External Flows

Patricio F. Mendez
University of Alberta
ChE 314

Contents

Introduction

The objective of this chapter is to predict the convection coefficient h. Because convection is the combination of conduction and advection, it relates to the fluid motion, involving consideration of fluid mechanics.

In our predictions, we will cover:

1. Mechanisms: natural and forced convection
2. Behavior: laminar and turbulent
3. Geometries: external (flat plate, cylinder, sphere, bank of tubes) and internal flow (tubes, plate "sandwich," and cavities)
4. Orientations: regarding flow and gravity

1 Boundary Layers

Boundary layers are thin layers surrounding the surface of a body immersed in a fluid. In your Fluid Mechanics class you were exposed to the viscous boundary layer; however there are other types too. There is also a thermal boundary layer that will be the focus of this class topic, and diffusion boundary layers, which you will study if you take a mass transfer class. Typically, boundary layers are considered in steady-state, and we will do so in this class too.

The behavior of the fluid surrounding the body can be laminar or turbulent. In turbulent flows, the turbulence introduces local advection (at the microscopic level) that enhances the transport properties (thermal conductivity, viscosity, diffusivity) as we saw when we first discussed the Peclet number. Laminar fluids are the simplest to understand and we will start with them.

2 Laminar Thermal Boundary Layers

The thermal boundary layer is a thin layer of fluid over the surface of a body, through which the dominant heat transfer mechanism is conduction. A thermal boundary layer is an "insulation blanket" around a body. The coefficient of convection h captures the heat transfer by conduction through the thermal boundary layer.

Some distance away from the surface, the fluid must be in motion (advection). If the motion is induced (e.g. by a fan, or flow in a pipe), this is called "forced convection." If the motion of the fluid is only due to buoyancy, then it is called "natural convection."

We see now why we have stated earlier that convection is not a heat transfer mechanism in itself, but in reality is a combination of convection and advection.

2.1 Origins of the convection coefficient

Let's look in detail the temperature distribution of the fluid in the boundary layer, illustrated in Figure 1. In this figure, x is the coordinate over the surface of the body in the direction of the fluid flow, and y is the coordinate perpendicular to the surface, with an origin at the surface.

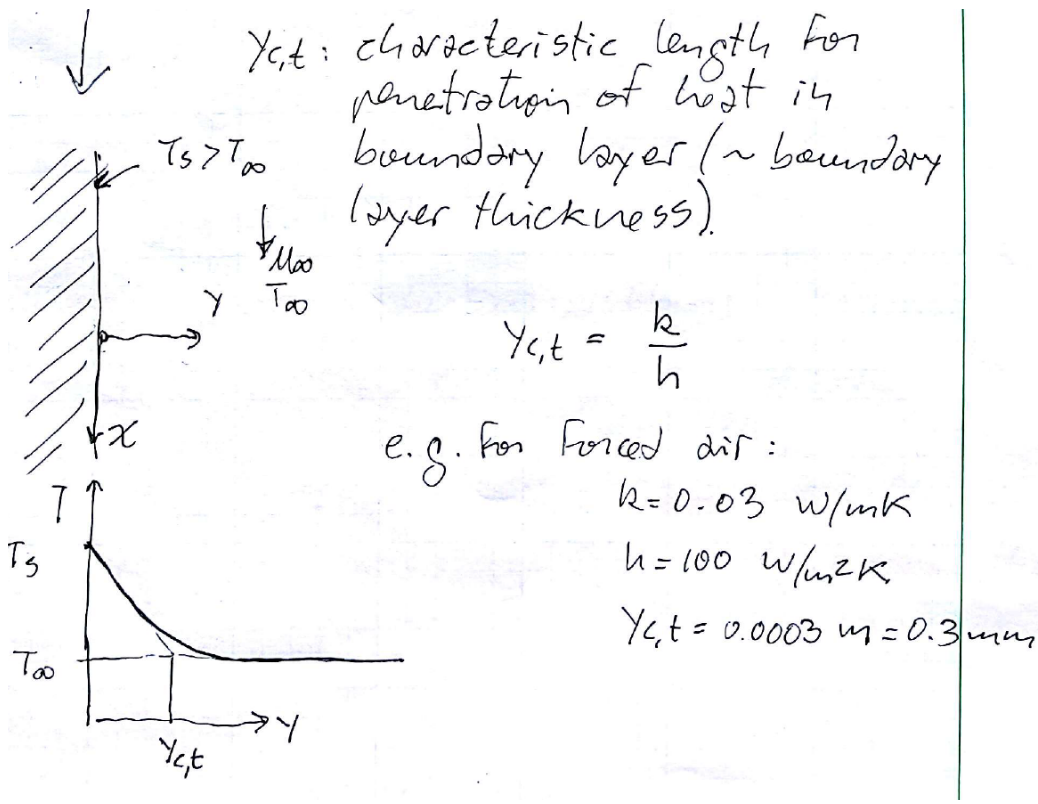


Figure 1: Schematic of the temperature distribution of a cooling case where $T_s > T_\infty$ and the characteristic length for heat penetration $y_{\mathrm{c,t}} = \frac{k}{h}$.

Heat exchanged via convection has the following expression:

$$q''_{\mathrm{conv}} = h(T_s - T_\infty) \tag{1}$$

but in reality, it is conduction through the "insulation blanket":

$$q''_{\mathrm{conv}} = q_{\mathrm{cond,BL}} = -k\frac{\partial T}{\partial x}\Big|_s = -k\frac{T_\infty - T_s}{y_{\mathrm{c,t}}} \tag{2}$$

where $y_{\mathrm{c,t}}$ is the characteristic length in the y direction for penetration of heat in boundary layer. In this class, we will define the thickness of the thermal boundary layer as $y_{\mathrm{c,t}}$, also called the "conduction thickness."

$$q''_{\mathrm{conv}} = h(T_s - T_\infty) = -k\frac{T_\infty - T_s}{y_{\mathrm{c,t}}} \tag{3}$$

$$y_{\mathrm{c,t}} = \frac{k}{h} \tag{4}$$

e.g. for forced air convection: $k = 0.03\frac{\mathrm{W}}{\mathrm{mK}}$ and $h = 100\frac{\mathrm{W}}{\mathrm{m^2K}}$, the thermal boundary layer thickness can be estimated as:

$$y_{\mathrm{c,t}} = \frac{k}{h} = \frac{0.03\frac{\cancel{\mathrm{W}}}{\cancel{\mathrm{m}}\cancel{\mathrm{K}}}}{100\frac{\cancel{\mathrm{W}}}{\mathrm{m}\cancel{^2}\cancel{\mathrm{K}}}} = 0.3\mathrm{mm}$$

The concept of boundary layer only makes sense when it is thin relative to the size of the body. If the boundary layer thickness were to be comparable or larger than the body, then the heat exchange would be dominated by conduction, and should e calculated using the 2D or 3D considerations we discussed before. The criterion of thin boundary layer is

$$y_{\mathrm{c,t}} \ll L_c \quad \text{or} \quad \frac{L_c}{y_{\mathrm{c,t}}} \gg 1 \tag{5}$$

where L_c is the characteristic length of body. Keeping in mind that $y_{\mathrm{c,t}} = \frac{k}{h}$:

$$\frac{L_c}{y_{\mathrm{c,t}}} = \frac{hL_c}{k} = \mathrm{Nu} \tag{6}$$

where Nu is a dimensionless group called "Nusselt number." Nu looks very much like Bi, but they have completely different meanings. In Nu, the characteristic length represents a length along the surface of a body, while for Bi it represents a length in the direction of heat penetration. Also, the thermal conductivity k for Nu corresponds to the fluid, while in Bi it corresponds to the body. We can state, then, that the concept of boundary layer is valid when the Nusselt number is much larger than 1.

As an example, consider the internal side of a 1 m tall window, $h \approx 10\frac{\mathrm{W}}{\mathrm{m^2K}}$, and for the thermal conductivity of air (k =0.03 W/m K), the Nusselt number is:

$$\mathrm{Nu} = \frac{L_c h}{k} = \frac{1\cancel{\mathrm{m}} \times 10\frac{\cancel{\mathrm{W}}}{\cancel{\mathrm{m^2K}}}}{0.03\frac{\mathrm{W}}{\cancel{\mathrm{mK}}}} = 333 \gg 1$$

confirming the boundary layer is thin.

3 Convection Coefficient for Laminar Flow

The thermal boundary layer involves fluid flow. In this class we will study flows against a surface, which involve a viscous boundary layer. We now have two boundary layers, thermal and viscous, which result in two extreme cases

1. Thermal boundary layer is much thicker than viscous boundary layer.
2. Thermal boundary layer is thinner or comparable to the viscous boundary layer.

For the first case, almost all the thermal boundary layer experiences the velocity of free flow. For the second case, the thermal boundary layer develops in the region where the flow is slowed down by the no-slip condition (flow velocity relative to the surface is zero in direct contact with the surface).

The calculation for each case is different. Let's start with the first case that is the simplest.

3.1 Thermal boundary layer much thicker than viscous boundary layer

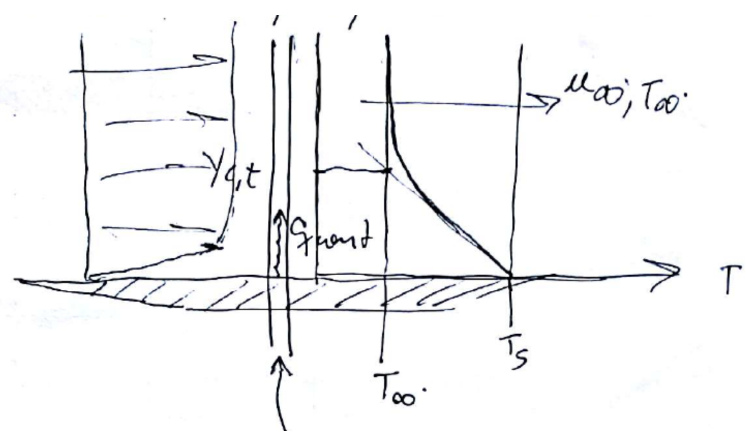


Figure 2: Schematic of the thermal boundary layer of a cooling case where $T_s > T_\infty$.

The boundary layer is thin in the y direction and large in the x direction. We have seen before that when the spect ratio is so large, conduction in the long direction (x) is negligible and only conduction in y matters. Because of the motion of the fluid, there is advection in the x direction. In this case we will build a control volume in Langrangian coordinates, which moves with the flow at velocity u_∞.

Because the control volume moves with the fluid, there is no advection in the energy balance, since no flow crosses the boundaries of the control volume. We are left then with only conduction in y for this control volume. Although the boundary layer is in steady-state, our Lagrangian control volume is not in steady-state, because it is moving with the flow and experiencing different temperatures as it moves.

If we consider that at time $t = 0$ the Lagrangian control volume is at $x = 0$, there is a relationship between position and time:

$$x = u_\infty t \tag{7}$$

As the control volume moves downstream, the governing equation will be

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{8}$$

which is the same equation we have seen in a semi-infinite solid, but applied to the fluid now. Initial temperature is T_∞, and the temperature far from the plate is also T_∞. The third boundary condition can vary, just as it happened for the semi-infinite solid.

Two common situations for the third boundary condition are constant plate temperature, and constant heat flux from the plate. We will consider the first case, of a plate a constant temperature, which is equivalent to the semi-infinite solid with constant surface temperature.

In this case, the penetration of heat into the control volume will be

$$y_{\mathrm{c,t}} = \sqrt{\pi}\sqrt{\alpha t} \tag{9}$$

which can incorporate the relationship between time and position of Equation 7, resulting in

$$y_{\mathrm{c,t}} = \sqrt{\pi}\sqrt{\frac{\alpha x}{u_\infty}} \tag{10}$$

and the following estimate of Nu:

$$\mathrm{Nu}_x = \frac{x}{y_{\mathrm{c,t}}} = \frac{1}{\sqrt{\pi}}\sqrt{\frac{u_\infty x}{\alpha}} \tag{11}$$

where $y_{\mathrm{c,t}}$ should always be much smaller than x for the concept of boundary layer to be applicable. This means the leading edge of the plate (when $x \to 0$) cannot be accurately captured by this analysis.

We can use Equation 4 to calculate the convection coefficient h

$$h = \frac{k}{y_{\mathrm{c,t}}} = \frac{k}{\sqrt{\pi}}\sqrt{\frac{u_\infty}{\alpha x}} \tag{12}$$

We see that the value of h is infinity at the location of $x = 0$. This is consistent with the infinite heat transfer rate at $t = 0$ in the semi-infinite solid.

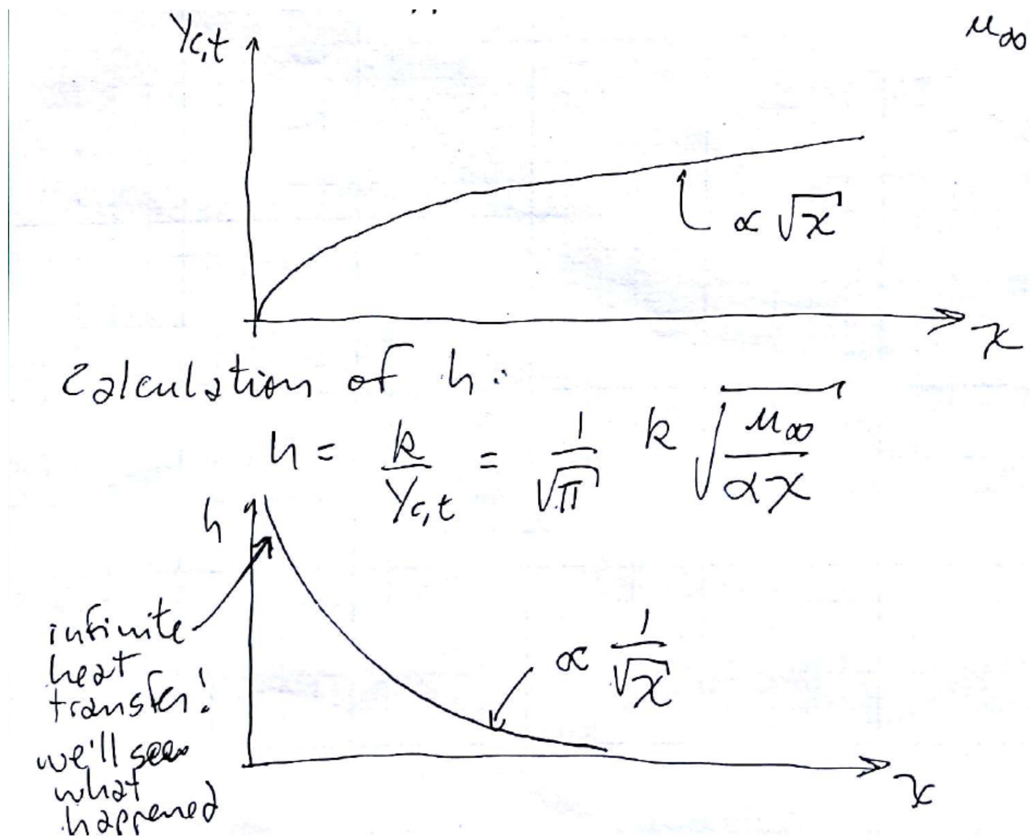


Figure 3: Schematic of the thermal boundary layer $y_{\mathrm{c,t}}$ and convection coefficient h as a function of x.

3.1.1 Tabulation based on Nu

Convection coefficients are typically tabulated in dimensionless form. The dimensionless number capturing the convection coefficient is the Nusselt number Nu we discussed earlier in Equation 13

$$\mathrm{Nu}_x = \frac{hx}{k} \tag{13}$$

where the subscript x indicates the characteristic length considered. The dependence of Nu is typically based on the Reynolds number Re

$$\mathrm{Re}_x = \frac{u_\infty x}{\alpha} \tag{14}$$

which relates the relative magnitude of inertial forces to viscous forces in the flow, and the Prandtl number Pr

$$\mathrm{Pr} = \frac{\nu}{\alpha} \tag{15}$$

which relates the fluids ability to transmit heat to its ability to transmit momentum through shear. Applying equations 13, 14, and 15 to 12 we obtain

$$\mathrm{Nu}_x = \frac{1}{\sqrt{\pi}}\mathrm{Re}^{1/2}\mathrm{Pr}^{1/2} = 0.564\mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/2} \tag{16}$$

which is the *exact* solution to the laminar thermal boundary layer when it is much thicker than the viscous boundary layer. It is remarkable we arrived to the exact solution without solving any differential equation. The traditional treatment of this problem involves solving a system of four linked non-linear partial differential equations.

3.1.2 Choice of fluid properties when they are not constant

If the thermal boundary layer (TBL) covers range from T_s to T_∞, what temperature to use for fluid properties in calculation?

- It will be said explicitly in the problem description (typically T_∞, and calculations involve a correction factor based on T_s).
- If not said, use the "Film Temperature":

$$T_F = \frac{T_s + T_\infty}{2} \tag{17}$$

3.1.3 Local and Average Convection Coefficients

When we calculate the heat transfer rate via convection using $q_{\mathrm{conv}} = Ah(T_s - T_\infty)$, an average h is used. However, we have seen that h varies with position and it is necessary to differentiate the local and average value:

Notation:

- $\bar{h}$: average convection coefficient
- h: local convection coefficient
- $\overline{\mathrm{Nu}} = \frac{\bar{h}L_c}{k}$: average Nusselt number

- $\mathrm{Nu} = \frac{hL_c}{k}$: local Nusselt number

For the thermal boundary layer studied:

$$h = \frac{k}{y_{\mathrm{c,t}}} = \frac{k}{\sqrt{\pi}}\sqrt{\frac{u_\infty}{\alpha x}} \tag{18}$$

$$\mathrm{Nu}_x = \frac{hx}{k} = \frac{x}{y_{\mathrm{c,t}}} = \frac{1}{\sqrt{\pi}}\mathrm{Pe}_x{}^{1/2} \tag{19}$$

calculation of the average convection coefficient over a length of L:

$$\bar{h}_{L_c} = \frac{1}{L_c}\int_0^{L_c} h dx = \frac{1}{L_c}\frac{k}{\sqrt{\pi}}\sqrt{\frac{u_\infty}{\alpha}}\int_0^{L_c}\frac{dx}{\sqrt{x}} \tag{20}$$

$$= \frac{1}{L_c}\frac{k}{\sqrt{\pi}}\sqrt{\frac{u_\infty}{\alpha}}\frac{x^{1/2}}{1/2}\Big|_0^{L_c} \tag{21}$$

$$\bar{h}_{L_c} = 2h_{L_c} \tag{22}$$

$$\overline{\mathrm{Nu}}_{L_c} = 2\mathrm{Nu}_{L_c} \tag{23}$$

An important conclusion from this analysis is that we should not confuse local and average values. Average values are used more often.

3.2 Laminar Viscous Boundary Layer

Fluid mechanics and heat transfer are very similar transport problems involving the diffusion of a quantity: thermal energy for heat transfer, and momentum for fluid flow.

For heat transfer the associated diffusivity is the thermal diffusivity $\alpha = k/(\rho c_p)$

For momentum transfer the associated diffusivity is the kinematic viscosity $\nu = \mu/\rho$, where the viscosity coefficient μ is typically independent of pressure but heavily dependent on temperature.

The ratio of the two diffusivities represent the ratio between the transport of heat and momentum. Both have units of m^2/s, resulting in a dimensionless number. Because of its importance, it is given a name: the Prandtl number:

$$\mathrm{Pr} = \frac{\nu}{\alpha} \tag{24}$$

Similarly, in mass transport the Schmidt number is defined similarly to Pr: $\mathrm{Sc} = \nu/D$.

The thickness of viscous boundary layer for incompressible flow was determined by Prandtl and his student PhD Blasius. The exact solution involves infinite series and is very tedious; so tedious that upon graduating Blasius decided to never return to research and became a teacher. From Blasius suffering, however, we can infer the following thickness for the viscous boundary layer (defined using the tangent at the surface, as we usually do in this class):

$$y_{\mathrm{c,v}} = 3\sqrt{\frac{\nu x}{u_\infty}} \tag{25}$$

$$\frac{x}{y_{\mathrm{c,v}}} = \frac{1}{3}\sqrt{\frac{u_\infty x}{\nu}} \approx 0.33{\mathrm{Re}_x}^{1/2} \tag{26}$$

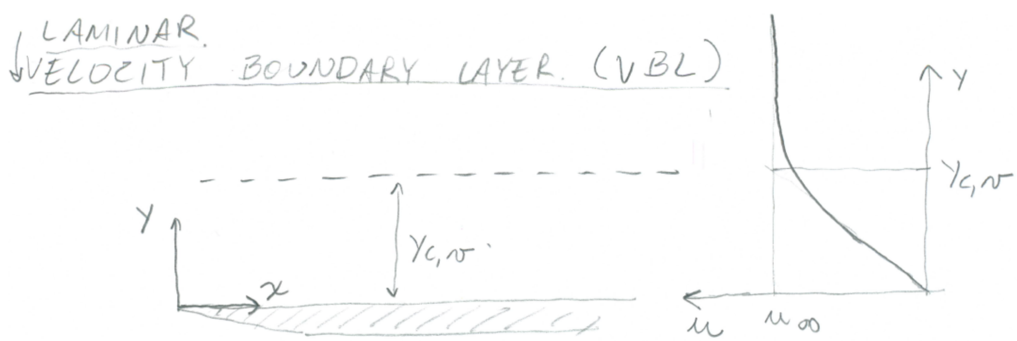


Figure 4: Schematic of laminar velocity boundary layer (VBL).

3.2.1 Definition of Boundary Layer Thickness

There are many different definitions, and all of them are of the same order of magnitude. We use $y_{\mathrm{c,t}}$ and $y_{\mathrm{c,v}}$ because:

$$\frac{\partial T}{\partial y} = \frac{T_\infty - T_{\mathrm{s}}}{y_{\mathrm{c,t}}}$$
$$\frac{\partial u}{\partial y} = \frac{u_\infty}{y_{\mathrm{c,v}}}$$

Most common definition of boundary layer in textbooks is δ such that $u(\delta) = 0.99u_\infty$ and the resulting boundary layer thickness $\delta = 5\sqrt{\frac{\nu x}{u_\infty}}$, as illustrated below:

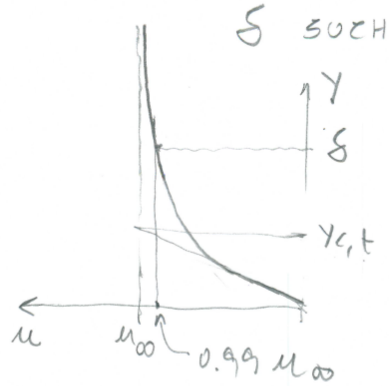


Figure 5: Schematic of the most common way of defining velocity boundary layer: $u(\delta) = 0.99u_\infty$.

Exactly same analysis applies to diffusion boundary layers replacing Pr by the Schmidt number Sc, which is typically $\gg 1$.

3.3 Termal boundary layer much thicker than viscous boundary layer

As derived before, for $y_{\mathrm{c,t}} \gg y_{\mathrm{c,v}}, T_s =$constant, and Pe $\gg 1$, the thickness of thermal boundary layer is:

$$y_{\mathrm{c,t}} = \sqrt{\pi}\sqrt{\frac{\alpha x}{u_\infty}} \approx 1.77\sqrt{\frac{\alpha x}{u_\infty}}$$
$$\frac{x}{y_{\mathrm{c,t}}} = \mathrm{Nu}_x = \frac{1}{\sqrt{\pi}}\mathrm{Re}_x{}^{1/2}\mathrm{Pr}^{1/2} \approx 0.564\mathrm{Re}_x{}^{1/2}\mathrm{Pr}^{1/2}$$

Now we can compare:

$$\frac{y_{\mathrm{c,t}}}{y_{\mathrm{c,v}}} = \frac{1.77}{3}\frac{\alpha}{\nu} \approx 0.6\mathrm{Pr}^{-1/2}$$

We can see that $y_{\mathrm{c,t}} \gg y_{\mathrm{c,v}}$ is valid only when $\mathrm{Pr} \ll 1$, which is the case of molten metals and plasmas. The exponent 1/2 in the Prandtl number is the telltale sign that the expression corresponds to low Pr, even in configurations different than a flat plate. Because this happens typically for molten metals, expressions with the 1/2 exponent in Pr are uncommon.

3.4 Thermal boundary layer comparable to the viscous boundary layer: Reynolds Analogy

For the case when $y_{\mathrm{c,t}} = y_{\mathrm{c,v}}$, both boundary layers are the same and have the same solution. This is called the Reynolds Analogy.

$$y_{\mathrm{c,t}} = 3\sqrt{\frac{\alpha x}{u_\infty}} \tag{27}$$

which is the exact solution to the problem, valid for Pe=1, which is close to the values for air and other common gases.

3.5 Thermal boundary layer much thinner than the viscous boundary layer

For the case when $y_{\mathrm{c,t}} \ll y_{\mathrm{c,v}}$, the Reynolds analogy still holds approximately, but the maximum velocity that the thermal boundary layer experiences is smaller than the velocity of free flow u_∞. We can do this analysis by comparing triangles ABC and ADE:

$$\frac{DE}{EA} = \frac{BC}{CA}$$
$$\frac{u_\infty}{y_{\mathrm{c,v}}} = \frac{u_c}{y_{\mathrm{c,t}}}$$
$$u_c = u_\infty \frac{y_{\mathrm{c,t}}}{y_{\mathrm{c,v}}}$$

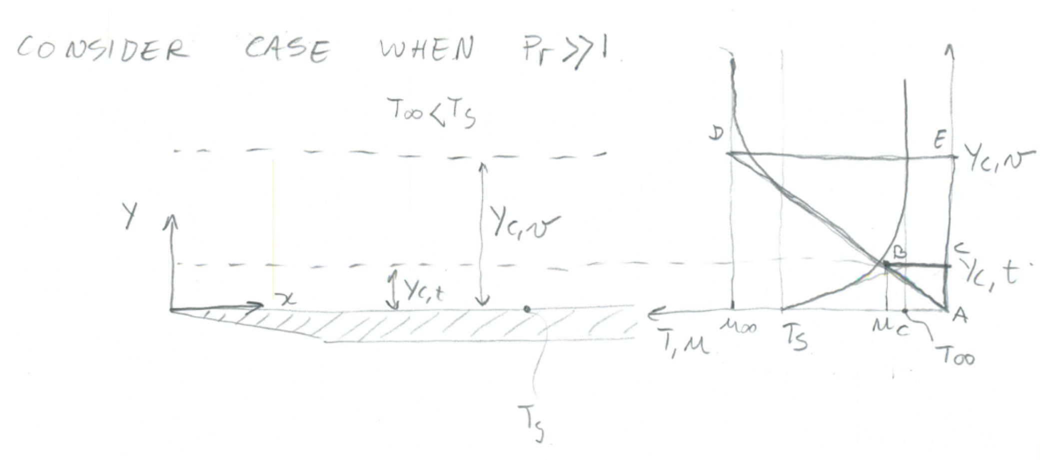


Figure 6: Schematic of velocity boundary layer and thermal boundary layer for $\mathrm{Pr} \gg 1$.

As $y_{\mathrm{c,v}} = 3\sqrt{\frac{\nu x}{u_\infty}}$ and $y_{\mathrm{c,t}} = 3\sqrt{\frac{\alpha x}{u_c}}$:

$$
\begin{aligned}
{y_{\mathrm{c,t}}}^2 &= 9\frac{\alpha x}{u_c} = 9\frac{\alpha x}{u_\infty}\frac{y_{\mathrm{c,v}}}{y_{\mathrm{c,t}}} \\
{y_{\mathrm{c,t}}}^3 &= 9\frac{\alpha x}{u_\infty} y_{\mathrm{c,v}} \\
\left(\frac{y_{\mathrm{c,t}}}{y_{\mathrm{c,v}}}\right)^3 &= 9\frac{\alpha x}{u_\infty y_{\mathrm{c,v}}^2} = \cancel{9}\frac{\alpha \cancel{x}}{\cancel{u_\infty}\cancel{9}}\frac{\cancel{u_\infty}}{\nu \cancel{x}} = \frac{1}{\mathrm{Pr}} \\
\frac{y_{\mathrm{c,t}}}{y_{\mathrm{c,v}}} &= \mathrm{Pr}^{-1/3} \text{ valid for } \mathrm{Pr} \gg 1 \text{ and } \mathrm{Pr} \approx 1
\end{aligned}
$$

calculation of h:

$$
h = \frac{k}{y_{\mathrm{c,t}}} = \frac{k}{y_{\mathrm{c,t}}}\frac{y_{\mathrm{c,v}}}{y_{\mathrm{c,v}}} = \frac{k}{y_{\mathrm{c,v}}}\frac{y_{\mathrm{c,v}}}{y_{\mathrm{c,t}}}
$$

combing $y_{\mathrm{c,v}} = 3\sqrt{\frac{\nu x}{u_\infty}}$ and $\frac{y_{\mathrm{c,t}}}{y_{\mathrm{c,v}}} = \mathrm{Pr}^{-1/3}$:

$$
h = \frac{k}{y_{\mathrm{c,t}}} = \frac{k}{3\sqrt{\frac{\nu x}{u_\infty}}}\mathrm{Pr}^{1/3}
$$

tabulating in the form of Nu_x:

$$\mathrm{Nu}_x = \frac{hx}{k} = \frac{1}{3}\sqrt{\frac{u_\infty x}{\nu}}\mathrm{Pr}^{1/3}$$
$$\approx 0.33\mathrm{Re}^{1/2}\mathrm{Pr}^{1/3}$$

which is the exact solution for Pr=1, and a close approximation for Pr$\gtrsim$1, which includes air, water and oil. The exponent 1/3 in the Prandtl number is the telltale that the expression corresponds to Pr$\gtrsim$1, even in configurations different than a flat plate. Because this happens typically for aqueaous liquids, gases, and oil, expressions with the 1/3 exponent in Pr are ubiquitous.

4 Turbulent Thermal Boundary Layers

Laminar flows are ordered and have clear streamlines while turbulent are chaotic, and microscopic motion enhances transport:

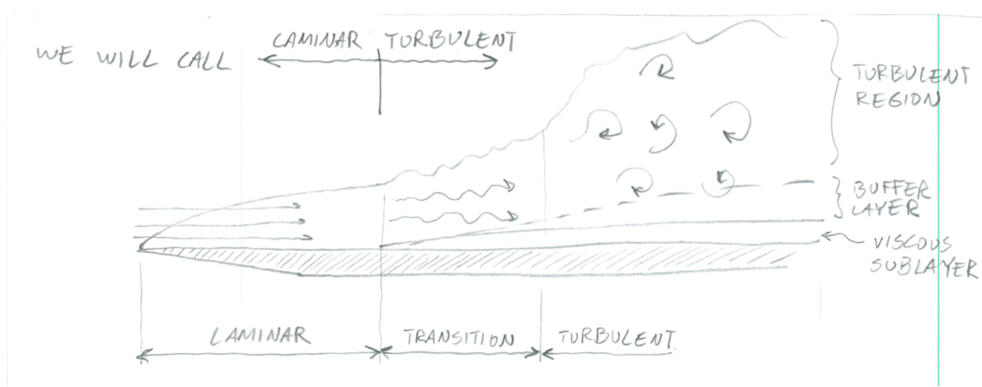


Figure 7: Schematic of the development of viscous boundary layers: laminar, transition region and turbulent

The onset of turbulence is when $\mathrm{Re}_x > \mathrm{Re}_{\mathrm{critical}}$.

The local Reynolds number is defined: $\mathrm{Re}_x = \frac{u_\infty x}{\nu}$ and $\mathrm{Re}_{\mathrm{critical}}$ depends on the geometry of the system and other factors. For a flat plate: $\mathrm{Re}_{\mathrm{critical}} = 5 \times 10^5$. (This can vary with other conditions.)

In the turbulent area, transport is dominated by advection from turbulent $\Rightarrow y_{\mathrm{c,t}} \approx y_{\mathrm{c,v}}, \delta_t \approx \delta$. The behavior of transport is much different in laminar and turbulent region.

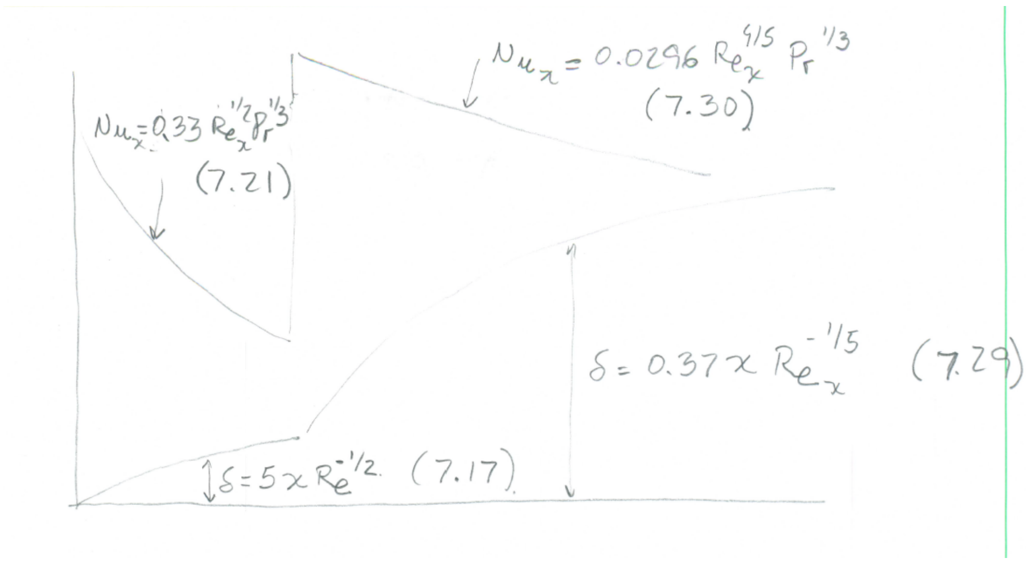


Figure 8: Schematic of the development of viscous boundary layers and variations on the local Nusselt number as a function of Re and Pr.

The average Nusselt number $\overline{\mathrm{Nu}}_L$ for combined laminar and turbulent boundary layers can be estimated as:

$$\overline{\mathrm{Nu}}_L = \left(0.037{\mathrm{Re}_L}^{4/5} - A\right) \mathrm{Pr}^{1/3} \quad \text{Eq. 7.31}$$
$$A = 0.037{\mathrm{Re}_{x,c}}^{4/5} - 0.664{\mathrm{Re}_{x,c}}^{1/2}$$

where $\mathrm{Re}_{x,c}$ is the critical Reynolds number, the same as the notation $\mathrm{Re}_{\mathrm{critical}}$.

5 External Flows

Considering common external flows in radial systems (e.g. cylinders or spheres):

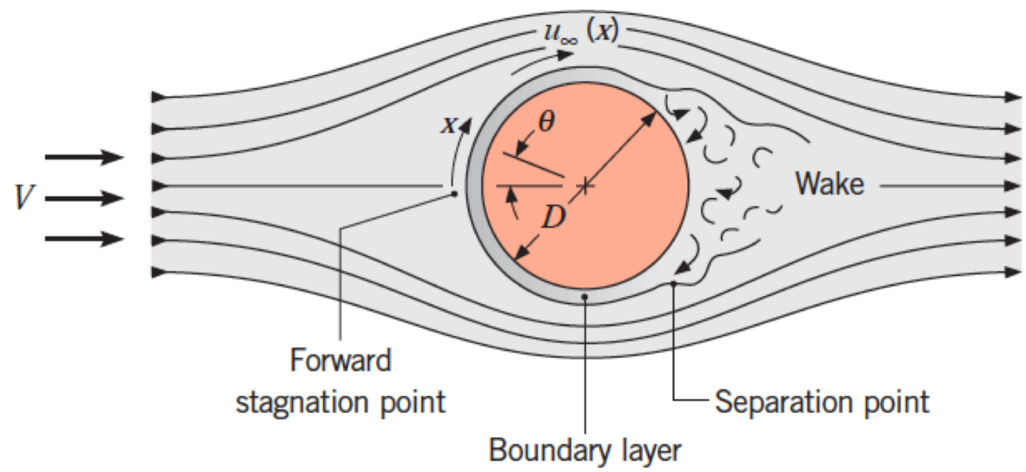


Figure 9: Schematic of boundary layer formation and separation on a circular cylinder in cross flow.

The occurrence of boundary layer transition is dependent on the Reynolds number defined as:

$$\mathrm{Re}_D = \frac{\rho V D}{\mu} = \frac{VD}{\nu}$$

Critical Re_D of the onset of the boundary layer transition: $\mathrm{Re}_D = 2 \times 10^5$

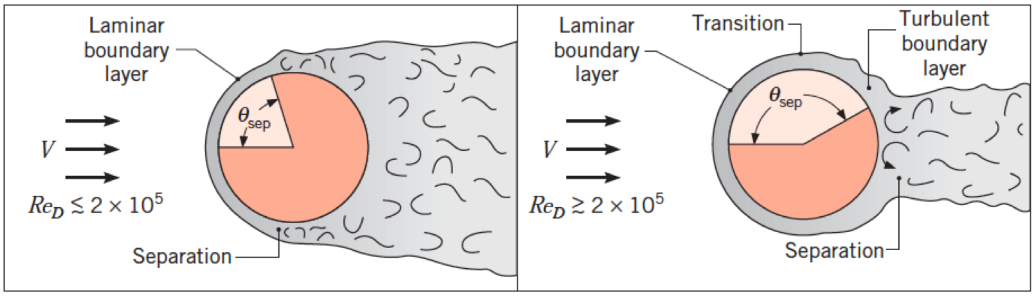


Figure 10: Schematic of the transition from laminar boundary layer to turbulent boundary layer and the effect of turbulence on separation.

The behavior of the external flows strongly influence the drag force acting on the cylinder F_D, which consists of two components: the boundary layer surface shear stress and the other part results from a pressure differential in the flow direction. In dimensionless form, drag coefficient C_D is defined as:

$$C_D = \frac{F_D}{A_f\,(\rho V^2/2)}$$

where A_f is the cylinder frontal area projected perpendicular to the velocity of the free stream. The influence of Re on C_D are presented below:

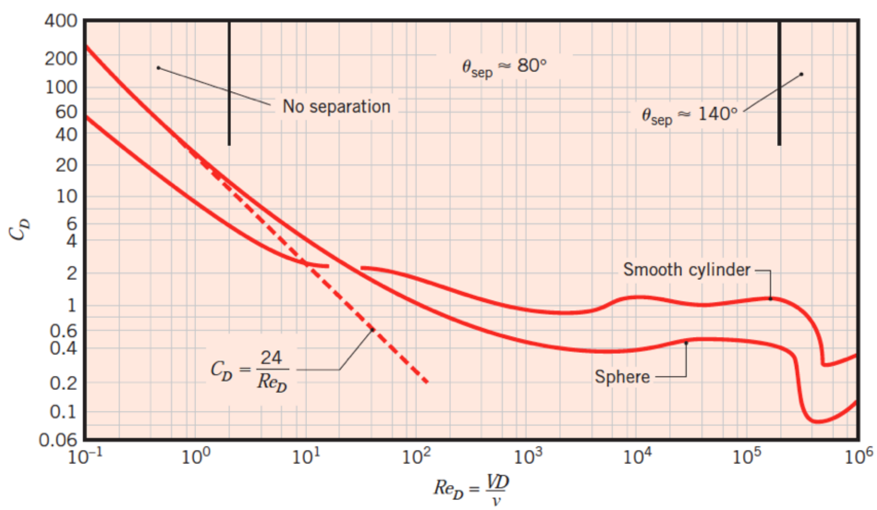


Figure 11: Drag coefficients as a function of Re for a smooth cylinder and a sphere, respectively.

5.1 Expression for the local Nusselt number

At the forward stagnation point ($\theta = 0$) for $\mathrm{Pr} \gtrsim 0.6$, the local Nusselt number has the following expression and is most accurate at low Reynolds number:

$$\mathrm{Nu}_D(\theta = 0) = 1.15 \mathrm{Re}_D{}^{1/2} \mathrm{Pr}^{1/3} \quad \text{Eq. (7.43)}$$

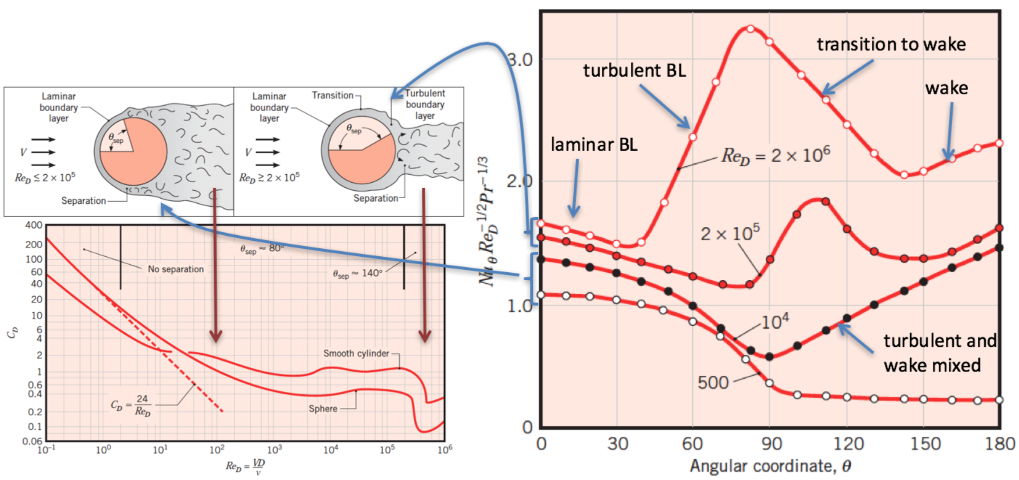


Figure 12: Local Nusselt number for airflow normal to a circular cylinder.

However, overall average conditions are more widely used in engineering calculation. An empirical correlation has been modified and is widely used for $\mathrm{Pr} \gtrsim 0.7$:

$$\overline{\mathrm{Nu}}_D = \frac{\bar{h}D}{k} = C \mathrm{Re}_D{}^m \mathrm{Pr}^{1/3} \quad \text{Eq. (7.44)}$$

where constants C and m can be checked from the below table for the circular cylinder in cross flow. (Table 7.2 in the textbook):

Re_D	C	m
0.4–4	0.989	0.330
4–40	0.911	0.385
40–4000	0.683	0.466
4000–40,000	0.193	0.618
40,000–400,000	0.027	0.805

Figure 13: Constants of Equation 7.44 for the circular cylinder in cross flow

Equation 7.44 can also be applied for external flows over cylinders of non-circular cross section with the characteristic length D and constants used in calculation:

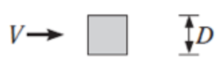
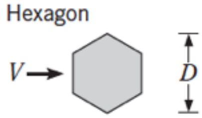
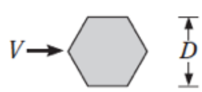
Geometry		Re_D	C	m
Square				
		6000–60,000	0.304	0.59
		5000–60,000	0.158	0.66
Hexagon				
		5200–20,400	0.164	0.638
		20,400–105,000	0.039	0.78
		4500–90,700	0.150	0.638
Thin plate perpendicular to flow				
	Front	10,000–50,000	0.667	0.500
	Back	7000–80,000	0.191	0.667

Figure 14: Constants of Equation 7.44 for the non-circular cylinders in cross flow of a gas

Note: in working with Equations 7.43 and 7.44, all properties are evaluated at the film temperature.

Another correlation of the average Nusselt number for the circular cylinder in cross form is valid for $0.7 \lesssim \text{Pr} \lesssim 500, 1 \lesssim \text{Re}_D \lesssim 10^6$ and has the following expression:

$$\overline{\text{Nu}}_D = \frac{\bar{h}D}{k} = C\text{Re}_D^m\text{Pr}^n\left(\frac{\text{Pr}}{\text{Pr}_s}\right)^{1/4} \quad \text{Eq. (7.45)}$$

where all properties are evaluated at T_∞, except Pr_s which is evaluated at T_s. Constants of n in Equation 7.45 is dependent on the value of Pr:

- If $\text{Pr} \lesssim 10, n = 0.37$
- If $\text{Pr} \gtrsim 10, n = 0.36$

Re_D	C	m
1–40	0.75	0.4
40–1000	0.51	0.5
10^3–2×10^5	0.26	0.6
2×10^5–10^6	0.076	0.7

Figure 15: Constants of Equation 7.45 for the circular cylinder in cross flow

5.2 Methodology of solving external flow problems

Step1 Select appropriate formula for geometry

Step2 Assess Re , Pr and other factors at appropriate reference temperature

Step3 Calculate Nu: don't confuse local Nu and average Nu

Step4 Calculate h: don't confuse local h and average h

5.2.1 Example of calculation of average convection coefficient

Calculate average h for a 20 mm diameter cylinder immersed in cross flow of water at 1m/s. Thermophysical properties for water: thermal conductivity $k = 0.6\frac{\text{W}}{\text{mK}}$, thermal diffusivity $\alpha = 0.15 \times 10^{-6}\frac{\text{m}^2}{\text{s}}$ and $\text{Pr} = 7$

Step1 Select appropriate formula for geometry
cylinder, $\mathrm{Pr} = 7 \gtrsim 0.1$ to calculate average convection coefficient⇒using Equation 7.44 $\overline{\mathrm{Nu}}_D = \frac{\bar{h}D}{k} = C\mathrm{Re}_D{}^m\mathrm{Pr}^{1/3}$

Step2 Assess Re , Pr and other factors at appropriate reference temperature

$$\mathrm{Pr} = 7$$

$$\nu = \alpha\mathrm{Pr} = 0.15 \times 10^{-6}\frac{\mathrm{m}^2}{\mathrm{s}} \times 7 = 1.05 \times 10^{-6}\frac{\mathrm{m}^2}{\mathrm{s}}$$

$$\mathrm{Re}_D = \frac{u_\infty D}{\nu} = \frac{1\frac{\mathrm{m}}{\mathrm{s}} \times 0.02\mathrm{m}}{1.05 \times 10^{-6}\frac{\mathrm{m}^2}{\mathrm{s}}} = 1.9 \times 10^4$$

According to Table 7.2, C=0.193 and m=0.618

Step3 Calculate Nu

$$\overline{\mathrm{Nu}}_D = 0.193\left(1.9 \times 10^4\right)^{0.618} \times 7^{1/3} = 163.0$$

Step4 Calculate h

$$\overline{\mathrm{Nu}}_D = \frac{\bar{h}D}{k} \Rightarrow \bar{h} = \overline{\mathrm{Nu}}_D\frac{k}{D} = \frac{163 \times 0.6\frac{\mathrm{W}}{\mathrm{mK}}}{0.02\mathrm{m}} = 4890\frac{\mathrm{W}}{\mathrm{m}^2\mathrm{K}}$$

5.3 Average Nusselt number for spheres

With constant properties:

$$\overline{\mathrm{Nu}}_D = 2 + 0.6\mathrm{Re}_D{}^{1/2}\mathrm{Pr}^{1/3} \quad \text{Eq. (7.49)}$$

With temperature-dependent properties:

$$\overline{\mathrm{Nu}}_D = 2 + \left(0.4\mathrm{Re}_D{}^{1/2} + 0.06\mathrm{Re}_D{}^{2/3}\right)\mathrm{Pr}^{0.4}\left(\frac{\mu}{\mu_s}\right)^{1/4} \quad \text{Eq. (7.48)}$$

where all properties except μ_s are evaluated at T_∞

5.4 Flow across banks of tubes

The geometric arrangement of a bank of tubes is typically, one fluid moves over the tubes while a second fluid of different temperature passes through the tubes. There are two common arrangement of banks: aligned or staggered in the direction of the fluid velocity. The configuration is characterized by the tube diameter D, transverse pitch S_T, and longitudinal pitch S_L measured between tube centers. Flow conditions within the bank are dominated by boundary layer separation effects, wake interactions, and the convection heat transfer. Flow conditions are quite different and strongly dependent on the bank arrangement.

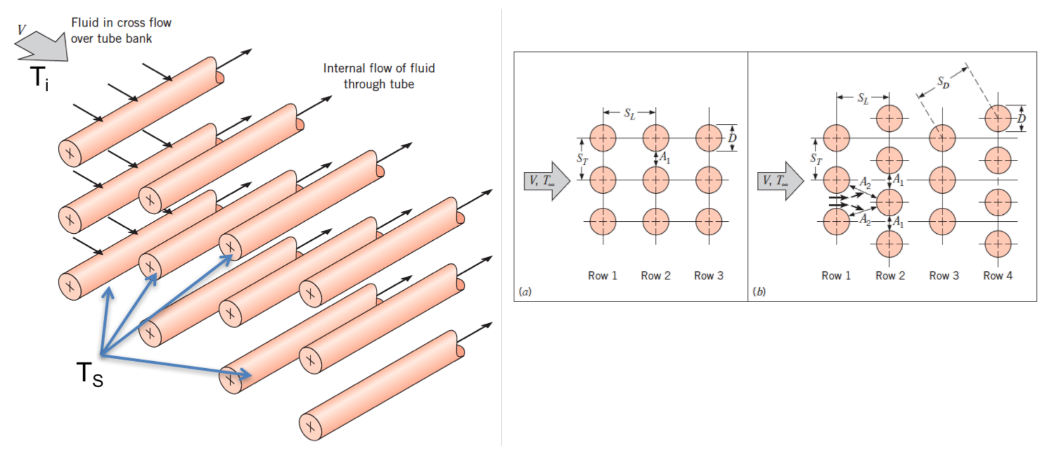


Figure 16: Schematic of a tube bank in cross flow and two common arrangements in a bank: aligned and staggered.

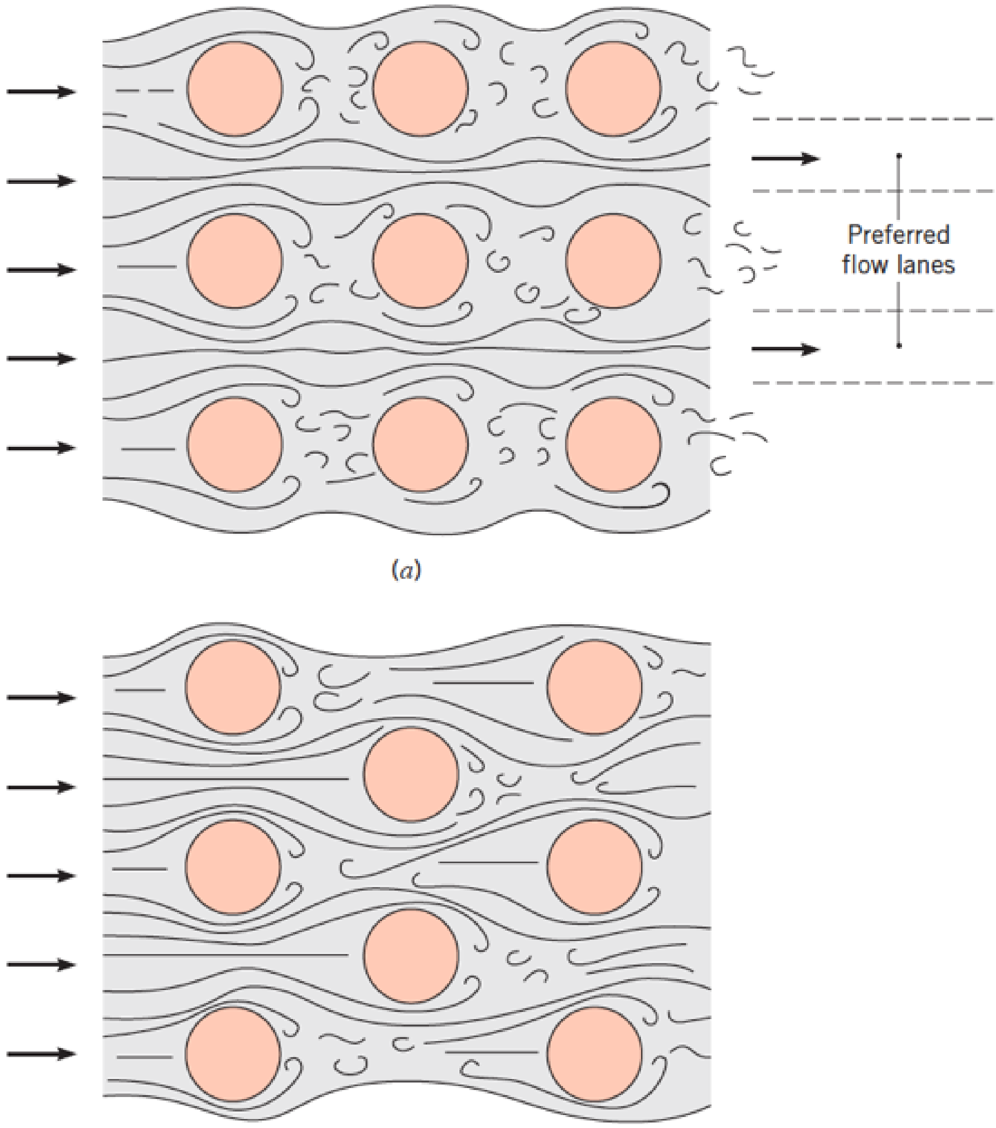


Figure 17: Flow conditions for aligned and staggered bank arrangements

The heat transfer rate per unit length q' of the tubes can be computed by:

$$q' = N\left(\bar{h}\pi D\Delta T_{\mathrm{lm}}\right) \quad \text{Eq. (7.56)}$$

where N is the total number of tubes in the bank and ΔT_{lm} is the log-mean temperature difference used to compute the appropriate temperature difference in calculation:

$$T_{\mathrm{lm}} = \frac{(T_s - T_i) - (T_s - T_o)}{\ln\left(\frac{T_s - T_i}{T_s - T_o}\right)} \quad \text{Eq. (7.54)}$$

where T_i and T_o are the temperatures of the fluid as it enters and leaves the bank, respectively. In order to calculate ΔT_{lm}, the outlet temperature T_o is needed and can be estimated by:

$$\frac{T_s - T_o}{T_s - T_i} = \exp\left(-\frac{\pi D N \bar{h}}{\rho V N_T S_T c_p}\right) \quad \text{Eq. (7.55)}$$

The average heat transfer coefficient for the entire tube bank has the following correlation valid for $N_L \gtrsim 20, 0.7 \lesssim \mathrm{Pr} \lesssim 500, 10 \lesssim \mathrm{Re}_{D,\max} \lesssim 2 \times 10^6$

$$\overline{\mathrm{Nu}}_D = C_1 \mathrm{Re}_{D,\max}^m \mathrm{Pr}^{0.36} \left(\frac{\mathrm{Pr}}{\mathrm{Pr}_s}\right)^{1/4} \quad \text{Eq. (7.50)}$$

where $\mathrm{Re}_{D,\max}$ is calculated based on the maximum velocity:

- For aligned arrangement: $V_{\max} = \frac{S_T}{S_T - D} V \quad \text{Eq. (7.52)}$
- For staggered arrangement, choose the higher one between

$$V_{\max} = \frac{S_T}{2\left(S_D - D\right)} V \quad \text{Eq. (7.53)}$$
$$V_{\max} = \frac{S_T}{S_T - D} V \quad \text{Eq. (7.52)}$$

Properties involved in Eq. 7.50 are evaluated at the film temperature $T_f = (T_i + T_o)/2$ except Pr_s, which is evaluated at the temperature of the surface T_s.

Table 7.5 **Constants of Equation 7.50 for the tube bank in cross flow [16]**

Configuration	$Re_{D,\max}$	C_1	m
Aligned	10–10^2	0.80	0.40
Staggered	10–10^2	0.90	0.40
Aligned	10^2–10^3	Approximate as a single (isolated) cylinder	
Staggered	10^2–10^3		
Aligned $(S_T/S_L > 0.7)$[a]	10^3–2×10^5	0.27	0.63
Staggered $(S_T/S_L < 2)$	10^3–2×10^5	$0.35(S_T/S_L)^{1/5}$	0.60
Staggered $(S_T/S_L > 2)$	10^3–2×10^5	0.40	0.60
Aligned	2×10^5–2×10^6	0.021	0.84
Staggered	2×10^5–2×10^6	0.022	0.84

[a]For $S_T/S_L < 0.7$, heat transfer is inefficient and aligned tubes should not be used.

Figure 18: Constants of Equation 7.50 for the tube bank in cross flow

If there are 20 or fewer rows of tubes ($\mathrm{Nu}_L \leqslant 20$), the average heat coefficient is typically reduced and a correction factor is necessary to capture the reduction:

$$\overline{\mathrm{Nu}}_D\Big|_{(\mathrm{Nu}_L<20)} = C_2\overline{\mathrm{Nu}}_D\Big|_{(\mathrm{Nu}_L\geqslant 20)} \quad \text{Eq. (7.51)}$$

where the correction factor C_2 can be found in Table 7.6:

Table 7.6 **Correction factor C_2 of Equation 7.51 for $N_L < 20$ ($Re_{D,\max} \gtrsim 10^3$) [16]**

N_L	1	2	3	4	5	7	10	13	16
Aligned	0.70	0.80	0.86	0.90	0.92	0.95	0.97	0.98	0.99
Staggered	0.64	0.76	0.84	0.89	0.92	0.95	0.97	0.98	0.99

Figure 19: Correction factor C_2 of Equation 7.51 for $\mathrm{Nu}_L < 20$

Summary

Correlation	Geometry	Re/Pr	Other Dimensionless	Temp.	Condition
$\delta = 5 \times \mathrm{Re}_x^{-1/2}$ (7.17)	Flat plate	Re<5×10^5	-	T_f	Laminar
$C_{f,x} = 0.664\mathrm{Re}_x^{-1/2}$ (7.18)	Flat plate	Re<5×10^5	-	T_f	Laminar, local
$\mathrm{Nu}_x = 0.332\mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3}$ (7.21)	Flat plate	Re<5×10^5 Pr $\gtrsim 0.6$	-	T_f	Laminar, local
$\delta_t = \delta\ \mathrm{Pr}^{-1/3}$ (7.22)	Flat plate	Re<5×10^5 Pr$\gtrsim 0.6$	-	T_f	Laminar
$\overline{C}_{f,x} = 1.328\mathrm{Re}_x^{-1/2}$ (7.24)	Flat plate	Re<5×10^5	-	T_f	Laminar, average
$\overline{\mathrm{Nu}}_x = 0.664\mathrm{Re}_x^{1/2}\mathrm{Pr}^{1/3}$ (7.25)	Flat plate	Re<5×10^5 Pr$\gtrsim 0.6$	-	T_f	Laminar, average
$\mathrm{Nu}_x = 0.564\mathrm{Pe}_x^{1/2}$ (7.26)	Flat plate	Re<5×10^5 Pr$\lesssim 0.05$	$\mathrm{Pe}_x \gtrsim 100$	T_f	Laminar, local
$C_{f,x} = 0.0592\mathrm{Re}_x^{-1/5}$ (7.28)	Flat plate	Re$\lesssim 10^8$	-	T_f	Turbulent, local
$\delta = 0.37 \times \mathrm{Re}_x^{-1/5}$ (7.29)	Flat plate	Re$\lesssim 10^8$	-	T_f	Turbulent
$\mathrm{Nu}_x = 0.0296\mathrm{Re}_x^{4/5}\mathrm{Pr}^{1/3}$ (7.30)	Flat Plate	Re$\lesssim 10^8$ $0.6 \lesssim$Pr$\lesssim 60$	-	T_f	Turbulent, local
$\overline{C}_{f,L} = 0.074\mathrm{Re}_L^{-1/5} - 1742\mathrm{Re}_L^{-1}$ (7.33)	Flat plate	Re$\lesssim 10^8$	-	T_f	Mixed, average
$\overline{\mathrm{Nu}}_L = (0.037\mathrm{Re}_L^{4/5} - 871)\mathrm{Pr}^{1/3}$ (7.31)	Flat plate	Re$\lesssim 10^8$ $0.6 \lesssim$Pr $\lesssim 60$	-	T_f	Mixed, average
$\overline{\mathrm{Nu}}_D = \mathrm{C}\mathrm{Re}_D^m\mathrm{Pr}^{1/3}$ (7.44) (Table 7.1)	Cylinder	$0.4\lesssim$Re $\lesssim 4 \times 10^5$ Pr$\gtrsim 0.7$	-	T_f	Average
$\overline{\mathrm{Nu}}_D = C\mathrm{Re}_D^m\mathrm{Pr}^n(\mathrm{Pr}/\mathrm{Pr}_s)^{1/4}$ (7.45) (Table 7.2)	Cylinder	$1\lesssim$Re$\lesssim 10^6$ $0.7\lesssim$ Pr$\lesssim 500$	-	T_∞	Average
$\overline{\mathrm{Nu}}_D = 0.3+[0.62\mathrm{Re}_D^{1/2}\mathrm{Pr}^{1/3}$ $\times[1+(0.4/\mathrm{Pr})^{2/3}]^{-1/4}]$ $\times[1+(\mathrm{Re}_D/282,000)^{5/8}]^{4/5}$ (7.46)	Cylinder	RePr$\gtrsim 0.2$	-	T_f	Average
$\overline{\mathrm{Nu}}_D = 2+(0.4\mathrm{Re}_D^{1/2}+0.06\mathrm{Re}_D^{2/3})\mathrm{Pr}^{0.4}$ $\times(\mu/\mu_s)^{1/4})$ (7.48)	Sphere	$3.5\lesssim$ Re$\lesssim$ 7.6×10^4 $0.71 \lesssim$ Pr$\lesssim 380$	$1.0\lesssim \mu/\mu_s \lesssim 3.2$	T_∞	Average
$\overline{\mathrm{Nu}}_D = 2 + 0.6\mathrm{Re}_D^{1/2}\mathrm{Pr}^{1/3}$ (7.49)	Falling drop	-	-	T_∞	Average
$\overline{\mathrm{Nu}}_D = C_1C_2\mathrm{Re}_{D,max}^m\mathrm{Pr}^{0.36}(\mathrm{Pr}/\mathrm{Pr}_s)^{1/2}$ (7.50), (7.51) (Tables 7.3, 7.4)	Tube bank	$10 \lesssim$Re$\lesssim 2 \times 10^6$ $0.7 \lesssim$Pr $\lesssim 500$	-	$\overline{T}$	Average
$\varepsilon\overline{j}_H = 2.06\mathrm{Re}_D^{-0.575}$ (7.72)	Packed bed of spheres	$90\lesssim$Re $\lesssim 4000$ Pr≈ 0.7	-	$\overline{T}$	Average

For tube banks and packed beds, properties are evaluated at the average fluid temperature, $\overline{T} = (T_i + T_o)/2$

Table 7.1: Constants of Equation 7.44 for the circular cylinder in cross flow

Re_D	C	m
0.4-4	0.989	0.330
4-40	0.911	0.385
40-4000	0.683	0.466
4000-40,000	0.193	0.618
40,000-400,000	0.027	0.805

Table 7.2: Constants of Equation 7.45 for the circular cylinder in cross flow

Re_D	C	m
1-40	0.75	0.4
40-1000	0.51	0.5
$10^3 - 2 \times 10^5$	0.26	0.6
$2 \times 10^5 - 10^6$	0.076	0.7

Table 7.3: Constants of Equation 7.50 for the tube bank in cross flow

Configuration	$Re_{D,max}$	C_1	m
Aligned	$10\text{-}10^2$	0.80	0.40
Staggered	$10\text{-}10^2$	0.90	0.40
Aligned	$10^2 - 10^3$	Approximate as a single	
Staggered	$10^2 - 10^3$	(isolated) cylinder	
Aligned (S_T/S_L>0.7)*	$10^3 - 2 \times 10^5$	0.27	0.63
Staggered (S_T/S_L<2)	$10^3 - 2 \times 10^5$	$0.35(S_T/S_L)^{1/5}$	0.60
Staggered (S_T/S_L>2)	$10^3 - 2 \times 10^5$	0.40	0.60
Aligned	$2 \times 10^5 - 2 \times 10^6$	0.021	0.84
Staggered	$2 \times 10^5 - 2 \times 10^6$	0.022	0.84

*For (S_T/S_L)<0.7,heat transfer is inefficient and aligned tubes should not be used.

Table 7.4: Correction factor C_2 of Equation 7.51 for Nu_L<20 ($\mathrm{Re}_{D,max} \gtrsim 10^3$)

Nu_L	1	2	3	4	5	7	10	13	16
Aligned	0.70	0.80	0.86	0.90	0.92	0.95	0.97	0.98	0.99
Staggered	0.64	0.76	0.84	0.89	0.92	0.95	0.97	0.98	0.99

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