely related to the matrix $S^X$ for regular VOAs, and although we cannot construct $S^X$ for irrational non $C_2$-cofinite VOAs, we can construct “regularized” quantum dimensions, which we expect to play the role of $S^X$ in the irrational non $C_2$-cofinite setting.

This thesis studies primarily the category $\mathcal{C}$ of finite dimensional highest weight modules for the unrolled restricted quantum group of $\mathfrak{sl}(2)$, $\mathcal{U}^H_q(\mathfrak{sl}(2))$, and its connections to the singlet VOA. The quantum group $\mathcal{U}^H_q(\mathfrak{sl}(2))$ is the associative algebra over $\mathbb{C}$ with five generators $\{E, F, K, K^{-1}, H\}$ and relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^{2}EK, \quadKF = q^{-2}FK, \quad[E,F] = \frac{K - K^{-1}}{q - q^{-1}},$$

$$HK^{\pm1} = K^{\pm1}H, \quad[H,E] = 2E, \quad[H,F] = -2F, \quad E^r = F^r = 0.$$

with additional coalgebra structure. It is conjectured that some subcategory of the representation category of the singlet is (at least) monoidally equivalent to $\mathcal{C}$, so we hope to use $\mathcal{U}^H_q(\mathfrak{sl}(2))$ as a
toy model to better understand the singlet. In this thesis, we motivate this conjecture and provide a new approach to obtaining the tensor structure of $\mathcal{C}$ which will be more useful when trying to understand the singlet (and perhaps more sophisticated quantum groups). Constantino, Geer, and Patureau-Mirand determined the tensor ring structure of $\mathcal{C}$ in [1] through the use of characters and other more direct means. They then used this to determine the open Hopf links. For regular VOAs, the Verlinde formula follows as a direct consequence of open Hopf links being representations of the tensor ring of modules. However, the tensor structure of the category of modules for the singlet is difficult to determine. We hope to use $\mathcal{U}_q^{H}(\mathfrak{sl}(2))$ as a toy model for the singlet, so it is natural to wonder if we could determine the open Hopf links without prior knowledge of tensor product decompositions for modules, and we will show that this is indeed possible.

It has been shown previously in examples that modular-like properties of characters [2,27-30] and their quantum dimensions [2,31] relate to the fusion ring of the singlet and other vertex (super) algebras. In particular, the asymptotic quantum dimensions of characters relate to representations of the tensor ring of modules for the singlet, and it was conjectured that these quantum dimensions should have a categorical interpretation. We provide this interpretation via Hopf link invariants of $\mathcal{U}_q^{H}(\mathfrak{sl}(2))$. This also gives a new interpretation from the perspective of quantum topology of the Jacobi variable introduced in [2] as a regularization parameter for the classical false theta functions.

1.2 Results

The proof of the Verlinde formula for regular VOAs from Hopf links requires the identity

$$S_{ij}^X / S_{0,j}^X = S_{ij}^\infty$$

(1)

where $S_{ij}^\infty$ is the closed Hopf link colored with the simple modules $M_i, M_j$ for a regular VOA. The primary results of this paper are to motivate the conjecture that some subcategory of the category of modules of $\mathcal{W}(2,2r - 1)$ is monoidally equivalent and to the category $\mathcal{C}$ of finite dimensional weight modules of $\mathcal{U}_q^H(\mathfrak{sl}(2))$ and to generalize equation (1), to improve upon existing techniques for computing Hopf link invariants for $\mathcal{C}$, and to extend Murakami’s definition of Alexander Invariant given in [8]. The first is done by constructing a map between simple modules of $\mathcal{U}_q^H(\mathfrak{sl}(2))$ and $\mathcal{W}(2,2r - 1)$, and giving a comparison of normalized open Hopf link invariants for $\mathcal{C}$ and the regularized quantum dimensions for modules of $\mathcal{W}(2,2r - 1)$. We show that the ring generated by normalized Hopf link invariants is identical to the ring generated by the regularized quantum dimensions and hence is isomorphic to the Verlinde algebra of characters for $\mathcal{W}(2,2r - 1)$ as follows:

The most important classes of modules in $\mathcal{C}$ are:

- $n+1$ dimensional simple modules $S_n$ with $n \in \{0, ..., r - 1\}$.
- $r$-dimensional modules $V_\alpha$ with $\alpha \in \mathbb{C}$.
• 2r-dimensional projective indecomposable modules $P_i$ with $i \in \{0, \ldots, r-1\}$.

• 1-dimensional modules $\mathbb{C}_{\ell r}^H$ with $\ell \in \mathbb{Z}$.

These modules are important because they appear in the classification of finite dimensional weight modules as shown by Constantino, Geer, and Patureau-Mirand [1]:

• Every simple module in $\mathcal{C}$ is isomorphic to $S_n \otimes \mathbb{C}_{\ell r}^H$ for some $n \in \{0, \ldots, r-1\}$ and $\ell \in \mathbb{Z}$ or $V_\alpha$ for some $\alpha \in \mathbb{C} - r\mathbb{Z}$.

• The module $P_i$ is projective and indecomposable, and any projective indecomposable weight module with integer highest weight $(\ell + 1)r - i - 2$ is isomorphic to $P_i \otimes \mathbb{C}_{\ell r}^H$.

A general classification for irreducible modules of the singlet VOA is not currently known, but the $\mathbb{Z}_{\geq 0}$-graded irreducible modules fall into one of two families:

• An uncountable family of typical modules $F_\lambda$, $\lambda \in \mathbb{C}$ with $\lambda \neq -\frac{t-1}{2}\sqrt{2r} + \frac{s-1}{\sqrt{2r}}$ for any $t \in \mathbb{Z}$ and $1 \leq s \leq r$.

• A countable family of atypical modules $M_{t,s}$ with $t \in \mathbb{Z}$ and $1 \leq s \leq r - 1$.

We first note that the endomorphism ring of $P_i \otimes \mathbb{C}_{\ell r}^H$ is two dimensional and spanned by $\{id, x_{i,\ell}\}$, and that $\Phi_{Z,P_i \otimes \mathbb{C}_{\ell r}^H} \in \text{End}(P_i \otimes \mathbb{C}_{\ell r}^H)$.

**Theorem 1.**

1. Let $\alpha \in (\mathbb{C} - \mathbb{Z}) \cup r\mathbb{Z}$. Then the map $\varphi : V_\alpha \mapsto F_{\frac{t-1}{2}\sqrt{2r}} \otimes \mathbb{C}_{\ell r}^H, S_i \otimes \mathbb{C}_{kr}^H \mapsto M_{1-k,i+1}$ between simple modules in $\overline{U}^H_q(\mathfrak{sl}(2))$ and $\mathcal{W}(2,2r-1)$, respectively, is a bijection and a morphism of rings up to equality of characters.

2. Let $\epsilon \in \mathcal{S}(k, j + 1 + p(k + 1))$, then

$$qdim[\varphi(X)^\epsilon] = \frac{t_{P_i} \otimes \mathbb{C}_{\ell r}^H(\Phi_{X,P_j \otimes \mathbb{C}_{\ell r}^H} \circ x_{j,\ell})}{t_{P_j} \otimes \mathbb{C}_{\ell r}^H(\Phi_{S_0,P_j \otimes \mathbb{C}_{\ell r}^H} \circ x_{j,\ell})}$$

and if $\text{Re}(\epsilon) > 0$ then

$$qdim[\varphi(X)^{-i\alpha \sqrt{2r}}] = \frac{t_{V_\alpha}(\Phi_{X,V_\alpha})}{t_{V_\alpha}(\Phi_{S_0,V_\alpha})}.$$
of regularized quantum dimensions and normalized traces of Hopf links is the irreducible non $C_2$-cofinite analogue of equation (1).

The second result is shown by deriving an isomorphism which allows us to determine the action of Hopf links on projective indecomposable modules from their action on simple modules. We associate to $\mathcal{O}_q^H(\mathfrak{sl}(2))$ a positive integer parameter $r$. For each fixed choice of this parameter, the ideal of projective modules in $C$ is generated by $r$-dimensional simple modules $V_\alpha$ with $\alpha \in (C \cup r\mathbb{Z}) - \mathbb{Z}$ and $2r$-dimensional projective indecomposable modules $P_i \otimes C^H_{\ell r}$ with $i \in \{0, \ldots, r-1\}$ and $\ell \in \mathbb{Z}$.

In a manner similar to the ideas of Murakami and Nagatomo [19,20], one can define a family of modules $X_\epsilon$ with $\epsilon \in (-\frac{1}{2}, \frac{1}{2})$ such that

$$X_\epsilon \cong \begin{cases} V_{1+i-r+\ell r+\epsilon} \oplus V_{-1-i+r+\ell r+\epsilon} & \text{if } \epsilon \neq 0 \\ P_i \otimes C^H_{\ell r} & \text{if } \epsilon = 0 \end{cases}$$

So the projective indecomposable modules can be thought of as the limit of a direct sum of simple modules. From this, and other considerations, we find our main result:

**Proposition 1.** Let $Z$ be a weight module in $C$. Then, the Hopf link $\Phi_{Z,P_i \otimes C^H_{\ell r}} = a_Z \text{Id}_{P_i \otimes C^H_{\ell r}} + b_Z x_i,\ell$ is given by:

$$a_Z = \lim_{\epsilon \to 0} (2\lambda_{Z,V_{1+i-r+\ell r+\epsilon}} - \lambda_{Z,V_{-1-i+r+\ell r+\epsilon}})$$

$$b_Z = \lim_{\epsilon \to 0} \frac{-1}{1 + \epsilon^2} (\lambda_{Z,V_{1+i-r+\ell r+\epsilon}} - \lambda_{Z,V_{-1-i+r+\ell r+\epsilon}})$$

where $\lambda_{Z,V_{1+i-r+\ell r+\epsilon}}$ and $\lambda_{Z,V_{-1-i+r+\ell r+\epsilon}}$ are the constants by which $\Phi_{Z,V_{1+i-r+\ell r+\epsilon}}$ and $\Phi_{Z,V_{-1-i+r+\ell r+\epsilon}}$ act respectively.

This proposition allows us to determine the action of the Hopf link $\Phi_{Z,-}$ for any weight module $Z \in C$ on the projective indecomposable modules from its action on the simple $V_\alpha$. The action of Hopf links on simple modules is easy to find as they act as scalars on them. Hence, we do not need to know the tensor structure of $C$ at all, a classification of its modules is sufficient.

The proof of the above proposition can be adapted to extend the definition of Murakami’s Alexander invariant $T_\lambda$ in [8]. This invariant is a framed version of the invariant defined in [7], and is constructed by assigning the braiding, twist, and duality morphisms to ribbon graphs in the usual way, and coloring the graph with the typical $\mathcal{O}_q^H(\mathfrak{sl}(2))$-modules $V_\alpha$. We show that this definition can be extended to allow the ribbon graphs to be colored with the projective indecomposable modules $P_i \otimes C^H_{\ell r}$ as follows:
Theorem 2. The colored \((1,1)\)-ribbon graph \(T_{P_i \otimes C_i^H}\) satisfies

\[
t_{P_i \otimes C_i^H}(T_{P_i \otimes C_i^H}) = \lim_{\epsilon \to 0} \left( t_{V_{1-i-r+\epsilon+\ell}}(T_{1-i-r+\ell+\epsilon}) + t_{V_{-1-i-r+\ell+\epsilon}}(T_{-1-i-r+\ell+\epsilon}) \right)
\]

and

\[
T_{P_i \otimes C_i^H} = a Id_{P_i \otimes C_i^H} + bx_{i,\ell}
\]

with coefficients

\[
a = \lim_{\epsilon \to 0} \frac{t_{V_{1-i-r+\ell+\epsilon}}(T_{1-i-r+\ell+\epsilon})}{d(V_{1-i-r+\ell+\epsilon})} = \lim_{\epsilon \to 0} \frac{t_{V_{1-i-r+\ell+\epsilon}}(T_{1-i-r+\ell+\epsilon})}{d(V_{1-i-r+\ell+\epsilon})}
\]

and

\[
b = \lim_{\epsilon \to 0} \frac{-1}{[1 + i][\epsilon]} \left( \frac{t_{V_{1-i-r+\ell+\epsilon}}(T_{1-i-r+\ell+\epsilon})}{d(V_{1-i-r+\ell+\epsilon})} - \frac{t_{V_{1-i-r+\ell+\epsilon}}(T_{1-i-r+\ell+\epsilon})}{d(V_{1-i-r+\ell+\epsilon})} \right)
\]

\[
= \frac{r}{2\pi i(1 + i)} \left( \frac{d}{d\lambda} \frac{t_{V_{\lambda}}(T_{\lambda})}{d(V_{\lambda})} \bigg|_{\lambda = i + i - 1} - \frac{d}{d\lambda} \frac{t_{V_{\lambda}}(T_{\lambda})}{d(V_{\lambda})} \bigg|_{\lambda = i + i + \ell} \right).
\]

2 Braided Tensor Categories

We will first recall the definition of a braided tensor category. Our main references for this section are [9] and [10]. A category \(C\) is a collection of objects \(\text{Ob}(C)\), and of morphisms \(\text{Hom}(C)\) such that for any morphism \(f \in \text{Hom}(C)\), there are associated objects \(s(f)\) and \(t(f)\) called the source and target of \(f\) respectively, and we use the notation \(f : s(f) \to t(f)\). A category is equipped with a composition operation \(f \circ g\) which is defined whenever \(s(f) = t(g)\). We require that the operation be associative and that for any \(V \in \text{Ob}(C)\), there is a morphism \(Id_V : V \to V\) such that \(f \circ Id_V = f\) and \(Id_V \circ g = g\) whenever \(f \in \text{Hom}(V, W)\), \(g \in \text{Hom}(U, V)\) for any \(U, W \in \text{Ob}(C)\). An isomorphism in a category is, as usual, a morphism \(f : U \to V\) such that there exists a morphism \(g : V \to U\) satisfying \(f \circ g = Id_U\) and \(g \circ f = Id_U\).

A tensor product on a category \(C\) is an operation \(\otimes : C \times C \to C\) such that

- For every pair of objects \(U, V \in \text{Ob}(C)\), there is an associated object \(U \otimes V \in \text{Ob}(C)\).
- For every pair of morphisms \(f : U_1 \to V_1, g : U_2 \to V_2\), there is an associated morphism \(f \otimes g : U_1 \otimes V_1 \to U_2 \otimes V_2\).
- For every additional pair of morphisms \(f' : V_1 \to W_1, g' : V_2 \to W_2\), we have
  \[
  (f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g).
  \]
- For every pair of objects \(U, V \in \text{Ob}(C)\), the identity morphism satisfies \(Id_{U \otimes V} = Id_U \otimes Id_V\).
An associativity constraint $a$ for a tensor product is a family of isomorphisms assigning an isomorphism $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ to each triple of objects $U, V, W \in \text{Ob}(C)$ such that the diagram

\[
(U \otimes V) \otimes W \xrightarrow{a_{U,V,W}} U \otimes (V \otimes W) \quad \xrightarrow{(f \otimes g) \otimes h} \quad (U' \otimes V') \otimes W' \xrightarrow{a_{U',V',W'}} U' \otimes (V' \otimes W')
\]

commutes for any morphisms $f, g, h$. The associativity constraint is said to satisfy the pentagon axiom if for every collection of objects $U, V, W, X \in \text{Ob}(C)$, the diagram

\[
((U \otimes V) \otimes W) \otimes X \xrightarrow{a_{U,V,W} \otimes \text{Id}_X} ((U \otimes V) \otimes W) \otimes X \quad \xrightarrow{a_{U,V,W,X}} \quad (U \otimes V) \otimes (W \otimes X) \quad \xrightarrow{\text{Id}_U \otimes a_{V,W,X}} \quad U \otimes ((V \otimes W) \otimes X)
\]

commutes. Let $I \in \text{Ob}(C)$ be the unit object in $C$. Then, a left unit constraint $l$ in $C$ is a natural family of isomorphisms assigning to each object $U$ an isomorphism $l_U : I \otimes U \to U$ such that for any morphisms $f : U \to V$, the diagram

\[
I \otimes U \xrightarrow{l_U} U \quad \xrightarrow{\text{Id}_I \otimes f} \quad I \otimes V \xrightarrow{l_V} V
\]

commutes. Right unit constraints are defined similarly. The associativity, left, and right constraints satisfy the triangle axiom if the diagram

\[
(U \otimes I) \otimes V \xrightarrow{a_{U,I,V}} U \otimes (I \otimes V) \quad \xrightarrow{r_U \otimes \text{Id}_V} \quad U \otimes V \quad \xrightarrow{\text{Id}_U \otimes l_V} \quad U \otimes V
\]

commutes for every pair of object $U, V \in \text{Ob}(C)$. Then, a tensor (or monoidal) category is a category $C$ equipped with a tensor product, associativity constraint satisfying the pentagon axiom, and left and right unit constraints satisfying the triangle axiom. The category is said to be strict if the associativity and unit constraints are the identity map. Strictness may sound like a very strong condition for a monoidal category, but it is not. Mac Lane’s coherence theorem (see [9] for proof) states that for every monoidal category, there exists an equivalent strict monoidal category. It is important to note that this is not true in general for categories with additional structure. The
A commutativity constraint is a family of isomorphisms $c_{U, V} : U \otimes V \to V \otimes U$ for each pair of objects $U, V \in \text{Ob}(C)$ such that

\[
\begin{array}{c}
U_1 \otimes V_1 \xrightarrow{c_{U_1, V_1}} V_1 \otimes U_1 \\
\downarrow f \otimes g \\
U_2 \otimes V_2 \xrightarrow{c_{U_2, V_2}} V_2 \otimes U_2
\end{array}
\]

commutes for any morphisms $f : U_1 \to U_2, g : V_1 \to V_2$. A braiding is a commutativity constraint which also satisfies the Hexagon Axiom, which is the commutativity of the following diagrams:

A twist in a tensor category is a family of morphisms $\theta = \{\theta_V : V \to V\}$ such that

$$\theta_{U \otimes V} = c_{V, U} \circ c_{U, V} \circ (\theta_U \otimes \theta_V).$$

and for any morphism $f : U \to V$, we have $\theta_V \circ f = f \circ \theta_U$. If for a given $V \in C$ there is an associated object $V^*$ and “duality morphisms”

$$\text{coev}_V : 1 \to V \otimes V^*, \quad \text{ev}_V : V^* \otimes V \to 1.$$

satisfying the commutative diagrams:
which are equivalent to the relations

\[(id_V \otimes \overline{\theta}_V) \circ a_{V,V^*,V} \circ (\overline{\coev}_V \otimes id_V) = id_V,\]
\[(\overline{\theta}_V \otimes id_{V^*}) \circ a_{V^*,V,V^*}^{-1} \circ (id_{V^*} \otimes \overline{\coev}_V) = id_{V^*},\]

then \(V^*\) is said to be left dual to \(V\). If such a dual exists for every \(V \in \mathcal{C}\) then \(\mathcal{C}\) is called left rigid. The duality morphisms are said to be compatible with the braiding and twist if the relation

\[(\theta_V \otimes id_{V^*}) \circ \overline{\coev}_V = (id_V \otimes \theta_{V^*}) \circ \overline{\coev}_V.\]

holds for all objects \(V\) in \(\mathcal{C}\). This compatibility ensures some nice properties for the category including a canonical isomorphism \((V^*)^* \simeq V\). Any category with this property is called pivotal. A category is called right rigid if it yields “right duality morphisms” given by

\[
\overline{\coev}_V := (Id_{V^*} \otimes \theta_V) \circ c_{V,V^*} \circ \overline{\coev}_V \in \text{Hom}(\mathbb{1}, V^* \otimes V),
\]
\[
\overline{\theta}_V := \overline{\theta}_V \circ c_{V,V^*} \circ (\theta_V \otimes Id_{V^*}) \in \text{Hom}(V \otimes V^*, \mathbb{1}).
\]

which satisfy the diagrams
which are equivalent to the relations

\[(\tilde{ev}_V \otimes id_V) \circ a_{V,V^*,V}^{-1} (id_V \otimes \tilde{ev}_V) = id_V,\]
\[(id_{V^*} \otimes \tilde{ev}_V) \circ a_{V^*,V,V^*} (\tilde{coev}_V \otimes id_{V^*}) = id_{V^*}.\]

A category is called rigid if it is both left and right rigid. A braided tensor category with twist and compatible duality is called ribbon.

### 2.1 Traces and Hopf Links

We will now construct the key objects of this thesis, the open Hopf links. The definitions in this subsection are taken from [1] and [37]. An ideal of \( \mathcal{C} \) is a subcategory \( \mathcal{I} \) which absorbs products and is closed under retracts. That is,

- If \( U \in \mathcal{I} \) and \( V \in \mathcal{C} \), then \( U \otimes V \in \mathcal{I} \).
- If \( U \in \mathcal{I} \) and \( \alpha : V \to U, \beta : U \to V \) morphisms such that \( \beta \circ \alpha = Id_V \), then \( V \in \mathcal{I} \).

**Definition 1.** For any \( U, V \in \mathcal{C} \) and any \( f \in \text{End}(U \otimes V) \), we define the left and right partial trace on \( \text{End}(V) \) and \( \text{End}(U) \) respectively as

\[
\text{ptr}_L(f) = (\tilde{ev}_U \otimes Id_V) \circ (Id_{U^*} \otimes f) \circ (\tilde{coev}_U \otimes Id_V) \in \text{End}(V),
\]
\[
\text{ptr}_R(f) = (Id_U \otimes \tilde{ev}_V) \circ (f \otimes Id_{V^*}) \circ (Id_U \otimes \tilde{coev}_V) \in \text{End}(U).
\]

A modified trace on \( \mathcal{I} \) is a family of linear functions indexed by the objects \( V \) in \( \mathcal{I} \)

\[\{t_V : \text{End}(V) \to \mathbb{1}\}\]
such that

- If $U \in \mathcal{I}$ and $V \in \mathcal{C}$ then for any $f \in \text{End}(U \otimes V)$,
  \[ t_{U \otimes V}(f) = t_U(\text{ptr}_R(f)) \]
- If $U, V \in \mathcal{I}$ then for any morphisms $f : V \to U$ and $g : U \to V$ we have,
  \[ t_V(g \circ f) = t_U(f \circ g) \]

The main focus of this paper will be on Hopf links:

**Definition 2.** If $U, V \in \mathcal{C}$, then the generalized Hopf link $\Phi_{U,V} \in \text{End}(V)$ is defined as

\[
\Phi_{U,V} = \text{ptr}_R(c_{U,V} \circ c_{V,U}) = (\text{Id}_V \otimes \widehat{\text{coev}}_U) \circ ((c_{U,V} \circ c_{V,U}) \otimes \text{Id}_{U^*}) \circ (\text{Id}_V \otimes \text{coev}_U)
\]

where $c_{U,V}$ is the braiding.

### 2.2 Ribbon Graphs and Graphical Calculus

In this section we review ribbon graphs and methods for constructing categorical morphisms from them. This is necessary for the construction of the Alexander invariant given by Murakami in [8]. Our primary references for this section are [9] and [10]. We begin by defining the fundamental constituents of a ribbon graph: bands, annuli, and coupons. A band is an embedding in $\mathbb{R}^3$ of the square $[0, 1] \times [0, 1]$, an annulus is an embedding of the cylinder $S^1 \times [0, 1]$, and a coupon is a band with a distinguished base. The image of the interval $\{1/2\} \times [0, 1]$ in a band is called the core of the band and the image of $S^1 \times \{1/2\}$ in an annulus is called the core of the annulus. Bands and annuli are called directed if their cores are oriented, and the orientation of the core is referred to as its direction.

**Definition 3.** Let $m$ and $n$ be nonnegative integers. Then a ribbon graph $T$ of type $(n, m)$ in $\mathbb{R}^3$ is an oriented surface $\Sigma$ embedded in $\mathbb{R}^2 \times [0, 1]$ which can be decomposed into a finite union of bands, coupons, and annuli such that:

- $\Sigma$ meets the planes $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ orthogonal along the line segments which are bases for certain bands in $\Sigma$:
  \[
  \{(i - (1/10), i + (1/10)) \times \{0\} \times \{0\} | i = 1, \ldots, n\},
  \]
  \[
  \{(j - (1/10), j + (1/10)) \times \{0\} \times \{1\} | j = 1, \ldots, m\}.
  \]
- All other bases of bands which are not in the above line segments are along coupons and the bands, coupons, and annuli are otherwise disjoint.
- The bands and annuli of $\Sigma$ are directed.
A ribbon graph is called homogeneous if in any neighbourhood of the above line segments the attached band is oriented upwards. A simple example of a (1,1)-ribbon graph containing bands, annuli, and a coupon is the graph

![Diagram of a (1,1)-ribbon graph](image)

For the sake of simplicity, from this point on we neglect to draw the width of bands and the boundary. Some examples of important ribbon graphs are crossings

\[ X^+ \quad X^- \]

and curves

\[ \cup \quad \cup \quad \cap \quad \cap \]

There is a way to represent morphisms in a category by "coloring" ribbon graphs with objects and morphisms. Let \( C \) be a strict tensor category. Then a morphism \( f : U \to V \) is represented graphically as

\[
    \begin{array}{c}
        V \\
        \downarrow f \\
        U
    \end{array}
\]

By labeling the arrows with \( U, V \) we are said to be coloring the bands with the objects \( U, V \in \text{Ob}(C) \), and the coupon with the morphism \( f \). When the crossings and curves are colored with objects \( U, V \), we denote them by \( X^+_{U,V}, X^-_{U,V}, \cup_U, \cap_U, \cap_U, \cap_U \). These ribbon graphs are important because the category of homogeneous ribbon graphs is generated by these six graphs and some relations between them (see [21]). Here, a ribbon graph is called homogeneous if the intersection points of bands with the boundaries have the same orientation. For example, the band with one twist

\[
    \begin{array}{c}
        \\\\\
    \end{array}
\]

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is not homogeneous, but the band with two twists

is homogeneous. The composition of two morphisms \( g_1 : U \to V \), \( g_2 : V \to W \) and tensor product of morphisms \( f_1 : U_1 \to V_1 \), \( f_2 : U_2 \to V_2 \) are represented as

\[
\begin{array}{ccc}
W & \overset{g_2 \circ g_1}{\longrightarrow} & W \\
\downarrow & & \downarrow \\
U & \overset{g_1}{\longrightarrow} & U \\
\end{array} = \begin{array}{ccc}
W & \overset{g_2}{\longrightarrow} & W \\
\downarrow & & \downarrow \\
U & \overset{g_1}{\longrightarrow} & U \\
\end{array}
\]

\[
\begin{array}{ccc}
V_1 \otimes V_2 & \overset{f_1 \otimes f_2}{\longrightarrow} & U_2 \otimes V_2 \\
\downarrow & & \downarrow \\
U_1 \otimes U_2 & \overset{f_1}{\longrightarrow} & U_1 \otimes V_1 \\
\end{array} = \begin{array}{ccc}
V_1 & \overset{f_1}{\longrightarrow} & U_1 \\
\downarrow & & \downarrow \\
V_2 & \overset{f_2}{\longrightarrow} & V_1 \\
\end{array}
\]

and a morphism on tensor products, \( f : U_1 \otimes \cdots \otimes U_m \to V_1 \otimes \cdots \otimes V_n \) can be represented as

\[
\begin{array}{ccc}
V_1 \otimes \cdots \otimes V_n & \overset{f}{\longrightarrow} & U_1 \otimes \cdots \otimes U_m \\
\downarrow & & \downarrow \\
V_1 \cdots V_n & \overset{f}{\longrightarrow} & U_1 \cdots U_m \\
\end{array}
\]

or

\[
\begin{array}{ccc}
U_1 \cdots U_m & \overset{f}{\longrightarrow} & V_1 \cdots V_n \\
\downarrow & & \downarrow \\
U_1 \cdots U_m & \overset{f}{\longrightarrow} & V_1 \cdots V_n \\
\end{array}
\]

If \( \mathcal{C} \) is a braided tensor category with braiding \( c_{V,W} \), then we represent the braiding by the crossing graphs \( X_{U,V}^+ \) and \( X_{U,V}^- \):

\[
\begin{array}{c}
c_{U,V} \quad \quad \quad \quad \quad c_{U,V}^{-1}
\end{array}
\]

The invertibility of the braiding is then represented as

\[
c_{U,V}^{-1} \circ c_{U,V} = Id_{U \otimes V} = c_{V,U} \circ c_{V,U}^{-1}
\]

\[
\begin{array}{c}
\begin{array}{c}
c_{U,V}^{-1} \circ c_{U,V} = Id_{U \otimes V} = c_{V,U} \circ c_{V,U}^{-1}
\end{array}
\end{array}
\]

and if \( f_1 : U_1 \to V_1 \) and \( f_2 : U_2 \to V_2 \) are morphisms, then the naturality of the braiding is given by

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
c_{U,V}^{-1} \circ c_{U,V} = Id_{U \otimes V} = c_{V,U} \circ c_{V,U}^{-1}
\end{array}
\end{array}
\end{array}
\]

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\[ c_{V_1,V_2} \circ (f_1 \otimes f_2) = (f_2 \otimes f_1) \circ c_{U_1,U_2} \]

The graphs for the left and right duality morphisms \( b_U, d_U, b'_U, d'_U \), are represented graphically by \( \cup_U, \bar{\cup}_U, \cap_U \), and \( \bar{\cap}_U \): 

\[ \begin{align*}
  & b_U & b'_U & d_U & d'_U \\
  & U & U & U & U
\end{align*} \]

and the twist is represented as 

\[ \varphi_U \]

An appropriately colored \((n,l)\)-ribbon graph can be reduced to a \((n-1,l-1)\) ribbon graph through the action of the partial trace functions described in section 3.1. The right partial trace acts on a colored ribbon graph by joining the right-most ends, and the left partial trace acts by joining the left-most ends. For example, the Hopf link is defined in section 3.1 as \( \Phi_{V,W} = \text{ptr}_R(c_{V,W} \circ c_{W,V}) \). Graphically, this is represented as 

\[ \begin{align*}
  & c_{V,W} \circ c_{W,V} & \Phi_{V,W} = \text{ptr}_R(c_{V,W} \circ c_{W,V}) \\
  & W & V & V & W
\end{align*} \]

Colored ribbon graphs will be used to construct Murakami’s Alexander invariants defined in [8].

3 The Singlet VOA

In this section we will recall the definition and elementary facts about VOAs and the singlet as they are described in [17] and [2] respectively.
3.1 Vertex Operator Algebras

Given a ring \( R \), we denote by \( R[z] \), \( R[[z]] \), and \( R((z)) \) the space of formal \( R \)-valued polynomials, Taylor series, and Laurent series respectively. That is,

\[
R[z] = \left\{ \sum_{i=0}^{n} r_i z^i | r_i \in R, n \in \mathbb{Z}_+ \right\}
\]
\[
R[[z]] = \left\{ \sum_{i=0}^{\infty} r_i z^i | r_i \in R \right\}
\]
\[
R((z)) = \left\{ \sum_{i=-m}^{\infty} r_i z^i | r_i \in R, m \in \mathbb{Z}_+ \right\}
\]

Let \( V \) be a complex vector space, and \( \text{End}(V) \) the collection of linear operators \( f : V \to V \). The formal power series

\[
A(z) = \sum_{i \in \mathbb{Z}} A_i z^{-i}
\]

with coefficients \( A_i \in \text{End}(V) \) is called a field if for all \( v \in V \), \( A(z)v \) is a Laurent series, so

\[
A(z)v = \sum_{i \in \mathbb{Z}} A_i(v)z^{-i} \in V((z)).
\]

This is equivalent to stating that for all \( v \in V \), \( A_n v = 0 \) for sufficiently large \( n(v) \). A \( \mathbb{Z}_+ \)-graded vector space is a vector space \( V \) such that \( V = \bigoplus_{i=0}^{\infty} V_i \) where each \( V_i \) is itself a vector space. A linear operator \( f \in \text{End}(V) \) on a graded vector space \( V \) is said to be homogeneous of degree \( m \) if \( f(V_m) \subset V_{n+m} \) for all \( n \in \mathbb{Z}_+ \), and we denote the degree by \( \text{deg} f \). A field \( A(z) \) is then said to be homogeneous of conformal dimension \( m \) if each operator \( A_i \) is homogeneous of degree \( m - i \).

**Definition 4.** Two fields \( A(z) \) and \( B(w) \) are said to be local iff there exists an \( N \in \mathbb{Z}_+ \) such that

\[
(z - w)^N [A(z), B(w)] := (z - w)^N (A(z)B(w) - B(w)A(z)) = 0.
\]

Equivalently, (see [17]) requiring that for every \( v \in V \) and \( \varphi \in V^* \), the Laurent series

\[
\varphi(A(z)B(w)v) = \sum_{i \in \mathbb{Z}_+} \left( \sum_{j \in \mathbb{Z}_+} \varphi(A_i(B_j(v))) z^{-i} \right) w^{-j}
\]
\[
\varphi(B(w)A(z)v) = \sum_{j \in \mathbb{Z}_+} \left( \sum_{i \in \mathbb{Z}_+} \varphi(B_j(A_i(v))) w^{-j} \right) z^{-i}
\]
are expansions in $C((z))((w))$ and $C((w))((z))$ respectively of the same element

$$f_{v,\varphi} \in C[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$$.

**Definition 5.** A vertex operator algebra (VOA) is a $\mathbb{Z}_+$-graded vector space $V = \bigoplus_{i=0}^{\infty} V_i$ where $\dim V_i < \infty$ for each $i \in \mathbb{Z}_+$ equipped with the following objects:

- A distinguished vector $|0\rangle \in V_0$ called the vacuum vector.
- A linear operator $T : V \to V$ of degree 1 called the translation operator.
- A linear operator $Y(-, z) : V \to \text{End} V[[z^{\pm 1}]]$ which sends each element $v \in V_m$ to a field of conformal dimension $m$: $Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1}$ with $\deg v(n) = -n + m - 1$.

These objects are subject to the following constraints:

- For any $v \in V$, we have $Y(v, z)|0\rangle \in V[[z]]$ where $Y(v, z)|0\rangle|z=0 = v$ and $Y(|0\rangle, z) = Id_V$.
- For any $v \in V$, $[T, Y(v, z)] = \partial_z Y(v, z)$ and $T|0\rangle = 0$.
- All fields $Y(v, z)$ are local with respect to each other.

Let $\mathfrak{h}$ denote the Heisenberg Lie algebra, so $\mathfrak{h}$ has vector space basis given by a central element $c$ and $b_n$ with $n \in \mathbb{Z}$. The commutation relations for $\mathfrak{h}$ are given by

$$[c, b_n] = 0 \text{ for all } n \in \mathbb{Z} \text{ and } [b_n, b_m] = n\delta_{n+m, 0}c.$$ 

Let $F_\lambda$ denote the usual Fock space of charge $\lambda$. As a vector space $F_\lambda$ is the $\mathbb{C}$-span of polynomials in variables $\{b_{-1}, b_{-2}, \ldots\}$ and $\mathfrak{h}$ acts on $F_\lambda$ by $b_0 = \lambda$, $b_n$ for $n < 0$ acts by multiplication, $c$ acts as the scalar $\lambda$, and $b_n$ for $n > 0$ acts as $\lambda n \frac{\partial}{\partial b_{-n}}$.

Let $p \geq 2$ be a positive integer and let $V_L$ be the lattice vertex algebra (see [17]) associated to the lattice $L = \sqrt{2p}\mathbb{Z}$.

$$V_L = \bigoplus_{\lambda \in \sqrt{2p}\mathbb{Z}} F_\lambda.$$ 

Let $e^\gamma$ denote the usual fields for lattice VOAs:

$$e^\gamma = S_\gamma z^{\gamma b_0} e^{\exp\left(-\gamma \sum_{n<0} \frac{b_n}{n} z^{-n}\right)} e^{\exp\left(-\gamma \sum_{n>0} \frac{b_n}{n} z^{-n}\right)},$$

where $S_\gamma$ is the shift operator $F_\lambda \mapsto F_{\lambda+\gamma}$. If $e^\gamma$ has Fourier expansion $e^\gamma(x) = \sum_{n \in \mathbb{Z}} e_0^\gamma x^{-n-1}$, define the “screening operator” $\check{Q} = e_0^{-\sqrt{2p}}$. The singlet VOA is then defined to be $\text{Ker} F_0 \check{Q}$, which we will denote by $W(2, 2p - 1)$.
3.2 Representations of $\mathcal{W}(2, 2p - 1)$

Let $\alpha_+ = \sqrt{2p}$, $\alpha_- = -\sqrt{2p}$, $\alpha_0 = \alpha_+ + \alpha_-$, and $\alpha_{r,s} = -\frac{r-1}{2} \sqrt{2p} + \frac{s-1}{\sqrt{2p}}$. If we assume $r \in \mathbb{Z}$ and $1 \leq s \leq p$, then when $s = p$, the $\mathcal{W}(2, 2r - 1)$-module $F_{\alpha_{r,s}}$ is irreducible as seen in [4]. For each $F_{\alpha_{r,s}}$ where $s \neq p$ we can associate an irreducible submodule $M_{r,s} := \text{soc}(F_{\alpha_{r,s}})$ which is defined to be the socle of $F_{\alpha_{r,s}}$.

A full classification of $\mathcal{W}(2, 2p - 1)$-modules is not known, but a classification for irreducible $\mathbb{Z} \geq 0$-graded modules was given in [5]. A $\mathbb{Z} \geq 0$-graded $\mathcal{W}(2, 2r - 1)$-module is called typical if it is also irreducible as a Virasoro module, and atypical if not. The irreducible $F_\lambda$ are typical while the $M_{r,s}$ are atypical.

The regularized characters of these modules given by a parameter $\epsilon$ are defined as

$$\text{ch}[F^\epsilon_\lambda](\tau) = e^{2\pi i (\lambda - \alpha_0/2)} \frac{e^{(\lambda - \alpha_0/2)^2}/2}{\eta(\tau)},$$

$$\text{ch}[M^\epsilon_{r,s}](\tau) = \sum_{n=0}^{\infty} \text{ch}[F^\epsilon_{\alpha_{r-2n-1,p-s}}](\tau) - \text{ch}[F^\epsilon_{\alpha_{r-2n-2,s}}](\tau).$$

Given the terms

$$S^\epsilon_{\lambda+\alpha_0/2,\mu+\alpha_0/2} = e^{2\pi i (\lambda - \mu)} e^{-2\pi i \lambda \mu},$$

$$S^\epsilon_{(r,s),\mu+\alpha_0/2} = -e^{-2\pi i (\frac{1}{2}(r-1)) \alpha_+ \mu} e^{\pi i (r-1) \alpha_+ \mu} \frac{\sin(\pi s \alpha_- (\mu + i\epsilon))}{\sin(\pi \alpha_+ (\mu + i\epsilon))},$$

the Verlinde algebra of characters $\mathcal{V}_{\text{ch}}$ is then defined (in [2]) to be the algebra whose vector space structure is generated by the characters $\text{ch}[V^\epsilon]$ with $V = F_\lambda, M_{r,s}$ and whose product is defined to be

$$\text{ch}[V^\epsilon_a] \times \text{ch}[V^\epsilon_b] := \int_{\mathbb{R}} \int_{\mathbb{R}} S^\epsilon_{a \rho, b \rho} \overline{S^\epsilon_{\rho \mu}} S^\epsilon_{(1,1) \rho} \text{ch}[F^\epsilon_\mu] d\mu d\rho.$$
\[
\text{ch}[F^e_{\lambda}] \times \text{ch}[F^e_{\mu}] = \sum_{l=0}^{p-1} \text{ch}[F_{\lambda+\mu+l\alpha_\lambda}],
\]
\[
\text{ch}[M^e_{r,s}] \times \text{ch}[F^e_{\mu}] = \sum_{l=-s+2}^{s} \text{ch}[F^e_{\mu+\alpha_{r,l}}],
\]
\[
\text{ch}[M^e_{r,s}] \times \text{ch}[M^e_{r',s'}] = \sum_{l=|s-s'|+1}^{s+s'-1} \text{ch}[M^e_{r+r'-1,l}] + \sum_{l=p+1}^{\min\{s+s'-1,p\}} (\text{ch}[M^e_{r+r'-2,l-p}] + \text{ch}[M^e_{r+r'-1,2p-l}] + \text{ch}[M^e_{r+r',l-p}]).
\]

The regularized asymptotic quantum dimensions are defined to be
\[
\text{qdim}[X^e] := \lim_{\tau \to 0^+} \frac{\text{ch}[X^e](\tau)}{\text{ch}[M^e_{1,1}](\tau)}.
\]

As in [31], we introduce
\[
B^r_\epsilon := -\min \left\{ \left| \frac{m}{\sqrt{2r}} - \text{Im}(\epsilon) \right| \mid m \in \mathbb{Z} - r\mathbb{Z} \right\}.
\]

For \(\text{Re}(\epsilon) > B^r_\epsilon\), the regularized asymptotic dimensions are given by
\[
\text{qdim}[F^e_{\lambda}] = d_\epsilon 2^{\lambda-a_0} \frac{\sin(-\pi\alpha_\lambda\epsilon i)}{\sin(\pi\alpha_\lambda i)} = d_\epsilon 2^{\lambda-a_0} \sum_{l=-p+1}^{p-1} q^{\alpha_{\lambda}}_\epsilon^{-l},
\]
\[
\text{qdim}[M^e_{r,s}] = d_\epsilon (r-1) \alpha_+ \frac{\sin(\pi s\alpha_-\epsilon i)}{\sin(\pi\alpha_- i)} = d_\epsilon (r-1) \alpha_+ \sum_{l=-s+1}^{s-1} q^{\alpha_{\lambda}}_\epsilon^{-l}.
\]

For \(\text{Re}(\epsilon) < B^r_\epsilon\), we have \(\text{qdim}[F^e_{\lambda}] = 0\), and for \(\epsilon \in S(k,m), k \in \mathbb{Z}, m = 0, ..., 2r-1\), we have
\[
\text{qdim}[M^e_{r,s}] = \begin{cases} 
(-1)^m (t-1) \frac{\sin(\pi ms/r)}{\sin(\pi m/r)} & \text{if } m \neq 0, r, \\
(-1)^{(m+1)(t-1)+m/r}(s-1) \frac{\sin(\pi s/r)}{\sin(\pi/r)} & \text{if } m = 0, r,
\end{cases}
\]

where
\[
S(k,m) = \left\{ \epsilon \in \mathbb{C} \mid k + \frac{2m-1}{4r} < \frac{\text{Im}(\epsilon)}{\sqrt{2r}} < k + \frac{2m+1}{4r} \right\}.
\]
The algebra of regularized quantum dimensions $Q$ is defined to be the algebra whose vector space structure is generated by the regularized quantum dimensions of typical and atypical modules with point-wise multiplication. It was shown in [2] that $V_{\text{ch}}$ and $Q$ are isomorphic. We will see that the normalized Hopf link invariants for $U^H_q(sl(2))$ are exactly the regularized quantum dimensions for $W(2,2r-1)$. The normalized Hopf link invariants determine much of the structure of the representation category for $U^H_q(sl(2))$ and it is conjectured ([2]) that $V_{\text{ch}}$ also determines much of the monoidal structure of the representation category of $W(2,2r-1)$, strongly suggesting that $U^H_q(sl(2))$ should make a good toy model for understanding representations of the singlet VOA.

4 The Unrolled Restricted Quantum Group of $sl(2)$ and its Representation Theory

We construct and review the unrolled restricted quantum group of $sl(2)$, $U^H_q(sl(2))$, as in [1]. Let $r \geq 2$ be a positive integer and $q = e^{\frac{2\pi i}{r}} \in \mathbb{C}$ a $2r^{th}$ root of unity. For any $x \in \mathbb{C}$, we fix the notation

\[
\{x\} = q^x - q^{-x}, \quad [x] = \frac{\{x\}}{\{1\}}, \quad \text{and} \quad \{x\}! = \{x\}\{x-1\}...\{1\}.
\]

4.1 Defining $U^H_q(sl(2))$

Let $U_q(sl(2))$ be the associative algebra over $\mathbb{C}$ with generators $K, K^{-1}, E, F$, and relations

\[
KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quadKF = q^{-2}FK, \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}.
\]

We can define a counit $\epsilon : U_q(sl(2)) \rightarrow \mathbb{C}$, coproduct $\Delta : U_q(sl(2)) \rightarrow U_q(sl(2)) \otimes U_q(sl(2))$, and antipode $S : U_q(sl(2)) \rightarrow U_q(sl(2))$ on $U_q(sl(2))$ by

\[
\Delta(K) = K \otimes K, \quad \epsilon(K) = 1, \quad S(K) = K^{-1},
\]

\[
\Delta(E) = 1 \otimes E + E \otimes K, \quad \epsilon(E) = 0, \quad S(E) = -EK^{-1},
\]

\[
\Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \epsilon(F) = 0, \quad S(F) = -KF.
\]

This gives $U_q(sl(2))$ the structure of a Hopf algebra. We define the unrolled quantum group of $sl(2)$, $U^H_q(sl(2))$, by extending $U_q(sl(2))$ through the addition of a fifth generator $H$ with relations
\[ HK^{\pm 1} = K^{\pm 1} H, \quad [H, E] = 2E, \quad [H, F] = -2F. \]

We extend the coproduct, counit, and antipode to this algebra by defining

\[ \Delta(H) = H \otimes 1 + 1 \otimes H, \quad \epsilon(H) = 0, \quad S(H) = -H. \]

The unrolled restricted quantum group \( U^H_q(\mathfrak{sl}(2)) \) is then obtained from \( U^H_q(\mathfrak{sl}(2)) \) by quotienting out the relations \( E^r = F^r = 0 \).

### 4.2 Representation theory of \( U^H_q(\mathfrak{sl}(2)) \)

We will now define the representation category of \( U^H_q(\mathfrak{sl}(2)) \) which we are interested in. Our main reference for this section is [1].

For any finite dimensional \( U^H_q(\mathfrak{sl}(2)) \)-module \( V \), an eigenvalue \( \lambda \in \mathbb{C} \) of \( H \) is called a weight of \( V \) and its associated eigenspace is called a weight space. Any \( v \in V \) in the eigenspace of \( \lambda \) \( (Hv = \lambda v) \) is called a weight vector of weight \( \lambda \). \( V \) is called a weight module if it is a direct sum of its weight spaces (\( H \) acts semi-simply) and the element \( K \in U^H_q(\mathfrak{sl}(2)) \) acts as \( q^H \) on \( V \) \( (Kv = q^Hv) \). Let \( \mathcal{C} \) be the category of finite dimensional weight \( U^H_q(\mathfrak{sl}(2)) \)-modules.

\( \mathcal{C} \) is a ribbon category: If \( V \) is an object in \( \mathcal{C} \) with basis \( \{v_1, \ldots, v_n\} \), then \( V \) has the natural dual vector space \( V^* = \text{Hom}_\mathbb{C}(V, \mathbb{C}) \) with basis \( \{v_1^*, \ldots, v_n^*\} \), and the left duality morphisms are

\[ \overrightarrow{\text{coev}}_V : \mathbb{C} \to V \otimes V^* \quad \text{and} \quad \overleftarrow{\text{ev}}_V : V^* \otimes V \to \mathbb{C} \]

where \( \overrightarrow{\text{coev}}(1) = \sum_i v_i \otimes v_i^* \), and \( \overleftarrow{\text{ev}}(f \otimes w) = f(w) \). Let \( v_1, \ldots, v_n \) denote a basis for a module \( V \), and let \( v = \sum_{j=1}^n \lambda_j v_i \) an element in \( V \). Then,

\[
(id_V \otimes \overleftarrow{\text{ev}}_V) \circ a_{V, V^*, V} \circ (\overrightarrow{\text{coev}}_V \otimes id_V)(1 \otimes v) = (id_V \otimes \overleftarrow{\text{ev}}_V) \circ a_{V, V^*, V} \sum_{i=1}^n (v_i \otimes v_i^*) \otimes v \\
= \sum_{i=1}^n \sum_{j=1}^n v_i \otimes \lambda_j v_i^*(v_j) \\
= \sum_{i=1}^n \lambda_i v_i \otimes 1 = v \otimes 1.
\]
and if \( f = \sum_{j=1}^{n} \lambda_j v_j^* \), then

\[
(\tilde{e}_V \otimes id_{V^*}) \circ a_{V^*,V^*}^{-1} \circ (id_{V^*} \otimes \tilde{\text{coev}}_V)(f \otimes 1) = (\tilde{e}_V \otimes id_{V^*}) \circ a_{V^*,V^*}^{-1} \sum_{i=1}^{n} f \otimes (v_i \otimes v_i^*)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_j v_j^*(v_i) \otimes v_i^*
\]

\[
= \sum_{j=1}^{n} \lambda_j v_j^* \otimes 1 = 1 \otimes f.
\]

Which shows that \( \mathcal{C} \) is left rigid. The braiding and twist were defined in [3] by Ohtsuki as follows:

For each pair of objects \( V, W \) in \( \mathcal{C} \) one can define an \( R \)-matrix operator on \( V \otimes W \) as

\[
R = q^{H \otimes H/2} \sum_{n=0}^{r-1} \frac{1}{\{n\}!} q^{n(n-1)/2} E^n \otimes F^n
\]  

(2)

where \( q^{H \otimes H/2}(v \otimes w) = q^{\lambda_v \lambda_w/2} v \otimes w \) for weight vectors \( v, w \) with weights \( \lambda_v \) and \( \lambda_w \). The braiding on \( \mathcal{C} \) is then given by a collection of maps \( c_{V,W} : V \otimes W \to W \otimes V \) where \( c_{V,W}(v \otimes w) = \tau(R(v \otimes w)) \) where \( \tau \) is the regular flip map \( w \otimes v \mapsto v \otimes w \). Ohtsuki defined an operator on each \( V \in \mathcal{C} \) as

\[
\theta = K^{r-1} \sum_{n=0}^{r-1} \frac{1}{\{n\}!} q^{n(n-1)/2} S(E^n) q^{-H^2/2} E^n.
\]  

(3)

The twist \( \theta_V : V \to V \) was then defined as the operator \( v \mapsto \theta^{-1} v \). \( \mathcal{C} \) also admits compatible right duality morphisms

\[
\tilde{\text{ev}}_V : V \otimes V^* \to \mathbb{C}, \quad \tilde{\text{ev}}_V(v \otimes f) = f(K^{1-r} v),
\]

\[
\tilde{\text{coev}}_V : \mathbb{C} \to V^* \otimes V, \quad \tilde{\text{coev}}(1) = \sum_i K^{r-1} v_i \otimes v_i^*.
\]

Showing \( \mathcal{C} \) is right dual is identical to the proof for left duality.\(^{l}\) The quantum dimension, \( \text{qdim}(V) \) of an object \( v \in \mathcal{C} \) is defined to be \( \text{qdim}(V) = \sum_{i=1}^{n} v_i^*(K^{1-r} v_i) \). Notice that this is the constant by which \( \tilde{\text{ev}}_V \circ \tilde{\text{coev}}_V \) acts on \( \mathcal{C} \).
4.3 Classification of simple and projective $\mathcal{U}^H_q(\mathfrak{sl}(2))$-modules

A classification of simple and projective $\mathcal{U}^H_q(\mathfrak{sl}(2))$-modules was given in [1] as follows: for any $n \in \{0, ..., r-1\}$ let $S_n$ be the simple highest weight $n+1$ dimensional module of weight $n$ with basis $\{s_0, ..., s_n\}$ and action

$$Fs_i = s_{i+1}, \quad Es_i = [i][n+1-i]s_{i-1}, \quad Hs_i = (n-2i)s_i, \quad Es_0 = Fs_n = 0.$$ 

For any $\alpha \in \mathbb{C}$, let $V_\alpha$ be the $r$-dimensional highest weight $\mathcal{U}^H_q(\mathfrak{sl}(2))$-modules of highest weight $\alpha + r - 1$ whose action is defined on a basis $\{v_0, ..., v_{r-1}\}$ as

$$Ev_i = [i][i-\alpha]v_{i-1}, \quad Fv_i = v_{i+1}, \quad Hv_i = (\alpha + r - 1 - 2i)v_i, \quad Ev_0 = Fv_{r-1} = 0.$$ 

$V_\alpha$ is called typical if $\alpha \in \bar{\mathbb{C}} := (\mathbb{C} - \mathbb{Z}) \cup r\mathbb{Z}$ and atypical otherwise. Notice that the typical $V_\alpha$ are simple since any element $v_i$ in the basis generates the entire module through the action of $E$ and $F$. If $V_\alpha$ is atypical, however, then we have $\alpha = rm + k$ for some $m \in \mathbb{Z}$ and $1 \leq k \leq r - 1$. Then,

$$Ev_k = -[k][k-(rm+k)]v_{k-1} = [k][rm]v_{k-1} = 0.$$ 

Since $\{rm\} = q^{rm} - q^{-rm} = (-1)^m - (-1)^{-m} = 0$. So, when $V_\alpha$ is atypical, it contains a simple submodule generated by the basis elements $\{v_k, v_{k+1}, ..., v_{r-1}\}$.

For any $\ell \in \mathbb{Z}$, let $\mathbb{C}_\ell$ be the one dimensional module on which $E, F$ act as zero and $H$ acts as $\ell r$. Then, we have the following proposition (proven in [1]):

**Proposition 2.** ● The typical $V_\alpha$ ($\alpha \in \bar{\mathbb{C}}$) are projective in $\mathcal{C}$.

● Every simple module in $\mathcal{C}$ is isomorphic to $S_n \otimes \mathbb{C}_\ell$ for some $n \in \{0, ..., r-1\}$ and $\ell \in \mathbb{Z}$ or $V_\alpha$ for some $\alpha \in \bar{\mathbb{C}}$.

Note that in [38], we consider an enlargement of the category $\mathcal{C}$ and the typical $V_\alpha$ are no longer projective in this larger category. For any $i \in \{0, ..., r-2\}$, let $P_i$ be the 2r-dimensional vector space over $\mathbb{C}$ with basis

$$\{w_i^H, w_{i-2}^H, ..., w_{i+2}^H, ..., w_{2r-2-i}^H, w_{i+2-2r}^H, ..., w_{i-2}^L, w_{i+2-2r}^S, ..., w_{i-2}^S\}.$$ 

Then $P_i$ is a well defined $\mathcal{U}^H_q(\mathfrak{sl}(2))$-module ([1]) with action

$$w_{i+2}^R = Ew_i^H, \quad w_i^S = Fw_{i+2}^R, \quad w_{i-2}^L = F^{i+1}w_i^H,$$
weight module with integer highest weight \((\gamma)\).

Proposition 3. The module \(P_i\) is projective and indecomposable, and any projective indecomposable weight module with integer highest weight \((\ell+1)r-i-2\) is isomorphic to \(P_i \otimes \mathbb{C}_{tr}^H\). The endomorphism ring of \(P_i \otimes \mathbb{C}_{tr}^H\) is two dimensional and we denote its basis \(\{\text{Id}, x_{i,\ell}\}\) where the action of \(x_{i,\ell}\) on \(P_i \otimes \mathbb{C}_{tr}^H\) is determined by \(x_{i,\ell}(w_i^H \otimes v) = w_i^S \otimes v\).

It was shown in [1] that there exists a modified trace \(\{t_V : \text{End}(V) \to \mathbb{C}\}_{V \in \text{Proj}}\) on the ideal of projective modules, denoted \(\text{Proj}\), in \(\mathcal{C}\). This trace is unique up to scalar multiples. In particular, there is a trace which acts as \(t_V(f) = (-1)^{r-1}\langle f \rangle\) on any element \(f \in \text{End}(V_0)\) where \(\langle f \rangle\) is the scalar by which \(f\) acts on \(V_0\). The modified quantum dimension is defined as

\[
d : \text{Ob}(\text{Proj}) \to \mathbb{C} \quad d(V) = t_V(\text{Id}_V).
\]

It was shown in [1] that for any typical \(V_\alpha\), we have

\[
d(V_\alpha) = (-1)^{r-1} \prod_{j=1}^{r} \frac{\{j\}}{\{\alpha + r - j\}} = (-1)^{r-1} \frac{\{\alpha\}}{\{r\alpha\}}.
\]

4.4 Characters and Tensor Decompositions

We define the character of a weight module \(V \in \mathcal{C}\) to be

\[
\chi(V) = \sum_\alpha \dim(V(\lambda))z^\lambda
\]

where \(z \in \mathbb{C}\) and \(V(\lambda)\) is the \(\lambda\)-eigenspace under the action of \(H\). Characters record the eigenspace structure of a module. Since all of the modules in \(\mathcal{C}\) are direct sums of their eigenspaces, the classification of projective modules above shows that projective modules are isomorphic if their
characters agree, which provides a nice method for determining tensor decompositions for projective modules. Let
\[ [k]_z = z^{k-1} + z^{k-3} + \ldots + z^{-(k-1)} = \frac{z^k - z^{-k}}{z - z^{-1}}. \]
Then, if \( \alpha \in \hat{\mathbb{C}} \) and \( i \in \{0, ..., r - 1\} \) we have
\[ \chi(V_\alpha) = z^\alpha[r]_z, \quad \chi(S_i) = [i + 1]_z, \quad \text{and} \quad \chi(P_i) = (z^{r-1-i} + z^{(1+i-r)})[r]_z. \]

**Proposition 4.** The complete list of tensor decompositions of isomorphism classes is the following:
For \( i, j \in \{0, ..., r\} \), we have
\[
S_i \otimes S_j = \begin{cases} 
  S_l & \text{if } i + j < r \\
  S_l \bigoplus_{l=|i-j|}^{r-4-i-j} P_l & \text{if } i + j \geq r
\end{cases}
\]
where "by 2" here means we only count every second term in the sum. If \( \alpha, \beta \in \hat{\mathbb{C}} \) such that \( \alpha + \beta \not\in \mathbb{Z} \) then
\[ V_\alpha \otimes V_\beta = \bigoplus_{l=1-r}^{r-1} V_{\alpha+\beta+t}. \]
When \( \alpha + \beta = n \in \mathbb{Z} \), set \( n = i + kr \) with \( i \in \{0, ..., r - 1\} \) and \( k \in \mathbb{Z} \). If \( \alpha \in \hat{\mathbb{C}} \) we have
\[
V_\alpha \otimes V_{n-\alpha} = \left( \bigoplus_{l=r-1-i}^{0} P_l \otimes \mathbb{C}_{kr}^H \right) \oplus \left( \bigoplus_{l=r-i}^{r-1} P_l \otimes \mathbb{C}_{(k+1)r}^H \right)
\]
and if \( \alpha \in \mathbb{C} - \mathbb{Z} \), then we have
\[ V_\alpha \otimes S_i = \bigoplus_{l=-i}^{i} V_{\alpha+t}, \quad \text{and} \quad V_{kr} \otimes S_i = V_0 \otimes S_i \otimes \mathbb{C}_{kr}^H. \]
where \( V_0 \otimes S_i = \bigoplus_{l=0}^{r-1} P_l \). The tensor decompositions for products of the projective \( P_i \) are

\[
P_i \otimes S_j = \left( \bigoplus_{l=0}^{r-1} P_l \right) \oplus \left( \bigoplus_{l=24-2-i-j}^{r-1} P_l \right) \oplus \left( \bigoplus_{l=r+i-j}^{r-1} P_l \right),
\]

\[
P_i \otimes P_j = \left( (\mathbb{C}_r^H \oplus \mathbb{C}_{-r}^H) \otimes (P_i \otimes S_{r-2-j}) \right) \oplus 2(P_i \otimes S_j),
\]

\[
V_0 \otimes P_j = \left( \bigoplus_{l=0}^{r-1} (\mathbb{C}_r^H \oplus \mathbb{C}_{-r}^H) \otimes P_l \right) \oplus \left( \bigoplus_{l=0}^{r-1} 2P_l \right),
\]

where we assume the index on the \( P_i \) to be in \( \{0, \ldots, r - 2\} \) while the index on the \( S_j \) are in \( \{0, \ldots, r - 1\} \). Finally, we have

\[
V_\alpha \otimes P_j = \bigoplus_{l=0}^{2(r-1)} V_{\alpha-j+l} \oplus \bigoplus_{l=0}^{2(r-1)} V_{\alpha+j-l}.
\]

Proof. All of these identities except for \( S_i \otimes S_j \) can be derived from their characters since every other product is projective. We only provide proofs for identities which are not given in [1].

\[
\chi(V_\alpha \otimes V_n - \alpha) = z^{i+kr}[r]_z^2 = z^{kr}(z^{r-1+i} + z^{r-3+i} + \ldots + z^{1-r+i})[r]_z = z^{kr} \left( (z^{1-r+i} + \ldots + z^{r-1-i}) + (z^{r+1-i} + \ldots + z^{r-1+i}) \right) [r]_z
\]

\[
= z^{kr} \left( \sum_{l=-1-i}^{0} (z^l + z^{-l}) \right) [r]_z + z^{(k+1)r}[i]_z \]

\[
= \sum_{l=-1-i}^{0} \chi(P_l \otimes \mathbb{C}^H_{kr}) + \chi(V_0 \otimes S_{i-1} \otimes \mathbb{C}^H_{(k+1)r}).
\]

If \( \alpha = kr \), we have \( \chi(V_{kr} \otimes S_i) = z^{kr}[r][i+1]_z = \chi(V_0 \otimes S_i \otimes \mathbb{C}^H_{kr}) \) and if \( \beta \in \mathbb{C} - \mathbb{Z} \) then

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χ(V_β ⊗ S_i) = z^β[i + 1]_z[r]_z = \sum_{l=-i}^{i} z^{β+l}[r]_z = \sum_{l=-i}^{i} χ(V_{β+l}).

Hence, for α ∈ Ĉ and n = i + kr with i ∈ {0, ..., r − 1} and k ∈ Z we have

\[ V_α ⊗ V_{n-α} = \left( \bigoplus_{l=-r-1-i}^{i} P_l \otimes \mathbb{C}_{kr}^H \right) \oplus \left( \bigoplus_{l=r-i}^{r-1} P_l \otimes \mathbb{C}_{(k+1)r}^H \right) \]

and for α ∈ C − Z, then we have

\[ V_α ⊗ S_i = \bigoplus_{l=-1}^{i} V_{α+l}, \]
\[ V_{kr} ⊗ S_i = V_0 ⊗ S_i \otimes \mathbb{C}_{kr}^H. \]

We can determine the product V_α ⊗ P_j for α ∈ C − Z from its character:

\[ \chi(V_α ⊗ P_j) = z^α[r]_z(z^{r-1-j} + z^{1+j-r})[r]_z \]
\[ = (z^{α-j} + z^{α-j+2} + ... + z^{α-j+2(r-1)})[r]_z + (z^{α+j} + z^{α+j-2} + ... + z^{α+j-2(r-1)})[r]_z \]
\[ = \sum_{l=0}^{2(r-1)} \chi(V_{α-j+l}) + \sum_{l=0}^{2(r-1)} \chi(V_{α+j-l}). \]

4.5 Hopf Link Computations

In this section we will construct the open Hopf links associated to Ĉ as was done in [1]. We will see that the method used to construct the open Hopf links on projective modules requires knowledge of the tensor ring structure. We will introduce later a new method for computing these Hopf links without using the tensor ring. Recall the Hopf links from definition 2:

\[ \Phi_{U,V} = \text{ptr}_R(c_{U,V} \circ c_{V,U}) = (Id_V \otimes \overline{\delta U}) \circ ((c_{U,V} \circ c_{V,U}) \otimes Id_U) \circ (Id_V \otimes \overline{\text{coev}}_U). \]

Proposition 5. Let i, j ∈ {0, ..., r − 2} and α, β ∈ Ĉ = (C \ Z) ∪ rZ. Then, we have the following:
\[ \Phi_{V_{j}, V_{a}} = \frac{(-1)^{r-1}r}{d(V_{a})}q^{\alpha \beta}Id_{V_{a}}, \quad \Phi_{P_{j} \otimes C_{H}, V_{a}} = \frac{(-1)^{r-1}r\ell \alpha q^{(r-1-i)\alpha} + q^{-(r-1-i)\alpha}}{d(V_{a})}Id_{V_{a}}, \]
\[ \Phi_{S_{i} \otimes C_{H}, V_{a}} = q^{\ell \alpha} \frac{(i+1)\alpha}{\{\alpha\}}Id_{V_{a}}, \quad \Phi_{V_{a}, P_{j} \otimes C_{H}} = (-1)\ell(r+1)+j q^{r\ell \alpha} r(q^{(r-j)\alpha} + q^{-(r-j)\alpha})x_{j, \ell}, \]
\[ \Phi_{P_{j} \otimes C_{H}', P_{j} \otimes C_{H}'} = (-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'} 2r(q^{(i+1)(j+1)} + q^{-(i+1)(j+1)})x_{j, \ell}, \]
\[ \Phi_{S_{i} \otimes C_{H}', P_{j} \otimes C_{H}'} = \frac{(-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}}{\{j+1\}} \left\{ (i+1)(j+1) \right\} Id_{P_{j} \otimes C_{H}'} \]
\[ + \frac{(-1)^{i(\ell+1)+(j+\ell r+1-r)\ell'}}{\{j+1\}} \left\{ (i+2)(j+1) \right\} - (i+2)\{i(j+1)\}x_{j, \ell}. \]

**Proof.** This proposition is a modification of Lemma 6.7 in [1] and the proof here will follow theirs closely adjusting it only where necessary.

Let \( V, W \in \mathcal{C} \) and \( w \in W \) a highest weight vector of weight \( \lambda \). Let \( \{v_{0}, ..., v_{s}\} \) be a basis for \( V \) and \( \lambda_{k} \) the weight of \( v_{k} \). Then,

\[ \Phi_{V, W}(w) = (Id_{W} \otimes \overline{ev}_{V}) \circ (c_{V, W} \otimes Id_{V'}) \circ (c_{W, V} \otimes Id_{V'}) \circ (Id_{W} \otimes coev_{V'})(w) \]
\[ = \sum_{k=0}^{s} (Id_{W} \otimes \overline{ev}_{V}) \circ (c_{V, W} \otimes Id_{V'}) \circ (c_{W, V} \otimes Id_{V'})(w \otimes v_{k} \otimes v_{k}^{*}) \]
\[ = \sum_{k=0}^{s} (Id_{W} \otimes \overline{ev}_{V}) \circ (\tau(R(R(w \otimes v_{k}))))(w \otimes v_{k}^{*}) \]
\[ = \sum_{k=0}^{s} (Id_{W} \otimes \overline{ev}_{V}) \circ (q^{H \otimes H}(w \otimes v_{k}) \otimes v_{k}^{*}) \]
\[ = \sum_{k=0}^{s} (Id_{W} \otimes \overline{ev}_{V}) \circ (q^{\lambda_{k}}(w \otimes v_{k}) \otimes v_{k}^{*}) \]
\[ = \sum_{k=0}^{s} q^{\lambda_{k}}w \otimes q^{(1-r)\lambda_{k}} \]
\[ = (\sum_{k=0}^{s} q^{(\lambda+1-r)\lambda_{k}})w. \]

Define for any \( \gamma \in \mathbb{C} \), \( \Psi_{\gamma} : \mathbb{Z}[z] \to \mathbb{C} \) by \( \Psi_{\gamma}(z^{s}) = q^{\gamma s} \). Then, one sees that \( \Phi_{V, W}(w) = \Psi_{\lambda+1-r}(\chi(V))w \). When \( W \) is simple we then have

\[ \Phi_{V, W} = \Psi_{\lambda+1-r}(\chi(V))Id_{W} \quad (4) \]
by Schur’s Lemma. Since $\alpha \in \mathbb{C} V_\alpha$ is typical, hence simple, we can compute the Hopf links on $V_\alpha$ directly.

$$
\Phi_{V_\beta, V_\alpha} = \Psi_\alpha(\chi(V_\beta)) Id_{V_\alpha} = \Psi_\alpha \left( \frac{z^r - z^{-r}}{z - z^{-1}} \right) Id_{V_\alpha} = \frac{\{r\alpha\}}{\{\alpha\}} q^{\alpha \beta} Id_{V_\alpha} = \frac{(-1)^r q^{-\alpha \beta}}{d(\alpha)} Id_{V_\alpha},
$$

$$
\Phi_{S_1, V_\alpha} = \Psi_\alpha(\chi(S_1)) Id_{V_\alpha} = \Psi_\alpha \left( \frac{z^{i+1} - z^{-(i+1)} + z^r - z^{-r}}{z - z^{-1}} \right) Id_{V_\alpha} = \frac{(i + 1)\alpha}{\{\alpha\}} Id_{V_\alpha},
$$

$$
\Phi_{P_i, V_\alpha} = \Psi_\alpha(\chi(P_i)) Id_{V_\alpha} = \Psi_\alpha \left( \frac{z^{r-1-i} + z^{-r+1+i}}{z - z^{-1}} \right) Id_{V_\alpha} = (-1)^{r-1} q^{(r-1-i)\alpha} + q^{-(r-1-i)\alpha} \frac{d(\alpha)}{Id_{V_\alpha}},
$$

$$
\Phi_{\mathbb{C}^H_{\ell r}, V_\alpha} = \Psi_\alpha(\chi(z^{\ell r})) Id_{V_\alpha} = q^{\ell r \alpha} Id_{V_\alpha}.
$$

For any $X, Y, Z \in \mathbb{C}$ we have $\Phi_{X \otimes Y, Z} = \Phi_{X, Z} \circ \Phi_{Y, Z}$ so this proves the proposition for $\Phi_{V_\beta, V_\alpha}$, $\Phi_{S_1 \otimes \mathbb{C}^H_{\ell r}, V_\alpha}$, and $\Phi_{P_i \otimes \mathbb{C}^H_{\ell r}, V_\alpha}$.

The endomorphism ring of $P_j \otimes \mathbb{C}^H_{\ell r}$ is two dimensional so any endomorphism of $P_j \otimes \mathbb{C}^H_{\ell r}$ has the form $a Id_{P_j \otimes \mathbb{C}^H_{\ell r}} + bx_j \epsilon$ for some $a, b \in \mathbb{C}$. So to determine $\Phi_{S_1, P_j \otimes \mathbb{C}^H_{\ell r}}$ it is enough to find $a$ and $b$. $w_j^S \otimes v \in P_j \otimes \mathbb{C}^H_{\ell r}$ is a highest weight vector, so

$$
\Phi_{S_1, P_j \otimes \mathbb{C}^H_{\ell r}} (w_j^S \otimes v) = \Psi_{j, \ell r+1-r} (\chi(S_1)) w_j^S \otimes v = (q^{j+\ell r+1-r} + q^{-(j+\ell r+1)\alpha})w_j^S \otimes v = (-1)^{1+1}(q^{(j+1) + q^{-(j+1)}})w_j^S \otimes v.
$$

Hence, $a = (-1)^{1+1}(q^{(j+1) + q^{-(j+1)}})$. $S_1$ has a basis $\{s_0, s_1\}$ where $E s_0 = 0$, $E s_1 = s_0$, $F s_0 = s_1$, $F s_1 = 0$, and $H s_i = (-1)^i s_i$. Recall that $c_{V, W} = \tau \circ R$ where $\tau$ is the flip map and $E^2 = F^2 = 0$ on $S_1$ so $R = q^{H \otimes H}/2 (Id \otimes Id + (q - q^{-1})E \otimes F)$ here. For some $X, Y \in P_j \otimes \mathbb{C}^H_{\ell r}$, we have

$$
c_{S_1, P_j \otimes \mathbb{C}^H_{\ell r}} \circ c_{P_j \otimes \mathbb{C}^H_{\ell r}, S_1} (w_j^H \otimes v \otimes s_0) = c_{S_1, P_j \otimes \mathbb{C}^H_{\ell r}} (q^{j+\ell r}/2) s_0 \otimes w_j^H \otimes v
$$

$$
+ (q - q^{-1})q^{\ell r-(j+\ell r+2)/2} s_1 \otimes w_j^H \otimes v
$$

$$
= (q^{j+\ell r}) w_j^H \otimes v \otimes s_0 + (q - q^{-1})^2 q^{\ell r-1} w_j^S \otimes v \otimes s_0 + X \otimes s_1
$$

$$
= (-1)^{\ell} (q^{j+\ell r}) w_j^H \otimes v \otimes s_0 + (q - q^{-1})^2 q^{-1} w_j^S \otimes v \otimes s_0 + X \otimes s_1
$$

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\[ c_{S_1, P_j \otimes C_H^{tr}} \circ c_{P_j \otimes C_H^{tr}, S_1}(w_j^H \otimes v \otimes s_1) = c_{S_1, P_j \otimes C_H^{tr}}(q^{-(j+\ell)r}/2s_1 \otimes w_j^H \otimes v) \]
\[ = (q^{-(j+\ell)r})w_j^H \otimes v \otimes s_1 + Y \otimes s_0 \]
\[ = (-1)^{\ell}q^{-j}w_j^H \otimes v \otimes s_1 + Y \otimes s_0. \]

Hence,

\[ \Phi_{S_1, P_j \otimes C_H^{tr}}(w_j^H \otimes v) = (Id_{P_j \otimes C_H^{tr}} \otimes \overline{ev}_S) \circ ((c_{S_1, P_j \otimes C_H^{tr}} \circ c_{P_j \otimes C_H^{tr}, S_1} \otimes Id_{S_1^*}) \circ (Id_{P_j \otimes C_H^{tr}} \otimes \overline{coev}_S)) (w_j^H \otimes v) \]
\[ = (Id_{P_j \otimes C_H^{tr}} \otimes \overline{ev}_S) \circ ((c_{S_1, P_j \otimes C_H^{tr}} \circ c_{P_j \otimes C_H^{tr}, S_1}(w_j^H \otimes v \otimes s_0)) \otimes s_0^*) \]
\[ + (Id_{P_j \otimes C_H^{tr}} \otimes \overline{ev}_S) \circ ((c_{S_1, P_j \otimes C_H^{tr}} \circ c_{P_j \otimes C_H^{tr}, S_1}(w_j^H \otimes v \otimes s_1)) \otimes s_1^*) \]
\[ = (-1)^{\ell}(q^jw_j^H \otimes v + (q - q^{-1})^2q^{-1}w_j^S \otimes v) \otimes s_0^*(K^{1-r}s_0) + X \otimes s_0^*(K^{1-r}s_1) \]
\[ + (-1)^{\ell}q^{-j}w_j^H \otimes v \otimes s_1^*(K^{1-r}s_1) + Y \otimes s_1^*(K^{1-r}s_0) \]
\[ = (-1)^{\ell+1}q(q^jw_j^H \otimes v + (q - q^{-1})^2q^{-1}w_j^S \otimes v) + (-1)^{\ell+1}q^{-j-1}w_j^H \otimes v \]
\[ = (-1)^{\ell+1}(q^{j+1} + q^{-j-1})w_j^H \otimes v + (-1)^{\ell+1}(q - q^{-1})^2w_j^S \otimes v. \]

For the sake of consistency with [1], we rescale \( x_{j, \ell} \) by a constant \( \frac{(q^{j+1} - q^{-j+1})^2}{(q - q^{-1})^2} \) so, we have \( a = (-1)^{\ell+1}(q^{j+1} + q^{-j-1}) \) and \( b = (-1)^{\ell+1}(q^{j+1} - q^{-j+1})^2 \). The \( S_i \) are classified up to isomorphism by their characters, so we obtain the relation \( S_1 \otimes S_i \cong S_{i+1} \oplus S_{i-1} \) from their characters. Therefore, we have

\[ \Phi_{S_1, P_j \otimes C_H^{tr}} \circ \Phi_{S_i, P_j \otimes C_H^{tr}} = \Phi_{S_1 \otimes S_i, P_j \otimes C_H^{tr}} = \Phi_{S_{i+1}, P_j \otimes C_H^{tr}} + \Phi_{S_{i-1}, P_j \otimes C_H^{tr}}. \]

This gives the recurrence relation

\[ \Phi_{S_{i+1}, P_j \otimes C_H^{tr}} = (a + bx_{j, \ell}) \Phi_{S_i, P_j \otimes C_H^{tr}} - \Phi_{S_{i-1}, P_j \otimes C_H^{tr}} \]

whose solution

\[ \Phi_{S_i, P_j \otimes C_H^{tr}} = (-1)^{i(j+1)} \frac{(j+1)!}{(j+1)!} \{(i+1)(j+1) + x_{j, \ell}(i((i+2)(j+1)) - (i+2)(i(j+1))\}} \]

can be found by induction. Since \( S_{r-1} = V_0 \), we have \( \Phi_{V_0, P_j \otimes C_H^{tr}} = (-1)^{r(j+1)+j+\ell}2rx_{j, \ell} \). As projective modules are classified up to isomorphism by their characters, we have the isomorphism of modules

\[ V_0 \otimes S_{r-i-1} = V_0 \oplus S_{r-i-3} \oplus P_i. \]

Hence, we obtain the relation

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Corollary 1. \( \Phi_{P_i, P_j \otimes C^H_{P_i}} = \Phi_{V_0, P_i \otimes C^H_{P_i}} \circ (\Phi_{S_{r-1-1}, P_j \otimes C^H_{P_i}} - \Phi_{S_{r-1-1}, P_j \otimes C^H_{P_i}}) \)

\[
\begin{align*}
\Phi_{P_i, P_j \otimes C^H_{P_i}} &= \Phi_{V_0, P_i \otimes C^H_{P_i}} \circ (\Phi_{S_{r-1-1}, P_j \otimes C^H_{P_i}} - \Phi_{S_{r-1-1}, P_j \otimes C^H_{P_i}}) \\
&= (-1)^{(j+1)+j+12} \left( \{ (r - i)(j + 1) \} - \{ (r - i - 2)(j + 1) \} \right) x_{j, \ell} \\
&= (-1)^{(j+1)+j+1} 2r(q^{r(i+1)}(j+1) + q^{-(i+1)(j+1)}) x_{j, \ell} \\
&= (-1)^{(j+1)2r(q^{r(i+1)}(j+1) + q^{-(i+1)(j+1)}) x_{j, \ell}}.
\end{align*}
\]

To complete the proof for \( \Phi_{P_i, P_j \otimes C^H_{P_i}} \), we need only compute \( \Phi_{C^H_{P_i}, P_j \otimes C^H_{P_i}} \) and take the appropriate compositions of Hopf links:

\[
\begin{align*}
\Phi_{C^H_{P_i}, P_j \otimes C^H_{P_i}}(w_i^H \otimes v) &= (Id_{P_j} \otimes C^H_{P_i} \otimes \hat{\epsilon} C^H_{P_i}) \circ (\Phi_{C^H_{P_i}, P_j \otimes C^H_{P_i}}(w_i^H \otimes v) \otimes w^*) \\
&= (Id_{P_j} \otimes C^H_{P_i} \otimes \hat{\epsilon} C^H_{P_i}) \circ (\Phi_{C^H_{P_i}, P_j \otimes C^H_{P_i}}(w_i^H \otimes v) \otimes w^*) \\
&= (Id_{P_j} \otimes C^H_{P_i} \otimes \hat{\epsilon} C^H_{P_i}) \circ (\Phi_{C^H_{P_i}, P_j \otimes C^H_{P_i}}(w_i^H \otimes v) \otimes w^*) \\
&= q^{(j+\ell r)\ell r}(w_j^H \otimes v) \otimes w^*(K^{1-r} w) \\
&= q^{(j+\ell r)\ell r}(w_j^H \otimes v) \otimes q^{\ell r(1-r)} \\
&= (-1)^{(j+\ell r+1-r)\ell r} w_j^H \otimes v.
\end{align*}
\]

Notice that the weight of any vector \( w_i^X \otimes v \in P_j \otimes C^H_{P_i} \) differs from the weight of \( w_j^H \otimes v \) by a multiple of 2, so the exponent of \(-1\) does not see this difference. Hence, we have

\[
\Phi_{C^H_{P_i}, P_j \otimes C^H_{P_i}} = (-1)^{(j+\ell r+1-r)\ell r} Id_{P_j} \otimes C^H_{P_i}.
\]

We delay the proof of \( \Phi_{V_0, P_j \otimes C^H_{P_i}} \) until after the following corollary.

**Corollary 1.** The modified trace of the Hopf links are given by:

\[
\begin{align*}
t_{V_0}(\Phi_{V_0, V_0}) &= (-1)^{r-1} q^{\alpha \beta}, \\
t_{V_0}(\Phi_{S_i \otimes C^H_{P_i}, V_0}) &= (-1)^{r-1} q^{r \alpha} \frac{\{ (i + 1) \alpha \}}{\{ \alpha \}}, \\
t_{P_i \otimes C^H_{P_i}}(\Phi_{V_0, P_i \otimes C^H_{P_i}}) &= t_{V_0}(\Phi_{P_i \otimes C^H_{P_i}, V_0}) = (-1)^{r-1} q^{r \alpha} (q^{(r-1-i) \alpha} + q^{-(r-1-i) \alpha}),
\end{align*}
\]
Proof. The Hopf links on $V_α$ are easy to compute:

\[
\begin{align*}
  t_{P_j \otimes \mathbb{C}^H_{\ell r}}(\Phi_{P_j \otimes \mathbb{C}^H_{\ell r}, P_j \otimes \mathbb{C}^H_{\ell r}}(\Phi_{P_j \otimes \mathbb{C}^H_{\ell r}, P_j \otimes \mathbb{C}^H_{\ell r}}(x_{j,\ell})) &= (-1)^{(i+r+1)\ell+(j+\ell r+1-r)\ell'+i+j+1} 2r(q^{(i+1)(j+1)} + q^{-(i+1)(j+1)}), \\
  t_{P_j \otimes \mathbb{C}^H_{\ell r}}(\Phi_{S_i \otimes \mathbb{C}^H_{\ell r}, P_j \otimes \mathbb{C}^H_{\ell r}}(x_{j,\ell})) &= (-1)^{(i+r+1)\ell+(j+\ell r+1-r)\ell'+i+j+1}(i+1)(q^{(i+1)(j+1)} + q^{-(i+1)(j+1)}), \\
  t_{P_j \otimes \mathbb{C}^H_{\ell r}}(\Phi_{S_i \otimes \mathbb{C}^H_{\ell r}, P_j \otimes \mathbb{C}^H_{\ell r}}(x_{j,\ell})) &= (-1)^{(i+r+1)\ell+(j+\ell r+1-r)\ell'+i+j+1} \frac{(i+1)(j+1)}{j+1}.
\end{align*}
\]

To compute the trace of the Hopf links on $P_j \otimes \mathbb{C}^H_{\ell r}$, we need to understand how the modified trace acts on the identity and $x_{j,\ell}$. Using the decomposition derived for $V_0 \otimes S_i$ in Proposition 4, one easily obtains the following isomorphism:

\[
V_0 \otimes S_{r-j-1} \otimes \mathbb{C}^H_{\ell r} \simeq V_0 \otimes S_{r-j-3} \otimes \mathbb{C}^H_{\ell r} \oplus P_j \otimes \mathbb{C}^H_{\ell r}.
\]

Using the properties of the modified trace given in definition 1, we see that the modified trace of the identities of these modules yields the relation

\[
d(P_j \otimes \mathbb{C}^H_{\ell r}) = d(V_0)(q \dim(S_{r-j-1} \otimes \mathbb{C}^H_{\ell r}) - q \dim(S_{r-j-3} \otimes \mathbb{C}^H_{\ell r}))
\]

where

\[
q \dim(S_n \otimes \mathbb{C}^H_{\ell r}) = \sum_{i=0}^{n} (s_i \otimes v)^*(K^{1-r}s_i \otimes v) = \sum_{i=0}^{n} q^{(n-2i+\ell r)(1-r)}
\]

\[
= q^{r(1-r)} \frac{n(1-r) + 1}{1} = (-1)^{(1-r)+n} \frac{n+1}{1}.
\]

Hence,
\[ d(P_j \otimes C^H_{\ell'}) = d(V_0)(q\text{dim}(S_{r-j-1} \otimes C^H_{\ell'})) - q\text{dim}(S_{r-j-3} \otimes C^H_{\ell'}) \]
\[
= \frac{(-1)^{r-1}}{1}((-1)^{(l-r)+r-j-1}\{r-j\} - (-1)^{(l-r)+r-j-3}\{r-j-2\})
\]
\[
= (-1)^{(l-r)+j}(q^{-r-j-1} + q^{-r-j-3} + \ldots + q^{-(r-j-3)}) - (q^{-r-j-3} + \ldots + q^{-(r-j-3)})
\]
\[
= (-1)^{(l-r)+j+1}(q^{j+1} + q^{-(j+1)}).
\]

Now, \( t_{P_j \otimes C^H_{\ell'}}(\Phi_{V_0, P_j \otimes C^H_{\ell'}}) = t_{V_0}(\Phi_{P_j \otimes C^H_{\ell'}, V_0}) \) gives the relation

\[ 2r(-1)^{\ell+j+1} t_{P_j \otimes C^H_{\ell'}}(x_{j,\ell}) = 2r(-1)^{r-1}. \]

Hence, \( t_{P_j \otimes C^H_{\ell'}}(x_{j,\ell}) = (-1)^{\ell+j+1} \). We can now compute the Hopf links on the \( P_j \otimes C^H_{\ell'} \) modules using Proposition 5:

\[ t_{P_j \otimes C^H_{\ell'}}(\Phi_{P_l \otimes C^H_{\ell'}, P_j \otimes C^H_{\ell'}}) = (-1)^{(i+1)+(j+\ell+1-r)\ell'} 2r(q^{(i+1)(j+1)} + q^{-(i+1)(j+1)})(-1)^{(r+1)+j+1}
\]
\[
= (-1)^{(i+r+1)+\ell+j+1+r+1}(q^{(i+1)(j+1)} + q^{-(i+1)(j+1)}),
\]

\[ t_{P_j \otimes C^H_{\ell'}}(\Phi_{S_i \otimes C^H_{\ell'}, P_j \otimes C^H_{\ell'}}) \]
\[
= \frac{(-1)^{(i+1)+(j+\ell+1-r)\ell'}}{j+1} \{(i+1)(j+1)\}(-1)^{(l-r)+j+1}(q^{j+1} + q^{-(j+1)})
\]
\[
+ \frac{(-1)^{(i+1)+(j+\ell+1-r)\ell'}}{j+1} \{(i+2)(j+1)\} - (i+2)(i(j+1))\}(1)^{(r+1)+j+1}
\]
\[
= \frac{(-1)^{(i+r+1)+\ell+j+1+r+1}(i+1)(j+1)}{(i+2)(j+1)\} - (i+2)(i(j+1))\}
\]
\[
= (-1)^{(i+r+1)+\ell+j+1+r+1}(i+1)(j+1),
\]

where the last equality is found by expanding the brackets and simplifying. The last trace is computed using the fact that \( x^2_{j,\ell} = 0 \).

\[ t_{P_j \otimes C^H_{\ell'}}(\Phi_{S_i \otimes C^H_{\ell'}, P_j \otimes C^H_{\ell'}} \circ x_{j,\ell}) = (-1)^{(l-r)+\ell+j+1+r+1}(i+1)(j+1)) \]
\[
= (-1)^{(l-r)+\ell+j+1+r+1}(i+1)(j+1)\] \( \frac{1}{j+1} \).
To find $\Phi_{V_\alpha,P_\ell\otimes \mathbb{C}_{\ell r}^H}$ we first compute the action of this Hopf link on the highest weight vector $w_j^S \otimes v \in P_\ell \otimes \mathbb{C}_{\ell r}^H$ of weight $j + \ell r$:

$$\Phi_{V_\alpha,P_\ell\otimes \mathbb{C}_{\ell r}^H}(w_j^S \otimes v) = \Psi_{j+\ell r+1-r}(\chi(V_\alpha))w_j^S \otimes v = q^{r\ell r + (j + \ell r + 1 - r)}a\{r(j + \ell r + 1 - r)\} = 0.$$  

So, we must have $a = 0$. We can now solve for $b$ using the symmetry property of the modified trace and the value of $t_{V_\alpha}(\Phi_{P_\ell\otimes \mathbb{C}_{\ell r}^H}, V_\alpha)$ computed above.

$$(-1)^{r-1}q^r \ell r q^{(r-j-1)\alpha} + q^{-(r-j-1)\alpha} = t_{P_\ell\otimes \mathbb{C}_{\ell r}^H}(\Phi_{V_\alpha,P_\ell\otimes \mathbb{C}_{\ell r}^H}) = t_{P_\ell\otimes \mathbb{C}_{\ell r}^H}(bx_j, \ell) = b(-1)^{(r+1)+j+1}$$

Hence, $\Phi_{V_\alpha,P_\ell\otimes \mathbb{C}_{\ell r}^H} = (-1)^{(r+1)+j}q^r \ell r q^{(r-j-1)\alpha} + q^{-(r-j-1)\alpha}x_j, \ell.$

5 Results

**Proposition 6.** We have for any $\alpha \in \{1, \ldots, r-1\}$ the short exact sequences of modules

$$0 \to V_{r-1-i+r} \to P_\ell \otimes \mathbb{C}_{\ell r}^H \to V_{1+i-r+\ell r} \to 0$$

$$0 \to S_{r-1-\alpha} \otimes \mathbb{C}_{\ell r}^H \to V_{\alpha+r} \to S_{\alpha-1} \otimes \mathbb{C}_{r(\ell+1)} \to 0$$

and corresponding identities of Hopf links for the projective modules $X = V_\alpha, P_\ell \otimes \mathbb{C}_{\ell r}^H$:

$$\Phi_{V_{\alpha+r},X} = \Phi_{S_{r-1-\alpha} \otimes \mathbb{C}_{\ell r}^H, X} + \Phi_{S_{\alpha-1} \otimes \mathbb{C}_{r(\ell+1)}}, X,$$

$$\Phi_{P_\ell \otimes \mathbb{C}_{\ell r}^H, X} = \Phi_{V_{r-1-i+r}, X} + \Phi_{V_{1+i-r+\ell r}, X}.$$  

**Proof.** $V_{\alpha+r}$ is atypical so it contains a highest weight submodule generated by standard basis vector $v_\alpha$ with basis $\{v_\alpha, \ldots, v_{r-1}\}$. This submodule has dimension $r - \alpha$ and highest weight $r - 1 - \alpha + \ell r$ so it is isomorphic to $S_{r-1-\alpha} \otimes \mathbb{C}_{\ell r}^H$. The quotient module $V_{\alpha+r} / < v_\alpha >$ has dimension $\alpha$ and highest weight $\alpha - 1 + (\ell + 1)r$ so it is isomorphic to $S_{\alpha-1} \otimes \mathbb{C}_{r(\ell+1)}$. We therefore have the short exact sequence

$$0 \to S_{r-1-\alpha} \otimes \mathbb{C}_{\ell r}^H \to V_{\alpha+r} \to S_{\alpha-1} \otimes \mathbb{C}_{r(\ell+1)} \to 0$$

The element $w_j^S \otimes v \in P_\ell \otimes \mathbb{C}_{\ell r}^H$ generates a highest weight submodule of dimension $i+1$ and highest weight $i+\ell r$ hence is isomorphic to the submodule $< v_{r-1-i} >$ of $V_{r-1-i+\ell r}$ as both of these modules are isomorphic to $S_i \otimes \mathbb{C}_{\ell r}^H$. So there is an isomorphism $\bar{\phi} : < v_{r-1-i} > \to < w_i^S \otimes v >$. We can ex-
tend this to an injective homomorphism of $V_{r-1+i+\ell r}$ by defining a map $\phi : V_{r-1+i+\ell r} \to P_i \otimes \mathbb{C}^H_{\ell r}$ by

$$\phi(v_n) = \begin{cases} w_j^R \otimes v & : n = 0 \\ (-1)^n \prod_{s=0}^{n-1} \gamma_{j,j-s} w_j^R \otimes v & : 0 < n < r-1-i \\ \tilde{\phi}(v_n) & : n \geq r-1-i \end{cases}$$

and extending linearly over sums, where we have set $j = r - 2 - i$ for simplicity. This map is clearly bijective and linear so we need only show it is a module morphism for $n < r - 1 - i$. When $n < r - 1 - i$ we have

$$\phi(Hv_n) = (2r - 2 - i + \ell r - 2n)\phi(v_n) = (-1)^n \prod_{s=0}^{n-1} \gamma_{j,j-s} H(w_j^R \otimes v) = H\phi(v_n),$$

$$\phi(Fv_n) = \phi(v_{n+1}) = (-1)^{n+1} \prod_{s=0}^{n} \gamma_{j,j-s} (w_j^R \otimes v) = (-1)^n \prod_{s=0}^{n-1} \gamma_{j,j-s} F(w_j^R \otimes v) = F\phi(v_n).$$

To show that $\phi$ commutes with the action of $E$ we first note the identity

$$[n][n-r+1+i-\ell r] = (-1)^{\ell+1}[r-n-1-i][n] = (-1)^{\ell+1}[j-(n-1)][n] = (-1)^{\ell+1}\gamma_{j,j-(n-1)}.$$

Hence,

$$\phi(Ev_n) = [n][n-r+1+i-\ell r]\phi(v_{n-1}) = (-1)^n \prod_{s=0}^{n-2} \gamma_{j,j-s} (-1)^{\ell} \gamma_{j,j-(n-1)} (w_j^R \otimes v)$$

$$= (-1)^n \prod_{s=0}^{n-1} \gamma_{j,j-s} E(w_j^R \otimes v) = E\phi(v_n).$$

This shows that $V_{r-1-i+\ell r}$ can be embedded in $P_i \otimes \mathbb{C}^H_{\ell r}$ by $\phi$. By a similar argument, one can see that $P_i \otimes \mathbb{C}^H_{\ell r}/\phi(V_{r-1-i+\ell r}) \cong V_{1+i-r+\ell r}$ which proves the second short exact sequence. The Hopf link identities are proven by direct computation using Proposition 5:

$$\Phi_{V_{r-1+i+\ell r}, V_\alpha} + \Phi_{V_{1+i-r+\ell r}, V_\alpha} - \Phi_{P_i \otimes \mathbb{C}^H_{\ell r}, V_\alpha} = \frac{(-1)^{r-1} q^{(r-1-i)\alpha}}{d(V_\alpha)} q^{(r-1-i)\alpha} + q^{(1+i-r)\alpha}$$

$$- \frac{(-1)^{r-1} q^{(r-1-i)\alpha}}{d(V_\alpha)} q^{(r-1-i)\alpha} + q^{-(r-1-i)\alpha}$$

$$= 0.$$
\[ \Phi_{S_r \otimes \mathbb{C}^H_{tr}} V_\alpha + \Phi_{S_{r-1} \otimes \mathbb{C}^H_{(\ell+1)r}} V_\alpha - \Phi_{V_{r-1+i'\ell'}} V_\alpha = \frac{q^{r+\ell}}{\alpha} \{ (i+1)\alpha \} + q^{r+\ell} \{ (r-i-1)\alpha \} - q^{(r-i)\alpha} \{ r\alpha \} \]

\[ = \frac{q^{r+\ell}}{\alpha} \{ (i+1)\alpha \} - \{ (i+1)\alpha \} - q^{(r-i)\alpha} (q^{\alpha} - q^{-\alpha}) \]

\[ = q^{r+\ell} \{ (r-i)\alpha \} (q^{\alpha} - q^{-\alpha}) \]

\[ = q^{r+\ell} \{ (r-i)\alpha \} (q^{\alpha} - q^{-\alpha}) \]

By expanding \( \Phi_{V_{r-1+i'\ell'}} P_j \otimes \mathbb{C}^H_{tr} + \Phi_{V_{r+i'\ell'}} P_j \otimes \mathbb{C}^H_{tr} \), we obtain

\[ (-1)^{(\ell+1)+i'\ell} r q^{r+\ell}(r-j+1)(r-j+1)\alpha x_j,\ell \]

\[ + (-1)^{(\ell+1)+i'\ell} r q^{r+\ell}(r-j+1)(r-j+1)\alpha x_j,\ell \]

\[ = (-1)^{(\ell+1)+i'\ell+1}(r-j+1)(r-j+1)\alpha x_j,\ell \]

\[ + (-1)^{(\ell+1)+(i'\ell+1)\alpha + i'\ell}(r-j+1)(r-j+1)\alpha x_j,\ell \]

\[ = (-1)^{(\ell+1)+(i'\ell+1)\alpha + i'\ell+1}2r(q^{(j+1)(1+i)} - q^{-(j+1)(1+i)}) x_j,\ell \]

\[ = \Phi_{P_j \otimes \mathbb{C}^H_{tr}}. \]

Recall that the Hopf links on \( P_j \otimes \mathbb{C}^H_{tr} \) have the form \( aId_{P_j \otimes \mathbb{C}^H_{tr}} + bx_{j,\ell} \). The coefficient of the identity of \( \Phi_{V_{r+i'\ell'}} P_j \otimes \mathbb{C}^H_{tr} \) is zero and the coefficient of the identity for \( \Phi_{S_{r-1} \otimes \mathbb{C}^H_{(\ell+1)r}} P_j \otimes \mathbb{C}^H_{tr} \) is

\[ \frac{(-1)^{(r-1)\alpha}(\ell+1)+(j+\ell+1)r^{(j+1)}}{(\ell+1)} \{ (r-\alpha)(j+1) \} + \frac{(-1)^{(r-1)\alpha}(\ell+1)+(j+\ell+1)r^{(j+1)}}{(\ell+1)} \{ (r-\alpha)(j+1) \} \]

\[ = - \frac{(-1)^{(r-1)\alpha}(\ell+1)+(j+\ell+1)r^{(j+1)}}{(\ell+1)} \{ (r-\alpha)(j+1) \} - \frac{(-1)^{(r-1)\alpha}(\ell+1)+(j+\ell+1)r^{(j+1)}}{(\ell+1)} \{ (r-\alpha)(j+1) \} \]

\[ = \frac{(-1)^{(r-1)\alpha}(\ell+1)+(j+\ell+1)r^{(j+1)}}{(\ell+1)} \{ (r-\alpha)(j+1) \} - \{ (r-\alpha)(j+1) \} = 0. \]

(Note that \( \alpha \) is an integer in this computation)

So, \( \Phi_{S_{r-1} \otimes \mathbb{C}^H_{tr}} P_j \otimes \mathbb{C}^H_{tr} + \Phi_{S_{r-1} \otimes \mathbb{C}^H_{(\ell+1)r}} - \Phi_{V_{r+i'\ell'}} P_j \otimes \mathbb{C}^H_{tr} = bx_{j,\ell} \) and we also have the identity:
\[
\frac{(-1)^{(r-1-\alpha)(\ell')+(j+\ell r+1+r)}}{\{j+1\}}((r-1-\alpha)\{(r+1-\alpha)(j+1)\} - (r+1-\alpha)\{(r-1-\alpha)(j+1)\}) + \frac{(-1)^{(a-1)(\ell)+1+(j+\ell r+1+r)}}{\{j+1\}}(\alpha-1)\{(\alpha+1)(j+1)\} - (\alpha+1)\{(\alpha-1)(j+1)\})
\]

\[
= \frac{(-1)^{(r-1-\alpha)(\ell)+(j+\ell r+1+r)}}{\{j+1\}}((r-1-\alpha)\{(1-\alpha)(j+1)\} - (r+1-\alpha)\{(-1-\alpha)(j+1)\}) + \frac{(-1)^{(r-\alpha-1)(\ell)+(j+\ell r+1+r)}}{\{j+1\}}(\alpha-1)\{(\alpha+1)(j+1)\} - (\alpha+1)\{(\alpha-1)(j+1)\})
\]

\[
= \frac{(-1)^{(r-1-\alpha)(\ell)+(j+\ell r+1+r)}}{\{j+1\}}((r-1-\alpha)\{(1-\alpha)(j+1)\} - (r+1-\alpha)\{(-1-\alpha)(j+1)\})
\]

\[
= (-1)^{(r-\alpha-1)(\ell)+(j+\ell r+1+r)}(\alpha-1)\{(\alpha+1)(j+1)\} - (\alpha+1)\{(\alpha-1)(j+1)\})
\]

\[
= (1-q^{\alpha+1}) + q^{-\alpha}(j+1).
\]

The left hand side of this equation is the constant coefficient for \(x_{j,\ell}\) in \(\Phi_{S_{r-1-\alpha} \otimes C_{\ell r}^H, P_j \otimes C_{\ell r}^H + \Phi_{S_{\alpha-1} \otimes C_{r+1}^H}}\) and the right hand side is the constant coefficient for \(x_{j,\ell}\) in \(\Phi_{V_{\alpha+\ell r}, P_j \otimes C_{\ell r}^H}\). This shows that \(b = 0\) and completes the proof.

\[\square\]

The equations

\[
0 \rightarrow V_{r-1-i+\ell r} \rightarrow P_i \otimes C_{\ell r}^H \rightarrow V_{i+\ell r} \rightarrow 0
\]

\[
0 \rightarrow S_{r-1-\alpha} \otimes C_{\ell r}^H \rightarrow V_{\alpha+\ell r} \rightarrow S_{\alpha-1} \otimes C_{(\ell+1) r}^H \rightarrow 0
\]

describe the structure \(P_i \otimes C_{\ell r}^H\) and \(V_{\alpha+\ell r}\) respectively in terms of simpler modules. It is easy to see from the definition that \(P_i \otimes C_{\ell r}^H\) has Loewy diagram as in Figure 1:

![Loewy diagram of \(P_i \otimes C_{\ell r}^H\)](image)

**Figure 1:** Loewy diagram of \(P_i \otimes C_{\ell r}^H\)

**Proposition 7.** The tensor ring of projective weight modules is determined up to isomorphism by
the modified traces of Hopf links and the isomorphism is given by \( \phi : V \mapsto \frac{t_{V_\alpha}(\Phi_{V,V_\alpha})}{d(V_\alpha)} \).

**Proof.** \( \phi \) is bijective and clearly preserves the additive structure as the trace is linear and \( \Phi_{V \oplus W,V_\alpha} = \Phi_{V,V_\alpha} + \Phi_{W,V_\alpha} \). So, \( \phi \) is an isomorphism if we have \( \phi(V \otimes W) = \phi(V) \phi(W) \) for any weight modules \( V, W \). This is equivalent to

\[
t_{V_\alpha}(\Phi_{V \otimes W,V_\alpha}) = \frac{t_{V_\alpha}(\Phi_{V,V_\alpha})t_{V_\alpha}(\Phi_{W,V_\alpha})}{d(V_\alpha)}.
\]

\( V_\alpha \) is simple so the Hopf links act as scalars and there exist \( \lambda_V, \lambda_W \in \mathbb{C} \) such that \( \Phi_{V,V_\alpha} = \lambda_V \text{Id}_{V_\alpha} \) and \( \Phi_{W,V_\alpha} = \lambda_W \text{Id}_{V_\alpha} \). Hence,

\[
t_{V_\alpha}(\Phi_{V \otimes W,V_\alpha}) = t_{V_\alpha}(\Phi_{V,V_\alpha} \circ \Phi_{W,V_\alpha}) = t_{V_\alpha}(\lambda_V \lambda_W \text{Id}_{V_\alpha})
= \lambda_V \lambda_W t_{V_\alpha}(\text{Id}_{V_\alpha})
= \lambda_V \lambda_W \frac{t_{V_\alpha}(\text{Id}_{V_\alpha})^2}{t_{V_\alpha}(\text{Id}_{V_\alpha})}
= t_{V_\alpha}(\lambda_V \lambda_W \text{Id}_{V_\alpha})
= \frac{t_{V_\alpha}(\Phi_{V,V_\alpha})t_{V_\alpha}(\Phi_{W,V_\alpha})}{d(V_\alpha)}
\]

which completes the proof. \( \square \)

### 5.1 Comparison

**Proposition 8.** Let \( \alpha \in \hat{\mathbb{C}} \). Then the map \( \varphi : V_\alpha \mapsto F_{\frac{\alpha+r-1}{\sqrt{2r}}} \), \( S_i \otimes \mathbb{C}_{H_r} \mapsto M_{1-l,i+1} \) extended linearly over direct sums for \( \alpha \in \hat{\mathbb{C}} \) is a morphism up to equality of characters.

**Proof.** We can compute the relations using the tensor decompositions computed in Proposition 4:

\[
\text{ch}[\varphi(V_\alpha \otimes V_\beta)] = \sum_{k=1-r \atop k+r=1 \mod 2}^{r-1} \text{ch}[\varphi(V_{\alpha+k})] = \sum_{k=1-r \atop k+r=1 \mod 2}^{r-1} \text{ch} \left[ F_{\frac{\alpha+r-1+k-k}{\sqrt{2r}}} \right]
= \sum_{k=1-r \atop k+r=1 \mod 2}^{r-1} \text{ch} \left[ F_{\frac{\alpha+r-1+k-k}{\sqrt{2r}}} \right]
= \sum_{l=0}^{r-1} \text{ch} \left[ F_{\frac{\alpha+r-1+k-k}{\sqrt{2r}}} \right]
= \text{ch} \left[ F_{\frac{\alpha+r-1+k-k}{\sqrt{2r}}} \right] \times \text{ch} \left[ F_{\frac{\beta+r-1+k-k}{\sqrt{2r}}} \right]
= \text{ch}[\varphi(V_\alpha)] \times \text{ch}[\varphi(V_\beta)],
\]
\[
\text{ch}[\varphi((S_i \otimes \mathbb{C}^H_{\ell r}) \otimes V_0)] = \sum_{l+i=0 \mod 2}^{i} \text{ch}[\varphi(V_{\alpha+\ell r+i})]
\]
\[
= \sum_{l+i=0 \mod 2}^{i} \text{ch}\left[\frac{F_{\alpha+\ell r+i} + \psi^i}{\sqrt{2^r}}\right]
\]
\[
= \sum_{l=-i}^{i+1} \text{ch}\left[\frac{F_{\alpha+i} + \psi^i}{\sqrt{2}^r}\right]
\]
\[
= \text{ch}[M_{1-\ell,i+1}] \times \text{ch}[\frac{F_{\alpha+i}}{\sqrt{2}^r}]
\]
\[
= \text{ch}[\varphi(S_i \otimes \mathbb{C}^H_{\ell r})] \times \text{ch}[\varphi(V_0)].
\]

The Hopf link identities \( \Phi_{V_{\alpha+\ell r}.X} = \Phi_{S_{r-1-i} \otimes \mathbb{C}^H_{\ell r}.X} + \Phi_{S_{\alpha-1} \otimes \mathbb{C}^H_{(\ell+i)r}.X} \) and \( \Phi_{P_1 \otimes \mathbb{C}^H_{\ell r}.X} = \Phi_{V_{r-1-i+\ell r}.X} + \Phi_{V_{1+i-\ell r}.X} \) motivate the definition

\[
\varphi(P_i \otimes \mathbb{C}^H_{\ell r}) = \varphi(V_{r-1-i+\ell r}) + \varphi(V_{1+i-\ell r})
\]
\[
= \varphi(S_i \otimes \mathbb{C}^H_{\ell r}) + \varphi(S_{r-1-i} \otimes \mathbb{C}^H_{(\ell+i-1)r}) + \varphi(S_{r-2-i} \otimes \mathbb{C}^H_{(\ell-1)r}) + \varphi(S_i \otimes \mathbb{C}^H_{\ell r})
\]
\[
= M_{1-\ell,i+1} \oplus M_{\ell,i+r-1} \oplus M_{2-\ell,r-i-1} \oplus M_{1-\ell,i+1}.
\]

It is easy to see using Proposition 4, the fact that \( S_{r-1} \) is projective, and projective modules being classified by characters that

\[
S_1 \otimes S_j = \begin{cases} 
S_1 & \text{if } j = 0 \\
S_{j+1} \oplus S_{j-1} & \text{if } 1 \leq j \leq r - 2 \\
P_{r-2} & \text{if } j = r - 1 
\end{cases}
\]

If \( j = 0 \), then \( \text{ch}[\varphi(S_1 \otimes S_0)] = \text{ch}[M_{1,2}] = \text{ch}[M_{1,2}] \times \text{ch}[M_{1,1}] = \text{ch}[\varphi(S_1)] \times \text{ch}[\varphi(S_0)] \). If \( 1 \leq j \leq r - 2 \), then we have

\[
\text{ch}[\varphi(S_1 \otimes \mathbb{C}^H_{\ell r} \otimes S_j \otimes \mathbb{C}^H_{\ell r})] = \text{ch}[\varphi(S_{j+1} \otimes \mathbb{C}^H_{(\ell+i)r}) + \varphi(S_{j+1} \otimes \mathbb{C}^H_{(\ell+i)r})]
\]
\[
= \text{ch}[M_{1-(\ell+i)r,j}] + \text{ch}[M_{1-(\ell+i)r,j+2}]
\]
\[
= \text{ch}[M_{1-\ell,i}] \times \text{ch}[M_{1-\ell,i+1}]
\]
\[
= \text{ch}[\varphi(S_1 \otimes \mathbb{C}^H_{\ell r})] \times \text{ch}[\varphi(S_j \otimes \mathbb{C}^H_{\ell r})].
\]
The more general case $\text{ch}[\varphi(S_i \otimes \mathbb{C}^H \otimes S_j \otimes \mathbb{C}^H)] = \text{ch}[\varphi(S_i \otimes \mathbb{C}^H)] \times \text{ch}[\varphi(S_j \otimes \mathbb{C}^H)]$ then follows inductively using the equation $S_1 \otimes S_i \otimes S_j = (S_{i+1} \otimes S_j) \oplus (S_{i-1} \otimes S_j)$.

$\square$

**Proposition 9.** The quantum dimensions of the typical and atypical modules for the singlet vertex algebra are in agreement with the modified traces of their corresponding $\overline{U}_q(\mathfrak{sl}_2)$ modules in the following sense:

$$\begin{align*}
\text{qdim}[F^{\beta\pm r-1}_{\beta r}] &= \frac{t_{V_\alpha}(\Phi_{\beta r},V_\alpha)}{t_{V_\alpha}(\Phi_{\beta r},V_\alpha)} \quad \text{qdim}[M^{\beta \pm r-1}_{1-\ell} | j + 1] = \frac{t_{V_\alpha}(\Phi_{S_j \otimes \mathbb{C}^H_{\ell r},V_\alpha})}{t_{V_\alpha}(\Phi_{S_j \otimes \mathbb{C}^H_{\ell r},V_\alpha})} \\
\text{qdim}[M^{\beta \pm r-1}_{1-\ell, i + 1}] &= \frac{t_{P_j \otimes \mathbb{C}^H_{\ell^r}}(\Phi_{S_j \otimes \mathbb{C}^H_{\ell^r},P_j \otimes \mathbb{C}^H_{\ell^r} \circ X_{j,\ell}})}{t_{P_j \otimes \mathbb{C}^H_{\ell^r}}(\Phi_{S_0 \otimes \mathbb{C}^H_{\ell^r} \circ X_{j,\ell}})} \text{ for } Re(\epsilon) < 0
\end{align*}$$

**Proof.** The proof follows directly by computation:

$$\begin{align*}
\text{qdim}[F^{\beta\pm r-1}_{\beta r}] &= e^{\pi i \alpha^2 (2 \beta + r - 1)} \left( e^{-\pi i \beta r} \right)^{\phi_\alpha} \frac{e^{-\pi \sqrt{2} + \frac{\alpha r}{\sqrt{2}}} \left( e^{-\pi \sqrt{2} + \frac{\alpha r}{\sqrt{2}} - e^{-\pi \sqrt{2} + \frac{\alpha r}{\sqrt{2}}}} \right)}{e^{-\pi \sqrt{2} + \frac{\alpha r}{\sqrt{2}}} \left( e^{-\pi \sqrt{2} + \frac{\alpha r}{\sqrt{2}} - e^{-\pi \sqrt{2} + \frac{\alpha r}{\sqrt{2}}}} \right)} \\
&= e^{-\pi i \phi_\alpha} \frac{e^{-\pi i \alpha r} - e^{-\pi i \phi_\alpha}}{e^{-\pi i \alpha r} - e^{-\pi i \phi_\alpha}} \\
&= q^{\phi_\alpha} \frac{\{r \phi_\alpha\}}{\{\phi_\alpha\}} \\
&= \frac{t_{V_\alpha}(\Phi_{\beta r},V_\alpha)}{t_{V_\alpha}(\Phi_{\beta r},V_\alpha)}
\end{align*}$$

$$\begin{align*}
\text{qdim}[M^{\beta \pm r-1}_{1-\ell} | j + 1] &= e^{\pi \beta r - \frac{\alpha^2}{\sqrt{2}}} \left( e^{-\pi (j+1) \frac{\beta^2 + \alpha^2}{\sqrt{2}}} - e^{\pi (j+1) \frac{\beta^2 + \alpha^2}{\sqrt{2}}} \right) \frac{\left( e^{-\pi (j+1) \frac{\beta^2 + \alpha^2}{\sqrt{2}}} - e^{\pi (j+1) \frac{\beta^2 + \alpha^2}{\sqrt{2}}} \right)}{\left( e^{-\pi (j+1) \frac{\beta^2 + \alpha^2}{\sqrt{2}}} - e^{\pi (j+1) \frac{\beta^2 + \alpha^2}{\sqrt{2}}} \right)} \\
&= e^{-\pi (j+1) \phi_\alpha} \frac{e^{-\pi (j+1) \phi_\alpha} - e^{-\pi \alpha^2 r}}{e^{-\pi (j+1) \phi_\alpha} - e^{-\pi \alpha^2 r}} \\
&= q^{\phi_\alpha} \frac{(j + 1) \phi_\alpha}{\phi_\alpha} \\
&= \frac{t_{V_\alpha}(\Phi_{S_j \otimes \mathbb{C}^H_{\ell r},V_\alpha})}{t_{V_\alpha}(\Phi_{S_j \otimes \mathbb{C}^H_{\ell r},V_\alpha})}
\end{align*}$$

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For \( \text{Re}(\epsilon) < 0 \),

\[
\frac{t_{P_j \otimes C_{t,r}}^2 \Phi_{S,\alpha} \circ t_{P_j \otimes C_{t,r}}^2 \circ x_j, t}{{t_{P_j \otimes C_{t,r}}^2 \Phi_{S,\alpha} \circ t_{P_j \otimes C_{t,r}}^2 \circ x_j, t}} = (-1)^{i(\ell + 1) + (j + \ell r + 1 - r)\epsilon} \frac{(i + 1)(j + 1)}{\{j + 1\}}
\]

\[
= (-1)^{(j + \ell r + 1 - r)\epsilon} \frac{(i + 1)(j + 1 + r(\ell + 1))}{\{j + 1 + r(\ell + 1)\}}
\]

\[
= \text{qdim}[M_{1-1,\iota+1}^\ell]
\]

with \( \epsilon \in S(\ell, j + 1 + r(\ell + 1)) \).

### 5.2 Hopf Links

Recall that the Hopf links on the simple \( V_\alpha \) are given by

\[
\Phi_{V_\beta, V_\alpha} = \frac{(-1)^{r-1} \text{d}(V_\alpha)}{q^{\alpha \beta} \text{d}(V_\alpha)} q^{r \alpha \beta} I_{dV_\alpha}
\]

\[
\Phi_{S,\alpha} \circ t_{P_j \otimes C_{t,r}}^2 = q^{r \alpha \beta} \frac{\{i + 1\} \text{d}(V_\alpha)}{\{\alpha\}}
\]

\[
\Phi_{P_j \otimes C_{t,r}, V_\alpha} = (-1)^{r-1} q^{r \alpha \beta} \frac{\{r(1 - i)\alpha\} + q^{-(r-1-i)\alpha}}{\text{d}(V_\alpha)} I_{dV_\alpha}
\]

**Theorem 3.** Let \( \epsilon \in (-\frac{1}{2}, \frac{1}{2}) \), \( \iota \in \{0, ..., r - 2\} \) and let \( \ell \in \mathbb{Z} \). Denote by \( X_\iota \) the module with vector space basis \( \{w^L_{i+2-2\iota}, w^L_{i+4-2\iota}, ..., w^L_{i-2}, w^L_i, w^H_i, ..., w^S_i, w^R_i, ..., w^R_{2r-2-\iota}\} \) and action given by

\[
w^R_{i+2} = (-1)^\ell Ew^H_i, \quad w^L_{i-2} = Fw^H_i, \quad Fw^R_{i+2} = w^S_i + [1 + i][\epsilon] w^H_i,
\]

\[
w^L_{i-2k} = F^kw^H_i \quad \text{and} \quad w^S_{i-2k} = F^kw^S_i \quad \text{for } k \in \{0, ..., i\},
\]

\[
w^L_{i+2-2\iota} = F^kw^L_i \quad \text{and} \quad w^R_{i+2+2\iota} = (-1)^k E^k w^R_{i+2} \quad \text{for } k \in \{0, ..., r - 2 - \iota\},
\]

\[
Hw^X_k = (k + \ell r + \epsilon)w^X_k, \quad Kw^X_k = (-1)^\ell q^{k+\epsilon} w^X_k, \quad \text{for } X \in \{L, H, S, R\},
\]

\[
Ew^R_k = (-1)^\ell w^R_{k+2}, \quad Fw^X_k = w^X_{k-2}, \quad \text{for } X \in \{L, S, H\},
\]

\[
Fw^S_i = [1 + i][\epsilon] w^L_{i-2}, \quad Ew^S_i = 2(-1)^{\ell+1}[1 + i][\epsilon] w^R_{i+2}, \quad Ew^R_{2r-2-\iota} = Fw^L_{i+2-2r} = 0,
\]
We will first prove the statement for $\epsilon \neq 0$. Let $\{x_0, \ldots, x_{r-1}\}$ denote the standard basis for $V_{1+i-r+\ell r+\ell}$. Define a new basis $\{w_i^{L}, w_i^{H}, w_i^{S}, w_{i+2}^{R}, \ldots, w_{2r-2-i}^{R}\}$ for $V_{1+i-r+\ell r+\ell} \otimes V_{1-i-r+\ell r+\ell}$ by

\[
\begin{align*}
  w_i^{H} &= 2x_k - \frac{1}{2[1+i][\epsilon]} y_{r-1-i+k}, \quad w_{i-2k}^{H} = -2[1+i][\epsilon]x_k + y_{r-1-i+k}, \\
  w_i^{L} &= 2x_{1+i+k}, \quad w_{i+2+2k}^{R} = \frac{-1}{2[1+i][\epsilon]} \left( \prod_{s=0}^{k} (1+i+s)[-s-\epsilon] y_{r-2-i-k} \right),
\end{align*}
\]

We will first prove the statement for $\epsilon \neq 0$. Recall that the action on the standard basis element $v_k \in V_\alpha$ is given by

\[
Hv_k = (\alpha + r - 1 - 2k)v_k, \quad Ev_k = [k][k-\alpha]v_{k-1}, \quad Fv_k = v_{k+1}.
\]

By direct computation, we obtain the following:

\[
\begin{align*}
  Ew_i^{H} &= 2Ex_0 - \frac{1}{2[1+i][\epsilon]} Ey_{r-1-i} = -\frac{[r-1-i][-\ell r-\epsilon]}{2[1+i][\epsilon]} y_{r-2-i} \\
  &= \frac{1}{2} (-1)^\ell y_{r-2-i} = (-1)^\ell w_{i+2}^{R}, \\
  Fw_{-i}^{H} &= 2Fx_i - \frac{1}{2[1+i][\epsilon]} Fy_{r-1} = 2x_{i+1} = w_{-i-2}^{L}, \\
  w_i^{S} + [1+i][\epsilon]w_i^{H} &= -2[1+i][\epsilon]x_0 + y_{r-1-i} + 2[1+i][\epsilon]x_0 - \frac{1}{2} y_{r-1-i} - \frac{1}{2} y_{r-1-i} = \frac{1}{2} Fy_{r-2-i} = Fw_{i+2}^{R}.
\end{align*}
\]

Hence, we have shown $w_{i+2}^{R} = (-1)^\ell Ew_i^{H}, w_{-i-2}^{L} = Fw_{-i}^{H}$, and $Fw_{i+2}^{R} = w_i^{S} + [1+i][\epsilon]w_i^{H}$. It is
easily seen that $Fw^X_k = w^X_{k-2}$ for all $X \in \{L, S, H\}$ and that

$$
Ew^R_{i+2+2k} = \frac{-1}{2[1+i][\epsilon]} \left( \prod_{s=0}^{k} [1+i+s][-s-\epsilon] \right) Ey_{r-2-i-k}
$$

$$
= \frac{r-2-i-k}{2[1+i][\epsilon]} \left( \prod_{s=0}^{k} [1+i+s][-s-\epsilon] \right) y_{r-2-i-(k+1)}
$$

$$
= \frac{(-1)^{ \ell \over 2}}{2[1+i][\epsilon]} \left( \prod_{s=0}^{k+1} [1+i+s][-s-\epsilon] \right) y_{r-2-i-(k+1)} = (-1)^{ \ell \over 2} w^R_{i+2+2(k+1)}
$$

so $Ew^R_{k} = (-1)^{ \ell \over 2} w^R_{k+2}$, which gives $w^R_{i+2+2k} = (-1)^{ \ell \over 2} F^k w^R_{i+2}$, $w^H_{i-2k} = F^k w^H_{i}$, $w^S_{i-2k} = F^k w^S_{i}$, and $w^L_{i-2-2k} = F^k w^L_{i-2}$.

H acts on a standard basis vector $v_k \in V_\alpha$ by $Hv_k = (\alpha + r - 1 - 2k)v_k$, so we have

$$
Hx_k = (1 + i - r + \ell \epsilon + \epsilon + r - 1 - 2k)x_k = (i - 2k + \ell \epsilon + \epsilon)x_k,
$$

$$
Hy_{r-1-i+k} = (-1 - i + r + \ell \epsilon + \epsilon + r - 1 - 2(r - 1 - i + k))y_{r-1-i+k} = (i - 2k + \ell \epsilon + \epsilon)y_{r-1-i+k},
$$

$$
Hx_{1+i+k} = (1 + i - r + \ell \epsilon + \epsilon + r - 1 - 2(1+i+k))x_{1+i+k} = (i - 2 - 2k + \ell \epsilon + \epsilon)x_{1+i+k},
$$

$$
Hy_{r-1-i-(k+1)} = (-1 - i + r + \ell \epsilon + \epsilon + r - 1 - 2(r - 1 - i - (k+1))y_{r-1-i-(k+1)},
$$

$$
= (i + 2 + 2k + \ell \epsilon + \epsilon)y_{r-1-i-(k+1)}.
$$

From this, it immediately follows that

$$
Hw^H_{i-2k} = (i - 2k + \ell \epsilon + \epsilon)w^H_{i-2k},
$$

$$
Hw^S_{i-2k} = (i - 2k + \ell \epsilon + \epsilon)w^S_{i-2k},
$$

$$
Hw^L_{i-2-2k} = (-i - 2 - 2k + \ell \epsilon + \epsilon)w^L_{i-2-2k},
$$

$$
Hw^R_{i+2+2k} = (i + 2 + 2k + \ell \epsilon + \epsilon)w^R_{i+2+2k},
$$

and $K$ acts as $q^H$, so we have shown that $Hw^X_k = (k + \ell \epsilon + \epsilon)w^X_k$ and $Kw^X_k = q^{k+\ell \epsilon}w^X_k = (-1)^{ \ell \over 2} q^{k+\ell \epsilon}w^X_k$ for all $X \in \{L, H, S, R\}$. It is easy to see that $Ew^R_{2r-2-i} = Fw^L_{i+2-2r} = 0$ as $Fx_{r-1} = Ey_0 = 0$. We also have
\[ Fw_i^S = -2[1 + i][\epsilon]Fx_i + Fy_{r-1} = -2[1 + i][\epsilon]x_{i+1} = -[1 + i][\epsilon]w_{i-2}^L, \]
\[ Ew_i^S = -2[1 + i][\epsilon]Ex_0 + Ey_{r-1-i} = [1 + i][\epsilon]y_{r-2-i} = 2(-1)^{\ell+1}[1 + i][\epsilon]w_{i+2}^R, \]
\[ Ew_{i-2-k} = 2Ex_{1+i+k} = 2[1 + i + k][r + k - \ell r - \epsilon]x_{i+k} = (-1)^{\ell+1}[1 + i + k][k - \epsilon]w_{i-2-2(k-1)}, \]
\[ Fw_{i+2+2k} = \frac{-1}{2[1 + i][\epsilon]} \left( \prod_{s=0}^{k-1} [1 + i + s][-s - \epsilon] \right) y_{r-1-i-k} \]
\[ = \frac{-[1 + i + k][k - \epsilon]}{2[1 + i][\epsilon]} \left( \prod_{s=0}^{k-1} [1 + i + s][-s - \epsilon] \right) y_{r-2-i-(k-1)} \]
\[ = -[1 + i + k][k + \epsilon]w_{i+2+2(k-1)}, \]
\[ (-1)\ell (w_{i+}^S + 2[1 + i][\epsilon]w_{i+}^H) = (-1)^{\ell} (-2[1 + i][\epsilon]x_i + y_{r-1} + 4[1 + i][\epsilon]x_i - y_{r-1}) \]
\[ = 2(-1)^{\ell}[1 + i][\epsilon]x_i = 2[1 + i][\epsilon]x_i = Ew_{i-2}^L. \]

From the definition of \( w_{i-2k}^H \) and \( w_{i-2k}^S \) it is easy to show that
\[ x_k = w_{i-2k}^H + \frac{1}{2[1 + i][\epsilon]} w_{i-2k}^S, \]
\[ y_{r-1-i+k} = 2w_{i-2k}^S + 2[1 + i][\epsilon]w_{i-2k}^H. \]

From this, we find
\[ Ew_{i-2k}^H = 2Ex_k - \frac{1}{2[1 + i][\epsilon]} Ey_{r-1-i+k} = 2[k][1 + i - k + \ell r + \epsilon]x_{k-1} - \frac{[1 + i - k][k - \ell r - \epsilon]}{2[1 + i][\epsilon]} y_{r-2-i+k} \]
\[ = 2[k][1 + i - k + \epsilon](-1)^{\ell} \left( w_{i-2(k-1)}^H + \frac{1}{2[1 + i][\epsilon]} w_{i-2(k-1)}^S \right) \]
\[ = \frac{[1 + i - k][k - \epsilon]}{2[1 + i][\epsilon]} (-1)^{\ell} \left( 2w_{i-2(k-1)}^S + 2[1 + i][\epsilon]w_{i-2(k-1)}^H \right) \]
\[ = (2[k][1 + i - k + \epsilon] - [1 + i - k][k - \epsilon])(-1)^{\ell} w_{i-2(k-1)}^H \]
\[ + \frac{[k][1 + i - k + \epsilon] - [1 + i - k][k - \epsilon]}{[1 + i][\epsilon]} (-1)^{\ell} w_{i-2(k-1)}^S. \]
Corollary 2. The action of the twist on $P_j \otimes C^H_{\ell \tau}$ is given by $a \text{Id}_{P_j \otimes C^H_{\ell \tau}} + bx_j, \ell$ where

$$a = q^{-(j+\ell \tau)(r-1-\frac{1}{2}(j+\ell \tau))}$$

$$b = -\frac{1}{r-j} \{1\}^2 \{r - 1 - j\} q^{-(j+\ell \tau)(r-1-\frac{1}{2}(j+\ell \tau))}$$

Proof. The twist acts as the inverse of

However, by expanding the brackets we get the identity

$$[k][1 + i - k + \epsilon] - [1 + i - k][k - \epsilon] = [1 + i][\epsilon].$$

Hence, the above equations give

$$w^H_{i-2k} = (2[k][1 + i - k + \epsilon] - [1 + i - k][k - \epsilon])(-1)^{\epsilon} w^H_{i-2(k-1)} + (-1)^{\epsilon} w^S_{i-2(k-1)},$$

$$w^S_{i-2k} = (2[1 + i - k][k - \epsilon] - [1 + i - k + \epsilon][k])(-1)^{\epsilon} w^S_{i-2(k-1)} + 2(-1)^{\epsilon + 1}[1 + i]^{2}[\epsilon]^{2} w^H_{i-2(k-1)},$$

as desired. This proves that $X_{\epsilon} = V_{1+i-r+\ell \tau+\epsilon} \oplus V_{-i-r+\ell \tau+\epsilon}$ when $\epsilon \neq 0$. As $\epsilon \to 0$, it is easy to see that the action on $X_0$ is exactly the action on $P_i \otimes C^H_{\ell \tau}$ (see [1]) by identifying $w^X_k \in X_0$ with $w^X_k \otimes v \in P_i \otimes C^H_{\ell \tau}$. Hence, we have shown

$$X_{\epsilon} = \begin{cases} 
V_{1+i-r+\ell \tau+\epsilon} \oplus V_{-i-r+\ell \tau+\epsilon} & \text{if } \epsilon \neq 0 \\
P_i \otimes C^H_{\ell \tau} & \text{if } \epsilon = 0
\end{cases}$$

\[ \square \]
\[ \theta = K^{r-1} \sum_{n=0}^{r-1} \frac{1}{n!} q^{n(n-1)/2} S(F^n) q^{-H^2/2} E^n \]

which acts as \( K^{r-1} q^{-H^2/2} \) on highest weight vectors. Endomorphisms act on simple modules by scalars, so it is enough to determine the scalar by which \( \theta \) acts on the highest weight vector in a simple module. So, \( \theta \) acts on \( V_{1+j-r+\ell r+\epsilon} \) and \( V_{-1-j+r+\ell r+\epsilon} \) as \( q^{(r-1)(j+\ell r+\epsilon)} q^{-(j+\ell r+\epsilon)^2/2} \) and \( q^{(r-1)(2r-2-j+\ell r+\epsilon)} q^{-(2r-2-j+\ell r+\epsilon)^2/2} \) respectively. These values agree in the limit \( \epsilon \to 0 \) since

\[
q^{(r-1)(2r-2-j+\ell r)} q^{-(2r-2-j+\ell r)^2/2} = q^{(2r-2-j+\ell r)(r-1-\frac{1}{2}(2r-2-j+\ell r))} = q^{-\frac{1}{2}(2r-2-j+\ell r)(-j+\ell r)} = q^{-\frac{1}{4}((2r-2)(-j+\ell r)+(-j+\ell r)^2)} = q^{(r-1)(j-\ell r)} q^{-(j+\ell r)^2/2}
\]

\[
= (-1)^{j-\ell r} q^{-(\frac{1}{2}j^2+\text{jitter}+\frac{1}{2}h^2 r^2)} = (-1)^{j+\ell r} q^{-(\frac{1}{2}j^2+\text{jitter}+\frac{1}{2}h^2 r^2)} = q^{(r-1)(j+\ell r)} q^{-(j+\ell r)^2/2}.
\]

Note that

\[
q^{(r-1)(j+\ell r+\epsilon)} q^{-(j+\ell r+\epsilon)^2/2} = q^{(j+\ell r+\epsilon)(r-1-\frac{1}{2}(j+\ell r+\epsilon)},
\]

\[
q^{(r-1)(2r-2-j+\ell r+\epsilon)} q^{-(2r-2-j+\ell r+\epsilon)^2/2} = q^{(j-\ell r+\epsilon)(r-1-(j-\ell r+\epsilon)/2)}.
\]

So, if we denote the action of \( \theta \) on \( P_j \otimes C_{\ell r}^H \) by \( a' \text{Id}_{P_j \otimes C_{\ell r}^H} + b' x_j, \ell \) then we have

\[
a' = \lim_{\epsilon \to 0} \left( 2q^{(j+\ell r+\epsilon)(r-1-(j+\ell r+\epsilon)/2)} - q^{(j-\ell r+\epsilon)(r-1-(j-\ell r+\epsilon)/2)} \right)
\]

\[
= \left( 2q^{(j+\ell r)(r-1-(j+\ell r)/2)} - q^{(j-\ell r)(r-1-(j-\ell r)/2)} \right) = q^{(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))}
\]

and

\[
b' = \lim_{\epsilon \to 0} \frac{-1}{[1+j]|\epsilon|} \left( q^{(j-\ell r-\epsilon)(r-1-(j-\ell r-\epsilon)/2)} - q^{(j+\ell r+\epsilon)(r-1-(j+\ell r+\epsilon)/2)} \right)
\]

We can compute \( b' \) using L’Hopital’s rule.

The derivative of the numerator is

\[
\ln q \left( (r - 1 - (j - \ell r - \epsilon)/2) - \frac{1}{2} (j - \ell r - \epsilon) \right) q^{(j-\ell r-\epsilon)(r-1-(j-\ell r-\epsilon)/2)} + \ln q \left( (r - 1 - (j + \ell r + \epsilon)/2) - \frac{1}{2} (j + \ell r + \epsilon) \right) q^{(j+\ell r+\epsilon)(r-1-(j+\ell r+\epsilon)/2)}
\]

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which gives

\[ 2\ln q(r - 1 - j)q^{(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))} \]

when evaluated at zero. The derivative of the denominator evaluated at zero is \( \frac{2[(1+j)\ln q}}{(1+j)} \), so

\[ \frac{\partial}{\partial j} = \frac{(1)^2}{(1+j)}(r - 1 - j)q^{(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))}. \]

If we denote the twist as \( a\text{Id}_{\mathcal{C}^H_{\ell \ell'}} + bx_{j,\ell} \), then since it is the inverse of \( \theta \), we have

\[ \text{Id}_{\mathcal{C}^H_{\ell \ell'}} = (a\text{Id}_{\mathcal{C}^H_{\ell \ell'}} + bx_{j,\ell}) \circ (a'\text{Id}_{\mathcal{C}^H_{\ell \ell'}} + b'x_{j,\ell}) = aa'\text{Id}_{\mathcal{C}^H_{\ell \ell'}} + (ab' + a'b)x_{j,\ell}. \]

Hence, \( aa' = 1 \) and \( ab' + a'b = 0 \). So, \( a = a'^{-1} = q^{-(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))} \) and

\[ b = -a'^{-1}ab' = -a^2b' = -\frac{(1)^2}{(1+j)}(r - 1 - j)q^{-(j+\ell r)(r-1-\frac{1}{2}(j+\ell r))}. \]

**Lemma 1.** \( \lim_{\epsilon \to 0} AX_\epsilon = A \lim_{\epsilon \to 0} X_\epsilon \quad \forall A \in \mathcal{U}_q^H(\mathfrak{sl}(2)) \)

**Proof.** The lemma holds for \( E, F, H \), and \( K \) by construction of \( X_\epsilon \) and hence holds for all polynomials in \( E, F, H \), and \( K \).

**Corollary 3.** The partial trace commutes with limits over functions in \( \text{End}(X_\epsilon) \).

**Proof.** The partial trace acts on elements of \( \text{End}(X_\epsilon) \) as a polynomial in \( E, F, H \), and \( K \) with complex coefficients hence commutes with limits by the Lemma.

**Proposition 10.** \( \lim_{\epsilon \to 0} d(X_\epsilon) = d(X_0) \)

**Proof.** Set \( \lambda = 1 + i - r \), then we have

\[
\lim_{\epsilon \to 0} d(X_\epsilon) = \lim_{\epsilon \to 0} d(V_{\lambda + \ell r + \epsilon} \oplus V_{-\lambda + \ell r + \epsilon}) = \lim_{\epsilon \to 0} (d(V_{\lambda + \ell r + \epsilon}) + d(V_{-\lambda + \ell r + \epsilon}))
\]

\[
= (-1)^{r-1} \lim_{\epsilon \to 0} \left( \frac{\{\lambda + \ell r + \epsilon\}}{\{r(\lambda + \ell r + \epsilon)\}} + \frac{\{-\lambda + \ell r + \epsilon\}}{\{r(-\lambda + \ell r + \epsilon)\}} \right)
\]

\[
= (-1)^{r-1} \lim_{\epsilon \to 0} \left( \frac{(q^{\lambda + \epsilon} - q^{-(\lambda + \epsilon)})q^{r(\lambda + \epsilon)} - q^{-r(\lambda + \epsilon)}}{(q^{r(\lambda + \epsilon)} - q^{-r(\lambda + \epsilon)})(q^{r(\lambda + \epsilon)} - q^{-r(\lambda + \epsilon)})} \right).
\]
The derivative of the numerator of the fraction evaluated at zero is given by
\[
\ln q \left( (q^λ + q^{-λ})(q^{-rλ} - q^rλ) + r(q^λ - q^{-λ})(q^{-rλ} + q^rλ) \right) \\
+ \ln q \left( (q^{-rλ} + q^rλ)(q^λ - q^{-λ}) + r(q^{-rλ} - q^rλ)(q^λ + q^{-λ}) \right) \\
= \ln q \left( 2(q^λ + q^{-λ})(q^{-rλ} - q^rλ) + 2r(q^λ + q^{-rλ})(q^λ - q^{-rλ} + q^λ - q^{-λ}) \right) \\
= 2\ln q(q^λ + q^{-λ})(q^{-rλ} - q^rλ) = 0.
\]

Where we have used the fact that \( q^r = q^{-r} \). The derivative of the denominator evaluated at zero is given by
\[
\ln q \left( r(q^λ + q^{-rλ})(q^{-rλ} - q^rλ) + r(q^{-rλ} - q^rλ)(q^λ + q^{-λ}) \right) = 2r\ln q(q^λ + q^{-rλ})(q^λ - q^{-rλ}) = 0.
\]

So, we apply L’Hopitals a second time and after simplifying we obtain a numerator of \( 8r(-1)^{1+i-r} (q^{1+i-r} + q^{-(1+i-r)}) \), and a denominator of \( 8r^2 \). Hence,
\[
\lim_{\epsilon \to 0} d(X_\epsilon) = (-1)^{f(r-1)+r-1} \frac{8r(-1)^{1+i-r}(q^{1+i-r} + q^{-(1+i-r)})}{8r^2} = (-1)^{f(r-1)+i+1}(q^i+1 + q^{-i-1}) = d(P_1).
\]

\( \square \)

**Proposition 11.** Let \( Z \) be a weight module in \( \mathcal{C} \). Then, the Hopf link \( \Phi_{Z,P_i \otimes C_{tr}^H} = a_Z Id_{P_i \otimes C_{tr}^H} + b_Z x_{i,t} \) is given by:
\[
a_Z = \lim_{\epsilon \to 0} \left( 2\lambda_Z v_{1+i-r+tr+\epsilon} - \lambda_Z v_{-1+i+r+tr+\epsilon} \right), \\
b_Z = \lim_{\epsilon \to 0} \frac{-1}{[1+i][\epsilon]} \left( \lambda_Z v_{-1-i+r+tr+\epsilon} - \lambda_Z v_{1+i-r+tr+\epsilon} \right)
\]

where \( \lambda_{Z,1+i-r+tr+\epsilon} \) and \( \lambda_{Z,-1+i+r+tr+\epsilon} \) are the constants by which \( \Phi_{Z,V_{1+i-r+tr+\epsilon}} \) and \( \Phi_{Z,V_{-1+i+r+tr+\epsilon}} \) act respectively. That is,
\[
\lambda_{Z,v_{1+i-r+tr+\epsilon}} = \Psi_{1+i-r+tr+\epsilon}(\chi(Z)) \\
\lambda_{Z,v_{-1+i+r+tr+\epsilon}} = \Psi_{-1-i-r+tr+\epsilon}(\chi(Z))
\]

**Proof.** Denote by \( \lambda_{Z,\gamma} \) the constant by which \( \Phi_{Z,V_{\gamma}} \) acts, and let \( a_\epsilon = 2, b_\epsilon = \frac{-1}{2[1+i][\epsilon]}, c = b^{-1}, d = 1 \) as in the proposition. By construction, \( \lim_{\epsilon \to 0} c_{Z,X_\epsilon \otimes C_{tr}^H} = c_{Z,P_i \otimes C_{tr}^H} \) and \( \lim_{\epsilon \to 0} c_{X_\epsilon \otimes C_{tr}^H} = c_{P_i \otimes C_{tr}^H} \) since the action on \( X_\epsilon \) converges to the action on \( P_1 \) in the limit \( \epsilon \to 0 \). Let \( \{x_0, \ldots, x_{r-1}\} \) denote the standard basis for \( V_{1+i-r+\epsilon} \) and \( \{y_0, \ldots, y_{r-1-i}\} \) the standard basis for \( V_{-1+i+r+\epsilon} \). Let \( w_i^H = a_\epsilon x_0 + b_\epsilon y_{-1-i} \) and \( w_i^S = c_\epsilon x_0 + d_\epsilon y_{-1-i} \) as in the proposition above. Then \( w_i^H \) generates
Hence, by Corollary

\[
\lim_{\epsilon \to 0} \Phi_{Z,X_i}(w_i^H \otimes v) = \lim_{\epsilon \to 0} \text{ptr}_R(c_{Z,X_i} \otimes c_{t_{li}}^H \circ c_{X_i} \otimes c_{t_{li}}^H, Z)(w_i^H \otimes v) = \text{ptr}_R(c_{Z,P_i} \otimes c_{t_{li}}^H \circ c_{P_i} \otimes c_{t_{li}}^H, Z)(w_i^H \otimes v) = \Phi_{Z,P_i}(w_i^H \otimes v).
\]

Notice that \(x_0 = d_i w_i^H - b_i w_i^S\) and \(y_{r-1-i} = a_i w_i^S - c_i w_i^H\). We can now compute the coefficients \(a, b\) of \(\Phi_{Z,P_i} = a_Z \text{Id}_{P_i} + b_Z x_i, 0\):

\[
\Phi_{Z,P_i}(w_i^H \otimes v) = \lim_{\epsilon \to 0} \Phi_{Z,X_i} \otimes c_{t_{li}}^H (w_i^H \otimes v) = \lim_{\epsilon \to 0} \Phi_{Z,X_i} \otimes c_{t_{li}}^H (a_i x_0 \otimes v + b_i y_{r-1-i} \otimes v)
\]

\[
= \lim_{\epsilon \to 0} \left( a_i \lambda_{Z,V_{1+i+r+tr+\epsilon}} x_0 \otimes v + b_i \lambda_{Z,V_{1+i-r+tr+\epsilon}} y_{r-1-i} \otimes v \right)
\]

\[
= \lim_{\epsilon \to 0} \left( a_i \lambda_{Z,V_{1+i-r+tr+\epsilon}} (d_i w_i^H - b_i w_i^S) \otimes v + b_i \lambda_{Z,V_{1+i-r+tr+\epsilon}} (a_i w_i^S - c_i w_i^H) \otimes v \right)
\]

\[
= \lim_{\epsilon \to 0} (a_i d_i \lambda_{Z,V_{1+i-r+tr+\epsilon}} - b_i c_i \lambda_{Z,V_{1+i-r+tr+\epsilon}}) w_i^H \otimes v
\]

\[
+ (a_i b_i \lambda_{Z,V_{1+i-r+tr+\epsilon}} - a_i c_i \lambda_{Z,V_{1+i-r+tr+\epsilon}}) w_i^S \otimes v
\]

\[
= \lim_{\epsilon \to 0} (2\lambda_{Z,V_{1+i-r+tr+\epsilon}} - \lambda_{Z,V_{1+i-r+tr+\epsilon}}) w_i^H \otimes v - \frac{1}{1 + i} (\lambda_{Z,V_{1+i-r+tr+\epsilon}} - \lambda_{Z,V_{1+i-r+tr+\epsilon}}) w_i^S \otimes v.
\]

\(V_{1+i-r+tr+\epsilon}\) and \(V_{1-i+r+tr+\epsilon}\) are simple, so

\[
\lambda_{Z,V_{1+i-r+tr+\epsilon}} = \Psi_{1+i-r+tr+\epsilon}(\chi(Z)),
\]

\[
\lambda_{Z,V_{1-i+r+tr+\epsilon}} = \Psi_{1-i+r+tr+\epsilon}(\chi(Z)).
\]

\(\Box\)

**Corollary 4.** The Hopf links \(\Phi_{Z,P_j} \otimes c_{t_{li}}^H\) with \(Z \in S_i, V_\alpha, P_i\) are given by \(\Phi_{Z,P_j} \otimes c_{t_{li}}^H = a_Z \text{Id}_{P_j} \otimes c_{t_{li}}^H + b_Z x_{j, l}\) where

\[
a_{P_i} = a_{V_\alpha} = 0, \quad a_{S_i} \otimes c_{t_{li}}^H = (-1)^{(i+1)+(1+j-r+tr)} \frac{(j+1)(j+1)}{j+1},
\]
and
\[
\begin{align*}
\lambda_{S_i, V_{1+j-r+\ell r+e}} & = \Psi_{1+j-r+\ell r+e}((\mathbb{C}^H_{\ell r})) = (-1)^{\ell}(1+j-r+\ell r+e), \\
\lambda_{S_i, V_{-i-j+r+\ell r+e}} & = \Psi_{-i-j+r+\ell r+e}((\mathbb{C}^H_{\ell r})) = (-1)^{\ell}(-1-j+r+\ell r+e), \\
\lambda_{P_i, V_{1+j-r+\ell r+e}} & = \Psi_{1+j-r+\ell r+e}((\mathbb{C}^H_{\ell r})) = q^{1+j-r+\ell r+e} \frac{(1+j-r+\ell r+e)}{1+j-r+\ell r+e}, \\
\lambda_{P_i, V_{-i-j+r+\ell r+e}} & = \Psi_{-i-j+r+\ell r+e}((\mathbb{C}^H_{\ell r})) = q^{(-1-j+r+\ell r+e)} \frac{1-j+r+\ell r+e}{-1-j+r+\ell r+e}, \\
\lambda_{P_i, V_{1+j-r+\ell r+e}} & = \Psi_{1+j-r+\ell r+e}((\mathbb{C}^H_{\ell r})) = q^{(r-i-1)(1+j-r+\ell r+e)} + q^{(1+i-r)(1+j-r+\ell r+e)} \frac{(1+j-r+\ell r+e)}{1+j-r+\ell r+e}, \\
\lambda_{P_i, V_{-i-j+r+\ell r+e}} & = \Psi_{-i-j+r+\ell r+e}((\mathbb{C}^H_{\ell r})) = q^{(r-i-1)(-1-j+r+\ell r+e)} + q^{(1+i-r)(-1-j+r+\ell r+e)} \frac{r(-1-j+r+\ell r+e)}{-1-j+r+\ell r+e}. 
\end{align*}
\]

Proof. For \( Z \in \{S_i, P_i, V_\alpha\} \), we have

\[
\begin{align*}
b_{S_i \otimes \mathbb{C}^H_{\ell r}} & = (-1)^{\ell(i+1)+\ell(1+j-r+\ell r)} \frac{i((i+2)(j+1)) - (i+2)(i(j+1))}{(j+1)^2(j+1)}, \\
b_{V_\alpha} & = (-1)^{\ell(i+1)+\ell+j} \frac{rq^{i\alpha}}{(j+1)^2}(q^{(r-1-j)\alpha} + q^{-(r-1-j)\alpha}), \\
b_{P_i \otimes \mathbb{C}^H_{\ell r}} & = (-1)^{\ell(i+1)+\ell(1+j-r+\ell r)} \frac{2r(q^{(i+1)(j+1)} + q^{-(i+1)(j+1)})}{(j+1)^2}.
\end{align*}
\]
Clearly, \( a_{V_n} = a_P = 0 \) since \( \frac{r((1+j-r+\ell))}{1+r+\ell} = \frac{r(1-j+r+\ell)}{-1-j+r+\ell} = 0 \), and

\[
a_{S_i} = \lim_{\epsilon \to 0} \left( 2\lambda_{S_i,V_{1+i+j-\epsilon}} - \lambda_{S_i,V_{1-i+j-\epsilon}} \right) = (-1)^{i(\ell+1)} \frac{(i+1)(j+1)}{j+1}.
\]

By L'Hôpital's rule, we can compute the other terms:

\[
b_{S_i} = \lim_{\epsilon \to 0} \frac{(-1)^{i(\ell+1)+1}}{[j+1][\epsilon]} \left( \frac{(i+1)(-1-j+\epsilon)}{-1-j+\epsilon} - \frac{(i+1)(1+j+\epsilon)}{1+j+\epsilon} \right)
= \lim_{\epsilon \to 0} \frac{(-1)^{i(\ell+1)+1}}{[j+1][\epsilon]} \left( \frac{(i+1)(-1-j+\epsilon)}{-1-j+\epsilon} \{1+j+\epsilon\} - \{(i+1)(1+j+\epsilon)\} \{1-j+\epsilon\} \right).
\]

The term in brackets can be expanded as

\[
(q^{-(i+1)(1+j-\epsilon)} - q^{(i+1)(1+j-\epsilon)})(q^{1+j+\epsilon} - q^{-1-j+\epsilon}) - (q^{(1+i)(1+j+\epsilon)} - q^{-(1+i)(1+j+\epsilon)})(q^{-1-j+\epsilon} - q^{1+j-\epsilon})
= q^{(i+2j)} - q^{(i+2j+1)} - q^{-(i+2)(j+1)} + q^{i(j+1)} q^{-i+2j} - q^{(i+2)(j+1)} q^i - q^{-(i+2)(j+1)} q^{ij} + q^{(i+2)(j+1)} q^{-i} - q^{i(j+1)} q^{-(i+2)}
\]

whose derivative evaluated at \( \epsilon = 0 \) is \( 2ln q ((i + 2)(j + 1)) - (i + 2)(i(j + 1)) \)

\[
\{\epsilon\} \{1-j+\epsilon\} \{1+j+\epsilon\} = \frac{1}{1} (q^\epsilon - q^{-\epsilon}) (q^{2\epsilon} - q^{2(j+1)} + q^{-2\epsilon})
= \frac{1}{1} (q^{3\epsilon} - q^{2(j+1)} q^{\epsilon} - q^{-2(j+1)} q^{\epsilon} + q^{2\epsilon} - q^{2(j+1)} q^{-\epsilon} - q^{-2(j+1)} q^{\epsilon} - q^{3\epsilon})
\]

whose derivative evaluated at \( \epsilon = 0 \) is

\[
\frac{1}{1} (4 - 2q^{2(j+1)} - 2^{-2(j+1)}) = -2ln q \frac{j+1}{1}.
\]

So,

\[
b_{S_i} = (-1)^{i(\ell+1)} \frac{i((i + 2)(j + 1)) - (i + 2)(i(j + 1))}{j+1}.
\]
Similarly, we find:

\[
\begin{align*}
\beta_{\alpha} &= \frac{(-1)^{r(\ell+1)+\ell+j}}{[j+1]^2} rq^{jk\alpha} (q^{(r-1-j)\alpha} + q^{-(r-1-j)\alpha}), \\
\beta_{P_i} &= \frac{(-1)^{i(\ell+1)}}{[j+1]^2} 2r(q^{(i+1)(j+1)} + q^{-(i+1)(j+1)}).
\end{align*}
\]

It is easy to show that \( a_{c^H_{\ell r'}} = (-1)^{\ell(1+j-r+\ell r')} \text{Id}_{P_j \otimes C_{\ell r}} \) and \( b_{c^H_{\ell r'}} = 0 \), so the proposition follows from \( \Phi_{X \otimes Y, P_j \otimes C_{\ell r}} = \Phi_{X, P_j} \otimes C_{\ell r} \).

\[
\square
\]

### 5.3 Alexander Invariants

Let \( L \) be a be a \((n, n)\)-ribbon graph colored by modules in \( C \) such that the bands at the \( i \) - th intersections are colored by the same module, and at least one of the colors is colored by a simple \( V_\lambda \). Let \( T_\lambda \) be the \((1,1)\)-ribbon graph obtained by closing the other ends through the action of the left and right partial trace. Then the re-normalized Reshetikhin-Turaev link invariant (see [6]) is

\[
F'(L) := t_{V_\lambda}(T_\lambda).
\]

Note that we are identifying \( T_\lambda \) with the corresponding morphism obtained by identifying tangles with combinations of the braiding, duality morphisms, and twist as described in section 3.2. These invariants were shown to coincide with Murakami’s Alexander invariants described in [8]. We now show that these results can be extended to any projective module, not just the simple \( V_\lambda \). Let \( L \) be constructed as above but with at least one of the colors being \( P_i \otimes C_{\ell r}^H \), and let \( T_{P_i \otimes C_{\ell r}^H} \) be the colored \((1,1)\)-ribbon graph obtained by closing the other ends, and let \( T_\lambda \) be the ribbon graph obtained by re-coloring the open \( P_i \otimes C_{\ell r}^H \)-colored band with \( V_\lambda \). Then,

**Theorem 4.** The colored \((1,1)\)-ribbon graph \( T_{P_i \otimes C_{\ell r}^H} \) satisfies

\[
t_{P_i \otimes C_{\ell r}^H}(T_{P_i \otimes C_{\ell r}^H}) = \lim_{\epsilon \to 0} (t_{V_1+i-r+\ell r+\epsilon}(T_{1+i-r+\ell r+\epsilon}) + t_{V_{-1-i-r+\ell r+\epsilon}}(T_{-1-i-r+\ell r+\epsilon}))
\]

and

\[
T_{P_i \otimes C_{\ell r}^H} = a \text{Id}_{P_i \otimes C_{\ell r}^H} + bx_{i,\ell r}
\]

with coefficients

\[
a = \lim_{\epsilon \to 0} \frac{t_{V_{-1-i-r+\ell r+\epsilon}}(T_{-1-i-r+\ell r+\epsilon})}{d(V_{-1-i-r+\ell r+\epsilon})} = \lim_{\epsilon \to 0} \frac{t_{V_1+i-r+\ell r+\epsilon}(T_{1+i-r+\ell r+\epsilon})}{d(V_{1+i-r+\ell r+\epsilon})}
\]
and

\[ b = \lim_{\epsilon \to 0} \frac{-1}{\pi i [1 + i][\epsilon]} \left( \frac{t_{V_{1+i-r+\ell r+\epsilon}} (T_{1-i-r+\ell r+\epsilon})}{d(V_{1+i-r+\ell r+\epsilon})} - \frac{t_{V_{1+i-r+\ell r+\epsilon}} (T_{1+i-r+\ell r+\epsilon})}{d(V_{1+i-r+\ell r+\epsilon})} \right) \]

\[ = \frac{r}{2\pi i (1 + i)} \left( \frac{d}{d\lambda} \left( \frac{t_{V_{1+i-r+\ell r+\epsilon}} (T_{}\lambda)}{d(V_{}\lambda)} \right) \right|_{\lambda=\ell r-r-i-1} - \frac{d}{d\lambda} \left( \frac{t_{V_{i}} (\lambda)}{d(V_{i})} \right) \right|_{\lambda=i+i-r+\ell r} \].

Before proving this theorem, we remark that this result nicely relates to the work of Murakami and Nagatomo on logarithmic link invariants for different quantum groups [8],[19],[20]. We also note that

\[ T_{\lambda} = \frac{t_{V_{\lambda}} (T_{\lambda})}{d(V_{\lambda})} Id_{\lambda}. \]

**Proof.** For the first statement, we use the identity \( \lim_{\epsilon \to 0} V_{1+i-r+\ell r+\epsilon} \oplus V_{1-i+r+\ell r+\epsilon} = P_{i} \otimes C^{H}_{\ell r} \) which gives

\[ t_{P_{i} \otimes C^{H}_{\ell r}} (T_{P_{i} \otimes C^{H}_{\ell r}}) = \lim_{\epsilon \to 0} t_{V_{1+i-r+\ell r+\epsilon} \oplus V_{1-i+r+\ell r+\epsilon}} (T_{V_{1+i-r+\ell r+\epsilon} \oplus V_{1-i+r+\ell r+\epsilon}}) \]

where we pulled the limit out of the function using the fact that limits commute with the partial trace and the action of \( \overline{U}_{q}^{H} (sl(2)) \). The relation follows since coloring with the direct sum of two objects \( X \oplus Y \) amounts to computing the sum of the individually colored components

\[ t_{X \oplus Y} (T_{X \oplus Y}) = t_{X} (T_{X}) + t_{Y} (T_{Y}). \]

For the second statement, since \( w_{i}^{H} \) generates \( P_{i} \otimes C^{H}_{\ell r} \), it is enough to find the action of \( T_{P_{i} \otimes C^{H}_{\ell r}} \) on \( w_{i}^{H} \). For this we compute the action of \( T_{V_{1+i-r+\ell r+\epsilon} \oplus V_{1-i+r+\ell r+\epsilon}} \) on \( w_{i}^{H} \) and then take the limit \( \epsilon \) to zero. Recall that we have \( w_{i}^{H} = a_{\epsilon} x_{0} + b_{\epsilon} y_{r-1-i} \) and \( w_{i}^{S} = c_{\epsilon} x_{0} + d_{\epsilon} y_{r-1-i} \) where \( x_{0} \), \( y_{r-1-i} \), \( a_{\epsilon} = 2 \), \( b_{\epsilon} = -\frac{1}{\pi i [1+i][\epsilon]} \), \( c_{\epsilon} = -2[1+i][\epsilon] \), and \( d_{\epsilon} = 1 \) are as in the construction of \( X_{\epsilon} \). Notice that \( x_{0} = d_{\epsilon} w_{i}^{H} - b_{\epsilon} w_{i}^{S} \) and \( y_{r-1-i} = a_{\epsilon} w_{i}^{S} - c_{\epsilon} w_{i}^{H} \). We can now compute the action of \( T_{P_{i} \otimes C^{H}_{\ell r}} \) on \( w_{i}^{H} \):

\[ T_{P_{i} \otimes C^{H}_{\ell r}} (w_{i}^{H}) = T_{P_{i} \otimes C^{H}_{\ell r}} \left( \lim_{\epsilon \to 0} (a_{\epsilon} x_{0} + b_{\epsilon} y_{r-1-i}) \right) = \lim_{\epsilon \to 0} (a_{\epsilon} T_{1+i-r+\ell r+\epsilon} (x_{0}) + b_{\epsilon} T_{1-i+r+\ell r+\epsilon} (y_{r-1-i})) \]

\[ = \lim_{\epsilon \to 0} \left( a_{\epsilon} \frac{t_{V_{1+i-r+\ell r+\epsilon}} (T_{1+i-r+\ell r+\epsilon})}{d(V_{1+i-r+\ell r+\epsilon})} (d_{\epsilon} w_{i}^{H} - b_{\epsilon} w_{i}^{S}) + b_{\epsilon} \frac{t_{V_{1-i+r+\ell r+\epsilon}} (T_{1-i+r+\ell r+\epsilon})}{d(V_{1-i+r+\ell r+\epsilon})} (a_{\epsilon} w_{i}^{S} - c_{\epsilon} w_{i}^{H})) \right). \]

It follows that

\[ a = \lim_{\epsilon \to 0} \left( a_{\epsilon} d_{\epsilon} \frac{t_{V_{1+i-r+\ell r+\epsilon}} (T_{1+i-r+\ell r+\epsilon})}{d(V_{1+i-r+\ell r+\epsilon})} - b_{\epsilon} c_{\epsilon} \frac{t_{V_{1-i+r+\ell r+\epsilon}} (T_{1-i+r+\ell r+\epsilon})}{d(V_{1-i+r+\ell r+\epsilon})} \right) \]

\[ = \lim_{\epsilon \to 0} \left( 2 \frac{t_{V_{1+i-r+\ell r+\epsilon}} (T_{1+i-r+\ell r+\epsilon})}{d(V_{1+i-r+\ell r+\epsilon})} - \frac{t_{V_{1-i+r+\ell r+\epsilon}} (T_{1-i+r+\ell r+\epsilon})}{d(V_{1-i+r+\ell r+\epsilon})} \right) \]

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and

\[
b = - \lim_{\epsilon \to 0} \left( a_{\epsilon} b_{\epsilon} \frac{t^{V_{1+i-r+\ell r+r}} (T_{1+i-r+\ell r+r})}{d(V_{1+i-r+\ell r+r})} - a_{\epsilon} b_{\epsilon} \frac{t^{V_{-1-i-r+\ell r+r}} (T_{-1-i-r+\ell r+r})}{d(V_{-1-i+r+\ell r+r})} \right)
= \lim_{\epsilon \to 0} \frac{1}{1 + i} \left( \frac{t^{V_{1+i-r+\ell r+r}} (T_{1+i-r+\ell r+r})}{d(V_{1+i-r+\ell r+r})} - \frac{t^{V_{-1-i-r+\ell r+r}} (T_{-1-i-r+\ell r+r})}{d(V_{-1-i+r+\ell r+r})} \right).
\]

Evaluating the limits (using L'Hôpital's rule for \( b \)) give the result.

\[\square\]

6 Future Work

There is a family of non \( C_2 \)-cofinite \( W \)-algebras denoted \( W^0(Q)_r \) where \( r \geq 2 \) and \( Q \) is the root lattice of a simply-laced simple Lie algebra \( g \). This family is a generalization of the singlet VOA \( \mathcal{M}(r) \). The representation categories for these \( W \)-algebras are expected to have connections to representation categories of certain unrolled quantum groups \( \widehat{U}_q^h(g) \) at a \( 2r \)-th root of unity. These quantum groups have tensor structure much more complicated than that of \( \widehat{U}_q^h(sl(2)) \) so direct methods for computing the tensor ring are difficult to manage. The Hopf link point of view provides a different approach to understanding these representation categories. In particular, this point of view may be of value to the study of \( \widehat{U}_q^h(sl(3)) \) where the tensor ring, and even module classifications are unknown.
7 References


[38] T. Creutzig, A. Milas, and M. Rupert - *Logarithmic link invariants of \( \mathcal{U}_q^H(\mathfrak{sl}(2)) \) and asymptotic dimensions of singlet vertex algebras*, arXiv:1605.05634

