Turbulence transition and internal wave generation in density stratified jets

B. R. Sutherland and W. R. Peltier

Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada

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The nonlinear evolution of an unstable symmetric jet in incompressible, density stratified fluid is simulated numerically. When \( N^2 \) is constant and near zero, like-signed vortices pair by way of an instability of the mean flow to a subharmonic disturbance with wavelength twice that of the most unstable mode of linear theory. For small but finite and constant values of \( N^2 \), however, the individual vortex cores are strained and vorticity is generated at small scales by the action of baroclinic torques. In this case, the mean flow of the fully evolved jet is stable to subharmonic disturbances. The linear stability of the two-dimensional nonlinear basic states to three-dimensional perturbations is examined in detail. From this stability analysis, it is inferred that jet flow with stratification characterized by constant \( N^2 \) is a poor candidate for IGW generation. However, the existence of an efficient mechanism whereby IGW may be radiated to infinity from the jet core is demonstrated via simulations initialized with a density profile such that \( N^2 = J \tanh^2(z/R) \). This mechanism is expected to be an important contributor to the wave field observed in a variety of geophysical circumstances.

I. INTRODUCTION

Since the initial demonstration by Hines\(^1\) that many of the irregular motions observed in the middle and upper atmosphere may be explained as due to the presence of vertically propagating internal gravity waves (IGW) generated at lower levels, there has been intense interest in the mechanisms whereby such IGW may be produced. The source of IGW responsible for much of the mixing observed in the stratosphere, for example, is believed to be associated with the "breaking" of internal waves forced by stratified shear flow over surface topography (e.g., Lilly\(^2\) and Peltier and Clark\(^3\)). However, topographically forced IGW are stationary with respect to the ground and therefore constitute only one component of the observed wave spectrum. This spectrum clearly includes constituents with nonzero horizontal phase velocity and it has long been suspected that this component could be generated in part through the parallel shear instability process. This idea has been pursued in the context of linear theory by Drazin \( et \ al.\)\(^4\) who classified as "unbound" those (neutraly stable) modes which propagate at infinity and which are modified by shear. Their analysis included an examination of internal gravity waves modified by coupling with unstable modes of the Bickley jet and hyperbolic tangent shear layer in fluid with \( N^2 \) constant. However, these authors focused primarily on the nature of modes with horizontal phase speeds greater than the maximum velocity or less than the minimum velocity of the parallel flow. Such waves clearly cannot be generated spontaneously by instability processes.\(^5\) IGW generation by more realistic jet profiles was studied by Mastrantonio \( et \ al.\)\(^6\) who investigated, using linear theory, the IGW that might grow through instability from an analytical approximation to the tropospheric jet stream, and further study has been pursued by Chimonas and Grant.\(^7\) None of these analyses were successful in providing models of stratified parallel flows that were capable of radiating significant IGW, according to the linear analyses on which they were based. However, as pointed out by Davis and Peltier,\(^8\) McIntyre and Weissman,\(^9\) and Fritts,\(^10\) the generation of propagating IGW by eddies which develop from the mean flow most probably involves strong nonlinear interactions and there is, therefore, a serious question as to the extent to which linear theory may reliably predict the viability of any hypothetical initial state as an internal wave generator. Spurred by the observation, to be reported below, that IGW may in fact be spontaneously generated by an unstable jet in (suitably) stratified fluid, a detailed analysis of the two-dimensional nonlinear evolution of such flows has been initiated, the present paper representing the first in this new series.

In one of the earliest experimental studies of the unstratified wake that develops in the lee of a flat plate, Sato and Kuriki\(^11\) observed the development of a double row of staggered vortices of opposite sign in flow with large Reynolds number (\( Re > 10^4 \)). This configuration of vortices was observed to persist until distorted by three-dimensional fluctuations, beyond which time the turbulent flow became fully developed. In both recent experiments and three-dimensional nonlinear numerical simulations of the unstratified wake problem, the structure of the three-dimensional perturbations whose development leads to the turbulence transition has been examined in detail.\(^12\)\(^\sim\)\(^15\)

Vortex pairing in free-mixing layer and jet flows has attracted a great deal of attention as this mechanism is clearly responsible for the thickening of the turbulent mixing region in moderate to large Reynolds number flow (Winant and Browand\(^16\)). Very detailed theories have now been developed to understand the pairing mechanism in both the unstratified and stratified free-mixing layer.\(^17\)\(^,\)\(^18\) In inviscid fluid it has also been shown that the Von Kármán vortex street is generally unstable [e.g., Saffman (Sec. 7.6) and Jiménez\(^19\)\(^,\)\(^21\)]. Though pairing is inhibited in low Reynolds number flow (\( Re < 100 \)), instability of
the Von Kármán-like vortex street to merging of vortices of like sign in moderate Reynolds number flow has been observed in early experiments by Taneda\textsuperscript{22} and more recently in wake flow restricted to two dimensions by Couder and Basdevant\textsuperscript{23} who studied the evolution of a wake in a soap film.

Though much attention has focused on the behavior of shear flow in stratified fluid (e.g., Thorpe\textsuperscript{24}), relatively few controlled experiments of plane jet flow in stably stratified fluid have been performed. Nonetheless, it is relevant to mention the work of Antonia et al.\textsuperscript{25} who examined the intrusion of a heated plane jet into ambient cool fluid. Over the vertical extent of the half-width of the jet, they observed the organization of large-scale structures in the form of a vortex street pattern. In particular, they measured significant coherent Reynolds stress and heat flux along the diverging separatrix which connects vortices of the same sign and this they identified with a temperature front. Though laboratory experiments on stratified fluids have frequently considered the dynamics of flows with shear coincident with steep density gradient, Delisi et al.\textsuperscript{26} have recently described the evolution of a vortex dipole in fluid with approximately linear vertical variations of density and velocity. The experiment on which they report is characterized by large Reynolds number and Richardson number near unity, and they observe the persistence of the vortex with vorticity equal to that of the background shear, whereas the vortex in which vorticity has the opposite sign is advected and strained by the shear reinforced vortex. As yet, no detailed experiments have been performed that consider the nature of the wave, mean-flow interaction in stratified jet flow with uniform $N^2$.

The evolution of an unstratified jet in two spatial dimensions was first simulated numerically by Zabusky and Deem\textsuperscript{27} who integrated the incompressible Navier–Stokes equations using a finite-difference method in a domain with periodic boundary conditions. The simulations were initialized with both Gaussian and Bickley jet profiles on which were superimposed a fluctuation with spatial structure determined by the fastest growing mode of linear stability theory. They observed the formation of vortices of opposite sign on either flank of the jet. Vortex pairing occurred in simulations initialized with many wavelengths of the primary instability. Similar observations have been made in numerical studies allowing for the development of spatial instability.\textsuperscript{28}

Results of the nonlinear numerical simulations of stratified jet flow in two spatial dimensions, which will be presented here for the first time, indicate that jet instability may exhibit a wide range of phenomena that has no counterpart in the free-stratified mixing layer. The theoretical model problem and the computational techniques to be employed in the simulations are presented in Sec. II. In Sec. III, results from simulations of the evolution of jet flows in fluid with constant $N^2$ are discussed. Denoting the bulk Richardson number by $J$, which is a characteristic value of $N^2$ (hence, for constant Brunt–Väisälä frequency $N^2=J$), variables are nondimensionalized so that the flow is unstable for values of $N^2=J \lesssim 0.127$. In the limit $J \to 0$, primary and secondary vortices develop on either flank of the jet and pairing between vortices of like sign ensues. These observations agree with the results first presented by Zabusky and Deem.\textsuperscript{27} In weakly stratified fluid for which $J$ is as small as 0.005, vortex cores are weakened by turbulent straining of vorticity gradients and pairing between the weak vortices of like sign that remain is inhibited. In simulations with moderate $J=0.02$, large vortices are destroyed by the strain field and strong vorticity is generated at small scales by the action of baroclinic torques. Examination of the energy transfer between waves and the mean flow and of the baroclinic conversion of energy between kinetic and available potential forms provides some insight into the mechanism of vortex pairing in weakly stratified fluid. Similar analysis coupled with an examination of the nature of the intervortex strain field provides some understanding of the process through which pairing is inhibited in moderately stratified flow.

In Sec. IV, the possibility of IGW radiation to infinity from an unstable stratified jet is examined. It is shown that such radiation may indeed occur, provided of course that the frequency of the disturbance is less than the local buoyancy frequency. The small scale structures characteristic of turbulent mixing regions are therefore poor candidates as radiative sources of IGW. It is demonstrated that the large-scale structures which develop from stratified jet flow characterized by constant $N^2=J=0.02$ are capable of generating IGW. These waves are weakly forced, however, and so have small amplitude. Indeed, if the two-dimensional flow was allowed to develop so as to access the third spatial degree of freedom then it would become more fully turbulent and IGW excitation would become even less prominent. We therefore present results from a three-dimensional linear stability analysis of the evolving two-dimensional stratified jet at various stages during the course of its temporal evolution. The results of our analysis of the stability of unstratified jet flow to spanwise perturbations explains the experimental and numerical results of Lasheras and Meiburg,\textsuperscript{14} though it is demonstrated that vortex core instabilities are not induced by the development of instabilities of the braids between vortex cores of like sign but rather that the two modes of three-dimensionalization develop in tandem. In moderately stratified fluid the growth of spanwise instabilities is much more intense, the three-dimensionally most unstable mode now being driven by the regions of intense shear that develop between the small-scale "filaments" of vorticity that are engendered by baroclinic torques.

The destruction of large-scale coherent structures in stratified fluid by processes in two and three dimensions has led us to consider IGW emission from jet flow restricted to two dimensions such that $N^2$ is assumed to be small through the region of enhanced velocity and large on the flanks. Specifically, an idealization is examined such that $N^2=J \tanh^2(z/R)$ and it is shown that large amplitude IGW are generated for a wide range of values of the bulk Richardson number $J$ and scale $R$ over which $N^2$ is depressed. It is argued that mixing processes in a shear layer may naturally adjust the background density varia-
tion so that $N^2$ is small over the vertical extent of the layer. Furthermore, the idealization above has some attractive theoretical merit on the basis of recently published linear theory. For example, IGW emission by a tanh shear layer in a density stratified background such that $N^2 = J + J_1 \tanh^2(z/R)$ has been examined in work by Lott et al. in which the linear stability of the flow is examined and analytical expressions for the marginal curves are found for $n=2$ and 4. Whether actual wave emission would in fact be realized in the presence of nonlinear interaction between a growing instability and the mean flow was, however, not addressed.

II. THEORETICAL PRELIMINARIES

The nonlinear evolution of the stratified jet flows that are of interest here will be analyzed by solving the Navier–Stokes equations in the usual Boussinesq approximation in a horizontally periodic channel with free slip upper and lower boundary conditions. The governing equations are expressed in dimensionless form with characteristic length scale $L$ equal to the jet width and characteristic velocity scale $U$ equal to the maximum velocity. The horizontal and vertical velocity fields $u$ and $w$, respectively, are written in dimensionless form by the substitutions $u \rightarrow \frac{u}{U}$ and $w \rightarrow \frac{w}{U}$. If $F$ is the characteristic scale over which the background density varies, then the density fluctuation $\rho'$ may be nondimensionalized by the substitution $\rho' \rightarrow \left(\frac{\rho_0 F}{\rho}\right) \rho'$ in which $\rho_0$ is the background density at some reference level. The nondimensional density fluctuation $\rho'$ then corresponds to the (dimensionless) vertical displacement of a fluid parcel.

The dimensionless forms of the conservation laws then become, dropping the primes on fluctuation quantities (e.g., Smyth and Peltier):

$$0 = u_x + w_z,$$  

$$\frac{Du}{Dt} = -\frac{p_x + 1}{Re} \nabla^2 u,$$  

$$\frac{Dw}{Dt} = -\frac{p_x - \rho_0}{Re} \nabla^2 w,$$  

$$\frac{D\rho}{Dt} = \frac{N^2}{J} \left(\frac{w}{U} + \frac{1}{Re Pr}\right) \nabla^2 \rho,$$  

in which $D/Dt = \partial/\partial t + u \cdot \nabla$ is the material derivative, $p$ is fluctuation pressure, and $N$ is the Brunt–Väisälä frequency defined by $N^2 = -J \rho_0/\partial z$, with $J = (g/\rho)(L/\rho)^2$ the bulk Richardson number and $g$ the acceleration due to gravity. In performing the two-dimensional nonlinear simulations it is convenient to eliminate the pressure field from (2) and (3) by employing a streamfunction vorticity formulation. The evolution equation for the spanwise component of the vorticity field is then

$$\frac{D\omega}{Dt} = \frac{p_x + 1}{Re} \nabla^2 \omega$$  

in which $\omega = u_z - w_x$. Since

$$\nabla^2 \psi = -\omega$$  

with the streamfunction $\psi$ such that $u = -\psi_x$ and $w = \psi$, the components of the velocity field required in (4) and (5) may be obtained by inverting the elliptic equation (6).

The three dimensionless parameters that appear in Eqs. (4) and (5) are the Reynolds number $Re = \frac{UL}{v}$, the Prandtl number $Pr = \nu/\kappa$, and the bulk Richardson number $J$, in which $v$ is the kinematic viscosity and $\kappa$ is the thermal diffusivity. Numerical simulations are performed with a moderately high Reynolds number, $Re = 600$, and with Prandtl number $Pr = 1$. The adjustable parameter of greatest interest for the purposes of the present study is the bulk Richardson number, which measures the stabilizing influence of the vertical density variation relative to the destabilizing influence of the shear.

The vertical profile of horizontal velocity is taken to be initially of the Bickley form, $U(z) = \text{sech}^2(z)$. The linear stability of a jet with this structure has been the subject of many previous theoretical analyses. In particular, Hazel first calculated the fastest growing modes of instability of the jet in fluid with constant $N^2$ and showed it to be unstable provided $0 < N^2 \leq 0.127$, in which $N^2$ is nondimensional with characteristic time $\frac{L}{U}$. Further details of the linear problem, especially concerning the question of absolute versus convective instability of the inviscid jet have been addressed more recently by Sutherland and Peltier. As well as studying the nonlinear evolution of the jet in fluid with constant $N^2 = J$, the case of variable $N^2$ will be examined in what follows for the choice $N^2 = J \tanh^2(z/R)$, in which $R$ is an adjustable length scale.

In order to initialize the nonlinear simulations, the background fields of density and horizontal velocity are perturbed by addition onto the basic state of a small-amplitude random component, as well as a fluctuation having the spatial structure of the fastest growing mode of linear theory determined on the basis of a Galerkin stability analysis employing finite Re and Pr (e.g., Klaassen and Peltier). The amplitude of the mode is prescribed such that the maximum vertical velocity in the perturbed flow is initially a small fraction of the characteristic speed. Before accepting numerical results concerning computed disturbance life cycles, it is ensured that the simulations adequately reproduce the exponential growth rate predicted by linear theory. This is done by comparing the linear growth rate to the initial perturbation growth rate $\sigma$ from the simulation by calculating $\sigma = 1/(2E) \mathrm{d}E/\mathrm{dt}$ from the evolving wave kinetic energy $E$.

Of Eqs. (4) and (5), only solutions that are periodic in the streamwise (horizontal) direction having wavelength $L_x$ and fundamental wave number $\alpha = 2\pi/L_x$ are considered. Accordingly, the horizontal structure of the dependent fields may be represented in a Fourier basis via

$$f(x,t) = \sum_{m=-M}^{M} f_m(z,t) \exp(i\alpha x),$$

in which $f$ may represent $\omega$ or $\rho$ and $M$ determines the limit of horizontal resolution of each field. The vertical dependence of the dependent variables is represented in finite difference form so that $\omega$ and $\rho$ are sampled at $P+1$
points \( z_0, \ldots, z_p \), at regularly spaced intervals spanning the channel of vertical extent \( L_z \), and vertical derivatives are replaced by their second-order finite difference equivalent

\[
\frac{\partial f_m(z_p, t)}{\partial z} = \frac{f_m(z_{p+1}, t) - f_m(z_{p-1}, t)}{2\Delta z}, \quad (8)
\]

in which \( \Delta z = L_z / P \). The resulting set of evolution equations is stepped forward in time using a leap-frog method with an Euler backstep taken at regular time intervals to minimize splitting errors. To ensure that the results of the simulations are not sensitive to the resolution parameters, simulations are performed for channels of varying width and the equations are integrated with varying spatial resolution always ensuring that the time step is sufficiently small to satisfy the CFL condition. For the simulations of uniformly stratified fluid presented here, the vertical grid spacing is taken to be \( dz = 0.12 \) and horizontal grid spacing is between \( dx = 0.10 \) and 0.11 depending on the value of \( J \) which determines the horizontal wavelength of the most unstable mode of linear theory. Tests demonstrating the adequacy of this resolution are presented in Sec. III.

In the mixed spectral and finite difference scheme modal budgets of quadratic quantities which are conserved in the absence of viscous and thermal diffusion may be assessed by means similar to those employed by Smyth and Peltier. As the simulated fields evolve, they are analyzed to ensure that the rate of energy loss is balanced by diffusion to machine precision. This and additional diagnostic analyses that are employed to understand the basic physical interactions governing flow dynamics are described in further detail in Sec. III.

III. MIXING IN CONSTANT \( N^2 \) JETS

For the purpose of comparison, in all of the simulations for which results are presented in this section the amplitude of the perturbation superposed initially on the background jet is prescribed so that the maximum vertical velocity is 0.05\%. Unless otherwise stated, the wave number of the most unstable mode of linear theory is taken to be twice that of the fundamental wave number fixed by the horizontal extent of the channel \( (L_x) \). Disturbances of the corresponding wavelength are said to correspond to wave number 2, whereas disturbances of horizontal wavelength \( L_x \) are said to correspond to wave number 1.

For unstratified flow \( (J=0) \), in a channel allowing only a single wavelength of the fastest growing mode to develop, the formation of two vortex sheets of oppositely signed vorticity on either flank of the jet is observed. Figure 1 shows contours of the vorticity field during the nonlinear evolution at times (in dimensionless units) \( t = 50 \) and 100. The contour interval in each diagram is 0.2 with solid (dashed) contours corresponding to positive (negative) vorticity. The vortices, which are well developed at time \( t = 100 \), are staggered such that a vortex center in one layer lies directly between the vortex centers of the other layer. A single vortex sheet evolves in a manner similar to that in a single isolated mixing layer, the vortices repeating the classic Kelvin “cat’s eye” pattern. Though the evolution is sensitive to the initial background noise that is imposed, it is not generally true that the final state of the system is dominated by a single dipole in a channel allowing the development of multiple wavelengths. In work not shown here, a single simulation for the long-time evolution of the jet in unstratified fluid with large Reynolds number \( (Re = 60,000) \) has been performed for which the Laplacian diffusion operator in (4) and (5) is replaced by a hyper-viscosity operator of the form. In a channel which supports four wavelengths of the most unstable mode, multiple dipoles were observed over long times. If the vortex sheet separation \( h \) is defined as the distance between horizontally averaged vorticity extrema of opposite sign shortly before pairing, then the ratio of \( h \) to the distance between like-signed vortices \( \lambda \) is approximately 0.5. This result is consistent with the previous numerical studies of Aref and Siggia who demonstrated the existence of stability regimes of two vortex sheets of opposite sign and of infinitesimal thickness. The regimes correspond to “pairing transitions” for \( h/\lambda > 0.6 \), “long-lived” stability for \( 0.6 < h/\lambda < 0.3 \), and “oscillatory modes” for \( h/\lambda < 0.3 \). “Long-lived” stability is, perhaps, a misnomer since in this case the vortex street is in a metastable state and susceptible to pairing instabilities which develop out of the background noise. Simulations in which Aref and Siggia explore this possibility are similar to ours, emphasizing the relative insensitivity of the qualitative features of the evolution of unstratified jet flow to the initial form of the horizontal velocity profile.

The evolution of the jet is remarkably different in a fluid that is even very weakly stratified. In Fig. 2 the vorticity fields are shown from simulations with \( J=0.001 \), 0.005, and 0.02 at time \( t=100 \). The interval between contours is 0.3 in all three diagrams. For \( J=0.001 \), vortex
centers are reasonably well defined as in the unstratified case, a structural characteristic that is destroyed with even modest enhancement of $J$ to $J=0.005$. Vortices which begin to develop in the latter simulation are rapidly strained leading to a strong cascade of enstrophy to small scales. Vorticity is also generated at intermediate scales by baroclinic torques. In simulations with $J=0.02$, the straining of vortex cores becomes more pronounced and although the flow appears "turbulent," energy is generally concentrated in the mean flow and in the wave number 2 mode and its harmonics. At time $t=200$ (Fig. 3), like-signed vortices pair in simulations with $J=0.005$, which will be shown to be a consequence of the transfer of energy into the wave number 1 mode, but pairing is strongly inhibited for larger values of $J$. Note that in order to enhance the details of the flow, the contour interval in each of the three diagrams in Fig. 3 is half that of the diagrams in Fig. 2.

High-resolution simulations with moderate and large Reynolds number have been performed to test the accuracy of the numerical model and to examine the effect on the flow evolution of lower kinematic viscosity. In Fig. 4 the vorticity fields of the standard simulation with $J=0.02$ and $Re=600$ at times $t=50$ and 100 are compared with the vorticity fields at the corresponding times calculated with $J=0.02$ at double the resolution for Reynolds number $Re=600$ and 1000. Equal contour intervals, including the zero contour, are shown in all six diagrams. The smooth variations of the zero contour demonstrate the lack of substantial background numerical noise. A comparison of the vorticity field of the high-resolution simulation with $Re=600$ at $t=50$ with the corresponding field of the standard simulation shows negligible differences. At $t=100$ similar large-scale features are apparent in the two simulations. The thin, elongated structures in the standard simulation which do not appear in the high-resolution simulation at this time evidently develop on a scale too fine to be advected adequately on the relatively coarse grid. However, the Reynolds number is sufficiently small to diffuse these structures without significantly affecting the large-scale flow evolution. As anticipated, in the high-resolution simulation with Reynolds number 1000 fine-scale features appear in the vorticity fields which are not apparent in the simulations with higher viscosity. Nonetheless, vortex core straining occurs and the large-scale structures that appear are qualitatively similar to those of the standard simulation. Thus it appears that the transition to "turbulence" at moderate Reynolds number is neither a numerical artifact nor a consequence of unphysically large viscosity.


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The sensitivity to the parameter $J$ of the basic features of flow evolution may be further examined by an analysis of the time scales of wave, mean-flow energy transfer and diffusion. In such an analysis the rate of diffusive loss of energy is separated from the rate of transfer of kinetic energy (KE) and available potential energy (APE) between eddies and the mean flow.

In two-dimensional, inviscid, stratified flow, it is well known that the sum of KE and APE is conserved [e.g., Gill (Sec. 7.8)]. In finite Reynolds and Prandtl number fluid with constant $N^2=J$, $KE=(u^2+w^2)/2$, $APE=Jp'^2/2$, and the equations for the rate of change of energy in these two forms are obtained by multiplying Eq. (2) by $u$, Eq. (3) by $w$, and Eq. (4) by $Jp'$. To these energy equations is applied the domain averaging operator.

$$\langle f(x,z,t) \rangle = \frac{1}{L_x} \int_{-L_z/2}^{L_z/2} dx \int_{L_z/2}^{-L_z/2} dz f(x,z,t),$$

and the resulting equations are normalized by twice the total energy $E$ to obtain

$$\sigma_{KE} = \sigma_B - \sigma_d KE,$$  \hspace{1cm} (10)

$$\sigma_{APE} = -\sigma_B - \sigma_d APE,$$  \hspace{1cm} (11)

in which $\sigma_f = \langle f \rangle / (2\langle E \rangle)$ denotes the inverse time scale for changes of the domain averaged quantity $f$ and $E=KE+APE$ is the total energy. The subscripts, $f$, represent the time rate of change of KE ($tKE=dKE/dt$) and APE ($tAPE=dAPE/dt$), the baroclinic conversion from APE to KE,

$$B = -Jwp',$$

and the rates of KE and APE loss due to viscous and thermal diffusion,

$$d KE = -\frac{1}{Re} \left( u \nabla^2 u + w \nabla^2 w \right)$$  \hspace{1cm} (13)

and

$$d APE = -\frac{J}{Re Pr} \rho' \nabla \cdot \rho'',$$  \hspace{1cm} (14)

respectively. The normalization, $2E$, is chosen so that the inverse time scale $\sigma$ provides a measure of KE and APE time scales which can be compared with the inertial time scales of the flow.

The sum of Eqs. (10) and (11) is the inverse time scale corresponding to the rate of change of average total energy, a quantity which clearly decreases only in consequence of viscous and thermal diffusion. The sums $\sigma_{KE} + \sigma_{APE}$ and $\sigma_d KE + \sigma_d APE$ are separately calculated during simulations so as to ensure balance between of total energy change and the diffusion rates.

The above described energy budget may be further disaggregated by examining the separate contributions to KE and APE by the eddies and the mean flow. The mean-flow kinetic energy (MKE) is generally defined in terms of the horizontally averaged velocity fields $\bar{u}$ and $\bar{w}$ so that MKE=$(\bar{u}^2 + \bar{w}^2)/2$. For spatially periodic channel flow, in particular, $\bar{w}=0$ so that MKE=$(\bar{u}^2)/2$. The eddy kinetic energy (EKE) is the difference between the total KE and MKE, i.e., EKE=$[(u-\bar{u})^2 + (w-\bar{w})^2]/2$. Similarly the mean flow and eddy APE are defined by MAPE=$J\bar{p}'^2/2$ and EAPE=$J(p'-\bar{p'})^2/2$, respectively.

To illustrate the impact of diffusion controlled dissipation on the flow, in Fig. 5(a) the horizontally averaged energy is shown for simulations with $J=0.001$, $0.005$, and $0.02$ as a function of time from $t=0$ to 200. Energy de-
creases monotonically in all three cases with the greatest amount of dissipation occurring in the case for \( J = 0.02 \) in which the flow is strongly mixed and energy cascades rapidly to small scales. Nonetheless, the energy at time \( t = 200 \) for \( J = 0.02 \) is more than two-thirds the energy of the flow for which \( J = 0.001 \). In Figs. 5(b)–5(d), the rate of change of MKE (heavy line) is compared continuously in time with the dissipation rate of MKE (light line) in simulations for \( J = 0.001, 0.005, \) and 0.02, respectively. The inverse time scale, \( \sigma_{MKE} = (d\text{MKE}/dt)/2\langle E \rangle \), which corresponds to the rate of energy loss from the mean flow due both to dissipation of the mean-flow energy and to transfer of energy from the mean flow to the eddies, is generally an order of magnitude larger than the inverse time scale for diffusion controlled dissipation \( \sigma_{MKE} = (d\text{MKE})/\langle E \rangle \), where the domain averaging operator \( \langle \rangle \) is defined by (9). The only exception occurs for the simulation with \( J = 0.02 \) for \( t \approx 100 \) at which time the flow becomes quasiparallel and the time scale of energy dissipation is comparable to that of the inertial time scale. These analyses serve to illustrate that the dynamics that govern large scale motions are not limited by dissipative effects.

In considering the transfer between the mean flow and waves of KE and APE, the rate of energy loss due to diffusion is subtracted from the total rate of change of energy and the result is normalized by twice the instantaneous total energy \( 1/\langle E \rangle \). Figure 6 shows the inverse time scale \( \sigma_{m} = \sigma_{m} - \sigma_{E} \) corresponding to transfer rates of KE in simulations for (a) \( J = 0.001 \), (b) 0.005, and (c) 0.02 and to transfer rates of APE in simulations for (d) \( J = 0.001 \), (e) 0.005, and (f) 0.02. The light line in each diagram corresponds to the normalized rate of transfer of eddy KE/
simulation for $J=0.001$, the analysis above may be further disaggregated by doing a spectral decomposition of the transfer rate information. In Fig. 7, wave number decompositions of the energy transfer data are therefore presented as a function of time for $J=0.001$. The curves in (a) describe KE transfers by wave, mean-flow interactions into waves with wave number 1 (medium line) and wave number 2 (heavy line). In (b), the curves describe KE transfers by wave–wave interactions into the mean flow (light line), to wave number 1 (medium line), and to wave number 2 (heavy line). Energy transfer into wave number 2 initially corresponds to the growth of the most unstable mode of linear theory. Energy transfer into wave number 1, which is evident at time $t=100$, indicates that the mechanism of vortex pairing is initiated by energy extracted from the mean flow and not from the mode of wave number 2.

For the first 100 time steps of the simulation, it is apparent that the dominant energy transfers occur between the mean flow and the mode with wave number 2. Vortex pairing, signaled by the transfer of energy to the wave-number 1 mode, occurs when kinetic energy is transferred by the mean flow to this wave (as demonstrated in detail in the case of the free mixing layer by Smyth and Peltier$^{36}$). At the time corresponding to the onset of vortex pairing, $t \approx 100$, no significant amount of energy is transferred to the wave number 1 mode by wave–wave interactions, though KE is extracted from the mean flow by the waves and KE is deposited into the wave number 1 mode by the mean flow. Contributions of energy to the pairing instability by transfers of APE are negligible at this time. Thus vortex pairing can be thought to occur due to instability of the horizontally averaged jet, modified by eddies.

To illustrate the sensitivity of the dynamics to baroclinic torques induced by increasing $J$, the results are presented of an analysis of the total enstrophy transfer rate. In a two-dimensional unstratified fluid, the change of the domain averaged enstrophy ($Z=\omega^2$) is balanced by the rate of viscous diffusion of $Z$. Multiplying Eq. (5) by $\omega$ and averaging over the domain and normalizing by $2Z$ gives

$$\sigma_{dz} = \sigma_{BZ} - \sigma_{dZ},$$

where $BZ = 2J\omega\rho'$ is the average increase in enstrophy due to the action of baroclinic torques and $dZ = -(2/Re)\omega \nabla^2 \omega$ is the loss of enstrophy due to viscous diffusion. Since enstrophy is conserved in two-dimensional unstratified flow, $\sigma_{BZ}$ (the inverse time scale of enstrophy increase due to the action of baroclinic torques) is a suitable measure of departure from enstrophy conservation. Figure 9 shows $\sigma_{BZ}$ for $t=0$ to 200. For $J=0.001$ (light line), the deviations from zero are small in comparison with the cases, where $J=0.005$ (medium line) and 0.02 (heavy line), that show a large increase in enstrophy when vortices are well developed at time $t=40$. Thus the case
by small-scale centers of vorticity can be studied using a

\[ J = 0.001 \] approximately conserves enstrophy so that the

stratified fluid with

\[ 0.005, \] vorticity generated by baroclinic torques is appre-

*•ciable. During the early stages of development of moder-

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these curves is that fluid is mixed more effectively in simulations with weak stratification. For $J=0.001$ and 0.005, the displacement of fluid is greatest during the initial development of eddies and the subsequent pairing events. For $J=0.02$, the fluid is displaced to a lesser extent by the development of eddies. As the flow evolves to a quasiparallel state, the displaced fluid relaxes such that $N^2$ is effectively reduced over the vertical extent of the region in which the shear of the mean flow is greatest. Figure 12 shows vertical profiles of the sum of background and horizontally averaged fluctuation density for $J=0.001$, 0.005, and 0.02 at time $t=200$ with $z$ ranging from $-15$ to 15.

The diagram emphasizes the extent of the large-scale mixing regions in weakly stratified fluid. For $J=0.001$ (light line), the density is uniform over a vertical range including the mixing region. Indeed, in the range about $z=4$ the fluid is convectively overturned. The comparable efficiency by which mass is mixed in weakly stratified fluid, as well as the tendency for jet flows in such fluid to support the evolution of large-scale eddies are properties which may be exploited when considering possible sources of IGW.

IV. EDDY EXCITATION OF IGW

As discussed in the Introduction, the source of IGW which are responsible for mixing in the middle and upper
The potential for IGW generation by hydrodynamic instability in a shear flow has been addressed by a number of authors (e.g., Davis and Peltier; Fritts; and McIntyre and Weissman), the last remarking that this process must be an essentially nonlinear one, since any instability that has a significant growth rate must remain vertically trapped.

For the stratified jet flows of interest here, IGW are observed in simulations with constant $N^2$ as shown, for example, in Fig. 13 which presents the perturbation density field at time 300 in the simulation with $J=0.02$. Contours in this diagram are shown with a very low interval of 0.02 ranging from $-0.05$ to 0.05 and high contour values are suppressed. For the purpose of this calculation, absorbing boundary conditions have been imposed to inhibit reflection of IGW from the channel walls. Damping has been incorporated by allowing the Reynolds number to vary linearly from $Re=600$ to 1 over the top and bottom 15% of the vertical extent of the domain. This absorbing sponge layer is admittedly crude and an exact radiative boundary condition would certainly be more appropriate though computationally prohibitive. However, it is well known on the basis of previous such analyses (e.g., Peltier and Clark) that a linear decrease of Reynolds number adequately and straightforwardly absorbs fluctuations propagating toward a boundary. For the simulations to be discussed below, furthermore, the time evolution is terminated when the perturbation density field exceeds 0.05 (in dimensionless units) at any point along the interior extreme of either the upper or lower sponge layer. IGW are not easily observed in these simulations since they are only weakly forced by the perturbations of the jet due to straining of the vortex centers. Perturbations manifest as large-scale vortices, which would provide adequate forcing of IGW, exist only in flow with weak stratification. However, it will be shown that for $N^2 < 0.005$ the fluid is incapable of supporting the propagation of large amplitude IGW.

By examining the dispersion relationship for IGW propagating in a continuously stratified medium in the Boussinesq approximation, some insight can be gained into the conditions under which energy may be transported vertically via propagating IGW. In terms of the ratio of the horizontal to the total wave number of the disturbance $\cos \Theta$, where $\Theta$ is the “take-off angle” of the wave, the dispersion relation is just

$$\omega^2 = N^2 \cos^2 \Theta. \tag{20}$$

From this simple relationship many interesting observations follow, all of which are of course well known. First, IGW with $|\omega| > |N|$ are evanescent, and second, the phase velocity $c_p$ of propagating IGW is normal to the group velocity $c_g$ with which energy is carried away from the disturbance source [e.g., Lighthill (Sec. 4.4)]. In particular, in fluid with uniform $N^2$, IGW with fixed horizontal wave number have the largest vertical component of group velocity for $\Theta = \tan^{-1}(1/21/2)$, which corresponds to a frequency of magnitude

$$\omega^* = (2/3)^{1/2} N. \tag{21}$$

If it is supposed that IGW are excited by the motion of large-scale eddies corresponding to a mode of wave number $\alpha$ and which move with a horizontal phase speed $c_{px}$, then the frequency at which IGW are forced in the ambient fluid is $\omega = \alpha c_{px}$. In dimensionless units, based on the results of linear theory for jet flow with constant $N^2=J$, the following estimates are made: $\alpha \approx 1$ and $c_{px} \approx 0.3$. Therefore, $J$ must be of order 0.1 for nonevanescent IGW. However, for $J > 0.127$ the jet flow is stable (q.v., Hazen) and it has been shown here that the large-scale vortices that develop in unstable jet flow in moderately stratified fluid are destroyed so that even this forcing mechanism is a poor candidate for IGW excitation. It is concluded that turbulent mixing processes, which are generally of high frequency and destroy large-scale coherent structures in the flow, inhibit the generation of IGW.

In realistic jet flow, turbulent mixing in three spatial dimensions further inhibits IGW excitation. It is therefore important to understand the stability of the two-dimensional coherent structures generated by the primary bifurcation of the jet flow to spanwise perturbations to these vortical structures as they evolve nonlinearly in time. To this end, an examination is made below of the linear stability to spanwise perturbations of the two-dimensional nonlinear basic states at fixed times during simulations of jet flow with constant $N^2$. In Sec. IV A, it is demonstrated that large-scale vortices in unstratified and weakly stratified fluid are unstable to spanwise undulations that are in phase with respect to vortices of opposite sign. The same mode of instability is responsible for the development of cross-stream vortex filaments in the braids located between vortices of like sign. The growth rate of the instability is marginally larger than that of the two-dimensional basic state and is least in unstratified flow. In moderately stratified fluid it is demonstrated that the most unstable disturbance into the third spatial dimension develops within the regions of intense turbulent straining and is much more vigorous than those that control transition in the unstratified case.
Supposing, therefore, that large-scale vortices in weakly stratified flow are relatively stable to spanwise perturbations, in Sec. IV B an efficient mechanism is presented by which IGW may be generated in stratified jet flow with height dependent $N^2$. Specifically, simulations are performed of jet instability in fluid with $N^2=J \tanh^2(z/R)$ and thereby the ability of such flows to efficiently launch IGW of large amplitude is demonstrated. This form of the variation selected for $N^2$ is clearly a theoretical idealization in which $N^2$ is small over the range including the inflection points in the velocity profile so that large-scale vortices may develop, and $N^2$ is large on either flank of the jet so that IGW may propagate vertically. Such an $N^2$ profile is not physically unreasonable, however, since mixing processes of the jet in constant $N^2$ fluid reduce density gradients in the vicinity of the critical layers and so locally reduce $N^2$, as has been shown previously. If the fluid is so mixed and the jet is caused to reintensify by reimposition of the same large-scale forcing that induced it initially, then the vertical density profile may provide a platform on which energetic vortices develop and force large-amplitude IGW in a manner similar to that which is simulated here.

A. Three-dimensional linear stability of two-dimensional nonlinear flows

The three-dimensional linear stability of slowly evolving two-dimensional stratified shear flow has been previously studied by Klaassen and Peltier$^{1,3,4,33}$ and Smyth and Peltier$^{18}$ for the free-mixing layer in the Kelvin–Helmholtz and Holmboe regimes, respectively. The reader is referred to these papers for details of the numerical methods that are applied here to the case of the stratified jet. The Floquet method developed in these papers assumes a modal form for the growth of three-dimensional disturbances with spanwise wave number $\beta$ superposed on the fields of velocity, fluctuation density, and pressure that might be assumed to be adequately approximated as temporally stationary (although see Smyth and Peltier$^{42}$ for methods whereby the latter assumption may be relaxed). Thus each fluctuating field component may be written

$$f(x,y,z,t) = F(x,z) + f'(x,z) \exp(i\beta y + \sigma t), \quad (22)$$

where $F$ represents the vertical cross section of the two-dimensional field generated by the simulation at a fixed time and $f'$ is the amplitude of the spanwise perturbation. Streamwise disturbances and total fields are assumed to be horizontally periodic with fundamental wavelength $L_x$. The assumption that $F$ is independent of $t$ is reasonable provided the growth rate $\sigma$ of the most unstable spanwise perturbation is large compared with the rate of evolution of the two-dimensional fields. Even if the rates are comparable, however, qualitative conclusions may be drawn.

Equation (22) is substituted into the Boussinesq equations and, keeping only terms of order $\epsilon$, the result is manipulated to give an eigenvalue problem for $\epsilon$ in terms of the growth rate $\sigma$ and the complex perturbation amplitudes of streamwise and vertical velocity $u'$ and $w'$, respectively, and of density fluctuation $\rho'$. For a background state at a fixed time characterized by streamwise and vertical velocity $U$ and $W$, respectively, and background and fluctuation density $\bar{\rho}$ and $\rho$, respectively, the linear stability equations are

$$\sigma u' = -U u' - W w' - p_x' + \frac{1}{Re} \nabla^2 u',$$

$$\sigma w' = -U w' - W u' - p_z' - \rho' + \frac{1}{Re Pr} \nabla^2 w',$$

$$\sigma \rho' = -U p_x' - W p_z' - \rho u' - \rho w' - \bar{\rho} w' + \frac{1}{Re Pr} \nabla^2 \rho'. \quad (23)$$

The perturbation pressure field $p'$ is given by the diagnostic equation

$$\nabla^2 p' = -J p' - 2(U u'_x + W w'_x + W u'_z + U w'_z). \quad (24)$$

The system of Eqs. (23) may be economically solved using the Galerkin method. The unknown fields are expanded in a set of orthogonal basis functions selected so as to satisfy the boundary conditions subject to which each field is to be determined, namely,

$$u' = \sum_{m=0}^{N_R} \sum_{k=-K}^{K} u_{km} C_{km},$$

$$w' = \sum_{m=0}^{N_R} \sum_{k=-K}^{K} w_{km} S_{km},$$

$$\rho' = \sum_{m=0}^{N_R} \sum_{k=-K}^{K} \rho_{km} S_{km}, \quad (25)$$

where

$$C_{km} = e^{i\alpha x} \cos(\gamma y z), \quad S_{km} = e^{i\alpha x} \sin(\gamma y z), \quad (26)$$

$$\alpha = \frac{2 \pi}{L_x}, \quad \gamma = \frac{2 \pi}{L_z} \text{ (for convenience, it is supposed that the vertical domain extends from } z=0 \text{ to } z=L_z \text{ and that the centerline of the jet is initially at } z=L_z/2).$$

Following Klaassen and Peltier,$^{23}$ the range of the sum of horizontal wave numbers is determined by the truncation scheme $K(m) = \lfloor (N_R - m)/2 \rfloor$, where $N_R$ is odd and $\lfloor x \rfloor$ denotes the largest integer not exceeding $x$. Limitations imposed by computational speed and memory restrict our examination of the stability of the two-dimensional fields to moderate resolution, $N_R$. The cited accuracy of the eigenvalues presented below is based on the convergence as a function of $N_R$. The expansions (25) are substituted into the perturbation equations (23) using (24) and the left-hand side of the system is diagonalized by taking the appropriate inner products. The result is a matrix eigenvalue problem in the form

$$M \nu^T = \sigma \nu^T \quad (27)$$

in which $\nu$ is a vector composed of the concatenation of the Fourier coefficients $\{u'_{km}, w'_{km}, \rho'_{km}\}$ and $M$ is the matrix of constant coefficients determined by the inner products, the elements of which are listed explicitly in Smyth and Peltier.$^{18}$

From the eigensolution, the modal structure of the perturbation can be reconstructed. Specifically, the structure of the perturbation kinetic energy field defined by
TABLE I. Growth rates for different spanwise wave numbers at time $t=50$ in simulations with $J=0.0$, 0.005, and 0.02. The top entry in each box is the growth rate of the most unstable mode and the bottom entry, where it appears, is the growth rate of the next most unstable mode. The complex part of the growth rate is the streamwise frequency. If the growth rate is pure real then the mode is a standing wave.

<table>
<thead>
<tr>
<th>$t=50$</th>
<th>$J=0.0$ ($N_R=29$)</th>
<th>$J=0.005$ ($N_R=27$)</th>
<th>$J=0.02$ ($N_R=25$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta=0.0$</td>
<td>0.063±0.169</td>
<td>0.067±1.144</td>
<td>0.140±0.169</td>
</tr>
<tr>
<td>$\beta=0.5$</td>
<td>0.058</td>
<td>0.068±0.01</td>
<td>0.142±0.080</td>
</tr>
<tr>
<td>$\beta=1.0$</td>
<td>0.083 (even)</td>
<td>0.083</td>
<td>0.163 (odd)</td>
</tr>
<tr>
<td>$\beta=1.5$</td>
<td>0.069 (odd)</td>
<td>0.079</td>
<td>0.158±0.083</td>
</tr>
<tr>
<td>$\beta=2.0$</td>
<td>0.084 (even)</td>
<td>0.091</td>
<td>0.172±0.088</td>
</tr>
<tr>
<td>$\beta=2.5$</td>
<td>0.070 (odd)</td>
<td>0.086</td>
<td>0.172±0.088</td>
</tr>
<tr>
<td>$\beta=3.0$</td>
<td>0.081 (even)</td>
<td>0.094</td>
<td>0.182 (odd)</td>
</tr>
<tr>
<td>$\beta=3.5$</td>
<td>0.069 (odd)</td>
<td>0.089</td>
<td>0.176±0.091</td>
</tr>
</tbody>
</table>

TABLE II. Growth rates for different spanwise wave numbers at time $t=100$ in simulations with $J=0.0$, 0.005, and 0.02.

<table>
<thead>
<tr>
<th>$t=100$</th>
<th>$J=0.0$ ($N_R=29$)</th>
<th>$J=0.005$ ($N_R=27$)</th>
<th>$J=0.02$ ($N_R=25$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta=0.0$</td>
<td>0.035</td>
<td>0.051±1.68</td>
<td>0.131±0.98</td>
</tr>
<tr>
<td>$\beta=0.5$</td>
<td>0.046</td>
<td>0.073</td>
<td>0.131±0.981</td>
</tr>
<tr>
<td>$\beta=1.0$</td>
<td>0.062</td>
<td>0.086</td>
<td>0.144 (even)</td>
</tr>
<tr>
<td>$\beta=1.5$</td>
<td>0.062</td>
<td>0.088</td>
<td>0.144 (even)</td>
</tr>
<tr>
<td>$\beta=2.0$</td>
<td>0.060</td>
<td>0.086</td>
<td>0.141 (even)</td>
</tr>
<tr>
<td>$\beta=2.5$</td>
<td>0.041</td>
<td>0.075</td>
<td>0.134 (odd)</td>
</tr>
<tr>
<td>$\beta=3.0$</td>
<td>0.052</td>
<td>0.080</td>
<td>0.137 (even)</td>
</tr>
<tr>
<td>$\beta=3.5$</td>
<td>0.029</td>
<td>0.073</td>
<td>0.123 (odd)</td>
</tr>
</tbody>
</table>

The growth rate $\beta$ is the growth rate of the most unstable mode and $\beta^*$ is the growth rate of the next most unstable mode.

$$\text{KE}' = \frac{1}{2} (u'^2 + v'^2 + w'^2)$$  \hspace{1cm} (28)

is examined to determine the source of energy which drives the growth of the mode. From the spanwise component of perturbation vorticity defined by

$$\omega' = u'_x - w'_x$$  \hspace{1cm} (29)

the development of spanwise undulations of vortex centres may be inferred. The amplitudes of the eigenfunctions shown are normalized so that the maximum value of KE' is unity.

Eigenvalues for increasing $\beta$ at times $t=50$ and 100 in simulations with $N^2=0.0, 0.005$, and 0.02 are given in Tables I and II. By noting the rate of convergence of the eigenvalues as $N_R$ is increased, the values quoted are accurately to within 2%. These values are calculated for simulations initialized in domains containing two wavelengths of the fastest growing mode of linear theory and vertical extent $L_z=10$ with $N_R=29$ for $N^2=0.0$, $N_R=27$ for $N^2=0.005$, and $N_R=25$ for $N^2=0.02$.

At time $t=50$ in the unstratified fluid simulation, the vortex sheet is well developed and pairing between vortices of like sign is not yet evident. The growth rate of the two-dimensional basic state at this time is $\sigma_{2D}=0.063$ which is smaller than the growth rate of the most unstable spanwise perturbation, $\sigma_{2D}=0.085$. However, the growth rates do not differ greatly and it is reasonable to suppose that the two-dimensional dynamics may persist for long times despite the presence of a three-dimensional disturbance. The growth rate of the three-dimensional disturbance is greatest for spanwise wave number $\beta^* \approx 1.3$, though the maximum is not sharply peaked (see Table I). For $J=0.005$, the spanwise perturbations generally develop at a faster rate and the wave number of the most unstable mode is greater. For $J=0.02$ the growth rates of the spanwise modes are generally more than twice as great as those of the modes with corresponding wave number for $J=0.0$. The wave number of the most unstable mode is not as great as in the $J=0.005$ case, however.

The characteristics of these disturbances at $t=50$ can be inferred from the contour plots in Fig. 14. The diagrams show the basic state vorticity ($\Omega$) in simulations for (a) $J=0$, (b) $J=0.005$, and (c) $J=0.02$. The contour fields of the spanwise perturbation vorticity ($\omega'$) of the most unstable mode are shown in (d), (e), and (f) and the kinetic energy of the corresponding modes ($\text{KE}'^*$) are shown in (g), (h), and (i). Contours of $\Omega$, $\omega'$, and $\text{KE}'^*$ are shown in intervals of 0.3, 2.0, and 0.25, respectively. (Note that the perturbation kinetic energy field, which is a quadratic quantity, is generally better resolved at intermediate values of $N_R$ than the basic state fields. Nonetheless, the finer details of the perturbation vorticity field are sufficiently well resolved for the purposes of the discussion here.) The mode which develops from the staggered vortex array in the nonlinear simulation of unstratified jet flow extracts energy from both the vortex cores and braids oriented between vortices of like sign. This result differs from those for the linear stability analyses of Kelvin–Helmholtz billows (i.e., Klaassen and Peltier\textsuperscript{[41]} in which it has been shown that three-dimensional perturbations of a single vortex sheet extract energy primarily from the braids. The result also requires a reinterpretation of the observations of Meiburg and Lasheras.\textsuperscript{[13]} Based upon their experiments, they explain that a braid centered instability developing in the region between vortex cores of like sign induces undulations in the cores. Our linear stability results show that undulations of the cores develop in tandem with braid modes. Indeed, perturbations with smaller spanwise wave number than $\beta^*$ are primarily core centered and perturbations with larger spanwise wave number are primarily braid centered. The perturbation vorticity field is such that undulations on vortices of opposite sign on either side of the jet maximum are in phase, and so correspond to an even (or sinuous) mode. The next most unstable mode, for
which the growth rates of modes for $\beta \approx \beta^*$ are shown in Table I, is braid and core centered like the even mode but it induces undulations of the vortex cores which are $180^\circ$ out of phase. In this case, the instability corresponds to an odd (or varicose) mode. Though the growth rate is smaller it may be observed nonetheless when forced explicitly, as demonstrated by Meiburg and Lasheras.\textsuperscript{13}

For $J=0.005$ at time $t=50$, the structure of the spanwise instability is qualitatively similar to that for $J=0.0$ at low wave numbers. The mode with the same wave number, as in the $J=0.0$ case, is driven by the mixed core-braid instability which is capable of extracting energy from the two-dimensional basic state into a mode of even (sinuous) symmetry driven by shear and buoyancy forces. However, the most unstable mode, the wave number of which is greater than that of the most unstable mode in the unstratified case, is driven primarily by the braid instability which extracts energy primarily from the regions of strong vorticity generated by baroclinic torques between the large-scale vortex cores. Like the $J=0.0$ case, the most unstable mode is a standing wave of even (sinuous) symmetry.

From diagrams (h) and (i) of Fig. 14, it is inferred that the most unstable mode which develops from the flow at time $t=50$ in the $J=0.02$ simulation is driven primarily by shear and buoyancy forces centered near the small-scale patches of baroclinically generated vorticity. The mode is a standing wave of odd (varicose) symmetry with spanwise wave number, $\beta^* \approx 2.1$, which extracts energy from the cores of small-scale vortices which are strained by the action of the large-scale motion.

In each of the three cases considered above, the growth rate of the most unstable mode is larger than the inertial rate of development of the basic state by roughly 30%. Thus it is reasonable to believe that the background state may continue to evolve as though unperturbed for moderately long times as there is no significant separation of time scales between the instability and the basic state. With this in mind the stability of the basic states of simulations is studied at time $t=100$. The diagrams (a) through (i) in Fig. 15 shown for time $t=100$ correspond to those of Fig. 14 and the intervals between contours are the same.

For $J=0.0$, the growth rate of the mode is nearly double that of the two-dimensional basic state. The mode extracts most of the perturbation energy from the braids between the vortices but, unlike the $t=50$ case, most of this energy is extracted from the saddle point located between the vortices of positive sign. The merging of the vortices of negative sign weakens the shear at the saddle point between them and thus reduces the strength of the three-dimensional instability. Thus pairing inhibits the development of three-dimensional braid modes, though long-wave core undulations prevail.

Though vortex pairing is not appreciable for $J=0.005$ at time $t=100$, the structure of the most unstable mode is qualitatively similar to that for $J=0.0$ in that the mode extracts energy from the region between vortices of positive...
The growth rate of this mode is larger than the most unstable mode for \( J = 0.0 \) and the spanwise wave number is larger.

For \( J = 0.02 \) at \( t = 100 \), the flow is characterized by turbulent patches wherein the flow is strained. The most unstable spanwise mode, which is a standing wave of even symmetry, extracts energy primarily from these regions. In comparison with the case of weakly stratified fluid, the growth rate of the three-dimensional perturbations is very much increased, though a smaller separation of time scales exists between the time scale of two-dimensional background flow evolution and three-dimensional perturbation.

It is therefore expected that the vortices generated in a stratified region will be strained both by the action of baroclinic torques and spanwise instability. The two-dimensional coherence of vortices in weakly stratified flow is destroyed by spanwise perturbations, though this process occurs on a larger time scale. As stated explicitly below, this suggests a highly plausible scenario in which strong internal wave emission from unstable jet flow may be realized.

**B. IGW emission in variable \( N^2 \) fluid**

Several simulations of the evolution of an unstable jet flow in variable \( N^2 \) fluid have been performed. As for the simulations in the preceding discussion, the vertical profile of horizontal velocity is assumed to be of the Bickley form. In these simulations, however, the background vertical density profile is defined such that \( N^2 = J \tanh^2(z/R) \), in which \( J \) is the bulk Richardson number and \( R \) denotes the length scale over which \( N^2 \) increases from 0 to \( J \) in the far field. In each simulation, the most unstable mode of linear theory is calculated and this mode is superposed on the background flow with amplitude set so that the maximum vertical velocity is initially 0.05. The horizontal extent of the channel is chosen to be equal to twice the wavelength of the most unstable mode of linear instability. To inhibit the reflection of waves from the upper and lower boundaries of the channel, motion near the boundaries is damped by the method described at the beginning of Sec. IV.

In these simulations, intense emission of IGW has been observed with \( J \) as low as 0.01 and as high as 1.0 for values of \( R \) between 3 and 5. These values do not represent extreme of the ranges of \( J \) and \( R \) which allow IGW emission, but they demonstrate the robustness of the emission mechanism that is being proposed. For this mechanism it is possible to estimate an optimal set of parameters which allow the largest vertical propagation of energy. In the central region of the jet, vortices develop in weakly stratified fluid when the above choice of the \( N^2 \) profile is made. The horizontal wave number of the street of vortices that develop is \( \alpha = 0.9 \) and these are advected at the speed at the injection point of the jet after the vortices are formed which is \( c^* = 0.3 \). Therefore, IGW are forced by the vortices with a frequency \( \omega^* = c^* \alpha = 0.3 \). If it is supposed that the flux of vertical energy is greatest when the vertical group velocity of IGW is largest in the uniform \( N^2 \) fluid on either flank of the jet, then from Eq. (21) the optimal value of \( J = (3/2)^{(1/2)} c^* \alpha = 0.15 \). Forcing of IGW is greatest when the vertical extent over which \( N^2 \) increases from 0 to \( J \) is comparable to the scale of the vortices, \( L_{r} \approx \pi / \alpha = 3 \).

Figure 16(a) shows the perturbation density field at time \( t = 100 \) shown over the entire domain of the channel with vertical extent \( L_z = 80 \) and horizontal extent \( L_x \approx 14 \) corresponding to two wavelengths of the most unstable mode of the initial flow. Contours increment by 0.1. (b) Reynolds stress profiles and (c) vertical profiles of horizontal phase speed (light line) and vertical phase speed (heavy line) are shown at the same time. The vertical axes of these graphs extend across the full extent of the channel. The phase velocity is calculated by assuming a modal form of the wave number 2 component of the fluctuation density field. The positive Reynolds stress and negative vertical phase speed on the upper flank of the jet are both indicative of upward radiating energy. Similarly energy is shown to radiate downwards from the bottom flanks of the jet. In (d)–(f) are shown the corresponding diagrams from the simulation at time \( t = 200 \). Wave packets on either flank of the jet are clearly independent of the central flow and continue to propagate outward.

**FIG. 16.** Results from simulations in variable \( N^2 = J \tanh^2(z/R) \) fluid for \( J = 0.1 \) and \( R = 3 \). In (a) are contours of fluctuation density at time \( t = 100 \) shown over the entire domain of the channel with vertical extent \( L_z = 80 \) and horizontal extent \( L_x \approx 14 \) corresponding to two wavelengths of the most unstable mode of the initial flow. Contours increment by 0.1. (b) Reynolds stress profiles and (c) vertical profiles of horizontal phase speed (light line) and vertical phase speed (heavy line) are shown at the same time. The vertical axes of these graphs extend across the full extent of the channel. The phase velocity is calculated by assuming a modal form of the wave number 2 component of the fluctuation density field. The positive Reynolds stress and negative vertical phase speed on the upper flank of the jet are both indicative of upward radiating energy. Similarly energy is shown to radiate downwards from the bottom flanks of the jet. In (d)–(f) are shown the corresponding diagrams from the simulation at time \( t = 200 \). Wave packets on either flank of the jet are clearly independent of the central flow and continue to propagate outward.

\[ r = \langle u' \omega' \rangle_x \]  
where \( \langle \cdot \rangle \) denotes the horizontal average [e.g., \( \langle f \rangle_x = (1/L_x) \int_0^{L_x} f \, dx \)]. The graph is shown on the same vertical scale as the contour plot and the horizontal axis ranges from \(-0.001\) to \(0.001\). The small horizontal scale is chosen to emphasize the vertical transport of horizontal momentum by IGW. On the top flank of the jet, the positive Reynolds stress is indicative of the upward transport of positive horizontal momentum. The Reynolds stress on the lower flank of the jet is negative, since forward moment-
tum is transported downward. The magnitude of $\tau$ is largest on either flank of the jet for $z \approx \pm 10$, the level which is taken to correspond to the center of the wave packet.

Because the internal waves are excited by the primary jet instability of wave number 2, the horizontal wavelength of the internal waves is half the horizontal extent of the channel. Indeed, it is possible to estimate the two-dimensional wave number and phase velocity from the vertical profile of the wave number 2 component of the fluctuation density $p_2(z)$ and its time derivative $p_2'(z)$. This calculation is described in detail in the Appendix. Figure 16(c) shows the vertical profile of the horizontal (light line) and vertical (heavy line) phase speed. The vertical axes range over the full vertical extent of the channel and the horizontal axis ranges from $-2$ to $2$. Because $c_{ph}$ fluctuates rapidly in regions where the flow evolves on short time and length scales, the vertical phase speed is plotted only over the vertical range corresponding to positive (negative) Reynolds stress on the upper (lower) flank of the jet. The two components of the phase speed vary rapidly near the top and bottom of the domain shown, these variations being greater at early times because of the numerical sensitivity which is introduced in the calculation of $c_{ph}$ for small $|\tilde{\omega}|$ and $|\tilde{\eta}|$ in (A4). The slow variation of these curves on either flank of the jet reflects the approximate constancy of $m$ and $\omega$ in these ranges and provides some empirical verification of the approximations $m \approx \bar{m}$ and $\omega = \bar{\omega}$, as per Eqs. (A2) and (A3). The horizontal phase speed of the wave number 2 mode over $-30 < z < -10$ and $10 < z < 30$ is $0.33 \pm 0.02$ which is approximately equal to the horizontal speed of the large eddies that propagate on either flank of the jet. The vertical phase speed exhibits a broad negative (positive) peak on the upper (lower) flank of the jet where IGW are most intense, near $z \approx \pm 8$. This observation is consistent with the requirement for downward phase propagation and upward group velocity of IGW being emitted from the jet.

Figures 16(d)–16(f) show perturbation density profiles, Reynolds stress, and horizontal and vertical phase speeds, respectively, for the same simulation at time $t=200$. The contour intervals and axes ranges are the same as the corresponding diagrams (a)–(c). The contour plot shows that the wave packet of IGW launched by the eddy disturbance of wave number 2 continues to propagate upward. In comparison with the flow at time $t=100$, the largest magnitude of the Reynolds stress on either flank of the jet occurs at a larger vertical distance from the jet at this time ($z \approx \pm 16$). Though $|\tau|$ is larger over a greater vertical range, the peak value of the Reynolds stress is not as large as that at time $t=100$. This is attributed to the effects of dispersion and diffusion. Comparing Figs. 16(c) and 16(f), the horizontal phase speed of the wave packet at time $t=200$ is $0.33 \pm 0.02$ over the ranges $-40 < z < -15$ and $15 < z < 40$ which is virtually unchanged from the speed over these vertical ranges at $t=100$. However, the peak negative (positive) vertical phase speed on the upper (lower) flank of the jet near $z \approx 20$ ($-20$) is smaller in absolute value than the peak value of the $t=100$ case.

To demonstrate the robustness of the IGW emission mechanism, the results of the simulation with $J=1.0$ and $R=5$ are presented in Fig. 17. Like Fig. 16, Figs. 17(a)–17(c) correspond to contours of fluctuation density, vertical profiles of Reynolds stress, and vertical profiles of horizontal and vertical phase speed, respectively, shown at time $t=50$. The vertical wave number of IGW in this case is larger and many vertical wavelengths of the radiating waves are apparent in both wave packets.

V. CONCLUSIONS

It has been demonstrated that the evolution of a symmetric jet in stratified fluid is remarkably different from that which occurs in unstratified fluid. For $N^2=J=0.005$ (constant), vortices which develop on either flank of the jet are strained by buoyancy forces and pairing is inhibited between the significantly weakened vortex cores. Between the large-scale vortex cores which develop on either flank of the jet in simulations for which $N^2=0.02$, small-scale centers of intense vorticity are generated by baroclinic torques and strong ("turbulent") mixing of the fluid occurs within small regions located near the large-scale eddies. The inhibition of pairing in the jet therefore occurs not only because buoyant restoring forces are larger and
inhibit the vertical motion of large-scale eddies, but also because the mean flow is significantly weakened by the turbulent mixing process.

Because vortices are intensely strained and thereby "shredded" even in moderately stratified fluid, the ambient stratified medium is only weakly forced by the eddies and so internal gravity waves are radiated into the external medium with small amplitude. An investigation of the stability of the two-dimensional flow to spanwise perturbations has shown that the small-scale structures in moderately stratified fluid are further susceptible to motion which is fully three dimensional. In reality, therefore, jet flow with uniform and moderate \( N^2 \) is a poor candidate as a source of IGW radiation. However, it has been demonstrated through investigations of simulations with \( N^2 = J \tanh^2(z/R) \) that internal gravity waves of significant amplitude may be generated by the large-scale eddies that develop in the weakly stratified region of the fluid where the shear is strong. This form of \( N^2 \) may be generated naturally by redistribution of the background vertical density variation by mixing processes in a jet whose strength is a slowly varying function of time. Internal gravity waves with nonstationary phase speed with respect to the ground have been observed in the atmosphere but attempts to explain their existence as a result of generation by shear instability have as yet failed. It is believed that this work advances a new and highly plausible explanation for the existence of such waves.

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APPENDIX: CALCULATION OF PHASE VELOCITY

It is supposed that the wave number \( n \) component of the fluctuation density \( \rho_n \) can be written in modal form, i.e.,

\[
\rho_n = \exp[i(k \cdot x - \omega t)],
\]

in which \( k = (k, m) \), with real and constant horizontal wave number \( k = n \alpha \) and complex vertical wave number \( m = m(z) \) and complex \( \omega = \omega(z) \). Both \( m \) and \( \omega \) are assumed to be only weakly \( z \) dependent in the vertical range of interest. Then the vertical wave number of the wave number 2 mode for each \( z \) is

\[
m \approx \tilde{m} + \omega t \frac{d\tilde{m}}{dz} = -t \frac{d}{dz} \left( \log |\rho_2| + t \arg \rho_2 \right) \tag{A2}
\]

and the frequency of this mode for each \( z \) is

\[
\omega \approx \tilde{\omega} + i\tilde{\omega} = \frac{\rho_2}{\rho_1} \tag{A3}
\]

Equations (A2) and (A3) are exact if \( m \) and \( \omega \) are independent of \( z \). However, the approximations are reasonable provided \( |d\tilde{m}/dz| \ll 1 \) and \( |d\tilde{\omega}/dz| \ll 1 \), a condition which is found to be valid over the range in which IGW exist with significant amplitude. From Eqs. (A2) and (A3) the horizontal and vertical components of the phase velocity \( c_p = (c_{px}, c_{pz}) \) for each \( z \) are defined to be the real parts of the frequency divided by the (real) horizontal and (complex) vertical components of the wave number 2 mode, respectively:

\[
c_p = \left( \frac{\tilde{\omega} - \tilde{\omega} \tilde{m} \cdot \tilde{e}_z}{2\pi} \right). \tag{A4}
\]


Without the introduction of small amplitude noise on the basic state fields, no pairing occurs in a channel of long horizontal extent. The amplitude of the noise is sufficiently small, however, so that the most unstable mode of linear theory initially grows at a rate predicted by the Galerkin analysis.


