Interfacial Mixing in Viscous Pipe Flows
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Abstract

We report on the results of laboratory experiments and numerical simulations examining the mixing between two viscous fluids in laminar and turbulent pipe flow. The work is motivated by an attempt to minimise both mixing and turbulent drag by identifying and characterising the turbulent transition. A series of experiments determine this transition as a function of the different viscosities of the two fluids. An extreme sensitivity to density differences is noted for viscous fluids in laminar pipe flow. An analytic model for turbulent mixing at an interface is derived and solved numerically.

1. Background

Oil is often transported through thousands of kilometers of pipeline between the supplier and consumer. The pipe diameters range from 30 to 120 cm (12 to 48 inches) with capacities ranging from 40,000 to 700,000 barrels per day* (1 U.S. barrel oil = 35.0 Canadian gallons = 159 L). A single pipeline may transport a range of oil products ranging from heavy to light crudes. The grades of oil are distinguished primarily by their viscosity\(^1\): heavy crudes have kinematic viscosities at 10\(^\circ\)C ranging from 143 to 350 cS, medium crudes from 35 to 76 cS, and light crudes from 5 to 30 cS. (1 cS, a ‘centiStoke’, is 0.01cm\(^2\)/s, approximately the kinematic viscosity of fresh water). Refined products can have still lower viscosities.

Because the number of pipelines is limited it is necessary to ship each grade of oil in batches. Typically a heavy crude batch train is followed by a medium crude batch train which is then followed by a light crude batch train. The order then repeats. The volume of each batch is approximately 100,000 barrels, which can occupy over 100 km of pipeline.

The oil is driven through the pipeline by a series of pumping stations, spaced approximately 50 km apart. The pressure at the discharging station is near the maximum allowed by the pipe and the pressure steadily decreases with distance from the station. The pressure decreases linearly with distance if the oil in the pipe is uniform, but the rate of decrease is greater for heavier (more viscous) grades of crude.

At present, the along-pipe pressure gradient is established so that the flow speed is approximately 1.3 m/s along the pipe (about 3 miles/hour). At these speeds the flow in the pipe, even for heavy crude oil, is turbulent. Pumping oil at turbulent velocities has a number of advantages. First, at the interface between batches, deposited materials such

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\(^{1}\) Heavier crudes also have a higher sulphur content than lighter grades.
as silt and wax are effectively scrubbed from the walls of the pipe reducing their rate of build up. Second, there is effectively less mixing at the interface between batches in a turbulent pipe flow compared with a laminar flow. This apparent contradiction can be explained as follows. In a laminar flow the fluid is effectively “stuck” to the side walls of the pipe. That is, the along-pipe velocity smoothly varies from no flow at the pipe walls to a maximum velocity at the centre of the pipe. Thus an intruding fluid will flow a significant distance along the centre of the pipe before the fluid at the side walls is eroded away by viscous stresses. On the other hand, if the flow is turbulent, the boundary layer nearest the pipe walls (where the flow velocity rapidly changes) is thin and the concentration of the mixture of fluids over any cross-section of the pipe is relatively uniform.

A significant disadvantage to pumping oil at turbulent velocities is that turbulence significantly increases the drag of the flow in the pipe: the fluid behaves as though it is more viscous. As a result, more energy is required to maintain the flow velocity at a specific speed.

One way to reduce operation costs would be to pump oil at rates that are slower, but not so slow that the flow is laminar and mixing between the batches is enhanced. Ideally one would like to pump the oil at an optimal rate in which the flow is transitional between laminar and turbulent. Thus there would be less drag while transient turbulent bursts would reduce along-pipe mixing.

It is not well understood how two fluids mix in such a transitional flow. For example, suppose two fluids flow through the pipe at the same rate but one fluid (e.g. heavy crude) is so viscous that it is in laminar flow while the viscosity of the other fluid (e.g. light crude) is so small that it flows turbulently. What is the structure of the interface between these fluids? What is the mixing length - the distance over which both fluids are intermixed in significant proportions?

To answer these questions, we have performed a series of laboratory experiments to measure at what velocities and relative viscosities the flow of both fluids is laminar, turbulent or transitional. The velocity profiles of the fluids in each regime has been determined, and from these the effective drag and mixing length may be estimated.

Another method to reduce drag has been explored: we examine how the flow velocity and structure changes as a fluid moves through pipe with walls of different roughness. Flow regimes have been examined in pipes that are uniformly roughened, lengthwise roughened and “rifled”.

In order to gain insight into interfacial mixing between two fluids in turbulent pipe flow, an approximate empirical equation has been derived and solved numerically.

2. Theoretical Preliminaries

Here the classical theories of transport of uniform fluid in circular pipe flow are reviewed. The discussion below summarizes analytic theories of laminar pipe flows and mixing in both laminar and turbulent pipe flows.

(a) Steady-State Laminar Flow

Whether a fluid flow is laminar or turbulent can be assessed by evaluating the Reynolds number, Re, defined here to be the ratio $UR/\nu$, where $U$ is the mean flow velocity, $R$ is the pipe radius, and $\nu$ is the kinematic viscosity of the fluid. Typically, for Reynolds numbers smaller than 2000 the flow is sufficiently slow or the fluid is sufficiently viscous
In the steady flow of a uniform fluid, the velocity profile is parabolic, as illustrated schematically in Figure 1a. Explicitly, the along-pipe velocity \( u_e \) as a function of the radius \( r \) from centre of the pipe is

\[
u_e(r) = U_{\text{max}} \left[ 1 - \left( \frac{r}{R} \right)^2 \right].
\]

where \( U_{\text{max}} = \frac{GR^2}{4\mu} \) is the maximum flow speed. The derivation of (1) is given in Appendix A. The maximum velocity, \( U_{\text{max}} \), is a function of \( G \), the along-pipe pressure gradient (which is assumed to be constant) and \( \mu \), the molecular viscosity of the fluid. The mean flow velocity in a circular pipe is \( \bar{U} = U_{\text{max}}/2 \).

If the pressure gradient is assumed to result from the difference in height, \( \Delta h \), between two reservoirs separated by the pipe length \( L \), then the maximum velocity can be written

\[
U_{\text{max}} = \frac{g\Delta h}{\nu L}
\]

in which \( g = 980 \, \text{cm/s}^2 \) is the acceleration of gravity.

(b) Starting Laminar Flow

If a uniform fluid in a pipe is initially stationary and an along-pipe pressure gradient is suddenly imposed at time \( t = 0 \), then the flow will gradually accelerate until the flow reaches the steady state given by (1).

The axisymmetric velocity profile over time \( t \), \( u(r,t) \), is given in the appendix by equation (27). The theory shows that the flow approaches steady state on a dominant time scale given by the first term in the series solution. Explicitly the steady state is approached after time \( \tau_1 \simeq 0.17R^3/\nu \). At this time the flow speed at the centre of the pipe is \( u(0,\tau_1) \simeq U_{\text{max}} \left[ 1 - 8 \frac{R^2}{2\frac{\pi^2 \tau_1}{11}} \right] \simeq 0.59 U_{\text{max}} \). Figure 2a shows velocity profiles at times \( \tau_1/2, \tau_1 \) and \( 2\tau_1 \) after the pressure gradient is applied.

In our experiments the velocity profile is measured indirectly by taking vertical time series. The arrival time is recorded of fluid starting at the valve and propagating a fixed distance, \( d \), along the pipe after the valve is opened. The result is compared with the predicted arrival time \( t = t(r,d) \) determined by implicitly numerically solving the equation: \( d = \int_0^t u(r,t) \, dt \), where \( u \) is given by (27). The predicted arrival times are shown in Figure 2b for the flow at a distance \( d = U_{\text{max}} \tau_1/2, U_{\text{max}} \tau_1 \) and \( U_{\text{max}} 2\tau_1 \).

(c) Laminar-Turbulent Transition

For Reynolds numbers much larger than 2000, the flow is turbulent. (Although under ideal conditions, the flow can be laminar at Reynolds numbers as high as 10,000.) In turbulent pipe flows, across-pipe mixing makes the (time-averaged) velocity profile approximately uniform over most of the cross-section of the pipe (for example, see Figure 1b). Only within thin boundary layers near the pipe walls does the average along-pipe speed decrease to zero.

The circumstance may arise in which two fluids flow through the pipe at the same velocity, and while one fluid is sufficiently viscous that it is in laminar flow, the other fluid is less viscous and turbulent. What then is the structure of the interface between these fluids? The structure of the interface in this transitional state is examined in detail here.
a) Laminar Pipe Flow

b) Turbulent Pipe Flow

FIG. 1. Schematic illustrating typical mean velocity profiles of a) laminar and b) turbulent pipe flow.

\[(d)\quad \text{Mixing Length}\]

In laminar flow, the mixing length between two fluids increases over time. To illustrate this, imagine a uniform fluid in steady, laminar flow which is dyed initially over the left half-length of the pipe (positions \(x < 0\)). Let \(c(x, t)\) be the concentration of dye over time, \(t\), at a position \(x\) along the pipe. Initially,

\[
c(x, 0) = \begin{cases} 
1 & x < 0, \\ 
0 & x > 0.
\end{cases}
\]  

(3)

Assuming the dye spreads as a consequence of the along-pipe flow alone (ignoring molecular diffusion), then using (1) gives

\[
c(x, t) = \begin{cases} 
1 & x < 0, \\ 
1 - \frac{x}{U_{\max} t} & 0 < x < U_{\max} t, \\ 
0 & x > U_{\max} t.
\end{cases}
\]  

(4)
The mixing length, $L_{99}$, is defined as the distance over which the concentration is both less than 99% and more than 0.01% pure. Thus, in the circumstance imagined above $L_{99} = 0.98 U_{\text{max}} t$; the mixing length grows linearly in time.

If the effects of molecular diffusion are added, the mixing length is found to depend upon an effective diffusion constant, $K_L$, which is a function of the diffusivity, the mean flow speed and the pipe radius (Taylor (1953)). Explicitly,

$$K_L = \frac{1}{48} \left( \frac{R^2 \overline{U}^3}{\nu} \right).$$  \hspace{1cm} (5)

Here $\overline{U} = U_{\text{max}}/2$ is the mean speed of the flow. Thus if the concentration is given initially by (3), then moving with the flow at speed $\overline{U}$ the mixing length is found to grow as $L_\sigma = \sqrt{2K_L \overline{U}}$, where $\sigma$ is the standard deviation. In terms of the 99% mixing length,
one finds
\[ L_{99} \approx 4.66L_{\sigma} \approx 0.95\sqrt{Re\sqrt{RU}}, \]  
where \( Re \) is the Reynolds number. Near the turbulent transition, for \( Re = 2000 \), \( L_{99} \approx 43\sqrt{RU} \).

In turbulent flow, the across-pipe mixing is controlled by turbulent diffusion, rather than shear. The growth of a turbulent mixing region with respect to a frame of reference moving with the mean speed of the flow, can be modelled by a diffusion process where the diffusion constant is a function of the mean flow speed and pipe radius (Taylor (1954)). Explicitly, the diffusion constant is
\[ K_T \approx 10.06RU. \]  
In the above example, the 99% mixing length, is
\[ L_{99} \approx 20.5\sqrt{RU}. \]  
Comparing (6) and (8) one finds that along-pipe turbulent mixing is less efficient than that due to diffusive laminar flow near the turbulent transition.

3. Experimental Set-up

A schematic showing the experimental set-up is shown in Figure 3. Two reservoirs are filled with solutions of different viscosities. The horizontal area of each reservoir is 59 by 59cm. The area is set to be sufficiently large that the depth of fluid in each reservoir changes negligibly during the course of a single experiment.

Typically one reservoir is filled with a glycerol solution, the concentration of which determines its kinematic viscosity, \( \nu \). Glycerol solutions can have viscosities ranging from 0.01 cm\(^2\)/s (1cS) for fresh water to over 1.0 cm\(^2\)/s (100cS) for an 80% glycerol solution. The second reservoir is filled with a salt (NaCl) water solution, the concentration of salt being set so that the densities of the two solutions are approximately equal. Hydrometers measure the density accurate to 0.0005 g/cm\(^3\). The kinematic viscosity of salt water varies negligibly for the small salt concentrations used in these experiments. For visualisation purposes, a small amount of dark blue dye is added to the glycerol solution.

The two reservoirs are connected to each other through 3.8 m of clear vinyl tubing and a 2 m long acrylic pipe. Both the tubing and pipe have a 1.27 cm (1/2 inch) inner diameter. A valve at one end of the pipe is closed while the reservoirs, tubing and pipe are filled with the glycerol and salt solution. One solution is filled in the reservoir to a greater height than the other so that when the valve is opened a constant pressure gradient is immediately established along the length of the pipe. The pressure gradient acts to accelerate the fluid through the pipe until viscous and turbulent stresses bring the flow to a mean steady state. The horizontal cross-sectional area of the two reservoirs is set to be so large that the height difference between the two fluids does not change by more than 0.2 cm over the course of an experiment. Thus, for typical initial height differences greater than 2 cm, the pressure gradient along the tube is approximately constant during an experiment.

The fluid in the reservoir with the greater height is referred to hereafter as the “intruding fluid” whereas the fluid in the reservoir with the lower height, which also fills the pipe initially, is referred to hereafter as the “ambient fluid”.

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FIG. 3. Set-up of experiment: once the valve is opened, a fluid of kinematic viscosity $\nu_i$ intrudes along an acrylic pipe filled initially with fluid of kinematic viscosity $\nu_a$. In the test section, the pipe is immersed in sugar solution of such large concentration that its index of refraction matches that of the acrylic pipe. The effects of parallax are thus reduced.

Experiments have also been performed in which the flow runs horizontally from one reservoir and leaves the pipe at an open end. In this circumstance, in which there are no bends in the pipe, the flow regimes and dynamics are not qualitatively different from those reported in the configuration shown in Figure 3.

To examine the flow structures, it is convenient to reduce the effect of parallax. When looking through the side of the pipe, the clear pipe wall acts as a lens that differentially magnifies and distorts images flow in the pipe. The effect of parallax is reduced by constructing a 20cm long rectangular trough around the pipe in a test section near its middle (see Figure 3). The section is filled with a 50% sugar solution whose index of refraction is close to that of the acrylic pipe ($n \approx 1.42$). Thus light follows an approximately straight path from the fluid in the pipe through the fluid in the trough to an observer.

Figure 4 shows images of the pipe before and after the sugar solution is added to the
a) No sugar solution added

b) Sugar solution added

FIG. 4. Test section a) before and b) after the addition of a saturated sugar to the trough surrounding the pipe. Parallax effects are eliminated in b).

test section. In a) the fluid in the tube appears to entirely fill the tube to the outer walls. (In particular, note the bubbles near the top left hand side of the image.) In b), with the effect of parallax eliminated, the thickness of the pipe walls is evident and the extent of the pipe interior is accurately represented.

The test section is illuminated from behind by a fluorescent light source placed approximately 0.5m away. Mylar (semi-translucent) paper is attached to the rear of the test section to provide an approximately uniform background illumination.

A series of experiments have been performed in a variety of acrylic pipes. Most experiments on flow structures are performed in a pipe with smooth inner walls. The
laminar to turbulent transition is examined for this pipe and also in acrylic pipes with uniformly roughened inner walls, lengthwise striations and “rifling” (spiral striations). In the last three cases, a 10 cm region in the middle of the pipe is left unroughened so that the flow patterns may clearly be observed in the test section. The roughness of the uniformly roughened pipe is $R_a \approx 3.30\mu m$ (130µin).

To record the flow structures, a digital camera is positioned between 1.5 and 2.0m from the pipe and is focussed on the 20cm wide test section. The images are later digitised, manipulated and analysed on a computer equipped with frame grabber card and flow visualisation software.

In some of the analyses reported here, vertical time series are extracted from the digitised movies of the experiments. In this procedure, the intensity of light from a vertical cross section of the pipe is recorded over time. The time difference between successive vertical “slices” may be as small as 0.017 seconds.

As the ambient fluid initially flows through the test section, there is little change in the light intensity. However, as the dyed, intruding fluid enters the test section, the intensity of light reaching the camera decreases. The vertical time series thus records the arrival time of intruding fluid reaching the test section. In a turbulent flow the intruding fluid is expected to reach the test section at the same time over the whole depth of the fluid, whereas in a laminar flow the fluid is expected to arrive first near the centre of the pipe and later near the pipe walls.

The varying intensity over time at a fixed vertical position also gives a measure of the degree of mixing between the two fluids at the interface. A gradual change in intensity would occur if the interface between the ambient and intruding fluid is diffuse. A rapid change would occur for a sharp interface.

4. Flow Regimes

A series of experiments have been performed in which a fluid of one viscosity $\nu_i$ intrudes into an ambient fluid of viscosity $\nu_a$. The density difference between the two fluids is negligible (less than 0.1 percent). The pressure gradient along the tube, and hence the velocity of fluid in the tube is increased by increasing the initial difference of depths, $\Delta h$, between the two fluids in the reservoirs.

We define a Reynolds number for the ambient fluid in the pipe to be

$$Re_a = \frac{\overline{U} R}{\nu_a} = \frac{R^3 g \Delta h}{8 \nu_a^2 L}. \quad (9)$$

Similarly, we define a Reynolds number for the intruding fluid to be

$$Re_i = \frac{\overline{U} R}{\nu_i} = \frac{R^3 g \Delta h}{8 \nu_i^2 L}. \quad (10)$$

Whether the pipe flow is laminar or turbulent may thereby be assessed by determining whether both Reynolds numbers are small or large, respectively. A series of experiments have been performed to establish the structure of the interface as a function of the Reynolds numbers of the two fluids.

(a) Time Series

Figure 5 shows vertical time series of uniform pipe flow in laminar, transitional and fully turbulent flow. The extent of the time axis varies between the three plots, which are
FIG. 5. Vertical time series of a) laminar, b) transitional and c) turbulent flow. The centre of the pipe is at $r = 0$. Time $t = 0$ corresponds to the time at which the valve at one end of the pipe is opened.

shown to emphasize the structure of the interface between the dyed and undyed fluid. In laminar flow, (Fig. 5a), the interface is curved. The flow propagates most quickly along the centre of the pipe and significant time passes before the dyed fluid approaches the side walls of the pipe. The mixing length in this case is quite long.

In turbulent flow, (Fig. 5c), the interface is almost vertical. This illustrates that
velocity of the flow across the pipe is almost uniform and that the mixing length is relatively short.

In the transitional case, (Fig. 5b), a curved interface near the centre of the pipe precedes rapid vertical mixing. This case represents the best compromise between energy savings by pumping the fluid at almost laminar flow speeds while, transient turbulent bursts enhance interfacial mixing.

\[ (b) \text{ Laminar-Turbulent Transition} \]

Figure 6a shows a scatter plot indicating the flow regimes in a smooth pipe. The open circles represent experiments whose flows are observed to be laminar. Crosses are plotted for experiments with fully turbulent flows and solid triangles are plotted for transitional cases. In experiments for which the ambient and intruding fluids have the same viscosity (those lying on the diagonal line in the figure) the flow is observed to be transitional for \( Re_c = 2100 \pm 500 \). When the viscosity of the two fluids differs, the flow is found to be laminar if the Reynolds numbers of both fluids are less than this critical value. It is interesting to note that if the intruding fluid, for example, has Reynolds number moderately smaller than \( Re_c \), then the interface remains laminar even if the ambient fluid is moderately larger than \( Re_c \). Likewise if the ambient fluid has Reynolds number less than \( Re_c \), then the interface remains laminar even if the intruding fluid has Reynolds number moderately larger than \( Re_c \).

In summary, the interface is generally stabilized by the presence of at least one fluid in laminar flow.

In the laminar regime the shape of the interface is found to be extremely sensitive to the density difference between the two fluids. Indeed, if the viscosity difference between the two fluids is large, the flow exhibits an instability by which the maximum speed of the intruding fluid near the interface is not along the centre of the pipe, but instead flows

\[ (c) \text{ Sensitivity to density differences} \]

In the laminar regime the shape of the interface is found to be extremely sensitive to the density difference between the two fluids. Indeed, if the viscosity difference between the two fluids is large, the flow exhibits an instability by which the maximum speed of the intruding fluid near the interface is not along the centre of the pipe, but instead flows
FIG. 6. Stability regimes of viscous flow in a) smooth and b) roughened pipes. Open circles, solid triangles and crosses are plotted for experiments in which the flow is laminar, transitional, and turbulent, respectively. The diagonal line indicates experiments for which the ambient and intruding fluid have the same viscosity.

close to the pipe bottom or close to the pipe top depending on whether the intruding fluid is more or less dense than the ambient fluid, respectively.

Figure 7 shows vertical time series taken from experiments in which a viscous fluid intrudes into a significantly less viscous fluid. Both fluids are in laminar flow.

In Figs. 7a and b, the intruding fluid is 20 times more viscous than the ambient. For the case shown in Fig. 7a, the intruding fluid is approximately 0.2(±0.05)% less dense than the ambient fluid. Despite the small density difference, there is an obvious asymmetry in the flow profile: the leading intrusion runs closer to the bottom than the top of the pipe.

The intruding fluid is 0.05% less dense for the case shown in Fig. 7b. This presumably negligible density difference gives rise to a qualitatively different flow profile. Ahead of the intrusion, suspended particles reveal that the ambient fluid has the symmetric profile
expected for uniform pipe flow. However, the interface between the ambient and the intrusion is asymmetric: it arrives first along the top half of the pipe and the fluid in the lower half of the pipe arrives at a much later time.

The results are similar if the viscosity difference between the intruding fluid and ambient fluid is not so large, as shown in Figs. 7c and d. In these cases the intruding fluid is 2.3 times more viscous than the ambient and the density differences are negligibly small, as indicated on the plots. One effect of smaller viscosity differences is that the mixing length, as measured by the time required before the fluid fills the pipe, is much smaller.
5. Nonlinear Dispersion in Turbulent Pipe Flow

A series of numerical experiments are performed to determine the mixing dynamics when one fluid follows another in turbulent flow through a pipe. Initially, the intruding fluid occupies $x < 0$ and the ambient fluid occupies $x > 0$. As time progresses the interface between the two fluids will be mixed by turbulent dispersion. A model has been developed to determine the turbulent mixing process using a turbulent dispersion coefficient that depends on the gradient of the concentration of the intruding fluid in the interfacial region.

(a) Governing Equations

Since across-pipe mixing in turbulent flows occurs over very fast time scales in relation to the mean flow velocity, the radial dependence of the interfacial mixing process can be neglected. The time-averaged vertical velocity is zero. Assuming perfect mixing of two miscible fluids, let $c(x,t)$ be the volume fraction of intruding fluid at position $x$ and at time $t$. The turbulent dispersion is then governed by the diffusion equation

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left( D \frac{\partial c}{\partial x} \right)$$

where $D$ is the dispersion coefficient and we are in a frame of reference moving with the mean velocity of the flow. We assume a turbulent dispersion coefficient of the form

$$D = K_1 + \frac{1}{2} K_2 \frac{\partial c}{\partial x}.$$  \(1\)

The equation for the dispersion is then

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left[ \left( K_1 + \frac{1}{2} K_2 \frac{\partial c}{\partial x} \right) \frac{\partial c}{\partial x} \right]$$

or

$$\frac{\partial c}{\partial t} = K_1 \frac{\partial^2 c}{\partial^2 x} + K_2 \frac{\partial^3 c}{\partial^3 x} \frac{\partial c}{\partial x}. $$

With $K_2 = 0$, equation (14) describes the usual “Fickian diffusion” in which $K_1$ represents the turbulent diffusion coefficient. The effect of the term involving $K_2$ is referred to hereafter as “gradient diffusion”.

(b) Exact Solution

If gradient diffusion is neglected ($K_2 = 0$) and the pipe is of infinite extent, then an exact solution exists for the Fickian diffusion equation given by

$$\frac{\partial c}{\partial t} = K_1 \frac{\partial^2 c}{\partial x^2}$$

with initial condition

$$c(x,0) = \begin{cases} 1 & x < 0, \\ 0 & x > 0, \end{cases}$$

(16)
and with boundary condition

\[ c(x, \infty) = \begin{cases} 1 & x \to -\infty, \\ 0 & x \to \infty. \end{cases} \]  

(17)

The solution to this set of equations is given in terms of the error function:

\[ c(x, t) = \frac{1}{2} - \frac{1}{2} \text{erf} \left( \frac{x}{\sqrt{4K_1t}} \right). \]  

(18)

Explicitly,

\[ c(x, t) = \frac{1}{2} - \frac{1}{\pi} \int_0^{\infty} e^{-r^2} \, dr. \]  

(19)

The mixing length can be determined from the gradient of the concentration: the position where the change in concentration exceeds a minimum threshold denotes the edge of the mixed region. A measure of the spread in time of the mixed region is given by the spread in time of the concentration gradient. In this case, the gradient is equal to a normal distribution with a standard deviation of \( \sigma = \sqrt{2K_1t} \).

\[ \frac{\partial c}{\partial x} = -\frac{1}{\sqrt{2\pi\sigma}} \exp \left( -\frac{x^2}{2\sigma^2} \right) \]  

(20)

The spread \( \sigma(t) \) of the concentration gradient, and therefore the spread of the mixed region, increases with time according to the power law \( \sigma \propto t^\gamma \) where \( \gamma = 0.5 \). The sharpness of the interface is measured by the kurtosis, which for a normal distribution is exactly zero.

\[ \text{Numerical Solution} \]

The fully nonlinear dispersion equation (14) is solved by numerical methods. The domain for the problem is discretized using \( N + 1 \) equal grid points of spacing \( \Delta x \), ranging from \( x_0 = -1 \) to \( x_N = 1 \). To model an effectively infinite domain, the turbulent interface is localized around \( x = 0 \) for small values of \( x \). Second-order finite difference equations are used to discretize the spatial derivatives and the equations are stepped forward in time by \( \Delta t \) using a leapfrog scheme with an Euler backstep at regular time intervals to minimize splitting errors.

The boundary conditions are

\[ c(-1, t) = 1, \quad c(1, t) = 0 \]  

(21)

and the smooth initial condition is

\[ c(x, 0) = \frac{1}{2} - \frac{1}{2} \text{erf} \left( \frac{x}{L} \right) \]  

(22)

This initial condition corresponds to the exact solution of the diffusion equation after time \( t_0 = \frac{L^2}{4K_1} \). The initial concentration and initial concentration gradient are illustrated in Figure 8.

The numerical solution is tested by comparing results from simulations with \( K_1 = 0 \) to the analytical solution. Simulations have been performed for varying \( \Delta x \), \( \Delta t \), and \( K_1 \) values and for varying \( L \) values in the initial condition. In all trials, the numerically
calculated value for the standard deviation $\sigma(t)$ of the concentration gradient $\frac{\partial c}{\partial x}$ agrees exactly with the analytical prediction of $\sigma = \sqrt{K_1 (t + t_0)}$.

The nonlinear equation is then numerically solved for the specified boundary conditions (21) and initial condition (22). The diffusion coefficient is set at $K_1 = 1$ and solutions are found for a range of values of the gradient diffusion coefficient $-1 \leq K_2 \leq 1$. Figures 9 a), b) and c) show the evolution in time of the concentration gradient, $\frac{\partial c}{\partial x}$, and the corresponding spread $\sigma(t)$ and sharpness (the kurtosis) of the mixed region for three extreme values of $K_2$. In the time series plots, the darker shades of gray denote increasingly negative values of the concentration gradient. The spread of the grey regions over time graphically demonstrates the spread of the interfacial mixed region. Positive values of $K_2$ result in a moderately smaller spread over time, whereas if $K_2$ is negative, the interface spreads at a rate moderately larger than that for Fickian diffusion. The sharpness of the interface is measured by increasingly positively values of the kurtosis (indicated by the scale on top of the three right-hand plots). Positive values of $K_2$ result
FIG. 9. The left-hand diagrams show time series showing the evolution of the concentration gradient profile computed using gradient diffusion coefficients a) $K_3 = -1$, b) 0 and c) $K_3 = 1$. The right-hand diagrams show over time the corresponding standard deviation (a measure of the width of the mixing region between the two fluids) and the kurtosis (a measure of the flatness or sharpness of the mixing region).

in interfaces sharper than those which would occur for Fickian diffusion. The interfaces are broader if $K_3$ is negative.

6. Conclusions

We have performed a range of experiments to determine at what Reynolds numbers the transition from laminar to turbulent flow occurs. In such a transitional state the flow has shortened mixing lengths compared with the those for laminar flow, and the drag is not so large as that for fully turbulent flow. In smooth pipes filled with homogeneous fluid, the critical Reynolds number is approximately $Re_c = 2000$. If one fluid is more viscous
than the other, the relationship between the Reynolds number of the ambient fluid, \( Re_a \), and that of the intruding fluid, \( Re_i \), is given approximately by

\[
Re_a = 2Re_c - Re_i
\]

If both the intruding and ambient fluids are in laminar flow and if one fluid is more viscous than the other, the interface between the two fluids is not axisymmetric; the maximum along-pipe velocity occurs either above or below the centre of the pipe. Whether above or below depends upon the relative density of the two fluids. For example, if the intruding fluid is as little as 0.05\% more dense than the ambient, than its leading edge flows predominantly along the lower half of the pipe. This asymmetric flow pattern is more pronounced if the viscosity difference between the two fluids is large.

As a result of this instability, for two fluids in laminar flow and with a large viscosity difference between them, the mixing length can be hundreds of times larger than that between the two fluids in turbulent flow.

A simple analytic model has been developed which examines mixing in turbulent pipe flow. It assumes that the turbulent diffusion coefficient is a function of not only the absolute viscosity but also the rate of change of the viscosity along the mixing region between two fluids. The former is the usual “Fickian diffusion”; the latter is referred to as “gradient diffusion”. Numerical solutions demonstrate that the interface between the two fluids remains sharp for longer times if the effect of gradient diffusion is comparable with the usual “Fickian diffusion”. More \textit{in situ} experiments must be performed to determine whether the analytic model is realistic and, if so, to determine the values of the gradient diffusion coefficient.

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A Theory of Uniform Laminar Pipe Flow

Here we review the classical theory for the steady state and time-dependent development of a laminar flow of uniform fluid in a pipe. The flow is assumed to be axisymmetric, governed entirely by a constant along-pipe pressure gradient that drives the flow and viscous diffusion that retards the flow through viscous stresses ultimately exerted from the pipe walls. The theory is reviewed in detail in Batchelor (Batchelor (1967)) §4.2.

In laminar flow the velocity of the fluid is given by $u = u(r, t)$, a function of time $t$ and the radial distance $r$ from the centre of the pipe. The fluid structure is prescribed by the momentum equation

$$
\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u,
$$

in which $\mu = \nu \rho_0$ is the molecular viscosity of the fluid. The pressure gradient term is assumed to be constant: $G = -\partial p/\partial x$. In polar co-ordinates for axisymmetric flows, the viscous diffusion term (the last term on the right-hand side of the equation) is

$$
\nu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right)
$$

Thus equation (23) is the forced diffusion equation:

$$
\frac{\partial u}{\partial t} - \nu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) = \frac{G}{\rho_0}
$$

Requiring that the flow speed is zero at the pipe walls, the steady state solution of (25) predicts that the flow should exhibit the parabolic velocity profile

$$
u_r(r, \infty) \equiv u_\infty(r) = \frac{G}{4\mu} (R^2 - r^2)
$$

The maximum flow speed is $U_{\text{max}} = GR^2/4\mu$, and the mean flow speed is $\overline{u} = U_{\text{max}}/2$.

Equation (23) also describes how a flow accelerates from zero velocity to the $u_\infty$ after a pressure gradient is suddenly imposed. The velocity profile is

$$
u(r, t) = u_\infty(r) - \frac{2GR^2}{\mu} \sum_{m=1}^{\infty} \frac{1}{\lambda_m^3 J_1(\lambda_m)} J_0(\lambda_m r/R) \exp(-t/\tau_m),
$$

in which $\tau_m = R^2/(\lambda_m^2 \nu)$, $J_0$ is the zero'th order Bessel function of the first kind. The constants $\lambda_m$ are the $m$'th zeroes of $J_0$ (i.e. $J_0(\lambda_m) = 0$ for $m = 1, 2, 3, \ldots$). Explicitly $\lambda_1 \approx 2.40$, $\lambda_2 \approx 5.52$, $\lambda_3 \approx 8.65$, etc.

This analysis shows that the flow approaches steady state on a dominant time scale $\tau_1 \approx 0.17 R^2/\nu$. In the laboratory experiments reported here with $R \approx 0.635\text{cm}$ and $\nu \approx 0.01\text{cm}^2/\text{s}$ (for water), the dominant time scale is approximately 7 seconds, independent of the imposed pressure gradient.
Bibliography

