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Discrete output regulator design for linear distributed parameter systems

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ABSTRACT

In this manuscript, we address discrete-time state and error feedback output regulator designs for a class of linear distributed parameter systems (DPS) with bounded input and output operators. By utilising the Cayley–Tustin bilinear transform, a linear infinite-dimensional discrete-time system is obtained without model spatial approximation or model order reduction. Based on the internal model principle, discrete state and error feedback regulators are designed. In particular, discrete Sylvester regulator equations are formulated, and their solvability is proved and linked to the continuous counterparts. To ensure the stability of the closed-loop system, the design of stabilising feedback gain and its dual problem of stabilising output injection gain design are provided in the discrete-time setting. Finally, three simulation examples including a first-order hyperbolic partial differential equation model and a 1-D heat equation with considerations of step-like, ramp-like and harmonic exogenous signals are shown to demonstrate the effectiveness of the proposed method.

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Distributed parameter systems; infinite-dimensional systems; discrete regulation; Sylvester equations; bilinear transform

1. Introduction

Output regulation has been an active research and practical application-driven topic during the past decades. The main idea of output regulation is to design regulators capable of stabilising unstable systems, tracking desired output references, and rejecting undesired disturbance signals simultaneously. In general, there are two distinct design problems associated with output regulation, i.e. the state feedback and output/error feedback regulator designs. In the case of the state feedback regulator design, the full state information is assumed to be available while only the output or error is known for the output/error feedback regulator design problems. Since the pioneering work of Francis and Wonham (1976), the internal model principle has initiated a plethora of important contributions in the output regulation theory of various linear and nonlinear finite-dimensional systems (Bengtsson, 1977; Byrnes et al., 1997; Davison, 1976; De Marco et al., 2018; Fridman, 2003; Gillella et al., 2014; Isidori & Byrnes, 1990; Johnson, 1971; Qiu et al., 2015; Serrani et al., 2001; Silva et al., 2016).

When it comes to complex dynamical systems in engineering applications which are often modelled by partial differential equations (PDE) and/or partial integral-differential equations (PIDE), the major challenge in regulator design is to account for the characteristics of infinite-dimensional systems and incipient efforts were made toward extending the output regulation theory in finite-dimensional systems to infinite-dimensional systems. In particular, the non-model based PI controllers were naturally explored as early regulators for stable distributed parameter systems with constant disturbance by Pohjolainen (1982), and later Kobayashi (1983) introduced the concepts of the state feedback and the output feedback in the design of regulators based on state-space representations. Isidori and Byrnes (1990) studied the output regulator design problem of finite-dimensional nonlinear systems by solving certain output regulator equations, which were generalised to infinite-dimensional systems with bounded control and observation operators in Byrnes et al. (2000), and recently further extended to an important class of regular linear systems by Natarajan et al. (2014). Within the regulator design approaches the distinct realisations can be grouped in the cases when the exo-system is finite-dimensional (Byrnes et al., 2000; Grasselli et al., 1996; Hämäläinen & Pohjolainen, 2000; Humaloja & Paunonen, 2017) or infinite-dimensional (Hämäläinen & Pohjolainen, 2000, 2010; Immonen, 2007; Immonen & Pohjolainen, 2005), which implies that reference and disturbance signals can take finite- or infinite-dimensional representation.

Except the existing contributions on geometric regulation (Aulisa & Gilliam, 2015), the backstepping approach (Krstic & Smyshlyaev, 2008) was introduced to solve output regulation problems leading to systematic regulator design methods. In particular, Aamo (2013) applied a backstepping method for disturbance rejection of a boundary controlled linear 2×2 hyperbolic system with co-located sensing and actuation, which was extended for the same type of system with interior domain disturbance by Anfinsen and Aamo (2015), $n+1$ systems by Hasan (2014) and more general $n+m$ heterodirectional first-order hyperbolic systems by Anfinsen and Aamo (2017) and Deutscher (2017). Recently, Bribiesca-Argomedo and Krstic (2015) introduced an integral transform

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into hyperbolic PDEs and proposed a backstepping-forwarding controller and observer for this class of hyperbolic PDEs with Fredholm integrals, which was further extended to the output regulation problem by Xu and Dubljevic (2017). As for parabolic systems, Deutscher (2015) first developed a backstepping-based regulator design approach for a scaler PDE system and the results were extended to output regulation problems of a 1-D Schrödinger equation by Zhou and Weiss (2017) and PIDE systems by Deutscher and Kerschbaum (2019). Recently, there are intense interests in cascade PDE systems by the use of backstepping approaches, including cascaded parabolic PDEs by Kang and Guo (2016), hyperbolic-parabolic PDE-PDE cascade by Gu and Wang (2018), ODE-PDE-ODE cascade systems by Deutscher and Gabriel (2018) and etc.

However, most of the existing work on the output regulation problem is conducted on the continuous-time setting, and relatively limited references are available on output regulation of discrete-time systems. Among these, discrete-time output regulation were considered for linear lumped parameter systems with input saturation by Mantri et al. (1997), piecewise-linear systems by Feng and Zhang (2006), and linear finite-dimensional multi-agent systems by Huang (2017). Nevertheless, there are even more scattered contributions when it comes to discrete-time output regulator design in infinite-dimensional systems. Among these, a simple sampled-data low-gain controller was proposed for approximate tracking and disturbance rejection of a class of exponentially stable well-posed infinite-dimensional systems (Ke et al., 2009). A sampled-data control problem of output tracking and disturbance rejection for unstable well-posed linear infinite-dimensional systems was considered with respect to constant disturbance and reference signals (Wakaiki & Sano, 2019), where a frequency-domain technique based on coprime factorisations approach was employed. A lifting technique was used to design a discrete-time feedback controller that achieves approximate robust output tracking and disturbance rejection in Paunonen (2017). Motivated by the fact that digital controllers and discrete-time systems are of great practical and theoretical interest, this manuscript addresses discrete-time output regulator design problem for linear distributed parameter systems with state-space models and consideration of general exogenous signals (including step-like, ramp-like and harmonic signals).

In particular, we consider a discrete-time output regulation design problem for linear distributed parameter systems driven by a finite-dimensional exo-system and extend key results of Byrnes et al. (2000) and Xu and Dubljevic (2016). More specifically, novel contributions of this work lie in the following aspects: (1) state and error feedback discrete regulators are designed for linear discrete-time distributed parameter systems by employing the Cayley–Tustin bilinear transform which preserves model structure and properties of linear infinite-dimensional continuous-time systems; (2) the discrete regulator equations are formulated and proved in the design of state and error feedback regulators, and the discrete state and error feedback regulator design problems are solvable if and only if the discrete regulator equations can be solved; (3) a 1-1 correspondence between the solutions of discrete regulator equations and the corresponding continuous regulator equations is established, implying that one can solve for the continuous Sylvester equations and utilise the results in a discrete regulator design and vice versa; (4) the non-resonance solvability conditions of the discrete regulator equations are provided and linked to the corresponding continuous regulator equations; (5) a novel way of determining discrete-time stabilising feedback gain (and its dual problem) is provided for the infinite-dimensional discrete-time systems using discrete-time Lyapunov and Riccati equations.

The rest of the manuscript is organised as follows: in Section 2, continuous-time infinite-dimensional plant and exogenous system are described and discretised in time by using the Cayley–Tustin transform. In Section 3, after revisiting some main results in the continuous-time state feedback regulator, the discrete-time state feedback regulator is designed and the solvability of the discrete Sylvester equations is proved and linked to its continuous counterpart. To ensure the stability of the closed-loop system, continuous- and discrete-time Lyapunov and Riccati equations are introduced to determine the discrete stabilising feedback gain along with their continuous parallels. In the same manner, a discrete error feedback regulator is formulated and the solvability of the corresponding regulator equations is proved, along with its continuous counterpart, and the dual problem of solving output injection gain is studied in Section 4. Finally, the results are shown to be applicable to a first-order hyperbolic PDE model and a heat equation model in Section 5, and conclusions are offered in Section 6.

We use the following notations in this manuscript. Assume that $\mathcal{X}$ and $\mathcal{V}$ are two Hilbert spaces and $\mathcal{A}:\mathcal{X}\mapsto\mathcal{V}$ is a linear operator from $\mathcal{X}$ to $\mathcal{V}$. $\mathcal{L}(\mathcal{X},\mathcal{V})$ denotes the set of linear bounded operators from $\mathcal{X}$ to $\mathcal{V}$. If $\mathcal{X}=\mathcal{V}$, we simply write $\mathcal{L}(\mathcal{X})$. The domain, spectrum, resolvent set and resolvent operator of a linear operator $\mathcal{A}$ are denoted as: $\mathcal{D}(\mathcal{A})$, $\sigma(\mathcal{A})$, $\rho(\mathcal{A})$, and $\mathcal{R}(s,\mathcal{A})=(sI-\mathcal{A})^{-1}$ with $s\in\rho(\mathcal{A})$, respectively. We denote the space $\mathcal{X}_1$ as the space $\mathcal{D}(\mathcal{A})$ with the norm $\|x\|_1=\|(\beta I-\mathcal{A})x\|$, and the space $\mathcal{X}_{-1}$ as the completion of $\mathcal{X}$ with the norm $\|z\|_{-1}=\|(\beta I-\mathcal{A})^{-1}z\|$, where $\forall\, x\in\mathcal{D}(\mathcal{A})$, $\forall\, z\in\mathcal{X}$, and $\beta\in\rho(\mathcal{A})$. The constructed space are linked by $\mathcal{X}_1\subset\mathcal{X}\subset\mathcal{X}_{-1}$, with each inclusion being dense and continuous embedding (Tucsnak & Weiss, 2009). In addition, the inner product is denoted by $\langle\cdot,\cdot\rangle$, and $L^2(0,l)^m$ with a positive integer m denotes a Hilbert space of a m-dimensional vector of the real functions that are square integrable over $[0,l]$ with a spatial length l.

2. Preliminaries

The plant – The following linear infinite-dimensional continuous-time system is considered:

$$\dot{x}(t)=\mathcal{A}x(t)+\mathcal{B}u(t)+\Xi d(t),\quad x(0)=x_0\in\mathcal{X} \tag{1a}$$

$$y_c(t)=\mathcal{C}x(t) \tag{1b}$$

where the spatial state $x(\cdot,t)\in\mathcal{X}$, with $\mathcal{X}=L^2((0,l),\mathbb{C})$ is being defined as a complex separable Hilbert space. $\zeta\in[0,l]\subset\mathbb{R}$ and $t\in[0,\infty)$ represent temporal and spatial coordinates. We denote the input $u(t)\in L^2_{loc}([0,\infty),U)$, the disturbance $d(t)\in L^2_{loc}([0,\infty),U_d)$, and the controlled output $y_c(t)\in L^2_{loc}([0,\infty),Y)$, where U, U_d and Y are finite-dimensional

Hilbert spaces. $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{X} \mapsto \mathcal{X}$ is an infinitesimal generator of a C_0-semigroup $\mathbb{T}_{\mathcal{A}}(t)$ on $\mathcal{X}$. The operators $\mathcal{B} \in \mathcal{L}(U, \mathcal{X})$, $\Xi \in \mathcal{L}(U_d, \mathcal{X})$, and $\mathcal{C} \in \mathcal{L}(\mathcal{X}, Y)$ are assumed to be bounded operators. We remark that our approach can be extended to the more general class of well-posed linear systems which include unbounded input and output operators in the sense of Weiss (1994a, 1994b). To address that, we need to introduce the spaces $\mathcal{X}_1$ and $\mathcal{X}_{-1}$ so as to define unbounded control, observation and disturbance operators as in Tucsnak and Weiss (2009, Pro. 2.10.3) and Salamon (1984).

Naturally, one can express the transfer functions as follows:

$$\mathcal{G}_c(s) = \mathcal{C}\mathcal{R}(s, \mathcal{A})\mathcal{B}, \quad s \in \rho(\mathcal{A}) \tag{2a}$$

$$\mathcal{T}_c(s) = \mathcal{C}\mathcal{R}(s, \mathcal{A})\Xi, \quad s \in \rho(\mathcal{A}) \tag{2b}$$

where $\mathcal{G}_c(s)$ and $\mathcal{T}_c(s)$ stand for transfer functions from $u(t)$ to $y_c(t)$ and from $d(t)$ to $y_c(t)$, respectively.

The plant discretisation in time – In order to address the issue of time discretisation, the energy and structure preserving Cayley–Tustin bilinear transform is applied to the linear infinite-dimensional continuous-time system (1) for a given time discretisation interval $\Delta t > 0$ as follows:

$$\frac{x(k\Delta t) - x((k-1)\Delta t)}{\Delta t} \approx \mathcal{A}\frac{x(k\Delta t) + x((k-1)\Delta t)}{2} + \mathcal{B}u(k\Delta t) + \Xi d(k\Delta t) \tag{3a}$$

$$y_c(k\Delta t) \approx \mathcal{C}\frac{x(k\Delta t) + x((k-1)\Delta t)}{2}, \quad x(0) = x_0,\ k \geq 1 \tag{3b}$$

As shown in Equation (3), the discretisation is performed based on the implicit mid-point integration rule, which does not rely on any spatial discretisation or model reduction, leading to a symmetric (in time) and symplectic discretisation scheme (Hairer et al., 2006). Through simple algebraic manipulation of Equation (3), one can obtain the infinite-dimensional discrete-time state-space model as:

$$x_k = \mathcal{A}_d x_{k-1} + \mathcal{B}_d u_k + \Xi_d d_k, \quad k \geq 1 \tag{4a}$$

$$y_{ck} = \mathcal{C}_d x_{k-1} + \mathcal{D}_d u_k + \Upsilon_d d_k \tag{4b}$$

where the discrete state, disturbance and output are denoted by x_k, d_k and y_{ck}. Additionally, the discrete input is given by $\frac{u_k}{\sqrt{\Delta t}} = \frac{1}{\Delta t}\int_{(k-1)\Delta t}^{k\Delta t} u(t)\,\mathrm{d}t$, and it can be shown that $\frac{u_k}{\sqrt{\Delta t}}$ converges to $u(t)$ on the interval $t \in ((k-1)\Delta t, k\Delta t)$ as $\Delta t \to 0$ (Havu & Malinen, 2007). Similar expressions hold for the discrete-time d_k and y_{ck}. The associated discrete-time operators are given as follows:

$$\begin{bmatrix} \mathcal{A}_d & \mathcal{B}_d & \Xi_d \\ \mathcal{C}_d & \mathcal{D}_d & \Upsilon_d \end{bmatrix} = \begin{bmatrix} -I + 2\delta\mathcal{R}(\delta, \mathcal{A}) & \sqrt{2\delta}\mathcal{R}(\delta, \mathcal{A})B & \sqrt{2\delta}\mathcal{R}(\delta, \mathcal{A})\Xi \\ \sqrt{2\delta}\mathcal{C}\mathcal{R}(\delta, \mathcal{A}) & \mathcal{G}_c(\delta) & \mathcal{T}_c(\delta) \end{bmatrix} \tag{5}$$

where $\mathcal{R}(\delta, \mathcal{A})$, $\mathcal{G}_c(\delta)$ and $\mathcal{T}_c(\delta)$ denote the resolvent operator $\mathcal{R}(s, \mathcal{A}) = (sI - \mathcal{A})^{-1}$, transfer functions $\mathcal{G}_c(s)$ and $\mathcal{T}_c(s)$ with s evaluated at $s = \delta = 2/\Delta t \in \rho(\mathcal{A})$. In addition, there are feedforward operators $\mathcal{D}_d$ and Υ_d appearing in the discrete-time setting (4) after applying Cayley–Tustin discretisation, which is not necessarily present in the continuous model (1b). The transfer functions of the discrete-time system (4) are given by

$$\mathcal{G}_d(z) = \mathcal{C}_d(zI - \mathcal{A}_d)^{-1}\mathcal{B}_d + \mathcal{D}_d, \quad z \in \rho(\mathcal{A}_d)\backslash\{-1\} \tag{6a}$$

$$\mathcal{T}_d(z) = \mathcal{C}_d(zI - \mathcal{A}_d)^{-1}\Xi_d + \Upsilon_d, \quad z \in \rho(\mathcal{A}_d)\backslash\{-1\} \tag{6b}$$

where $\mathcal{G}_d(z)$ and $\mathcal{T}_d(z)$ are transfer functions from u_k to y_{ck} and from d_k to y_{ck}, respectively. Based on the well-known bilinear mapping $z = \frac{\delta+s}{\delta-s}$ and $s = \frac{z-1}{z+1}\delta$ (Curtain & Oostveen, 1997) (taken as: $\delta = 1$), the discrete- and continuous-time transfer functions are linked as

$$\mathcal{G}_d(z) = \mathcal{G}_c\left(\frac{z-1}{z+1}\delta\right),$$

$$\mathcal{T}_d(z) = \mathcal{T}_c\left(\frac{z-1}{z+1}\delta\right), \quad z \in \rho(\mathcal{A}_d)\backslash\{-1\} \tag{7a}$$

$$\mathcal{G}_c(s) = \mathcal{G}_d\left(\frac{\delta+s}{\delta-s}\right), \quad \mathcal{T}_c(s) = \mathcal{T}_d\left(\frac{\delta+s}{\delta-s}\right), \quad s \in \rho(\mathcal{A})\backslash\{\delta\} \tag{7b}$$

By the Cayley–Tustin bilinear transformation, the open right-half plane $\mathbb{C}^+ = \{s \in \mathbb{C} : \Re(s) > 0\}$ is mapped into the exterior of the unit disc $\mathbb{D}^+ = \{z \in \mathbb{C} : |z| > 1\}$ and vice versa. Based on that, a 1-1 correspondence of stability, admissibility, controllability and observability between continuous- and discrete-time systems has been established in terms of Lyapunov and Riccati equations by Curtain and Oostveen (1997). In this paper, we will explore the 1-1 equivalence of discrete- and continuous-time regulation problems in terms of Sylvester equations.

Remark 2.1: In the discrete-time model (4), all discrete-time operators are bounded and defined as: $\mathcal{A}_d \in \mathcal{L}(\mathcal{X})$, $\mathcal{B}_d \in \mathcal{L}(U, \mathcal{X})$, $\Xi_d \in \mathcal{L}(U_d, \mathcal{X})$, $\mathcal{C}_d \in \mathcal{L}(\mathcal{X}, Y)$, $\mathcal{D}_d \in \mathcal{L}(U, Y)$, $\Upsilon_d \in \mathcal{L}(U_d, Y)$, see details in Paunonen (2017).

Exo-system – In order to generate disturbance and reference signals, a finite-dimensional exogenous system (exo-system in short) is introduced as follows:

$$\dot{q}(t) = Sq(t), \quad q(0) = q^0 \in \mathbb{C}^{n_q} \tag{8a}$$

$$d(t) = Fq(t), \quad y_r(t) = Qq(t) \tag{8b}$$

where q, d and y_r represent the state, disturbance and reference signals of the continuous-time exo-system. In addition, $q(t) \in \mathbb{C}^{n_q}$, S, F and Q have compatible dimensions.

Assumption 2.1: *$S : \mathcal{D}(S) : \mathbb{C}^{n_q} \to \mathbb{C}^{n_q}$ is a matrix with all eigenvalues on the imaginary axis and S has two candidates S_m and S_n. S_m has distinct eigenvalues and is diagonalisable with dimension of n_q, while S_n is a nilpotent matrix of dimension 2 as $S_n = \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$. Hence, this design of S accounts for the modelling of step-like, ramp-like, and harmonic exogenous signals.*

Exo-system discretisation – A discrete exo-system is formulated to generate discrete-time disturbance and output reference signals as follows:

$$q_k = S_d q_{k-1}, \quad q_0 = q^0 \in \mathbb{C}^{n_q} \tag{9a}$$
$$d_k = F_d q_k, \quad y_{rk} = Q_d q_k, \quad k \geq 1 \tag{9b}$$

where q_k, d_k and y_{rk} are the state, disturbance and output reference signals in the discrete-time setting. Specifically, S_d is the discrete state evolution matrix and obtained by discretising the corresponding continuous evolution matrix S using the Cayley–Tustin transform as

$$S_d = -I + 2\delta_0(\delta_0 I - S)^{-1}, \quad \delta_0 \in \rho(S) = \mathbb{C}\backslash\sigma(S) \tag{10a}$$
$$F_d = F, \quad Q_d = Q \tag{10b}$$

where $\delta_0 = 2/\Delta T$ with ΔT defined as the discretisation time for the exo-system and we assume $\Delta T = \Delta t$ for simplicity, which implies that $\delta = \delta_0$.

By the light of Assumption 2.1, we have two candidates for S_d as $S_d = S_m^d$ or $S_d = S_n^d$, which implies that all the eigenvalues of S_d are on the unit circle boundary on the complex plane. In particular, $S_i^d = -I + 2\delta(\delta I - S_i)^{-1}, i = m, n,$ and S_m^d has all distinct eigenvalues and $S_n^d = [\begin{smallmatrix} 1 & 0 \\ \frac{2}{\delta} & 1 \end{smallmatrix}]$. Hence, S_d is capable of generating step-like, ramp-like, and harmonic signals in the discrete-time setting. Thus, the correspondence between the continuous- and discrete-time exogenous systems is established in order to link the solvability of discrete Sylvester equations to the continuous counterparts.

Remark 2.2: We need discretise F and Q as below using Cayley–Tustin discretisation approach in order to ensure the corresponding relationship of discrete- and continuous-time error feedback regulators.

$$S_d = -I + 2\delta(\delta I - S)^{-1}, \quad \delta \in \rho(S) = \mathbb{C}\backslash\sigma(S) \tag{11a}$$
$$F_d = \sqrt{2\delta}F(\delta I - S)^{-1}, \quad Q_d = \sqrt{2\delta}Q(\delta I - S)^{-1} \tag{11b}$$

Corollary 2.1: *With diagonalisable S and S_d (i.e. $S = S_m$ and $S_d = S_m^d$), for each eigenpair (λ_i^s, ϕ_i^s) of S, the associated eigenpair (λ_i^d, ϕ_i^d) of S_d is given by $\lambda_i^d = -1 + 2\delta(\delta - \lambda_i^s)^{-1}$ and $\phi_i^d = \phi_i^s$ where $\delta \in \rho(S)$; With non-diagonalisable S and S_d (i.e. $S = S_n$ and $S_d = S_n^d$), we have the multiplicity of eigenvalues $\lambda^s = 0$ and $\lambda^d = 1$ (for simplicity we drop the subscript i in this case), which induces a standard eigenvector $S\phi_1^s = \lambda^s\phi_1^s$ or $S_d\phi_1^d = \lambda^d\phi_1^d$ and a generalised eigenvector as $S\phi_2^s = \lambda^s\phi_2^s + \phi_1^s$ or $S_d\phi_2^d = \lambda^d\phi_2^d + \phi_1^d$ by using the chain rule. Furthermore, we suppose $\lambda_i^s \neq \delta$ and $\lambda_i^d \neq -1$, which can always be ensured by Assumption 2.1 and a proper choice of the discretisation interval.*

For clarification, we introduce the following stability concepts.

Definition 2.1: The C_0-semigroup $\mathbb{T}_{\mathcal{A}}(t)$ on $\mathcal{X}$ is exponentially stable if there exist positive constants M and α such that

$$\|\mathbb{T}_{\mathcal{A}}(t)\| \leq M\,\mathrm{e}^{-\alpha t} \quad \forall\, t \in \mathbb{R}^+$$

and it is strongly stable if $\|\mathbb{T}_{\mathcal{A}}(t)x\| \to 0$ as $t \to \infty$ for all $x \in \mathcal{X}$. $\mathcal{A}_d$ is power stable if there exist positive constants M and $\gamma < 1$ such that

$$\|\mathcal{A}_d^k\| \leq M\gamma^k \quad \forall\, k \in \mathbb{N}$$

and $\mathcal{A}_d$ is strongly stable if $\mathcal{A}_d^k x \to 0$ as $k \to \infty$ for all $x \in \mathcal{X}$.

Theorem 2.1 (Curtain & Oostveen, 1997, The. 2.9)**:** *Suppose that $\Sigma(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ and $\Sigma_d(\mathcal{A}_d, \mathcal{B}_d, \mathcal{C}_d, \mathcal{D}_d)$ are continuous- and discrete-time analogues. Then $\Sigma_d(\mathcal{A}_d, \mathcal{B}_d, \mathcal{C}_d, \mathcal{D}_d)$ is strongly stabilisable (detectable) if and only if $\Sigma(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ is strongly stabilisable (detectable).*

Normally, it is more favourable to show the 1-1 correspondence between the power stability of $\mathcal{A}_d$ and exponential stability of $\mathbb{T}_{\mathcal{A}}(t)$. However, it needs more rigorous assumption on the boundedness of $\mathcal{A}$ on $\mathcal{X}$ (Curtain & Rodriguez, 1994, Lem. 4.4) or the semigroup generated by $\mathcal{A}$ and $\mathcal{A}^{-1}$ (Guo & Zwart, 2006, The. 4.4), which needs more technical treatments and it will not be considered in this manuscript.

Throughout this manuscript, we make some general assumptions as below:

Assumption 2.2: *The spectrum of S is included in the resolvent set of $\mathcal{A}$, i.e. $\sigma(S) \subset \rho(\mathcal{A})$.*

By applying the Cayley–Tustin transform with some proper time discretisation interval, we can ensure that the spectrum of S_d is contained in the resolvent set of $\mathcal{A}_d$, i.e. $\sigma(S_d) \subset \rho(\mathcal{A}_d)$.

Assumption 2.3: *The pair $(\mathcal{A}, \mathcal{B})$ is exponentially stabilisable.*

By Theorem 2.1 and Assumption 2.3, we induce that the pair $(\mathcal{A}, \mathcal{B})$ is strongly stabilisable, which further implies that $(\mathcal{A}_d, \mathcal{B}_d)$ is strongly stabilisable.

Assumption 2.4: *The pair $([\begin{smallmatrix} \mathcal{A} & P \\ 0 & S \end{smallmatrix}], [\mathcal{C} \quad -Q])$ is exponentially detectable and there exists $\mathcal{G}_2 = [G_1; G_2] \in \mathcal{L}(Y, \Omega)$, where Ω is a Hilbert space with $\Omega = \mathcal{X} \oplus \mathbb{C}^{n_q}$ such that*

$$\begin{bmatrix} \mathcal{A} & P \\ 0 & S \end{bmatrix} - \mathcal{G}_2[\mathcal{C} \quad -Q] = \begin{bmatrix} \mathcal{A} - G_1\mathcal{C} & P + G_1 Q \\ -G_2\mathcal{C} & S + G_2 Q \end{bmatrix}$$

generates an exponentially stable C_0-semigroup, where $P = \Xi F$.

By Theorem 2.1 and Assumption 2.4, we can induce that the pair $([\begin{smallmatrix} \mathcal{A}_d & P_d \\ 0 & S_d \end{smallmatrix}], [\mathcal{C}_d \quad \Theta_{cd} - Q_d])$ is strongly detectable and there exists $\mathcal{G}_{2d} \in \mathcal{L}(Y, \Omega)$, such that

$$\begin{bmatrix} \mathcal{A}_d & P_d \\ 0 & S_d \end{bmatrix} - \mathcal{G}_{2d}[\mathcal{C}_d \quad \Theta_{cd} - Q_d] = \begin{bmatrix} \mathcal{A}_d - G_{1d}\mathcal{C}_d & P_d - G_{1d}(\Theta_{cd} - Q_d) \\ -G_{2d}\mathcal{C}_d & S_d - G_{2d}(\Theta_{cd} - Q_d) \end{bmatrix}$$

is a strongly stable operator, where $\mathcal{G}_{2d} = [G_{1d}; G_{2d}]$, $P_d = \Xi_d F_d$ and $\Theta_{cd} = \Upsilon_d F_d$. It can be proved by showing that $([\begin{smallmatrix} \mathcal{A}_d & P_d \\ 0 & S_d \end{smallmatrix}], [\mathcal{C}_d \quad \Theta_{cd} - Q_d])$ corresponds to the discrete-time operator $([\begin{smallmatrix} \mathcal{A} & P \\ 0 & S \end{smallmatrix}], [\mathcal{C} \quad -Q])$ by using the Cayley–Tustin transform.

In order to show the solvability of the regulator equations, we introduce the concept of transmission zero. Under the consideration of finite-dimensional input and output spaces, we make the following definition as in Byrnes et al. (2000).

Definition 2.2: $s_0 \in \mathbb{C}$ is a transmission zero of continuous-time plant (1) if $\det \mathcal{G}_c(s_0) = 0$, and $z_0 \in \mathbb{C}$ is a transmission zero of discrete-time plant (4) if $\det \mathcal{G}_d(z_0) = 0$.

3. State feedback regulation

In this section, with the full state information being provided, a discrete-time state feedback regulator is designed for the discrete system (4) and it is presented in parallel with its continuous analogue for comparison. Based on the Cayley–Tustin discretisation, we prove the solvability of the discrete output regulator equations and provide a 1-1 correspondence between the solutions of discrete- and continuous-time regulator equations.

3.1 Continuous-time state feedback regulator

To proceed with the discrete-time state-feedback regulator design, we briefly revisit the corresponding continuous-time counterpart in this section.

For simplicity, the continuous-time state feedback regulator design problem is reviewed as follows. A continuous-time state-feedback regulator is designed for the system (1) by finding a control law having the following form:

$$u(t) = Kx(t) + Lq(t) \tag{12}$$

where $K \in \mathcal{L}(\mathcal{X}, U)$, $L \in \mathcal{L}(\mathbb{C}^{n_q}, U)$ such that the following conditions hold.

[c1]: The closed-loop system operator $\mathcal{A} + \mathcal{B}K$ generates an exponentially stable C_0-semigroup.
[c2]: For the closed-loop system, the output tracking error $e(t) = y_c(t) - y_r(t) \to 0$ with $t \to +\infty$ for any given initial conditions of $x_0 \in \mathcal{X}$ and $q^0 \in \mathbb{C}^{n_q}$.

To determine the control law (12), the following theorem is often utilised.

Theorem 3.1: *Let Assumptions 2.2 and 2.3 hold. The continuous-time state feedback regulation problem is solvable if and only if there exist mappings $\Pi \in \mathcal{L}(\mathbb{C}^{n_q}, \mathcal{X})$ with $\Pi D(S) \subset D(\mathcal{A})$ and $\Gamma \in \mathcal{L}(\mathbb{C}^{n_q}, U)$ such that the following Sylvester equations hold* (Byrnes et al., 2000, The. IV.1):

$$\Pi S = \mathcal{A}\Pi + \mathcal{B}\Gamma + P \tag{13a}$$

$$\mathcal{C}\Pi = Q \tag{13b}$$

where $P = \Xi F$, and $L = \Gamma - K\Pi$ can be utilised for computing the control input $u(t) = Kx(t) + Lq(t)$.

3.2 Discrete-time state feedback regulator

A discrete state feedback regulator is designed for the discrete system (4) in a discrete-time setting by satisfying the following conditions:

[C1]: The closed-loop system operator $\mathcal{A}_d + \mathcal{B}_d K_d$ is strongly stable.
[C2]: For the closed-loop system, the output tracking error $e_k = y_{ck} - y_{rk} \to 0$ with $k \to +\infty$ for any given initial conditions of $x_0 \in \mathcal{X}$ and $q^0 \in \mathbb{C}^{n_q}$.

Discrete-time regulator design – a full state feedback: With full state information of plant and exo-system being available, the discrete state feedback regulator design problem is addressed by finding a discrete regulator in the following form:

$$u_k = K_d x_{k-1} + L_d q_k \tag{14}$$

where $K_d \in \mathcal{L}(\mathcal{X}, U)$, $L_d \in \mathcal{L}(\mathbb{C}^{n_q}, U)$ such that [C1] and [C2] hold.

To address the discrete-time state feedback regulator design problem, we propose the following theorem.

Theorem 3.2: *Under Assumptions 2.2 and 2.3, the discrete state feedback regulation problem is solvable if and only if there exist mappings $\Pi_d \in \mathcal{L}(\mathbb{C}^{n_q}, \mathcal{X})$ and $\Gamma_d \in \mathcal{L}(\mathbb{C}^{n_q}, U)$ such that the following discrete Sylvester equations hold*:

$$\Pi_d S_d = \mathcal{A}_d \Pi_d + (\mathcal{B}_d \Gamma_d + P_d) S_d \tag{15a}$$

$$Q_d S_d = \mathcal{C}_d \Pi_d + (\mathcal{D}_d \Gamma_d + \Theta_{cd}) S_d \tag{15b}$$

where $P_d = \Xi_d F_d$, $\Theta_{cd} = \Upsilon_d F_d$, and $L_d = \Gamma_d - K_d \Pi_d S_d^{-1}$ can be utilised to compute the state feedback control law u_k in Equation (14).

Proof: First, let us prove the sufficiency. Plugging Equation (14) into the discrete system (4) leads to the closed-loop model as follows:

$$x_k = (\mathcal{A}_d + \mathcal{B}_d K_d) x_{k-1} + (\mathcal{B}_d L_d + P_d) q_k \tag{16}$$

To ensure [C1], the operator $\mathcal{A}_d + \mathcal{B}_d K_d$ needs to be strongly stable and the discrete-time solution takes the following form:

$$x_k = (\mathcal{A}_d + \mathcal{B}_d K_d)^k x_0 + \sum_{m=1}^{k} (\mathcal{A}_d + \mathcal{B}_d K_d)^{m-1} (\mathcal{B}_d L_d + P_d) q_{k+1-m} \tag{17}$$

By substituting Equations (9) and (15) into Equation (17), one gets

$$x_k = (\mathcal{A}_d + \mathcal{B}_d K_d)^k x_0 + \sum_{m=1}^{k} (\mathcal{A}_d + \mathcal{B}_d K_d)^{m-1} [\mathcal{B}_d (\Gamma_d - K_d \Pi_d S_d^{-1}) + P_d] q_{k+1-m}$$

$$
\begin{aligned}
&= (\mathcal{A}_d + \mathcal{B}_d K_d)^k x_0 \\
&\quad + \sum_{m=1}^{k} (\mathcal{A}_d + \mathcal{B}_d K_d)^{m-1} [(\mathcal{B}_d \Gamma_d + P_d) S_d \\
&- \mathcal{B}_d K_d \Pi_d)] q_{k-m} \\
&= (\mathcal{A}_d + \mathcal{B}_d K_d)^k x_0 \\
&\quad + \sum_{m=1}^{k} (\mathcal{A}_d + \mathcal{B}_d K_d)^{m-1} [\Pi_d S_d - (\mathcal{A}_d + \mathcal{B}_d K_d)\Pi_d)] q_{k-m} \\
&= (\mathcal{A}_d + \mathcal{B}_d K_d)^k x_0 \\
&\quad + \sum_{m=1}^{k} (\mathcal{A}_d + \mathcal{B}_d K_d)^{m-1} \Pi_d q_{k+1-m} \\
&\quad - \sum_{m=2}^{k+1} (\mathcal{A}_d + \mathcal{B}_d K_d)^{m-1} \Pi_d q_{k+1-m} \\
&= (\mathcal{A}_d + \mathcal{B}_d K_d)^k (x_0 - \Pi_d q_0) + \Pi_d q_k
\end{aligned} \tag{18}
$$

Moreover, the discrete tracking error can be expressed as

$$
\begin{aligned}
e_k &= y_{ck} - y_{rk} \\
&= \mathcal{C}_d x_{k-1} + \mathcal{D}_d u_k + \Theta_{cd} q_k - Q_d q_k \\
&= (\mathcal{C}_d + \mathcal{D}_d K_d) x_{k-1} + (\mathcal{D}_d L_d + \Theta_{cd} - Q_d) q_k \\
&= (\mathcal{C}_d + \mathcal{D}_d K_d)(\mathcal{A}_d + \mathcal{B}_d K_d)^{k-1}(x_0 - \Pi_d q_0) \\
&\quad + [(\mathcal{C}_d + \mathcal{D}_d K_d)\Pi_d + (\mathcal{D}_d L_d + \Theta_{cd} - Q_d) S_d] q_{k-1}
\end{aligned} \tag{19}
$$

Since $\mathcal{A}_d + \mathcal{B}_d K_d$ is a strongly stable operator, we have that $(\mathcal{A}_d + \mathcal{B}_d K_d)^k x \to 0$ as $k \to +\infty$ for all $x \in \mathcal{X}$. Therefore, x_k converges to $\Pi_d q_k$ in Equation (18) and the discrete tracking error e_k goes to zero in Equation (19) as $k \to +\infty$, which is guaranteed by the discrete Sylvester equations (15a)–(15b).

Now, we focus on the proof of the necessity and let us construct the following extended closed-loop system:

$$
\begin{bmatrix} x_k \\ q_k \end{bmatrix} = \begin{bmatrix} \mathcal{A}_d + \mathcal{B}_d K_d & (\mathcal{B}_d L_d + P_d) S_d \\ 0 & S_d \end{bmatrix} \begin{bmatrix} x_{k-1} \\ q_{k-1} \end{bmatrix} \tag{20}
$$

It is straightforward to conclude the solution of Equation (20) by induction as follows:

$$
\begin{bmatrix} x_k \\ q_k \end{bmatrix} = \begin{bmatrix} (\mathcal{A}_d + \mathcal{B}_d K_d)^k x_0 \\ + \sum_{m=1}^{k} (\mathcal{A}_d + \mathcal{B}_d K_d)^{m-1} (\mathcal{B}_d L_d + P_d) q_{k+1-m} \\ S_d^k q_0 \end{bmatrix} \tag{21}
$$

Given that $\mathcal{A}_d + \mathcal{B}_d K_d$ is strongly stable, $(\mathcal{A}_d + \mathcal{B}_d K_d)^k x_0 \to 0$ as $k \to +\infty$ and Equation (21) indicates that $\left[\begin{smallmatrix} x_k \\ q_k \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \Pi_d q_k \\ q_k \end{smallmatrix}\right]$ as $k \to +\infty$ and $\Pi_d \in \mathcal{L}(\mathbb{C}^{n_q}, \mathcal{X})$. To determine Π_d, we can construct the dynamical evolution of $\left[\begin{smallmatrix} x_k \\ q_k \end{smallmatrix}\right] - \left[\begin{smallmatrix} \Pi_d q_k \\ q_k \end{smallmatrix}\right]$ as the following homogeneous difference equation:

$$
\begin{bmatrix} x_k \\ q_k \end{bmatrix} - \begin{bmatrix} \Pi_d q_k \\ q_k \end{bmatrix} = \begin{bmatrix} \mathcal{A}_d + \mathcal{B}_d K_d & (\mathcal{B}_d L_d + P_d) S_d \\ 0 & S_d \end{bmatrix} \times \left(\begin{bmatrix} x_{k-1} \\ q_{k-1} \end{bmatrix} - \begin{bmatrix} \Pi_d q_{k-1} \\ q_{k-1} \end{bmatrix} \right) \tag{22}
$$

where the initial condition is defined as $\left[\begin{smallmatrix} x_0 \\ q_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \Pi_d q_0 \\ q_0 \end{smallmatrix}\right] \in \Omega$ with $\Omega = \mathcal{X} \oplus \mathbb{C}^{n_q}$. The first component in Equation (22) leads to $(\mathcal{A}_d + \mathcal{B}_d K_d)\Pi_d + (\mathcal{B}_d L_d + P_d) S_d = \Pi_d S_d$ which is identical to discrete-time Sylvester equation (15a). Furthermore, the discrete tracking error is described as

$$
\begin{aligned}
e_k &= y_{ck} - y_{rk} \\
&= \mathcal{C}_d x_{k-1} + \mathcal{D}_d u_k + \Theta_{cd} q_k - Q_d q_k \\
&= (\mathcal{C}_d + \mathcal{D}_d K_d) x_{k-1} + (\mathcal{D}_d L_d + \Theta_{cd} - Q_d) q_k \\
&= [\mathcal{C}_d + \mathcal{D}_d K_d \quad (\mathcal{D}_d L_d + \Theta_{cd} - Q_d) S_d] \begin{bmatrix} x_{k-1} \\ q_{k-1} \end{bmatrix} \\
&\to [(\mathcal{C}_d + \mathcal{D}_d K_d)\Pi_d + (\mathcal{D}_d L_d + \Theta_{cd} - Q_d) S_d]\, q_{k-1} \\
&\quad (\text{as } k \to +\infty)
\end{aligned} \tag{23}
$$

To realise perfect tracking, it is necessary to ensure that $(\mathcal{C}_d + \mathcal{D}_d K_d)\Pi_d + (\mathcal{D}_d L_d + \Theta_{cd} - Q_d) S_d = 0$, which implies Equation (15b) by substituting $L_d = \Gamma_d - K_d \Pi_d S_d^{-1}$. ■

3.3 Link between continuous and discrete regulator equations

The solutions of the proposed discrete-time regulator equations are linked to the associated continuous analogues, based on Theorems 3.1 and 3.2 and Cayley–Tustin bilinear transform.

Before we proceed with the theorem showing the equivalent link, let us propose the following proposition:

Proposition 3.1: *Suppose that we have these 'second-order' transfer functions (or equivalently the derivatives of transfer functions $\mathcal{G}_c(s)$, $\mathcal{G}_d(z)$, $\mathcal{T}_c(s)$, and $\mathcal{T}_d(z)$) for the continuous- and discrete-time system (1) and (4) accordingly as*

$$
\mathcal{G}_c^{(2)}(s) = \mathcal{C}(sI - \mathcal{A})^{-2}\mathcal{B}, \quad s \in \rho(\mathcal{A}) \backslash \{\delta\} \tag{24a}
$$

$$
\mathcal{G}_d^{(2)}(z) = \mathcal{C}_d(zI - \mathcal{A}_d)^{-2}\mathcal{B}_d, \quad z \in \rho(\mathcal{A}_d) \backslash \{-1\} \tag{24b}
$$

$$
\mathcal{T}_c^{(2)}(s) = \mathcal{C}(sI - \mathcal{A})^{-2}\Xi, \quad s \in \rho(\mathcal{A}) \backslash \{\delta\} \tag{24c}
$$

$$
\mathcal{T}_d^{(2)}(z) = \mathcal{C}_d(zI - \mathcal{A}_d)^{-2}\Xi_d, \quad z \in \rho(\mathcal{A}_d) \backslash \{-1\} \tag{24d}
$$

Using the Cayley–Tustin bilinear transform, it can be proved that the following relationships hold:

$$
\mathcal{G}_d^{(2)}(z) = \frac{(\delta - s)^2}{2\delta} \times \mathcal{G}_c^{(2)}(s) \tag{25a}
$$

$$
\mathcal{T}_d^{(2)}(z) = \frac{(\delta - s)^2}{2\delta} \times \mathcal{T}_c^{(2)}(s) \tag{25b}
$$

Proof: By substituting the discrete operators $(\mathcal{A}_d, \mathcal{B}_d, \mathcal{C}_d, \Xi_d)$ given by Equation (5) and $z = \frac{\delta+s}{\delta-s}$ into (24b), one can obtain

$$\begin{aligned} \mathcal{G}_d^{(2)}(z) &= \sqrt{2\delta}\mathcal{C}(\delta I - \mathcal{A})^{-1}[2\delta(\delta - s)^{-1} - 2\delta(\delta I - \mathcal{A})^{-1}]^{-2} \\ &\quad \sqrt{2\delta}(\delta I - \mathcal{A})^{-1}\mathcal{B} \\ &= \frac{(\delta - s)^2}{2\delta} \times \mathcal{C}(sI - \mathcal{A})^{-2}\mathcal{B} \\ &= \frac{(\delta - s)^2}{2\delta} \times \mathcal{G}_c^{(2)}(s) \end{aligned} \tag{26}$$

In the similar manner, the proof of Equation (25b) can be completed. ∎

To reveal the relationship between (Γ, Π) and (Γ_d, Π_d), we provide the following theorem.

Theorem 3.3: *Let Assumptions* 2.1–2.3 *hold. By Cayley–Tustin transform* (5) *and* (10), *the solutions of continuous- and discrete-time Sylvester equations are linked by*

(a) *For diagonalisable* $S = S_m, S_d = S_m^d$

$$\Gamma_d = \Gamma \tag{27a}$$

$$\Pi_d \phi_i^s = \frac{\delta + \lambda_i^s}{\sqrt{2\delta}} \Pi \phi_i^s = \frac{\sqrt{2\delta}\lambda_i^d}{\lambda_i^d + 1} \Pi \phi_i^s \tag{27b}$$

(b) *For non-diagonalisable* $S = S_n, S_d = S_n^d$

$$\Gamma_d = \Gamma \tag{27c}$$

$$\Pi_d \phi_1^s = \sqrt{\frac{\delta}{2}} \Pi \phi_1^s, \quad \Pi_d \phi_2^s = \sqrt{\frac{\delta}{2}} \Pi \phi_2^s + \frac{1}{\delta}\sqrt{\frac{\delta}{2}} \Pi \phi_1^s \tag{27d}$$

Proof: Under the stated assumptions, we have that Theorems 3.1 and 3.2 hold. We first consider diagonalisable S and S_d, namely $S = S_m$, $S_d = S_m^d$. Based on simple manipulations of discrete Sylvester equations (15) on the eigenpair (λ_i^d, ϕ_i^d) of S_d, the discrete regulator gains (Γ_d, Π_d) can be found as

$$\Pi_d \phi_i^d = \lambda_i^d (\lambda_i^d I - \mathcal{A}_d)^{-1} (\mathcal{B}_d \Gamma_d + P_d) \phi_i^d \tag{28a}$$

$$\Gamma_d \phi_i^d = [\mathcal{G}_d(\lambda_i^d)]^{-1} [Q_d - \mathcal{T}_d(\lambda_i^d) F_d] \phi_i^d \tag{28b}$$

where $\mathcal{G}_d(\lambda_i^d)$ and $\mathcal{T}_d(\lambda_i^d)$ are discrete-time transfer functions $\mathcal{G}_d(z)$ (from u_k to y_{ck}) and $\mathcal{T}_d(z)$ (from d_k to y_{ck}) with z evaluated at $z = \lambda_i^d$. Since $\lambda_i^d \in \rho(\mathcal{A}_d)\backslash\{-1\}$, $\mathcal{G}_d(\lambda_i^d)$ and $\mathcal{T}_d(\lambda_i^d)$ are always solvable.

Similarly, one can solve for the continuous regulator gains (Γ, Π) from continuous Sylvester equations (13) as below

$$\Pi \phi_i^s = (\lambda_i^s I - \mathcal{A})^{-1} (\mathcal{B}\Gamma + P) \phi_i^s \tag{29a}$$

$$\Gamma \phi_i^s = [\mathcal{G}_c(\lambda_i^s)]^{-1} [Q - \mathcal{T}_c(\lambda_i^s) F] \phi_i^s \tag{29b}$$

where $\mathcal{G}_c(\lambda_i^s)$ and $\mathcal{T}_c(\lambda_i^s)$ are continuous-time transfer functions $\mathcal{G}_c(s)$ (from $u(t)$ to $y_c(t)$) and $\mathcal{T}_c(s)$ (from $d(t)$ to $y_c(t)$) with s evaluated at $s = \lambda_i^s$. Since $\lambda_i^s \in \rho(\mathcal{A})\backslash\{\delta\}$, $\mathcal{G}_c(\lambda_i^s)$ and $\mathcal{T}_c(\lambda_i^s)$ are always solvable.

To proceed with the proof, one need to show the following relationships between the continuous- and discrete-time transfer functions:

$$\mathcal{G}_c(\lambda_i^s) = \mathcal{G}_d(\lambda_i^d), \quad \mathcal{T}_c(\lambda_i^s) = \mathcal{T}_d(\lambda_i^d) \quad \forall\, \lambda_i^s \in \sigma(S),\ \forall\, \lambda_i^d \in \sigma(S_d) \tag{30}$$

Under Assumptions 2.1–2.2 and Corollary 2.1, one can deduce that $\lambda_i^d = \frac{\delta+\lambda_i^s}{\delta-\lambda_i^s} \in \sigma(S_d) \subset \rho(\mathcal{A}_d)$ (and $\lambda_i^d \neq -1$ since $-1 \notin \sigma(S_d)$) which coincides with the bilinear mapping $z = \frac{\delta+s}{\delta-s}$ with $z = \lambda_i^d$ and $s = \lambda_i^s$. Combining Equation (7), we can infer Equation (30). Note that $F = F_d$ and $Q = Q_d$, so one can finally conclude that $\Gamma = \Gamma_d$.

With the relationship between λ_i^s and λ_i^d shown in Corollary 2.1, we establish the correspondence between Π and Π_d in Equation (29a) and Equation (28a) as follows:

$$\begin{aligned} \Pi_d \phi_i^d &= \lambda_i^d (\lambda_i^d I - \mathcal{A}_d)^{-1} (\mathcal{B}_d \Gamma_d + P_d) \phi_i^d \\ &= \lambda_i^d [2\delta(\delta - \lambda_i^s)^{-1} - 2\delta(\delta I - A)^{-1}]^{-1} \\ &\quad \sqrt{2\delta}(\delta I - A)^{-1} (\mathcal{B}\Gamma + P) \phi_i^d \\ &= \lambda_i^d \sqrt{2\delta}^{-1} (\delta - \lambda_i^s)(\lambda_i^s I - \mathcal{A})^{-1} (\mathcal{B}\Gamma + P) \phi_i^s \\ &\quad (\text{with } \phi_i^d = \phi_i^s) \\ &= \lambda_i^d \sqrt{2\delta}^{-1} (\delta - \lambda_i^s) \Pi \phi_i^s \\ &= \frac{\sqrt{2\delta}\lambda_i^d}{\lambda_i^d + 1} \Pi \phi_i^s \end{aligned} \tag{31}$$

Therefore, in the case that S and S_d are diagonalisable the solutions of Sylvester regulator equations in continuous- and discrete-time settings are related by Equation (27a)–(27b).

Then we consider non-diagonalisable S and S_d, i.e. $S = S_n$ and $S_d = S_n^d$. By recalling Assumption 2.1, we can show that there are two eigenvectors associated with the eigenvalue 0 of S, and two eigenvectors associated with the eigenvalue 1 of S_d. Considering the multiplicity of eigenvalues 0 and 1, there are a standard eigenvector and a generalised eigenvector associated with S and S_d respectively.

From Corollary 2.1, we have the following relationship of the first (standard) eigenvector in S and S_d:

$$S\phi_1^s = \lambda^s \phi_1^s, \quad S_d \phi_1^d = \lambda^d \phi_1^d \tag{32}$$

In this case, the solutions of continuous and discrete Sylvester regulator equations are related by Equation (27a)–(27b). The proof is the same as the previous one so it is omitted. More specifically, due to $\lambda^s = 0$ and $\lambda^d = 1$, we have the following for the first (standard) eigenpair:

$$\Gamma_d = \Gamma \tag{33a}$$

$$\Pi_d \phi_1^s = \sqrt{\frac{\delta}{2}} \Pi \phi_1^s \tag{33b}$$

Now, we need to fully consider the action of Π and Π_d on the generalised eigenvectors of S and S_d, namely ϕ_2^s and ϕ_2^d as follows:

$$\Pi_d S_d \phi_2^d = \mathcal{A}_d \Pi_d \phi_2^d + (\mathcal{B}_d \Gamma_d + P_d) S_d \phi_2^d \quad (34a)$$

$$Q_d S_d \phi_2^d = \mathcal{C}_d \Pi_d \phi_2^d + (\mathcal{D}_d \Gamma_d + \Theta_{cd}) S_d \phi_2^d \quad (34b)$$

By substituting $S_d \phi_2^d = \lambda^d \phi_2^d + \phi_1^d$ into Equation (34), a directly algebraic manipulation leads to

$$\begin{aligned} \Pi_d \phi_2^d &= (\lambda^d I - \mathcal{A}_d)^{-1} (\mathcal{B}_d \Gamma_d + P_d) \lambda^d \phi_2^d \\ &\quad - (\lambda^d I - \mathcal{A}_d)^{-1} \lambda_d^{-1} \mathcal{A}_d \Pi_d \phi_1^d \quad (35a) \\ \Gamma_d \phi_2^d &= [\mathcal{G}_d(\lambda^d)]^{-1} [Q_d - \mathcal{T}_d(\lambda^d) F_d] \phi_2^d \\ &\quad + [\mathcal{G}_d(\lambda^d)]^{-1} [\mathcal{G}_d^{(2)}(\lambda^d) \Gamma_d + \mathcal{T}_d^{(2)}(\lambda^d) F_d] \phi_1^d \quad (35b) \end{aligned}$$

where $\mathcal{G}_d^{(2)}(\lambda^d)$ and $\mathcal{T}_d^{(2)}(\lambda^d)$ represent the discrete-time 'second-order' transfer functions $\mathcal{G}_d^{(2)}(z)$ and $\mathcal{T}_d^{(2)}(z)$ with z evaluated at $z = \lambda^d$. Similarly, by inserting the generalised eigenvector ϕ_2^s (with $S\phi_2^s = \lambda^s \phi_2^s + \phi_1^s$) in the continuous-time Sylvester regulator equations (13), it is straightforward to attain

$$\begin{aligned} \Pi \phi_2^s &= (\lambda^s I - \mathcal{A})^{-1} (\mathcal{B}\Gamma + P) \phi_2^s - (\lambda^s I - \mathcal{A})^{-1} \Pi \phi_1^s \quad (36a) \\ \Gamma \phi_2^s &= [\mathcal{G}_c(\lambda^s)]^{-1} [Q - \mathcal{T}_c(\lambda^s) F] \phi_2^s \\ &\quad + [\mathcal{G}_c(\lambda^s)]^{-1} [\mathcal{G}_c^{(2)}(\lambda^s) \Gamma + \mathcal{T}_c^{(2)}(\lambda^s) F] \phi_1^s \quad (36b) \end{aligned}$$

where $\mathcal{G}_c^{(2)}(\lambda^s)$ and $\mathcal{T}_c^{(2)}(\lambda^s)$ denote continuous-time 'second-order' transfer functions $\mathcal{G}_c^{(2)}(s)$ and $\mathcal{T}_c^{(2)}(s)$ with s evaluated at $s = \lambda^s$. Applying the link (24) between discrete- and continuous-time 'second-order' transfer functions, one can readily conclude that $\Gamma_d = \Gamma$ (i.e. Equation (27c)) holds.

To show the relationship between Π_d and Π, we can substitute Equation (28a) with $\phi_i^d = \phi_1^d$ into Equation (35a) as follows:

$$\begin{aligned} \Pi_d \phi_2^d &= (\lambda^d I - \mathcal{A}_d)^{-1} (\mathcal{B}_d \Gamma_d + P_d) \lambda^d \phi_2^d \\ &\quad - (\lambda^d I - \mathcal{A}_d)^{-1} \mathcal{A}_d (\lambda^d I - \mathcal{A}_d)^{-1} (\mathcal{B}_d \Gamma_d + P_d) \phi_1^d \\ &= \lambda^d (\lambda^d I - \mathcal{A}_d)^{-1} (\mathcal{B}_d \Gamma_d + P_d) \phi_2^d \\ &\quad + (\lambda^d I - \mathcal{A}_d)^{-1} (\mathcal{B}_d \Gamma_d + P_d) \phi_1^d \\ &\quad - (\lambda^d I - \mathcal{A}_d)^{-1} \Pi_d \phi_1^d \quad (37) \end{aligned}$$

Based on the chain rule of the generalised eigenvectors ϕ_2^s and ϕ_2^d shown in Corollary 2.1, we have the following:

$$\phi_2^d = \frac{(\delta - \lambda^s)(\delta - S)}{2\delta} \phi_2^s = \frac{(\delta - \lambda^s)^2}{2\delta} \phi_2^s - \frac{\delta - \lambda^s}{2\delta} \phi_1^s \quad (38)$$

which can be further substituted in Equation (37). Through simple algebraic manipulation, one can rewrite Equation (37) as

$$\begin{aligned} \Pi_d \phi_2^s &= \sqrt{\frac{\delta}{2}} (-\mathcal{A})^{-1} (\mathcal{B}\Gamma_d + P) \phi_2^s \\ &\quad + \frac{2}{\delta} \sqrt{\frac{\delta}{2}} (-\mathcal{A})^{-1} (\mathcal{B}\Gamma_d + P) \phi_1^s \\ &\quad - \frac{1}{\delta} (-\mathcal{A})^{-1} (\delta I - \mathcal{A}) \Pi_d \phi_1^d \\ &= \sqrt{\frac{\delta}{2}} \Pi \phi_2^s + \sqrt{\frac{1}{2\delta}} \Pi \phi_1^s \quad (39) \end{aligned}$$

The last expression is induced by the use of Equation (36a). This completes the whole proof. ■

Remark 3.1: With the 1-1 correspondence (27), it can be seen that the solutions of discrete Sylvester equations (15) and their continuous counterparts (13) are linked via the Cayley–Tustin bilinear transform. Hence, one can solve for (Γ_d, Π_d) from discrete-time Sylvester equations to attain (Γ, Π) for continuous-time regulator design and vice versa.

Regarding the solvability of the regulator equations, the non-resonance conditions of finite-dimensional systems have been generalised for the continuous-time linear infinite-dimensional systems in Byrnes et al. (2000). Following that, we establish the non-resonance conditions for the discrete state feedback regulator equations (15).

Lemma 3.1: *Let Assumptions* 2.1–2.3 *hold. The regulator equations* (13) *are solvable for every choice of P and Q if and only if no eigenvalue of S is a transmission zero of continuous-time plant* (1), *i.e.* $\det \mathcal{G}_c(\lambda_i^s) \neq 0,\ \forall\ \lambda_i^s \in \sigma(S)$.

Proof: For the diagonalisable S having all eigenvalues on the imaginary axis (namely $S = S_m$), the proof is shown in Byrnes et al. (2000, Corollary V.1). For the case of non-diagonalisable S, i.e. $S = S_n$, we observe that Π is always solvable under stated assumptions and the solvability of Γ depends on the invertibility of $\mathcal{G}_c(\lambda^s)$ (indeed $\lambda^s = 0$ and for simplicity we drop the subscript i in λ_i^s in this case) as shown in Equation (36) that is same as the case of diagonalisable S but through more complicated manipulation. ■

In a similar manner, we can prove the non-resonance solvability criteria for discrete-time regulator equations (15) with a proper choice of the time discretisation interval.

Corollary 3.1: *Let Assumptions* 2.1–2.3 *hold. The regulator equations* (15) *are solvable for every choice of P_d and Q_d if and only if no eigenvalue of S_d is a transmission zero of discrete-time plant* (4), *i.e.* $\det \mathcal{G}_d(\lambda_i^d) \neq 0, \forall\ \lambda_i^d \in \sigma(S_d)$.

By combining Lemma 3.1 and Corollary 3.1, we show that the non-resonance conditions stay invariant under the Cayley–Tustin transformation.

Theorem 3.4: *Let Assumptions* 2.1–2.3 *hold. The non-resonance conditions in Lemma* 3.1 *and Corollary* 3.1 *are equivalent under* *the Cayley–Tustin bilinear transformation, and regulator equations* (15) *are solvable if and only if regulator equations* (13) *are solvable.*

Proof: Under Assumptions 2.1–2.2 and Corollary 2.1, we have $\lambda_i^d = \frac{\delta+\lambda_i^s}{\delta-\lambda_i^s} \in \sigma(S_d) \subset \rho(\mathcal{A}_d)$, $\lambda_i^s = \frac{\lambda_i^d-1}{\lambda_i^d+1}\delta \in \sigma(S) \subset \rho(\mathcal{A})$, where $\delta \notin \sigma(S)$ and $-1 \notin \sigma(S_d)$ with a proper choice of δ, which indicates $\mathcal{G}_c(\lambda_i^s) = \mathcal{G}_d(\lambda_i^d)$, for all $\lambda_i^s \in \sigma(S)$ and $\lambda_i^d \in \sigma(S_d)$ by using the 1-1 correspondence in continuous- and discrete-time transfer functions (7). Thus, we have $\det \mathcal{G}_c(\lambda_i^s) \neq 0$, $\forall\ \lambda_i^s \in \sigma(S)$ if and only if $\det \mathcal{G}_d(\lambda_i^d) \neq 0$, $\forall\ \lambda_i^d \in \sigma(S_d)$. The proof is completed by combining Lemma 3.1 and Corollary 3.1. ■

3.4 The stabilising feedback control gain

The 1-1 correspondence of exponential (strong) stability of the pairs $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{A}_d, \mathcal{B}_d)$ has been addressed by Curtain and Oostveen (1997). In this work, we will provide a novel way to determine the discrete stabilising feedback controller gain by finding its correspondence relationship with the associated continuous counterpart.

Lemma 3.2: *Given that $\mathcal{A}_c$ is an infinitesimal generator of the C_o-semigroup $\mathbb{T}_{\mathcal{A}_c}(t)$ on the Hilbert space, $\mathbb{T}_{\mathcal{A}_c}(t)$ is exponentially stable if and only if there exists a non-negative self-adjoint operator Q_c such that (Curtain & Zwart, 1995, The. 5.1.3):*

$$\mathcal{A}_c^* Q_c + Q_c \mathcal{A}_c + M_c = 0, \quad \text{on } \mathcal{D}(\mathcal{A}_c) \tag{40}$$

with $Q_c(\mathcal{D}(\mathcal{A}_c)) \subset \mathcal{D}(\mathcal{A}_c^*)$, where M_c is a positive definite design parameter.

With $\mathcal{A}_c = \mathcal{A} + \mathcal{B}K$, Lemma 3.2 is linked to the following theorem.

Theorem 3.5: *Let Assumption 2.3 hold. If there exists a non-negative self-adjoint operator Q_c that solves the following algebraic Riccati equation* (Xu & Dubljevic, 2016, The. 1):

$$\mathcal{A}^* Q_c + Q_c \mathcal{A} + M_c - 2Q_c \mathcal{B}\mathcal{B}^* Q_c = 0, \quad \text{on } \mathcal{D}(\mathcal{A}) \tag{41}$$

where M_c is a positive definite design parameter such that $Q_c(\mathcal{D}(\mathcal{A})) \in \mathcal{D}(\mathcal{A}^*)$ and the stabilising feedback control gain is $K = -\mathcal{B}^* Q_c$, then the closed-loop system is exponentially stable, i.e. $\mathcal{A} + \mathcal{B}K$ generates an exponentially stable C_0-semigroup. It can be shown that Equation (41) *is equivalent to Equation* (40) *by taking $K = -\mathcal{B}^* Q_c$ and $Q_c(\mathcal{D}(\mathcal{A})) \in \mathcal{D}(\mathcal{A}^*)$, see* Xu and Dubljevic (2016)*. Motivated by this, we aim at proposing a novel way to determine the discrete stabilising control gain K_d.*

By Curtain and Zwart (1995, Exe. 4.30), it can be shown that the operator Q_c that is the solution of the continuous-time Lyapunov equation (40) and Riccati equation (41) coincides with the solution Q_{cd} of the discrete-time Lyapunov equation:

$$\mathcal{A}_{cd}^* Q_{cd} \mathcal{A}_{cd} - Q_{cd} + M_{cd} = 0, \quad \text{on } \mathcal{X} \tag{42}$$

where M_{cd} is a positive definite design parameter such that $Q_{cd} \in \mathcal{L}(\mathcal{X})$.

To show that the discrete- and continuous-time Lyapunov equations share the same solution, we propose the following proposition.

Proposition 3.2: *Let Assumption 2.3 hold. Given that $M_c = \mathcal{C}^* N_c \mathcal{C}$, $M_{cd} = \mathcal{C}_{cd}^* N_c \mathcal{C}_{cd}$, $\mathcal{A}_{cd} = -I + 2\delta(\delta - \mathcal{A}_c)^{-1}$ and $\mathcal{C}_{cd} = \sqrt{2\delta}\mathcal{C}(\delta - \mathcal{A}_c)^{-1}$ by the Cayley–Tustin transform, the discrete Lyapunov equation* (42) *and its continuous version* (40) *share the same solution, i.e. $Q_c = Q_{cd}$, where N_c is a positive definite design parameter.*

Proof: Stemming from the continuous Lyapunov equation (40), we can demonstrate the following:

$$\begin{aligned}
&\mathcal{A}_c^* Q_c + Q_c \mathcal{A}_c + \mathcal{C}^* N_c \mathcal{C} = 0 \Leftrightarrow \\
&-2\delta(\delta I - \mathcal{A}_c)^* Q_c - 2\delta Q_c(\delta I - \mathcal{A}_c) \\
&\quad + 4\delta^2 Q_c + 2\delta \mathcal{C}^* N_c \mathcal{C} = 0 \Leftrightarrow \\
&-2\delta Q_c(\delta I - \mathcal{A}_c)^{-1} - 2\delta[(\delta I - \mathcal{A}_c)^{-1}]^* Q_c \\
&\quad + 4\delta^2[(\delta I - \mathcal{A}_c)^{-1}]^* Q_c(\delta I - \mathcal{A}_c)^{-1} \\
&\quad + 2\delta[(\delta I - \mathcal{A}_c)^{-1}]^* \mathcal{C}^* N_c \mathcal{C}(\delta I - \mathcal{A}_c)^{-1} = 0 \Leftrightarrow \\
&[-I + 2\delta(\delta I - \mathcal{A}_c)^{-1}]^* Q_c[-I + 2\delta(\delta I - \mathcal{A}_c)^{-1}] \\
&\quad - Q_c + \mathcal{C}_{cd}^* N_c \mathcal{C}_{cd} = 0 \Leftrightarrow \\
&\mathcal{A}_{cd}^* Q_c \mathcal{A}_{cd} - Q_c + \mathcal{C}_{cd}^* N_c \mathcal{C}_{cd} = 0
\end{aligned}$$

The last expression implies that $Q_c = Q_{cd}$. ■

Then, we further investigate the link between solutions of the discrete-time Lyapunov equation and Riccati equation by the following corollary.

Corollary 3.2: *Let Assumption 2.3 hold. If there exist a non-negative operator Q_{cd} that solves the following discrete-time algebraic Riccati equation*:

$$\begin{aligned}
&\mathcal{A}_d^* Q_{cd} \mathcal{A}_d - Q_{cd} + \mathcal{C}_d^* N_c \mathcal{C}_d \\
&\quad - K_d^* \left(2I + \mathcal{B}_d^* Q_{cd} \mathcal{B}_d + \mathcal{D}_d^* N_c \mathcal{D}_d\right) K_d = 0, \quad \text{on } \mathcal{X}
\end{aligned} \tag{43}$$

where N_c is a positive definite design parameter, then the strongly stabilising feedback control gain is $K_d = -(I + \mathcal{B}_d^* Q_{cd} \mathcal{B}_d + \mathcal{D}_d^* N_c \mathcal{D}_d)^{-1}(\mathcal{B}_d^* Q_{cd} \mathcal{A}_d + \mathcal{D}_d^* N_c \mathcal{C}_d)$.

As for the proof, one can take $\mathcal{A}_{cd} = \mathcal{A}_d + \mathcal{B}_d K_d$, $\mathcal{C}_{cd} = \mathcal{C}_d + \mathcal{D}_d K_d$ and $K_d = -(I + \mathcal{B}_d^* Q_{cd} \mathcal{B}_d + \mathcal{D}_d^* N_c \mathcal{D}_d)^{-1}(\mathcal{B}_d^* Q_{cd} \mathcal{A}_d + \mathcal{D}_d^* N_c \mathcal{C}_d)$ in Equation (42) which can be further simplified as Equation (43).

Remark 3.2: The continuous- and discrete-time stabilising feedback control gains are given by: $K = -\mathcal{B}^* Q_c$ and $K_d = -(I + \mathcal{B}_d^* Q_{cd} \mathcal{B}_d + \mathcal{D}_d^* N_c \mathcal{D}_d)^{-1}(\mathcal{B}_d^* Q_{cd} \mathcal{A}_d + \mathcal{D}_d^* N_c \mathcal{C}_d)$ where $Q_c = Q_{cd}$, so we can solve for discrete Q_{cd} and apply it for the construction of continuous K, and vice versa.

4. Error feedback regulation

In this section, after a brief review of the continuous-time error feedback regulator design, the discrete-time error feedback regulator design is proposed. More specifically, the discrete error feedback output regulator equations are constructed and proved.

4.1 Continuous-time error feedback regulator

In comparison to the discrete error feedback regulator design, the continuous counterpart is reviewed shortly. A continuous-time error feedback regulator design is achieved by finding a regulator taking the following form:

$$\dot{r}(t) = \mathcal{G}_1 r(t) + \mathcal{G}_2 e(t), \quad r(0) = r_0 \tag{44a}$$

$$u(t) = Hr(t) \tag{44b}$$

where $r(t) \in \Omega = \mathcal{X} \oplus \mathbb{C}^{n_q}$ for $t \in [0, +\infty)$, $\mathcal{G}_1 \in \mathcal{L}(\Omega)$, $\mathcal{G}_2 \in \mathcal{L}(Y, \Omega)$ and $H \in \mathcal{L}(\Omega, U)$, and only the error signal $e(t)$ is known in order to satisfy the following conditions:

(c3) The system

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}Hr(t) \tag{45a}$$

$$\dot{r}(t) = \mathcal{G}_2 \mathcal{C}x(t) + \mathcal{G}_1 r(t) \tag{45b}$$

is exponentially stable when $q \equiv 0$, which implies $[\mathcal{A} \;\; \mathcal{B}H; \mathcal{G}_2\mathcal{C} \;\; \mathcal{G}_1]$ is an infinitesimal generator of an exponentially stable C_0-semigroup.

(c4) The tracking error $e(t) \to 0$ as $t \to +\infty$ for any given $x_0 \in \mathcal{X}$, $r_0 \in \Omega$ and $q^0 \in \mathbb{C}^{n_q}$.

To solve the continuous-time error feedback regulator design problem, the following theorem is often utilised.

Theorem 4.1: *Let Assumptions 2.2–2.4 hold. The continuous-time error feedback regulation problem is solvable if and only if there exist mappings $\Pi \in \mathcal{L}(\mathbb{C}^{n_q}, \mathcal{X})$ with $\Pi D(S) \subset D(\mathcal{A})$ and $\Gamma \in \mathcal{L}(\mathbb{C}^{n_q}, U)$ such that the following Sylvester equations hold* (Byrnes et al., 2000, The. IV.2):

$$\Pi S = \mathcal{A}\Pi + \mathcal{B}\Gamma + P \tag{46a}$$

$$\mathcal{C}\Pi = Q \tag{46b}$$

where $P = \Xi F$, and $L = \Gamma - K\Pi$. With Π and Γ, the error feedback regulator is found by

$$\dot{r}(t) = \mathcal{G}_1 r(t) + \mathcal{G}_2 e(t) \tag{47a}$$

$$u(t) = Hr(t) \tag{47b}$$

where $r(t) \in \Omega = \mathcal{X} \oplus \mathbb{C}^{n_q}$ and

$$\mathcal{G}_1 = \begin{bmatrix} \mathcal{A} + \mathcal{B}K - G_1\mathcal{C} & P + \mathcal{B}(\Gamma - K\Pi) + G_1 Q \\ -G_2\mathcal{C} & S + G_2 Q \end{bmatrix}$$

$$\mathcal{G}_2 = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H = [K \quad \Gamma - K\Pi]$$

Here, $K \in \mathcal{L}(\mathcal{X}, U)$, $G_1 \in \mathcal{L}(Y, \mathcal{X})$ and $G_2 \in \mathcal{L}(Y, \mathbb{C}^{n_q})$ such that $K \in \mathcal{L}(\mathcal{X}, U)$ is an exponentially stabilising feedback gain for the pair $(\mathcal{A}, \mathcal{B})$ and $\mathcal{G}_2 = [\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}]$ is an exponentially stabilising output injection gain for the pair $([\begin{smallmatrix} \mathcal{A} & P \\ 0 & S \end{smallmatrix}], [\mathcal{C} \quad -Q])$.

4.2 Discrete-time error feedback regulator

Discrete-time error feedback regulator design: Find a regulator having the following form:

$$r_k = \mathcal{G}_{1d} r_{k-1} + \mathcal{G}_{2d} e_{k-1}, \quad k \geq 1 \tag{48a}$$

$$u_k = H_d r_k \tag{48b}$$

where $r_k \in \Omega = \mathcal{X} \oplus \mathbb{C}^{n_q}$, Ω is a Hilbert space, $\mathcal{G}_{1d} \in \mathcal{L}(\Omega)$, $\mathcal{G}_{2d} \in \mathcal{L}(Y, \Omega)$ and $H_d \in \mathcal{L}(\Omega, U)$, where only the error signal e_k is available, such that the following conditions hold:

(C3) The system

$$x_k = \mathcal{A}_d x_{k-1} + \mathcal{B}_d H_d r_k \tag{49a}$$

$$r_{k+1} = \mathcal{G}_{2d}\mathcal{C}_d x_{k-1} + (\mathcal{G}_{1d} + \mathcal{G}_{2d}\mathcal{D}_d H_d) r_k \tag{49b}$$

is strongly stable when $q_k \equiv 0$, which means $[\mathcal{A}_d \;\; \mathcal{B}_d H_d; \mathcal{G}_{2d}\mathcal{C}_d \;\; (\mathcal{G}_{1d} + \mathcal{G}_{2d}\mathcal{D}_d H_d)]$ is a strongly stable operator.

(C4) The tracking error $e_k \to 0$ as $k \to +\infty$ for any given $x_0 \in \mathcal{X}$, $r_0 \in \Omega$ and $q^0 \in \mathbb{C}^{n_q}$.

To address the discrete-time error feedback regulation problem, we propose the following theorem.

Theorem 4.2: *Under Assumptions 2.2–2.4, the discrete error feedback regulation problem is solvable if and only if there exist mappings $\Pi_d \in \mathcal{L}(\mathbb{C}^{n_q}, \mathcal{X})$ and $\Gamma_d \in \mathcal{L}(\mathbb{C}^{n_q}, U)$ such that the following discrete Sylvester equations hold*:

$$\Pi_d S_d = \mathcal{A}_d \Pi_d + (\mathcal{B}_d \Gamma_d + P_d) S_d \tag{50a}$$

$$Q_d S_d = \mathcal{C}_d \Pi_d + (\mathcal{D}_d \Gamma_d + \Theta_{cd}) S_d \tag{50b}$$

where $P_d = \Xi_d F_d$, $\Theta_{cd} = \Upsilon_d F_d$, and the discrete error feedback control law u_k can be computed as follows:

$$r_k = \mathcal{G}_{1d} r_{k-1} + \mathcal{G}_{2d} e_{k-1}, \quad k \geq 1 \tag{51a}$$

$$u_k = H_d r_k \tag{51b}$$

where $r_k \in \Omega = \mathcal{X} \oplus \mathbb{C}^{n_q}$ and

$$\mathcal{G}_{1d} = \begin{bmatrix} \mathcal{A}_d + \mathcal{B}_d K_d - G_{1d}\iota & P_d + \mathcal{B}_d L_d - G_{1d}\nu \\ -G_{2d}\iota & S_d - G_{2d}\nu \end{bmatrix} \tag{52a}$$

$$\mathcal{G}_{2d} = \begin{bmatrix} G_{1d} \\ G_{2d} \end{bmatrix}, \quad H_d = [K_d \quad L_d] \tag{52b}$$

where $\iota = \mathcal{C}_d + \mathcal{D}_d K_d$, $\nu = \mathcal{D}_d L_d + \Theta_{cd} - Q_d$, $L_d = \Gamma_d - K_d \Pi_d S_d^{-1}$, $K_d \in \mathcal{L}(\mathcal{X}, U)$, $G_{1d} \in \mathcal{L}(Y, \mathcal{X})$ and $G_{2d} \in \mathcal{L}(Y, \mathbb{C}^{n_q})$, such that $\mathcal{A}_d + \mathcal{B}_d K_d$ is a strongly stable operator and $\mathcal{G}_{2d} = [\begin{smallmatrix} G_{1d} \\ G_{2d} \end{smallmatrix}]$ is a strongly stabilising output injection gain for the pair $([\begin{smallmatrix} \mathcal{A}_d & P_d \\ 0 & S_d \end{smallmatrix}], [\mathcal{C}_d \quad \Theta_{cd} - Q_d])$.

Proof: Let us prove sufficiency first. In the regulator Equation (51), one can take $r_k = [\begin{smallmatrix}\hat{x}_{k-1}\\ \hat{q}_k\end{smallmatrix}] \in \Omega$ as the estimated plant and exogenous states, which leads to

$$\begin{aligned}
\begin{bmatrix}\hat{x}_{k-1}\\ \hat{q}_k\end{bmatrix}
&= \mathcal{G}_{1d}\begin{bmatrix}\hat{x}_{k-2}\\ \hat{q}_{k-1}\end{bmatrix} + \mathcal{G}_{2d}e_{k-1}\\
&= \begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}\begin{bmatrix}\hat{x}_{k-2}\\ \hat{q}_{k-1}\end{bmatrix}\\
&\quad + \begin{bmatrix}G_{1d}\\ G_{2d}\end{bmatrix}[\mathcal{C}_d x_{k-2} + \mathcal{D}_d u_{k-1} + \Theta_{cd}q_{k-1} - Q_d q_{k-1}]\\
&= \begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}\begin{bmatrix}\hat{x}_{k-2}\\ \hat{q}_{k-1}\end{bmatrix} + \begin{bmatrix}G_{1d}\mathcal{D}_d\\ G_{2d}\mathcal{D}_d\end{bmatrix}u_{k-1}\\
&\quad + \begin{bmatrix}G_{1d}\mathcal{C}_d & G_{1d}(\Theta_{cd} - Q_d)\\ G_{2d}\mathcal{C}_d & G_{2d}(\Theta_{cd} - Q_d)\end{bmatrix}\begin{bmatrix}x_{k-2}\\ q_{k-1}\end{bmatrix}\\
&= \begin{bmatrix}G_{11} + G_{1d}\mathcal{D}_d K_d & G_{12} + G_{1d}\mathcal{D}_d L_d\\ G_{21} + G_{2d}\mathcal{D}_d K_d & G_{22} + G_{2d}\mathcal{D}_d L_d\end{bmatrix}\begin{bmatrix}\hat{x}_{k-2}\\ \hat{q}_{k-1}\end{bmatrix}\\
&\quad + \begin{bmatrix}G_{1d}\mathcal{C}_d & G_{1d}(\Theta_{cd} - Q_d)\\ G_{2d}\mathcal{C}_d & G_{2d}(\Theta_{cd} - Q_d)\end{bmatrix}\begin{bmatrix}x_{k-2}\\ q_{k-1}\end{bmatrix}
\end{aligned} \tag{53}$$

$$u_k = [K_d \quad L_d]\begin{bmatrix}\hat{x}_{k-1}\\ \hat{q}_k\end{bmatrix} \tag{54}$$

where $\mathcal{G}_{1d} = [\begin{smallmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{smallmatrix}]$. Then, we substitute u_k in Equation (54) back into the plant (4a) and exo-system (9a) as follows:

$$\begin{bmatrix}x_{k-1}\\ q_k\end{bmatrix} = \begin{bmatrix}\mathcal{A}_d & P_d\\ 0 & S_d\end{bmatrix}\begin{bmatrix}x_{k-2}\\ q_{k-1}\end{bmatrix} + \begin{bmatrix}\mathcal{B}_d K_d & \mathcal{B}_d L_d\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_{k-2}\\ \hat{q}_{k-1}\end{bmatrix} \tag{55}$$

By combining Equations (53) and (55) and introducing two estimation errors $e_{x_{k-1}} = x_{k-1} - \hat{x}_{k-1}$ and $e_{q_k} = q_k - \hat{q}_k$, one can attain:

$$\begin{aligned}
\begin{bmatrix}e_{x_{k-1}}\\ e_{q_k}\end{bmatrix} &= \begin{bmatrix}\mathcal{A}_d - G_{1d}\mathcal{C}_d & P_d - G_{1d}(\Theta_{cd} - Q_d)\\ -G_{2d}\mathcal{C}_d & S_d - G_{2d}(\Theta_{cd} - Q_d)\end{bmatrix}\begin{bmatrix}x_{k-2}\\ q_{k-1}\end{bmatrix}\\
&+ \begin{bmatrix}(\mathcal{B}_d - G_{1d}\mathcal{D}_d)K_d - G_{11} & (\mathcal{B}_d - G_{1d}\mathcal{D}_d)L_d - G_{12}\\ -G_{2d}\mathcal{D}_d K_d - G_{21} & -G_{2d}\mathcal{D}_d L_d - G_{22}\end{bmatrix}\\
&\quad \begin{bmatrix}\hat{x}_{k-2}\\ \hat{q}_{k-1}\end{bmatrix}
\end{aligned} \tag{56}$$

Through direct calculation, one can have the following homogeneous difference equation for describing the error evolution dynamics:

$$\begin{bmatrix}e_{x_{k-1}}\\ e_{q_k}\end{bmatrix} = \begin{bmatrix}\mathcal{A}_d - G_{1d}\mathcal{C}_d & P_d - G_{1d}(\Theta_{cd} - Q_d)\\ -G_{2d}\mathcal{C}_d & S_d - G_{2d}(\Theta_{cd} - Q_d)\end{bmatrix}\begin{bmatrix}e_{x_{k-2}}\\ e_{q_{k-1}}\end{bmatrix} \tag{57}$$

With Theorem 2.1 and Assumption 2.4, one can readily conclude that $[\begin{smallmatrix}e_{x_{k-1}}\\ e_{q_k}\end{smallmatrix}]$ converges to zero as $k \to +\infty$, and obtain the following by combining Equations (56) and (57):

$$\begin{aligned}
\mathcal{G}_{1d} &= \begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}\\
&= \begin{bmatrix}\mathcal{A}_d + \mathcal{B}_d K_d - G_{1d}\iota & P_d + \mathcal{B}_d L_d - G_{1d}\nu\\ -G_{2d}\iota & S_d - G_{2d}\nu\end{bmatrix}
\end{aligned} \tag{58}$$

where $\iota = \mathcal{C}_d + \mathcal{D}_d K_d$, $\nu = \mathcal{D}_d L_d + \Theta_{cd} - Q_d$. Finally, from Equation (54) we have $u_k \to K_d x_{k-1} + L_d q_k$ since $\hat{x}_{k-1} \to x_{k-1}$ and $\hat{q}_k \to q_k$ as $k \to +\infty$. Based on the proof of Theorem 3.2, we can conclude that the tracking error $e_k \to 0$ as $k \to +\infty$ and the error feedback regulation problem is solvable given that Sylvester equations (50) hold.

Now we prove the necessity. Similarly, let us consider $r_k = [\begin{smallmatrix}\hat{x}_{k-1}\\ \hat{q}_k\end{smallmatrix}] \in \Omega$, $e_{x_{k-1}} = x_{k-1} - \hat{x}_{k-1}$ and $e_{q_k} = q_k - \hat{q}_k$. Substituting Equations (51) and (52) into the extended system (55) leads to Equation (57). Then one can plug the expression of u_k in Equation (51b) into the plant system (4), and induce that

$$\begin{aligned}
x_k &= \mathcal{A}_d x_{k-1} + \mathcal{B}_d(K_d\hat{x}_{k-1} + L_d\hat{q}_k) + P_d q_k\\
&= (\mathcal{A}_d + \mathcal{B}_d K_d)x_{k-1} - \mathcal{B}_d K_d e_{x_{k-1}}\\
&\quad + (\mathcal{B}_d L_d + P_d)q_k - \mathcal{B}_d L_d e_{q_k}
\end{aligned} \tag{59}$$

By combining Equations (57) and (59), we denote $\Phi_{k-1} = [x_{k-1}; e_{x_{k-1}}; e_{q_k}]$ and obtain

$$\Phi_k = \mathcal{A}_1\Phi_{k-1} + P_1 q_k + \mathcal{B}_1 u_k \tag{60a}$$

$$q_k = S_d q_{k-1} \tag{60b}$$

$$e_k = \mathcal{C}_1\Phi_{k-1} + \mathcal{D}_d u_k + (\Theta_{cd} - Q_d)q_k \tag{60c}$$

where $\mathcal{B}_1 = [0; 0; 0]$, $\mathcal{C}_1 = [\mathcal{C}_d \ \ 0 \ \ 0]$, $P_1 = [\mathcal{B}_d L_d + P_d; 0; 0]$ and

$$\mathcal{A}_1 = \begin{bmatrix}\mathcal{A}_d + \mathcal{B}_d K_d & -\mathcal{B}_d K_d & -\mathcal{B}_d L_d\\ 0 & \mathcal{A}_d - G_{1d}\mathcal{C}_d & P_d - G_{1d}(\Theta_{cd} - Q_d)\\ 0 & -G_{2d}\mathcal{C}_d & S_d - G_{2d}(\Theta_{cd} - Q_d)\end{bmatrix}$$

Along this line, we can define an extended mapping $\Pi_1 = [\Pi_d; 0; 0] : \mathbb{C}^{n_q} \mapsto \mathcal{X} \oplus Y \oplus \mathbb{C}^{n_q}$ and then apply Theorem 3.2 to design a state feedback regulator for the system (60) as: $u_k = K_1\Phi_{k-1} + (\Gamma_1 - K_1\Pi_1 S_d^{-1})q_k$ where $K_1 = [K_d \ \ 0 \ \ 0]$ and $\Gamma_1 = \Gamma_d$. Then the following Sylvester equations are obtained for solving the corresponding operators Π_1 and $\Gamma_1 \in \mathcal{L}(\mathbb{C}^{n_q}, Y)$:

$$\Pi_1 S_d = \mathcal{A}_1\Pi_1 + (\mathcal{B}_1\Gamma_1 + P_1)S_d \tag{61a}$$

$$Q_d S_d = \mathcal{C}_1\Pi_1 + (\mathcal{D}_d\Gamma_1 + \Theta_{cd})S_d \tag{61b}$$

Under Assumptions 2.2–2.4 and Theorem 2.1, it is apparent that $\mathcal{A}_1$ is strongly stable, and hence $\mathcal{A}_1 + \mathcal{B}_1 K_1$ is strongly stable due to $\mathcal{B}_1 = 0$. Taking the first components of Equations (61a) and (61b) into consideration, one can obtain

$$\Pi_d S_d = (\mathcal{A}_d + \mathcal{B}_d K_d)\Pi_d + (\mathcal{B}_d L_d + P_d)S_d \tag{62a}$$

$$Q_d S_d = \mathcal{C}_d\Pi_d + (\mathcal{D}_d\Gamma_d + \Theta_{cd})S_d \tag{62b}$$

which further indicates Equation (50) with $L_d = \Gamma_d - K_d\Pi_d S_d^{-1}$. ■

Corollary 4.1: *Let Assumptions* 2.1–2.4 *hold. With Cayley–Tustin transform* (5) *and* (11), *the corresponding relationships between* (Γ_d, Π_d) *and* (Γ, Π) *are established as below for the continuous- and discrete-time error feedback regulator designs.*

(a) *For diagonalisable* $S = S_m, S_d = S_m^d$

$$\Gamma_d = \frac{\sqrt{2\delta}}{\delta - \lambda_i^s}\Gamma \tag{63a}$$

$$\Pi_d \phi_i^s = \lambda_i^d \Pi \phi_i^s \tag{63b}$$

(b) *For non-diagonalisable* $S = S_n, S_d = S_n^d$

$$\Gamma_d \phi_1^s = \sqrt{\frac{2}{\delta}}\Gamma\phi_1^s, \quad \Gamma_d \phi_2^s = \sqrt{\frac{2}{\delta}}\Gamma\phi_2^s + \frac{1}{\delta}\sqrt{\frac{2}{\delta}}\Gamma\phi_1^s \tag{63c}$$

$$\Pi_d \phi_i^s = \Pi \phi_i^s \tag{63d}$$

Proof: Under stated assumptions, we have that Theorems 4.1 and 4.2 hold. The next proof is similar to Theorem 3.3, so it is omitted. ■

We note that under Assumptions 2.1–2.4, the solvability of the state feedback regulator problem is equivalent to that of the error feedback regulator problem in a continuous- or discrete-time setting. Along this line, we can establish the same non-resonance solvability criteria for continuous-time error feedback regulator equations (46) and discrete-time error feedback regulator equations (50) as in Lemma 3.1 and Corollary 3.1 by including Assumption 2.4. Thus we can further show that the solvability of discrete-time error feedback regulator equations is equivalent to that of the continuous analogues.

Corollary 4.2: *Under Assumptions* 2.1–2.4, *we have the following assertions:* (a) *regulator equations* (46) *are solvable if and only if no eigenvalue of S is a transmission zero of continuous-time plant* (1), *i.e.* $\det \mathcal{G}_c(\lambda_i^s) \neq 0, \forall\ \lambda_i^s \in \sigma(S)$; (b) *regulator equations* (50) *are solvable if and only if no eigenvalue of* S_d *is a transmission zero of discrete-time plant* (4), *i.e.* $\det \mathcal{G}_d(\lambda_i^d) \neq 0, \forall\ \lambda_i^d \in \sigma(S_d)$; (c) *the non-resonance conditions* (a) *and* (b) *are equivalent under the Cayley–Tustin bilinear transformation; and* (d) *regulator equations* (50) *are solvable if and only if regulator equations* (46) *are solvable.*

Proof: The proof is similar to Lemma 3.1, Corollary 3.1 and Theorem 3.4 under Assumptions 2.1–2.4, so we omit it. ■

4.3 Discrete stabilising output injection gain

In this section, we will provide a new way to solve for the discrete stabilising output injection gain $\mathcal{G}_{2d}$. First, let us revisit some main results on stabilisation of continuous- and discrete-time systems from Curtain and Zwart (1995) and Xu and Dubljevic (2016).

Lemma 4.1: *Given that* $\mathcal{A}_o$ *is an infinitesimal generator of the* C_o*-semigroup* $\mathbb{T}_{\mathcal{A}_o}(t)$ *on the Hilbert space,* $\mathbb{T}_{\mathcal{A}_o}(t)$ *is exponentially stable if and only if there exists a non-negative self-adjoint operator* Q_o *such that (Curtain & Zwart,* 1995*)*

$$\mathcal{A}_o Q_o + Q_o \mathcal{A}_o^* + M_o = 0, \quad on\ \mathcal{D}(\mathcal{A}_o^*) \tag{64}$$

with $Q_o(\mathcal{D}(\mathcal{A}_o^*)) \subset \mathcal{D}(\mathcal{A}_o)$*, where* M_o *is a positive definite design parameter.*

Given that $\mathcal{A}_o = [\begin{smallmatrix} \mathcal{A} & P \\ 0 & S \end{smallmatrix}] - \mathcal{G}_2[\mathcal{C} \quad -Q]$, we adopt the following theorem from Xu and Dubljevic (2016) to solve for $\mathcal{G}_2$.

Theorem 4.3: *Let Assumption* 2.4 *hold. If there exist non-negative self-adjoint operators* Θ_1 *and* Θ_2 *that solve the following algebraic Riccati equations* (Xu & Dubljevic, 2016, The. 5):

$$\mathcal{A}\Theta_1 + \Theta_1\mathcal{A}^* - 2\Theta_1\mathcal{C}^*\mathcal{C}\Theta_1 + M_{o1} = 0 \tag{65a}$$

$$P\Theta_2 + 2\Theta_1\mathcal{C}^*Q\Theta_2 + M_{o2} = 0 \tag{65b}$$

$$S\Theta_2 + \Theta_2 S^* - 2\Theta_2 Q^* Q\Theta_2 + M_{o3} = 0 \tag{65c}$$

where M_{o1} *and* M_{o3} *are positive definite operator and matrix, respectively, and* M_{o2} *is determined based on solutions of* Θ_1 *and* Θ_2 *to ensure that* $[\begin{smallmatrix} M_{o1} & M_{o2} \\ M_{o2}^* & M_{o3} \end{smallmatrix}]$ *is positive definite and* $\Theta_1(\mathcal{D}(\mathcal{A}^*)) \subset \mathcal{D}(\mathcal{A})$*, then* $\mathcal{G}_2 = [\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}] = [\begin{smallmatrix} \Theta_1\mathcal{C}^* \\ -\Theta_2 Q^* \end{smallmatrix}]$ *is an exponentially stabilising output injection gain.*

By Curtain and Zwart (1995, Exe. 4.30), it can be shown that the operator Q_o that solves the continuous Lyapunov equation (64) coincides with the solution Q_{od} (namely $[\Theta_1 \ \ 0; 0 \ \ \Theta_2] = [\Theta_{1d} \ \ 0; 0 \ \ \Theta_{2d}]$ as shown in Equation (67)) of the following discrete-time Lyapunov equation:

$$\mathcal{A}_{od} Q_{od} \mathcal{A}_{od}^* - Q_{od} + M_{od} = 0, \quad \text{on } \mathcal{X} \tag{66}$$

where M_{od} is a positive definite design parameter and $Q_{od} \in \mathcal{L}(\mathcal{X})$ is a non-negative self-adjoint operator. To prove that, one can take $M_o = \mathcal{B}N_o\mathcal{B}^*$ in Equation (64), $M_{od} = \mathcal{B}_{od}N_o\mathcal{B}_{od}^*$, $\mathcal{B}_{od} = \sqrt{2\delta}(\delta - \mathcal{A}_o)^{-1}\mathcal{B}$ and $\mathcal{A}_{od} = -I + 2\delta(\delta - \mathcal{A}_o)^{-1}$ by using the Cayley–Tustin transform, where N_o is a positive definite design parameter.

Given that $\mathcal{A}_{od} = [\begin{smallmatrix} \mathcal{A}_d & P_d \\ 0 & S_d \end{smallmatrix}] - \mathcal{G}_{2d}[\mathcal{C}_d \quad \Theta_{cd} - Q_d]$, we provide the following discrete Riccati equations for solving the discrete stabilising output injection gain $\mathcal{G}_{2d}$.

Corollary 4.3: *Let Assumption* 2.4 *hold. If there exist the non-negative operators* Θ_{1d} *and* Θ_{2d} *that solve the following discrete-time algebraic Riccati equations*

$$\mathcal{A}_d\Theta_{1d}\mathcal{A}_d^* - \Theta_{1d} + P_d\Theta_{2d}P_d^* - G_{1d}(2R_1 + \mathcal{C}_d\Theta_{1d}\mathcal{C}_d^* + \Omega_d\Theta_{2d}\Omega_d^*)G_{1d}^* + M_{d1} = 0 \tag{67a}$$

$$(P_d - G_{1d}\Omega_d)\Theta_{2d}(S_d - G_{2d}\Omega_d)^* - (\mathcal{A}_d - G_{1d}\mathcal{C}_d)\Theta_{1d}\mathcal{C}_d^*G_{2d}^* + M_{d2} = 0 \tag{67b}$$

$$S_d\Theta_{2d}S_d^* - \Theta_{2d} - G_{2d}(2R_2 + \mathcal{C}_d\Theta_{1d}\mathcal{C}_d^* + \Omega_d\Theta_{2d}\Omega_d^*)G_{2d}^* + M_{d3} = 0 \tag{67c}$$

where R_1, R_2, M_{d3} *and* M_{d1} *are positive definite matrices and operator, respectively, and* M_{d2} *is determined such that* $[\begin{smallmatrix} M_{d1} & M_{d2} \\ M_{d2}^* & M_{d3} \end{smallmatrix}]$

is positive definite, then

$$\mathcal{G}_{2d} = \begin{bmatrix} G_{1d} \\ G_{2d} \end{bmatrix} = \begin{bmatrix} (\mathcal{A}_d\Theta_{1d}\mathcal{C}_d^* + P_d\Theta_{2d}\Omega_d^*) \\ \times(R_1 + \mathcal{C}_d\Theta_{1d}\mathcal{C}_d^* + \Omega_d\Theta_{2d}\Omega_d^*)^{-1} \\ S_d\Theta_{2d}\Omega_d^*(R_2 + \mathcal{C}_d\Theta_{1d}\mathcal{C}_d^* + \Omega_d\Theta_{2d}\Omega_d^*)^{-1} \end{bmatrix}$$

is a strongly stabilising output injection gain, where $\Omega_d = \Theta_{cd} - Q_d$.

Proof: Given $Q_{od} = \text{bdiag}(\Theta_{1d}, \Theta_{2d})$, $M_{od} = [\begin{smallmatrix} M_{d1} & M_{d2} \\ M_{d2}^* & M_{d3} \end{smallmatrix}]$ and $\mathcal{A}_{od} = [\begin{smallmatrix} \mathcal{A}_d & P_d \\ 0 & S_d \end{smallmatrix}] - \mathcal{G}_{2d}[\mathcal{C}_d \ \ \Theta_{cd} - Q_d]$ with

$$\mathcal{G}_{2d} = \begin{bmatrix} G_{1d} \\ G_{2d} \end{bmatrix} = \begin{bmatrix} (\mathcal{A}_d\Theta_{1d}\mathcal{C}_d^* + P_d\Theta_{2d}\Omega_d^*) \\ \times(R_1 + \mathcal{C}_d\Theta_{1d}\mathcal{C}_d^* + \Omega_d\Theta_{2d}\Omega_d^*)^{-1} \\ S_d\Theta_{2d}\Omega_d^*(R_2 + \mathcal{C}_d\Theta_{1d}\mathcal{C}_d^* + \Omega_d\Theta_{2d}\Omega_d^*)^{-1} \end{bmatrix},$$

simple algebraic manipulation of Equation (66) leads to Equation (67). ■

Remark 4.1: To solve the algebraic Riccati equations (67), we provide the following steps:

(1) Initialise R_1, R_2, M_{d1} and M_{d3}.
(2) Solve for Θ_{1d} and Θ_{2d} in Equations (67a) and (67c) using numerical iteration methods (e.g. Newton-Kleinman iteration method Gibson & Rosen, 1988; Hewer, 1971; Kleinman, 1974).
(3) Find M_{d2} by substituting Θ_{1d} and Θ_{2d} into Equation (67b), and check the positive definiteness of $M_{od} = [\begin{smallmatrix} M_{d1} & M_{d2} \\ M_{d2}^* & M_{d3} \end{smallmatrix}]$ and Q_{od} =bdiag$(\Theta_{1d}, \Theta_{2d})$, if not return to repeat steps (1)–(2).

Remark 4.2: Given that $M_o = \mathcal{B}N_o\mathcal{B}^*$ and $M_{od} = \mathcal{B}_{od}N_o\mathcal{B}_{od}^*$, the continuous- and discrete-time stabilising output injection gains $\mathcal{G}_2$ and $\mathcal{G}_{2d}$ can be linked by Theorem 4.2 and Corollary 4.3. Thus one can solve for $\mathcal{G}_{2d}$ and then apply the result $(\Theta_{1d}, \Theta_{2d})$ for the construction of $\mathcal{G}_2$, and vice versa.

5. Simulation

To verify the effectiveness and applicability of the proposed discrete-time regulator design methods, we provide three examples including two state feedback regulator designs for a first-order hyperbolic PDE (non-spectral system) with considerations of harmonic and polynomial exogenous signals respectively, and an error feedback regulator design for a 1-D heat equation (spectral system) to realise set-point reference control.

5.1 Example 1: state feedback regulator design for a first-order hyperbolic PDE (non-spectral system) with consideration of harmonic reference and disturbance

Let us consider a tubular reactor system described by a first-order hyperbolic partial differential equation model as follows:

$$z_t(\zeta, t) = -vz_\zeta(\zeta, t) + \psi(\zeta)z(\zeta, t) + b(\zeta)u(t) + f(\zeta)d(t) \tag{68a}$$

$$z(0, t) = 0, \quad z(\zeta, 0) = z_0(\zeta) \tag{68b}$$

$$y_c(t) = C_c z(\zeta, t) \tag{68c}$$

where $\zeta \in [0, 1]$ and $t \in [0, +\infty)$ stand for spatial and temporal coordinates, respectively. We consider bounded input and disturbance operators as $b(\zeta) = 1, f(\zeta) = 0.5$ with model parameters $v = 2$ and $\psi(\zeta) = \sinh(\zeta)$. A bounded output operator is considered as $C_c := \int_0^1 \frac{1}{2\epsilon_c}\mathbf{1}_{[\zeta_c-\epsilon_c, \zeta_c+\epsilon_c]}(\cdot)\, d\zeta$, where $\mathbf{1}_{[a,b]}(\zeta)$ denotes the spatial shaping function:

$$\mathbf{1}_{[a,b]}(\zeta) = \begin{cases} 1, & \zeta \in [a, b] \\ 0, & \text{otherwise} \end{cases}$$

The system operator $\mathcal{A}$ is defined as: $\mathcal{A} := -v\frac{\partial}{\partial\zeta} + \sinh(\zeta)$ with the domain $\mathcal{D}(\mathcal{A}) = \{\phi(\zeta) \in \mathcal{X} | \phi(\zeta)$ is absolutely continuous, $\frac{d\phi(\zeta)}{d\zeta} \in \mathcal{X}$ and $\phi(0) = 0\}$. In addition, $\zeta_c = 0.5, \epsilon_c = 0.01$, and $z_0(\zeta) = 2\sin(2\pi\zeta)$.

In this example, we focus on harmonic reference and disturbance signals, so the discrete-time exo-system is designed with the diagonalisable $S_d = S_m^d = [0.9969, 0.0784; -0.0784, 0.9969]$ (with the continuous counterpart $S = S_m = [0, 0.05\pi; -0.05\pi, 0]$), $q^0 = [0; 1]$, $Q_d = [2, 0]$, and $F_d = [0, 1]$. Along this line, the discrete disturbance and reference signals are generated as: $d_k = \cos(0.025k\pi)$ and $y_{rk} = 2\sin(0.025k\pi)$. Apparently, we have $\sigma(S) \subset \rho(\mathcal{A})$ ensuing that Assumption 2.2 holds. Then, the state feedback regulator is constructed using the procedures shown in Table 1.

Through discrete Sylvester equations (15), the discrete feedforward gain can be solved as $\Gamma_d = [7.6629, -0.3479]$ and Π_d can be obtained as spatial functions correspondingly. To ensure the stability of the closed-loop system, the state feedback stabilising gain is solved using Equation (41) as bellow

$$2\frac{dQ_c(\zeta)}{d\zeta} + \sinh(\zeta)Q_c(\zeta) + Q_c(\zeta)\sinh(\zeta) + M - 2Q_c(\zeta)^2 = 0, \quad Q_c(1) = 0 \tag{69}$$

where M is a design parameter which is chosen as 0.001 in this example. It is straightforward to solve this ODE by the finite difference method, from which we obtain the feedback control law $u_k = K_d x_{k-1} + L_d q_k$ by performing Steps 3–4.

Table 1. Construction of the discrete state feedback regulator.

Algorithm 1: Discrete state feedback regulator.
Step 1: Solve discrete Sylvester equations (15) for Γ_d and Π_d
Step 2: Solve for continuous stabilising feedback gain Q_c in Equation (41), and then obtain K_d to stabilise the discrete operator $\mathcal{A}_d + \mathcal{B}_d K_d$
Step 3: Determine $L_d = \Gamma_d - K_d\Pi_d S_d^{-1}$ and simulate the exo-system and plant
Step 4: Construct the discrete state feedback regulation law (14) and apply to the discrete system (4)

Table 2. Comparison of regulation performance with different time discretisation intervals.

Δt (s)	0.375	0.475	0.50	0.525	0.625
Absolute tracking error $\lvert e_k \rvert$	0.0016	0.0009	0.0070	0.0022	0.0100

After 80 s of simulation, the regulation results are illustrated in Figure 2. Specifically, the simulated time and spatial intervals are set as 0.5 s and 0.01. With the control action applied, the closed-loop state profile can follow the sinusoidal reference trend and reject undesired cosine disturbance as well. As shown in Figure 2(b), it can be seen that controlled output y_c tracks the desired sinusoidal reference rapidly and the tracking error converges to zero under the closed-loop control.

In addition, the influence of the choice of the sampling time on the output regulation performance is investigated below. With the same regulation objective and overall simulation time of 80 s, different simulated time intervals (with $\pm 5\%$ and $\pm 25\%$ based on the chosen 0.50 s) are implemented. As illustrated in Table 2, it is apparent that the tracking error increases approximately with the increase of discretisation time interval and overall the relative error stays within a reasonable range (around 0.01).

5.2 Example 2: state feedback regulator design for a first-order hyperbolic PDE (non-spectral system) with consideration of step-like and ramp-like reference and disturbance

In this example, we construct another state feedback regulator for the same first-order hyperbolic PDE model (68) with the same parameters considered in the first example. Differently to Example 1, we aim at tracking ramp-like and step-like references by considering a non-diagonal exo-system in this example.

In this case, we consider the continuous exo-system with the non-diagonalisable $S = S_n = [0, 0; 1, 0]$, and the discrete counterpart $S_d = S_n^d = [1, 0; 0.5, 1]$ with $\Delta t = 0.5$ and $q^0 = [1; 0]$. By designing $Q_d = [1, 1]$ for $0 \leq k \leq 60$ and $Q_d = [15, 0]$ for $61 \leq k \leq 160$, the output reference signal is generated as

$$y_{rk} = \begin{cases} 1 + 0.5k, & 0 \leq k \leq 60 \\ 15, & 61 \leq k \leq 160 \end{cases}$$

In addition, a constant disturbance $d_k = 0.5$ is considered in this example with $F_d = [0.5, 0]$. By revisiting discrete Sylvester equations (15), the discrete feedforward gain can be solved as $\Gamma_d = [3.5819, 3.8319]$ for $0 \leq k \leq 60$, $\Gamma_d = [57.2292, 0]$ for $61 \leq k \leq 160$. Using the same stabilising gain calculated in the first example, the feedback control law $u_k = K_d x_{k-1} + L_d q_k$ can be computed by implementing Steps 2–4 given in Table 1. The initial condition in this case is taken as: $z_0(\zeta) = 6\sin(3\pi\zeta)$.

After 80 s of simulation, the output regulation performance is illustrated in Figure 3. Using the constructed state feedback regulator, the output is steered to track the ramp-like and step-like reference signals and reject the undesired step disturbance simultaneously. In particular, the tracking error converges to zero rapidly as shown in Figure 3(b).

Table 3. Construction of the discrete error feedback regulator.

Algorithm 2: Discrete error feedback regulator.
Step 1: Solve discrete Sylvester equations (50) for Γ_d and Π_d
Step 2: Solve for discrete stabilising output injection gain Θ_{1d} and Θ_{2d} in Equation (67), and then obtain $\mathcal{G}_{2d}$
Step 3: Determine $\mathcal{G}_{1d}$ and simulate r_k system (51a)
Step 4: Construct the discrete error feedback control law (51b) and apply it to the discrete system (4)

5.3 Example 3: error feedback regulator design for a 1-D heat equation (spectral system) with set-point reference control

In this case, we consider a 1-D heated bar model described by a parabolic PDE with Newman boundary conditions as follows:

$$x_t(\zeta, t) = x_{\zeta\zeta}(\zeta, t) + b(\zeta)u(t) + f(\zeta)d(t) \tag{70a}$$

$$x_\zeta(0, t) = 0 = x_\zeta(1, t) \tag{70b}$$

$$y_c(t) = C_c x(\zeta, t) \tag{70c}$$

where $\zeta \in [0, 1]$ and $t \in [0, +\infty)$ represent spatial and temporal coordinates, respectively. In addition, we consider spatially distributed actuation and disturbance that are characterised by: $b(\zeta) = \frac{1}{2\varepsilon_b}\mathbf{1}_{[\zeta_b - \varepsilon_b, \zeta_b + \varepsilon_b]}(\zeta)$ and $f(\zeta) = 1$. The goal is to regulate the output y_c with $C_c := \int_0^1 \frac{1}{2\varepsilon_c}\mathbf{1}_{[\zeta_c - \epsilon_c, \zeta_c + \epsilon_c]}(\cdot)\,d\zeta$. More specifically, we consider $\zeta_b = 0.5$, $\varepsilon_b = 0.3$, $\zeta_c = 0.99$ and $\epsilon_c = 0.01$.

It is apparent that the original state evolution operator $\mathcal{A} := \frac{\partial^2}{\partial \zeta^2}$ has the eigenvalues $\lambda_n = -(n\pi)^2$, $n \in \mathbb{N}$, which violates the Assumption 2.2 since $0 \in \sigma(\mathcal{A})$. To address this issue, we first stabilise the system by introducing a stabilising gain K as $K\Phi = -\beta\langle\Phi, \mathbf{1}\rangle$, with $\beta > 0$ (Byrnes et al., 2000). For the stabilised system $\mathcal{A}_c := \frac{\partial^2}{\partial \zeta^2} + b(\zeta)K$ with $0 \notin \sigma(A)$, we have $\sigma(s) \subset \sigma(\mathcal{A}_c)$ so Assumption 2.2 is satisfied. The rest design of the discrete error feedback regulator follows the steps as shown in Table 3.

To track and reject step signals, we set $S = 0$ (hence $S_d = 1$), $F_d = 1$ and $Q_d = 3$. Based on the discrete output regulator equations (50), we obtain: $L_d = \Gamma_d = 8.7859$ and $H_d = [\mathbf{0}\;\; 8.7859]$ due to $K_d = 0$ and $\mathbf{0}$ is a zero vector with proper dimension. By solving Riccati equations (67), we can determine the discrete stabilising output injection gains $\mathcal{G}_{2d}$ and $\mathcal{G}_{1d}$, with $R_1 = R_2 = M_{3d} = 1$ and $M_{1d} = \mathbf{1}$ where $\mathbf{1}$ is an identity matrix with proper dimension. Then, one can simulate r_k system (51) to generate control trajectory u_k, which is plugged into the original discrete plant (4). As shown in Figure 1, the closed-loop output y_c follows the set-point and the tracking error converges to zero rapidly. In addition, the discretisation time interval is 0.2 s and the number of spatial nodes is 1001.

Remark 5.1: As proposed in this work, the discrete-in-time regulator design provides a novel way to directly realise digital regulator design in a late-lumping manner. Most importantly, intrinsic properties of linear continuous systems are fully preserved in the discrete-time systems by the use of Cayley–Tustin transform. Therefore, the discrete-in-time regulator

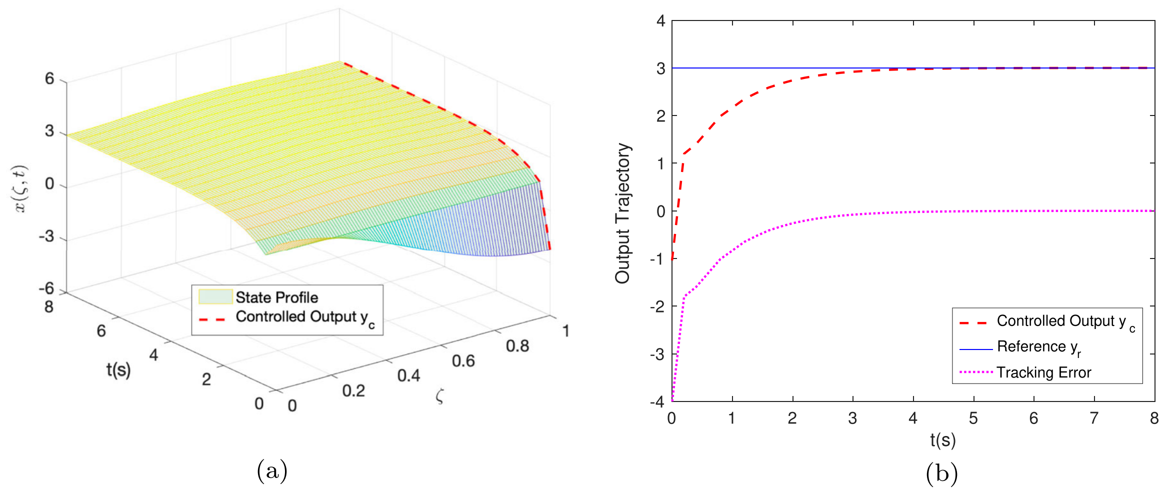

Figure 1. Output regulation of the heat equation. (a) State regulation performance. (b) Output regulation performance.

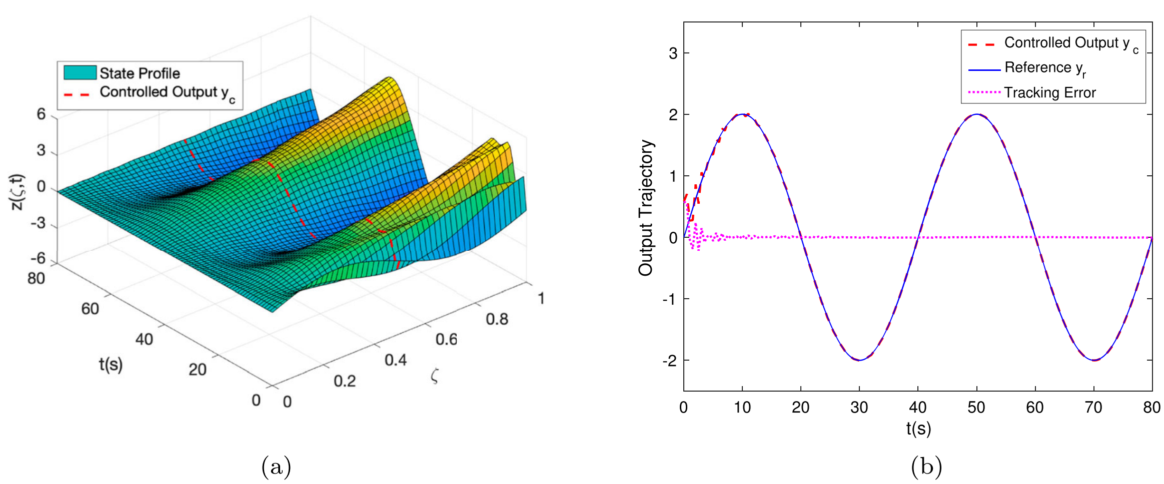

Figure 2. Output regulation of the transport equation with harmonic reference and disturbance signals. (a) State regulation performance. (b) Output regulation performance.

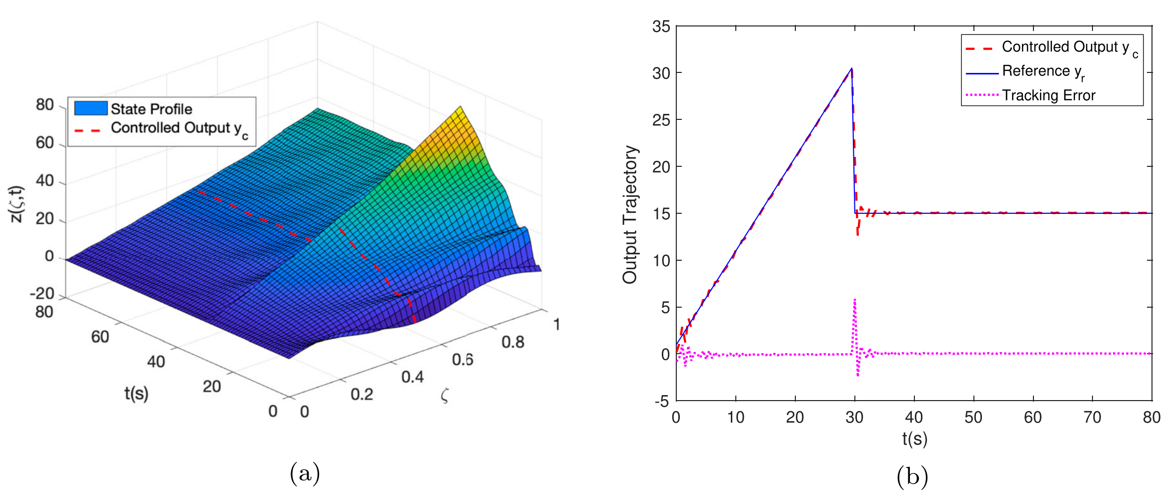

Figure 3. Output regulation of the transport equation with polynomial reference and disturbance signals. (a) State regulation performance. (b) Output regulation performance.

design method is beneficial to the corresponding design of continuous-time systems by using the equivalent relationships between continuous- and discrete-time systems established in this work.

Remark 5.2: Spatial discretisation may have a potential effect on the time-discretised stability in the numerical realisation stage. Particularly, dissipation and dispersion errors may be induced by an improper choice of spatial and temporal discretisation intervals, which may eventually influence time-discretised stability. To reduce the dissipation and dispersion errors, the spatial and temporal discretisation intervals need to be chosen small enough, see details in Thomas (2013, Cha. 7.2, 7.3).

6. Conclusion

In this work, discrete-time state and error feedback output regulators are designed for a class of linear distributed parameter systems with bounded input and output operators. The Cayley–Tustin bilinear transform is used for model discretisation without spatial approximation or model order reduction. Based on the discretised plant and exogenous systems, discrete-time Sylvester equations are formulated and solved for discrete state and error feedback regulator designs. The solvability of discrete-time regulator equations is proved and linked to the associated continuous version. Given the solutions of discrete-time Sylvester regulator equations, one can attain the corresponding continuous solutions and vice versa. To stabilise the closed-loop systems, a novel way to determine the stabilising output injection gain (and its dual problem) is provided. Finally, three output regulators are designed and simulated, including harmonic, step-like and ramp-like reference control for a first-order hyperbolic PDE system and set-point tracking for a 1-D heat equation, which verify the feasibility of the proposed method. The discrete-in-time design and continuous-time design can be beneficial to each other in determining the stabilising controller gain, stabilising output injection gain and regulator gain. Hence, the proposed discrete-in-time design has the potential to be utilised in digital control applications, such as sampled-data control of distributed parameter systems.

Disclosure statement

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