Shoaling internal solitary waves

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Received 29 March 2013; revised 18 June 2013; accepted 27 June 2013.

[1] The evolution and breaking of internal solitary waves in a shallow upper layer as they approach a constant bottom slope is examined through laboratory experiments. The waves are launched in a two-layer fluid through the standard lock-release method. In most experiments, the wave amplitude is significantly larger than the depth of the shallow upper layer so that they are not well described by Korteweg-de Vries theory. The dynamics of the shoaling waves are characterized by the Iribarren number, Ir, which measures the ratio of the topographic slope to the square root of the characteristic wave slope. This is used to classify breaking regimes as collapsing, plunging, surging, and nonbreaking for increasing values of Ir. For breaking waves, the maximum interface descent, $H_i^*$, is predicted to depend upon the topographic slope, $s$, and the incident wave’s amplitude and width, $A_{iw}$ and $L_{iw}$, respectively, as $H_i^* \approx \sqrt{4sA_{iw}L_{iw}}$. This prediction is corroborated by our experiments. Likewise, we apply simple heuristics to estimate the speed of interface descent, and we characterize the speed and range of the consequent upslope flow of the lower layer after breaking has occurred.


1. Introduction

[2] Large amplitude surface waves in relatively shallow water are generally classified as solitary waves, in homage to the first reported single hump-shaped surface wave generated by a towed barge on the River Severn [Russell, 1844]. Within estuaries and the coastal oceans, internal solitary waves are manifest as large amplitude undulations of the pycnocline, which separates a surface layer from relatively more dense underlying fluid. They can be generated by river plumes [Nash and Moum, 2005] and by tidal flow over steep topographic features such as underwater sills and the continental shelf [Osborne and Burch, 1980; Apel et al., 1985; New and Pingree, 1992; Farmer and Armi, 1999; Pinkel, 2000; New and DaSilva, 2002; Zhao and Alford, 2006; Klymak et al., 2006; Li and Farmer, 2011; Alford et al., 2011]. Though studied in part for their (admittedly relatively small) contribution to ocean mixing on a global scale, internal solitary waves have a substantial impact upon marine ecosystems where they shoal by resuspending nutrients into the fluid column [Xu et al., 2012]. Also, the stresses associated with shoaling internal solitary waves can redistribute sediments and degrade the support of pipelines that link offshore oil drilling sites to refineries at the coast.

[3] The prototypical solitary wave results from a balance between weak nonlinearity, which tends to steepen it, and dispersion, which tends to flatten it. The combination of these effects is captured by the Korteweg-de Vries (KdV) equation [Korteweg and de Vries, 1895] which, assuming quiescent upstream and downstream conditions, predicts that the interfacial displacement due to the waves takes the form of a squared hyperbolic secant.

[4] Whether for a two-layer fluid or for Boussinesq waves in continuously stratified fluid, KdV theory requires the amplitude to be large, but not so large that it is comparable to the shallow-layer depth [Benney, 1966; Whitham, 1974; Sutherland, 2010]. The experiments of Grue et al. [1999] indicate that KdV is useful for amplitudes up to 40% the depth of the shallow layer. However, oceanic internal solitary waves are frequently observed to have vertical displacements from tens to hundreds of meters amplitudes, many times that of the shallow-layer depth [Helfrich and Melville, 2006]. Under these circumstances, the wave structure deviates significantly from that predicted by KdV theory, and the dependence of speed and width upon amplitude is different. For example, in laboratory experiments, Stamp and Jacka [1995] and Grue et al. [1999] found that the crest of large amplitude internal solitary waves tended to flatten and their lateral extent increased with amplitude. While these dynamics are well captured by the Dubreil-Jacotin-Long (DJL) equation [Dubreil-Jacotin, 1937; Long, 1953, 1956; Lamb, 2002;
White and Helfrich, 2008], except in special circumstances, this equation does not have analytic solutions that explicitly reveal the relationship between the speed and width of solitary waves upon the amplitude.

[KdV theory, its weakly nonlinear relatives and DJL theory all assume a rectilinear geometry in which the depth of the ambient is constant. Although the KdV equation can be modified to include slow variations of the background properties [Helfrich et al., 1984; Helfrich and Melville, 1986; Grimshaw et al., 1998; El et al., 2007, 2012], an understanding of shoaling, overturning, and breaking internal solitary waves lies beyond the capacity of analytic theory. Instead, insight into these dynamics has been gained through a combination of laboratory experiments and fully nonlinear numerical simulations.

[8] Early laboratory studies into internal solitary wave shoaling examined solitary waves in an approximately two-layer fluid approaching a shelf, passing over a slope in the transition from deep to shallow ambient fluid. In the experimental setup of Kao et al. [1985], the upper layer was always more shallow than the lower layer whether in the deep ambient or over the shelf, so that a solitary wave of depression could exist in both the deep and shallow ambient. They observed wave steepening in the lee of the approaching wave and consequent partial reflection and transmission of the wave from the shelf, with some irreversible energy loss due to mixing. The experiments of Helfrich and Melville [1986] likewise examined solitary waves shoaling onto a shelf, but their setup was designed to examine the transition of the wave from one of depression in the deep ambient (in which the upper-layer fluid was relatively shallow) to one of elevation over the shelf (in which the upper-layer fluid was relatively deep). Beyond the prediction of KdV theory adapted for slowly varying topography [Helfrich et al., 1984], their experiments revealed instabilities and mixing when the waves transformed from the depressed to elevated form.

[7] Experiments examining the approach of a solitary wave of depression toward a uniform slope [Helfrich, 1992] showed that the lower-layer ambient was drawn downslope while the lee of the wave steepened. Eventually this evolved to create “boluses” of dense fluid that propagated upslope in similar manner to the production of boluses by solitary waves of elevation incident upon a slope [Wallace and Wilkinson, 1988]. However, the downslope lower-layer flow induced by the leading edge of the wave that opposed the wave’s advance permitted multiple boluses to form from just a single incident wave. The study by Helfrich [1992] focused upon the number and sizes of boluses formed by the shoaling solitary wave. Relevant to the experiments presented here, Helfrich also compared the location of the maximum breaking depth to the wave’s width, finding that the ratio of wave amplitude to the breaking depth was approximately 0.4 ± 0.1, exhibiting a small decreasing trend as the width of the incident wave increased relative to the horizontal extent of the slope in the lower layer. Because the solitary waves were generated by a flap mechanism the incident wave amplitudes were not so large compared to the shallow-layer depth, and so resembled KdV-like waves. The range of slopes examined was also limited to s = 0.034, 0.050, and 0.067.

[8] Larger amplitude internal solitary waves in two-layer fluid were created in laboratory experiments using a lock-release mechanism [Michallet and Barthélémy, 1998; Grue et al., 2000]. This method was used to examine the energetics of shoaling waves on a constant slope by comparing the energies of the incident and reflected waves and by measuring the overall change in the potential energy of the stratified ambient [Michallet and Ivey, 1999]. The experiments examined shoaling on three slopes: s = 0.069, 0.169, and 0.214. They found that up to 25% of the wave energy was irreversibly lost due to mixing, with the greatest loss occurring when the width of the wave was approximately half the total length of the slope.

[9] Using a tilted tank to generate a solitary wave train from an internal seiche [Boegman et al., 2005b], the shoaling of the wavetrain on a constant slope with s = 0.097 and 0.145 was examined [Boegman et al., 2005a]. They noted that the mechanism for wave breaking was different depending upon the relative magnitudes of the topographic slope and the incident wave slope. Borrowing from the nomenclature used to describe breaking surface waves, they classified breaking regimes in terms of either surging or plunging breakers.

[10] We have adapted and extended this classification scheme here, as illustrated in Figure 1. At very steep topographic slopes, waves reflect with no evidence of overturning at the slope or aloft (Figure 1a). For moderately more shallow slopes, overturning can occur at the slope itself forming a “surging breaker” (Figure 1b). Peculiar to internal waves, the downslope flow beneath the leading flank of the approaching wave can become so strong relative to the

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**Figure 1.** Schematics illustrating different breaking mechanisms for solitary waves of depression approaching a constant slope. Sketches are representative of the flow around the time of maximum breaking depth.
lower-layer flow beneath the trailing flank of the wave that the flow separates from the boundary and reconnects lower down. This forms an across-slope vortex with counter-clockwise vorticity for a rightward-propagating incident wave. If the interface in the lee of the wave is not vertical when this “separation bubble” occurs, it is classified as a collapsing breaker (Figure 1c). If the slope is very shallow, the leeward interface can overturn within the body of the fluid even as the bottom fluid continues to flow downslope toward the incident wave. This is referred to as a plunging breaker (Figure 1e). As our experiments show, a separation bubble at the boundary can develop at the same time that the trailing flank of the wave overturns. This we refer to here as a collapsing-plunging breaker (Figure 1d).

Defining $s$ to be the topographic slope and $L_{sw}$ and $A_{sw}$ to be the characteristic width and vertical displacement amplitude, respectively, of the incident wave, Boegman et al. [2005a] classified the character of internal solitary wave breaking in terms of the (internal) Iribarren number, $Ir$:

$$Ir \equiv \frac{s}{\sqrt{A_{sw}/L_{sw}}}.$$  \hspace{1cm} (1)

Their study assumed a KdV structure for the incident waves to define the wave extent $L_{sw} = L_{sw,KdV} \propto A_{sw}^{1/3}$. With this definition, they found that solitary waves formed plunging breakers if $0.45 \leq Ir \leq 0.75$, and collapsing breakers if $Ir \geq 0.75$.

They found that the mixing efficiency was greatest for plunging breakers. They also formulated an empirical prediction for the interface descent, $H_i$, at the maximum breaking depth. This was given in terms of $A_{sw}$, $L_{sw,KdV}$, and the horizontal extent of the slope in the lower layer, $L_1$ ($L_1 = H_1/s$, in which $H_1$ is the depth of the lower layer). In particular, for relatively shallow slopes, they found (adapted from equation (15) of Boegman et al. [2005a])

$$H_i \approx 7.1A_{sw}\left(\frac{L_{sw,KdV}}{L_1}\right)^{0.52}.$$  \hspace{1cm} (2)

Numerical simulations restricted to two dimensions further explored the dynamics of internal solitary waves shoaling. This has been done with realistic stratification and topography similar to that found in the Andaman and Sulu Seas [Vlasenko and Hutter, 2002] and more recently has been examined for an approximately two-layer fluid and constant topographic slopes between $s = 0.01$ and 0.3 [Aghsae et al., 2010]. In the latter case, the initial conditions were set by solutions of the DJL equation for a solitary wave over a flat bottom having amplitudes from one fifth to double the (shallow) upper-layer depth. Although the two-dimensional simulations could not capture the effects of mixing in fully three-dimensional flows [Fringer and Street, 2003; Venayagamoorthy and Fringer, 2007], they could be used to classify the breaking regimes and to gain insight into the instability mechanisms during breaking.

Aghsae et al. [2010] delineated the regimes for surging, collapsing, and plunging breakers by comparing the time scales for the downslope flow, separation-bubble formation, and leeward interface steepening. Comparing these time scales helped to explain why a particular breaking mechanism occurred. Analysis of their simulations revealed more complicated relationships of breaking regimes upon topographic and wave slopes. Generally, they observed that incident solitary waves plunged, collapsed, or surged upon a slope as $Ir$ increased. However, the range separating the regimes depended upon the magnitude of the topographic slope.

They also determined an empirical relationship for the breaking depth. Defining the wave extent, $L_{sw,t}$, to be the area divided by amplitude and assuming $L_{sw,t} \ll L_1$, their expression reduced to (adapted from equation (5.1) of Aghsae et al. [2010])

$$H_i \approx 7.1A_{sw}\left(\frac{L_{sw,t}}{L_1}\right)^{0.28}.$$  \hspace{1cm} (3)

The discrepancy in the exponent between equations (3) and (2) may be attributed to the explicit measurement of $L_{sw,t}$ in the numerical simulations and also because the simulations were designed so that only a single wave of depression impacted upon the slope.

The breaking depth predictions (equations (2) and (3)) both depend upon the length of the slope, $L_1$, a quantity that is difficult to estimate in realistic oceanographic circumstances for which the bottom topography varies in slope from the abyss to coast. One aim of this study is to formulate a prediction for breaking depth that depends upon more accessible parameters as might be measured, for example, in field situations.

In this work, we perform laboratory experiments to reexamine the problem of internal solitary wave shoaling on a constant slope with the intent of broadening the parameter range explored earlier in both experiments and simulations. We generate waves with amplitudes up to 3.5 times the shallow layer depth, well beyond the KdV regime. So that the results can be readily applied to interpret ocean observations, we classify the wave breaking regimes in terms of incident solitary wave characteristics that can be measured by surface observations and vertical time series constructed, for example, from traversing conductivity-temperature-depth (CTD) probes or moored thermistor chains [e.g., van Haren and Gostiaux, 2012]. Our analyses focus on predicting the maximum breaking depth and the consequent upslope propagation of the lower layer as a bolus.
examine the time scale for breaking, and we measure the upslope advance of the leading bolus. The implications for interpreting ocean observations and predicting wave-breaking locations are discussed in section 5.

2. Experiment Setup

[21] Internal solitary waves were generated in an approximately two-layer salt-stratified fluid through the standard lock-release method [Michallet and Ivey, 1999; Grue et al., 2000]. Experiments were performed in short and long tanks. One was $L = 1.97$ m long, 0.49 m high, and 0.18 m wide. The other was the same tank used by Michallet and Ivey [1999] in their study of internal solitary wave generation and dissipation. Its interior was 6.90 m long, 0.25 m high, and 0.25 m wide. In both tanks, a constant slope of length $L_s$ and height $H_s$ was established at one end of the tank, as illustrated in Figure 2. The slopes examined ranged between $s = H_s/L_s = 0.143$ and 0.417.

[22] The tanks were first filled with salt water of density $\rho_0$. Fresh water of density $\rho_f$ was then poured slowly into a sponge float creating a layer of depth $H_f$ on top of the salt water. In all experiments, the density difference between the salt and fresh water ranged from $\Delta \rho \equiv \rho_f - \rho_0 \approx 10$ to 20 kg/m$^3$.

[23] In the short-tank experiments, the interface was marked by dye as the fresh water was first poured into the sponge float on top of the salt-water layer. In some of these experiments, fine kaolin clay was deposited on the slope to help visualize boundary layer separation. After filling, a gate was inserted a fixed distance from the endwall opposite the slope forming a lock of width $L_L = 0.286$m. The bottom of the gate was situated a few centimeters above the bottom to allow salt water to pass easily underneath it. Fresh water was then added to the surface water between the gate and endwall so increasing the extent of the surface fresh water to a depth $H_s$. When done, the total depth of the ambient was measured to be $H$. In these experiments, $H$ ranged between 0.43 and 0.46 m with the upper-layer ambient ranging from $H_0 = 0.05$ to 0.15 m.

[24] In the long-tank experiments, the lower layer was dyed before clear fresh water was layered on top. In these experiments, poppy seeds (of mean density 1065 g/m$^3$ and diameter 0.001 m) were sprinkled on the slope to visualize boundary layer separation. After filling, a false endwall was inserted to full depth midway along the tank, a distance $L$ from the slope end of the tank. The distance from the false endwall to the start of the slope was fixed to be $L - L_L = 2.60$ m, and the length of the lock was set to be either $L_L = 0.40$ or 0.80 m. The total ambient depth ranged between $H = 0.16$ and 0.18 m with the upper-layer depth ranging between $H_0 = 0.01$ and 0.04 m.

[25] Before the start of an experiment, the ambient density profile was measured using a conductivity probe (MSCTI, Precision Measurement Engineering), which was calibrated by measuring the conductivity of prepared salt-water solutions whose densities were measured using an Anton Paar DMA5000 densitometer. These measurements were used to provide an accurate measurement of the fresh water layer depth, $H_0$, and the thickness of the interface between salt and fresh water. Specifically, $H_0$ was determined to be the distance from the surface to the depth in the fluid where the density was the average of the fresh and salt water: $(\rho_f + \rho_0)/2$. The thickness of the interface $2H_I$ was measured as the distance over which the density changed from $(3\rho_f + \rho_0)/4$ to $10$ to $(\rho_f + 3\rho_0)/4$. In most experiments reported upon here the half-thickness, $H_I$, was less than 0.004 m. However, in the classification of types of wave breaking, additional data were included from some experiments where $H_I$ was as large as 0.02 m.

[26] At the start of an experiment, the gate was rapidly extracted vertically. The fresh water layer in the lock then slumped upward into the shallower fresh water ambient, thus launching a hump-shaped internal solitary wave of depression, which propagated rightward toward the slope.

[27] Up to four runs were performed in the long-tank experiments before the two-layer ambient was recreated as described previously. After two runs, the fresh water layer had deepened and the interface had thickened as a result of mixing by the shoaling solitary waves. Between the second and third run, the false endwall was temporarily extracted so that the relatively deep fresh water layer in the test section flowed rearward as a bore into the previously sealed-off section of the tank. Before the bore returned into the test section, the false endwall was reinserted. In this way, the surface layer shallowed by as much as half and the interface between fresh and salt water was thinner.

[28] In the short-tank experiments, a single camera (Canon EOS Rebel T3i) was situated 3 m in front of the tank with the lens (Canon EFS 18–35 mm) situated at midpoint of the tank. This recorded movies of the evolution of the generation, propagation, and shoaling waves.

[29] In the long-tank experiments, two digital cameras separately recorded the steadily propagating incident solitary wave and the shoaling wave on the slope. The incident wave was recorded by a camera (Nikon D90) situated 2.5 m from the side of the tank with the lens (Nikon DX SWM ED Aspherical) at the same height as the interface. In most experiments, the midpoint of the field of view was 1 m to the right of the gate. The second camera (Canon EOS Rebel T3i) was situated 2.5 m from the side of the tank in front of the midpoint of the slope with the lens (Canon EFS 18–35mm) situated at the height of the slope at its midpoint.

[30] The movies were processed and analyzed using Matlab. The methodology used to measure the properties of

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**Figure 2.** Experiment setup showing the definition of length scales and densities used to characterize the control variables of the experiment. Note that the length of the tank is much longer than the height, as indicated by the zigzag at the tank bottom at the mid-section of the schematic. The slope is indicated in black, salt water in dark gray, and fresh water in light gray.
the propagating and shoaling solitary waves are described in the next section.

3. Qualitative Results

[31] Here we present snapshots and time-series images extracted from typical experiments. These were chosen to illustrate how quantitative measurements were taken and to show how the breaking mechanism of the shoaling wave depended upon the magnitude of the slope and incident wave amplitude and extent.

3.1. Solitary Wave Properties

[32] Figure 3a shows a snapshot of a rightward-propagating solitary wave of depression. The field of view ranges over the ambient fluid depth and in the horizontal over a distance between −0.70 and −0.30 m to the left from the base of the slope (i.e., the point where the slope reaches the bottom of the tank). In this experiment, the significant deepening of the fresh water layer from \( H_0 = 0.016 \text{ m} \) to \( H_0 + A_{sw} = 0.063 \text{ m} \) is evidence of a large amplitude wave whose description lies beyond the range of validity of KdV theory [Stamp and Jacka, 1995; Grue et al., 1997, 1999]. Indeed, the snapshot reveals a flattened trough whose structure differs qualitatively from the squared hyperbolic secant structure of KdV waves. In the quantitative analysis that follows we also show that the dependence of width upon amplitude significantly deviates from KdV behavior.

[33] The corresponding vertical time series in Figure 3b is constructed by stacking from left to right vertical slices at \( x = -0.70 \text{ m} \) taken through a succession of snapshots. Here, \( t = 0 \) corresponds to the time when the gate is extracted. As is the case in most experiments with lock length \( L_t = 0.40 \text{ m} \), the leading solitary wave had a smaller trailing wave. We found that the number of trailing waves increases with lock length, consistent with Maxworthy [1980]. In experiments with a lock length of \( L_t = 0.80 \text{ m} \), up to three trailing waves were observed.

[34] The solitary wave amplitude \( A_{sw} \) was determined to be the difference between the maximum interface depth and the initial depth, \( H_0 \), of the interface before the experiment began, as indicated in Figure 3a.

[35] As well as vertical time series, we constructed horizontal times series (not shown) by stacking from bottom to top successive horizontal slices at a depth midway between the initial interface depth and the maximum interface depth (i.e., a distance \( H_0 + A_{sw}/2 \) below the interface). By tracking the rightward advance of the interface in these time series, we computed the speed, \( C_{sw} \), of the solitary wave.

[36] The half width, \( L_{sw} \), of the solitary wave was measured from the vertical time series by determining the time for the interface to deepen from \( H_0 + A_{sw}/2 \) to \( H_0 + A_{sw} \) and multiplying this result by \( C_{sw} \). This is illustrated in Figure 3b.

[37] For the solitary wave shown in Figure 3, we found \( A_{sw} = 0.047 \text{ m}, L_{sw} = 0.150 \text{ m}, \) and \( C_{sw} = 0.070 \text{ m/s} \).

3.2. Shoaling Solitary Waves

[38] Figure 4 shows snapshots from four experiments in which solitary waves were incident upon a range of relatively steep and shallow slopes. These illustrate four distinct behaviors. While not shown here, if the slope is very steep relative to the slope of the incident wave, the wave reflects with little to no overturning of the interface. In Figure 4a, the wave slope compared to the bottom slope is such that the wave develops into a surging breaker in which the interface is vertical at the slope itself at the maximum breaking depth. In this case, a small amount of mixing now occurs at the slope itself as the interface begins to propagate back upslope.

[39] If the topographic slope is moderately smaller (Figure 4b), the lower layer now separates from the boundary as it flushes downslope under the leading flank of the incident wave. This collapsing breaker forms a separation bubble below the point where the interface intersects the boundary. As is clearer in movies of the experiment (provided as supporting information), the flow in the separation

Figure 3. (a) Snapshot and (b) vertical time series of an internal solitary wave taken from an experiment in the long tank with \( \rho_1 = 1012.1 \text{ kg/m}^3 \), \( H = 0.164 \text{ m} \), \( H_0 = 0.016 \text{ m} \), \( H_t = 0.10 \text{ m} \), and \( L_t = 0.40 \text{ m} \). The horizontal position in Figure 3a is shown with respect to the location of the base of the slope off-image to the right. Time in Figure 3b is given with respect to the time at which the gate is extracted. The vertical axes in Figures 3a and 3b are the same. Note the vertical black section between \( x = -0.45 \text{ m} \) and \( -0.41 \text{ m} \) in Figure 3a is the result of a supporting column outside the tank blocking the back lighting. The horizontal white dashed line in Figure 3a indicates the depth of the undisturbed interface and the amplitude of the wave is indicated by the arrow. The white arrow in Figure 3b indicates the time for the interface to deepen from \( A_{sw}/2 \) to \( A_{sw} \).
bubble is counterclockwise. This gives rise to substantial resuspension of clay particles on the slope.

In an experiment with the same slope but with steeper incident wave (Figure 4c), the separation of the flow from the slope is more clearly evident, and this is characteristic of collapsing breakers. In this case, the rear flank of the wave simultaneously overturns within the body of the fluid, characteristic of a plunging breaker. Such a collapsing-plunging breaker acts to mix the upper and lower-layer fluid substantially in the region between the slope and the depth of the undisturbed interface. At later times, the gravitational collapse of this mixed patch acts to thicken the interface between the surface and deep ambient.

If the slope is shallower still (Figure 4d), the rear flank of the wave overturns even as the lower-layer fluid continues to be drawn downslope beneath the leading flank of the incident wave. It resembles plunging surface waves in that the lee of the incident wave “pours” over the interface to make a tube-like structure. For this plunging breaker, mixing occurs initially within the body of the fluid. However, at later times a separation bubble can still form on the slope and the resulting turbulent upslope flow can result in substantial mixing and sediment resuspension [e.g., see Boegman and Ivey, 2009, Figure 8].

These classifications of the breaking regimes may be delineated in terms of the Iribarren number (1), as described in section 4.2. To examine the deepening of the interface and consequent runup along the slope of the lower-layer fluid after maximum deepening occurs, we constructed vertical and along-slope time series.

The development and consequent evolution of a collapsing-plunging breaker in a long-tank experiment is shown through a sequence of snapshots in Figure 5. As the wave front approaches the slope, the interface between fresh and salt water forms a sloping front parallel to the slope. This front descends as fresh water piles into the right corner of the tank over the slope and salt water flushes downslope to the left (Figure 5b). The confluence of this downslope bottom flow with the incoming flow aloft, in turn, leads to separation of the flow and turbulence where the interface touches the bottom slope at the location of its...
steepest descent (Figure 5c). Thereafter, the interface rises back up the slope as a front resembling that of a gravity current (indicated by an arrow in Figure 5d).

[44] Particles seeded on the slope are observed to be resuspended into the fluid column near the separation point both at the position of maximum interface descent and during the consequent upslope flow. Sediment transport and resuspension is the subject of separate research. Here our interest is with the occurrence of boundary-layer separation and the characteristics of the interface descent and the upslope flow that follows.

[45] Along-slope and vertical time series were constructed from movies of the flow, as shown in Figure 6. The along-slope time series was constructed by taking diagonal slices through images a short distance (typically 0.005–0.010 m) above the slope and stacking these in time. These revealed the rapid downslope advance of the interface as the wave approached the slope. Figure 6a shows that it moved downslope by approximately 0.20 m to $d_{nn} = 0.13$ m. After this point of maximum descent had been reached, this time series also showed the relatively slow upslope advance of the interface. The interface moved upslope at constant speed for some distance before decelerating and ultimately halting. In some experiments, multiple boluses of lower-layer fluid were observed to advance up the slope. In these cases, we focus our analysis upon the front associated with the leading bolus alone.

[46] Fitting a line to the location of the interface as it now rises upslope gives a measure of the upslope interface position versus time, as shown in Figure 7. The best fit line to the points along the front at early times is given by $d = 0.0266t - 0.0626$. The points show that the front rises at constant speed, $C_u \approx 0.0266$ m/s, until time $t \approx 13$ s after which it decelerates at an approximately constant rate as given by the best fit quadratic $d = -0.00248t^2 + 0.0911t - 0.483$.

[47] Having thus determined the along-slope distance of maximum interface descent, $d_{nn}$, we also determined the location at which to construct the vertical time series. For example, the time series in Figure 6b was constructed by stacking vertical slices taken at the horizontal position $x_{nn} \equiv d_{nn}/\sqrt{1 + s^2}$, in which $s$ is the slope. The time series shows that the interface descends at an almost constant speed until reaching the slope at its point of maximum descent, a distance 0.09 m below the original interface height. Fitting a line to the location of the interface in the vertical time series as it descends gives a measure of the vertical descent speed, $W_i$. By understanding what parameters determine the value of $W_i$, we are able to predict the time scale for breaking.

4. Quantitative Results

[48] We performed over 60 experiments examining shoaling solitary waves on a slope. We begin by examining the properties of the incident solitary wave measured 0.5–1.0 m to the left of the base of the slope. This distance was sufficiently close to the slope itself that dissipation of the propagating wave due to viscosity was considered negligible between the measurement and breaking locations [Michallet and Ivey, 1999]. The amplitude, $A_{sw}$, half width at half maximum, $L_{sw}$, and speed, $C_{sw}$, are measured as they depend upon the lock parameters.

[49] Thereafter, we present the analyses of solitary wave shoaling. Rather than lock parameters, which are specific to the experimental configuration, and so that the results may be applied to ocean observations, the breaking regimes, maximum interface descent and speed, and the subsequent

![Figure 6](image.png)

**Figure 6.** (a) Diagonal time series taken along a slice parallel to the slope 0.010 m above it and (b) vertical time series taken above the position of maximum interface deepening. Both are shown for the experiment shown in Figure 5 with parameters given in Figure 3 and with slope $s = 0.417$. Time $t = 0$ corresponds to the moment the interface maximum displacement passes over the base of the slope and $d$ measures the distance upslope from its base.
upsslope flow are characterized entirely in terms of the characteristics of the incident solitary waves and bottom slope.

4.1. Solitary Wave Properties

[50] The structure of the solitary wave was set by the depth, \( H_t \), and horizontal extent, \( L_s \), of the upper layer lock fluid as well as by the characteristic ambient depth, as measured by the harmonic mean of the upper and lower-layer depths of a two-layer fluid:

\[
\mathbf{T} = \frac{H_0(H - H_0)}{H}.
\] (4)

[51] In practice, we found that the amplitude, width, and speed of the wave were set by the difference between the lock depth and the depth of the upper-layer fluid: \( H_t - H_0 \). The lock length determined the number of generated waves. For experiments with \( L_d = 0.28 \) and 0.40m, a single hump-shaped disturbance was generated with a small trailing wave. With \( L_d = 0.80 \)m, three to four waves were generated.

[52] Figure 8 shows how the amplitude, width, and speed of the solitary wave depended upon the depth of the upper-layer lock fluid. The errors in the measurements result from uncertainty in locating the middle of the interface, both in the lock and ambient fluid, as the wave passed the test section. To reduce these errors, here we plot only the results for those experiments in which the initial interface half thickness was less than 0.004 m. Despite some scatter in the data, we found that the amplitude and width of the solitary wave vary approximately linearly with the depth of the upper-layer lock fluid. Explicitly, the best fit lines through the origin that pass through the points in Figures 8a and 8b reveal that

\[
A_{sw} \approx 0.45(\pm 0.02) (H_t - H_0) \text{ and } L_{sw} \approx 1.68(\pm 0.06) (H_t - H_0),
\] (5)

where \( A_{sw} \) and \( L_{sw} \) are both approximately proportional to \( H_t - H_0 \) and so are approximately proportional to each other: \( L_{sw} \approx 3.7A_{sw} \). In part for this reason, we conclude that the generated waves were generally of such large amplitude that they are not well described by KdV theory. This theory predicts that the width, \( L_{sw,KdV} \) (defined as area over amplitude), depends upon its amplitude, \( A \), according to (e.g., see equation (4.85) and accompanying text in Sutherland [2010])

\[
L_{sw,KdV} = 4\sqrt{\frac{\mathbf{T}^3}{3A}}.
\] (6)

[53] That is, the width of a KdV solitary wave decreases as the inverse square root of the amplitude. A departure from KdV theory is expected because in most of our experiments the solitary wave amplitude is comparable to or larger than the characteristic depth \( \mathbf{T} \).

[54] As shown in Figure 8c, the speed of very large amplitude solitary waves is typically larger than the shallow water linear wave speed given by

\[
C_0 = \sqrt{g'\mathbf{T}}.
\] (7)

in which \( g' \equiv g(\rho_t - \rho_0)/\rho_t \) is the reduced gravity. Qualitatively, this is consistent with KdV theory, which predicts the speed, \( C_{sw,KdV} \), exceeds \( C_0 \) by an amount that increases with the wave amplitude:

\[
C_{sw,KdV} = C_0\left[1 + \frac{1}{2\mathbf{T}}\right].
\] (8)

[55] However, a best fit line through our data gives

\[
C_{sw} \approx C_0\left[1 + 0.052(\pm 0.004)\frac{H_t - H_0}{\mathbf{T}}\right] \approx C_0\left[1 + 0.12\frac{A_{sw}}{\mathbf{T}}\right].
\] (9)
in which the last expression makes use of the first expression in equation (5). Thus, the solitary wave speed is relatively slower than predicted by the corresponding KdV wave speed. This results from the nonnegligible flow in the lower layer that opposes the advance of the solitary wave.

[56] The properties of the solitary waves in our experiments are consistent with Stamp and Jacka [1995] and Grue et al. [1999]. They serve as a reminder that it is erroneous to infer the wavepacket width and speed from its amplitude using KdV theory if the wave amplitude is comparable to or larger than the shallow-layer depth. For oceanic solitary waves observed by means of a thermistor chain, the amplitude can be measured directly. The width is best estimated as we have done in these experiments by measuring the time for the interface to descend from half to full maximum and multiplying this by the wave speed. In the ocean, the speed itself can be determined by the advance of smooth and rough surface signatures induced by the waves, which can be observed onboard ship, by satellites or by pairs of moorings situated a known distance apart.

[57] In the analysis that follows, the measured solitary wave properties are used to characterize the dynamics of shoaling on a slope.

4.2. Breaking Regimes

[58] The dynamics of the shoaling wave depend upon the values of the bottom slope relative to the slope of the incoming solitary wave front. This is shown in Figure 9. For experiments with prescribed topographic slope, s, and measured incident wave slope $A_{sw}/L_{sw}$, Figure 9a plots different symbols depending upon whether the waves reflect with no significant breaking or whether they break as surging, plunging, or collapsing waves. Superimposed on this plot are lines of constant (internal) Iribarren number [Boegman et al., 2005a], defined by equation (1). We found that waves reflected with little to no overturning if $\text{Ir} > 1.5$. Collapsing and plunging breakers formed for lower Ir. Predominantly collapsing breakers occurred if $0.7 < \text{Ir} < 1.0$ and plunging breakers occurred predominantly if $\text{Ir} < 0.3$.

[59] For comparison with the experiments of Boegman et al. [2005a] and the numerical simulations of Aghsae et al. [2010] (see Figure 17 in Aghsae et al. [2010]), the breaking regime data are replotted as slope versus Iribarren number in Figure 9b. Qualitatively, we find the same trend in which waves change from collapsing to plunging to surging as Ir increases. However, the critical Iribarren numbers separating the breaking regimes are different in our experiments. In part, this is because the wave extent, $L_{sw,A}$, defined in the simulations was based upon the area of the wave divided by the amplitude. This value is larger than the half width at half maximum, $L_{sw}$, defined in our experiments. Partially to correct for this, the Iribarren number, $\text{Ir}_2$, plotted along the abscissae in Figure 9b is defined in terms of $L_{w} = 2L_{sw} \approx L_{sw,A}$. Consistent with Aghsae et al. [2010], we find collapsing and plunging breakers for $\text{Ir}_2$ in a range between 0.4 and 1.2 for slopes below $s = 0.30$. Because our classification of collapsing and plunging breakers was based upon observations of the interface shape at the point of maximum breaking, we were unable to delineate the different regimes as rigorously as could be done from the examination of time scales in the simulations.

4.3. Breaking Depth

[60] Particularly in experiments with plunging-collapsing breakers, we found that the shoaling solitary wave evolved so that the interface was nearly vertical where it intersected the slope at its point of maximum descent (e.g., see Figures 4c and 5c). We use this observation to estimate the maximum vertical displacement, $H_{f}$, of the interface. Assuming the fresh water that pools in the rightmost corner of the tank has a right-angled triangular shape, the volume per unit width of this fluid below the initial depth of the tank is

**Figure 9.** (a) Breaking regimes showing breaking characteristics observed in several experiments as they depend upon the topographic slope $s$ and the characteristic incident wave slope $A_{sw}/L_{sw}$. The dashed lines show values of constant Iribarren number as indicated. (b) The breaking regimes are plotted on a regime diagram with topographic slope and Iribarren number [after Aghsae et al., 2010, Figure 17]. The symbols indicating the breaking regimes for both plots are indicated in the legend in Figure 9b. Collapsing-plunging breakers have a circle and cross superimposed.
$H/L_2$, in which $L_2 = H_i/s$ is the horizontal distance from the point of maximum descent to where the undisturbed interface intersects the slope. This we equate to the approximate area (volume per across-slope width) of the leading incident solitary wave, $A_{sw}L_{sw} \approx 2A_{sw}L_{sw}$. Thus, we predict the maximum interface descent to be

$$H_i^* = \sqrt{\frac{4sA_{sw}L_{sw}}{H}}. \tag{10}$$

[61] Figure 10a compares this prediction with the measured maximum depth of descent of the interface, $H_i$. Points are plotted only for those experiments with initial interface half thickness smaller than 0.004 m. Considering the crude approximation that the incident solitary wave area is given by $2A_{sw}L_{sw}$, equation (10) well predicts the breaking depth for wave shoaling on both moderate and shallow slopes. The prediction succeeds less well for steep slopes ($s > 0.4$) because the upper-layer fresh water does not pool in the corner with a right-angled triangular shape in advance of surging breakers and nonbreaking reflected waves. For shallow slopes, which are more representative of the ocean, the prediction works quite well because in most experiments incident waves shoal as plunging or collapsing breakers.

[62] The prediction (equation (10)) differs from both the empirical prediction (equation (2)) determined by the laboratory experiments of Boegman et al. [2005a] and the empirical prediction (equation (3)) determined by the numerical simulations of Aghsaeie et al. [2010]. In both, the breaking depth was assumed to depend upon the horizontal distance, $L_i = H_i/s$, from the base of the slope to the point where the undisturbed interface intersected the slope. But one should not expect the dynamics of shoaling to depend upon the total length of the slope in the bottom layer: in the limit of an infinitely long (not necessarily shallower) slope, an incident solitary wave would still act to deepen the interface to a finite extent relative to its incident amplitude.

[63] For comparison with the results of Helfrich [1992], Boegman et al. [2005a], and Aghsaeie et al. [2010], Figure 10b plots the ratio of incident wave amplitude to maximum interface displacement depth against the ratio of the width of the wave to the extent of the slope in the lower layer. For best comparison with Aghsaeie et al. [2010], we define the extent of the wave to be $L_{sw} = 2L_{sw}$. Consistent with earlier studies, we find that $A_{sw}/H_i$ holds values in a range about 0.6 ± 0.1 for small $L_i/L_{sw}$. However, we do not find a decreasing trend for larger $L_i/L_{sw}$. In part, this is because our study focused upon larger amplitude incident waves and in many of our experiments the slope was steeper than $s = 0.2$.

4.4. Vertical Descent Speed

[64] Vertical time series taken at the location along the slope of maximum interface descent showed that the initial descent speed was constant almost until the interface descended to the slope itself. We predict this speed by following the reasoning used above to estimate the maximum interface descent. We estimate the characteristic time for descent based upon the time for the incoming solitary wave to traverse its whole extent (approximately $2\alpha L_{sw}$, with $\alpha$ being an order unity constant) while moving at speed $C_{sw}$. Hence, using equation (10), an estimate of the descent speed is

$$W_i^* = \frac{H_i}{2\alpha L_{sw}/C_{sw}} = \frac{1}{\alpha} C_{sw} \sqrt{\frac{A_{sw}}{L_{sw}}} \tag{11}$$

[65] Figure 11 compares the measured descent speed, $W_i$, against $W_i^*$ for a range of experiments restricted to those on slopes shallower than $s = 0.42$ and with initial ambient interface half thickness less than 0.004 m. Although there is some scatter, we find $W_i \approx 1.01(\pm 0.05)W_i^*$ if we set $\alpha = 1.6$.

4.5. Upslope Bolus Propagation

[66] Finally, we measured the upslope propagation of the leading bolus. In the experiments of Helfrich [1992], the leading bolus was found to propagate initially at a speed, $C_{bol}$, proportional to the shallow water speed computed for the undisturbed ambient at the depth of maximum interface descent, $C_{bol\delta}$. Explicitly, the shallow water speed is given
by equation (7) in which the characteristic depth at the breaking point is \( H = H_0 + H_s \). Whereas Helfrich [1992] found that \( C_{0,brk} \approx 0.6 C_0 \), this relation does not hold in our experiments with larger amplitude waves and steeper slopes. This is illustrated in Figure 12a, which shows \( C_{0,brk} \) holds a range of values between 0.02 and 0.03 m/s, with no obvious dependence upon \( C_0,brk \).

When it reaches its maximum depth, the interface associated with plunging and surging breakers is approximately vertical. Ignoring the ambient motion, we consider the extent to which the bolus may be treated as an upslope-propagating gravity current. We imagine the bolus originates from a partial-depth lock release at the point of maximum breaking in which the lock depth is \( H_i \) and the total depth is \( H_i + H_0 \). The initial speed of a gravity current of depth \( h \) in a relatively deep ambient is \( \sqrt{g' h/2} \). If we assume the current depth is half the depth of dense fluid in the imagined lock (a result predicted by Benjamin [1968] for energy-conserving gravity currents), then

\[
C_{gc} = \frac{1}{2} \sqrt{g' H_i}.
\]

This is the speed of a current moving along a horizontal bottom. Crudely, we correct for the fact that the current is moving upslope by computing the speed in the upslope direction. Thus, we propose that the gravity current speed up a slope \( s \), which originates from a partial-depth lock release of depth \( H_i \) is

\[
C_{gc,s} = \frac{1}{2} \sqrt{\frac{g' H_i}{1 + s^2}}.
\]

Figure 12b plots the bolus speed against \( C_{gc,s} \), with both quantities being given relative to the shallow water speed \( C_0 \) given by equation (7), in which \( H \) is given by equation (4). Plotting the results in this way does reveal a tentative relationship between the observed bolus speed and the predicted gravity current speed, although the bolus speed is found to be about half the gravity current speed.

Particularly for collapsing and plunging breakers, the dynamics of the bolus are significantly different from those of a gravity current. In particular, during the generation of the bolus, significant across-slope vorticity is clearly associated with it, whereas a gravity current head is assumed to be irrotational.

Wallace and Wilkinson [1988] proposed that the speed of the bolus should depend upon upslope distance as a power law with exponent \( \phi \). Their expression was rewritten by Helfrich [1992] in the form

\[
C_b = C_0 (1 - d/db)^\phi,
\]

in which \( db \) is the along-slope distance from the surface to the position where the bolus begins to decelerate, with deceleration beginning at \( d = 0 \). In their experiments,
Wallace and Wilkinson [1988] found $\phi = 0.60 \pm 0.05$ for slopes with $s = 0.030$. For boluses on a constant slope with a relatively more shallow upper layer, Helfrich [1992] found better agreement between theory and experimental measurements using $\phi \approx 0.25$. That said, the theoretical prediction underestimated the observed speeds soon after bolus formation and it overpredicted the speeds as the bolus came to halt.

[73] In our analysis of along-slope time series, we accurately and continuously measured the along-slope bolus position versus time, as illustrated in Figures 6a and 7. The results consistently showed that the bolus did not immediately begin to decelerate upon its formation. Instead it propagated along-slope at constant speed until it had risen from a depth $H_f$ to a depth $H_b$. Only then did deceleration commence. The constant-speed regime preceding deceleration is also evident in Helfrich [1992, Figure 13].

[74] From the along-slope position at which the bolus first began to decelerate, we determined its corresponding depth $H_b$. For example, in Figure 7, the along-slope distance for transition, $d_b$, occurs where the linear and quadratic curves are tangent. Knowing $d_b$, $H_b$ is easily determined by geometry. Figure 13a plots the rise height $H_b - H_f$ versus the maximum interface depth, $H_f$. The only experiments reported upon here had an interface half thickness less than 0.004 m and the incident wave slope was sufficiently large to form a plunging breaker. The plot shows that the bolus propagates upward more than half the depth of the maximum interface descent before it begins to decelerate.

[75] The along-slope time series shows that when the bolus does begin to slow down it does so with constant deceleration, $C'_b$. A simple analysis of the deceleration that does not invoke friction, Richardson numbers, or shape factors, as in Wallace and Wilkinson [1988], examines the dependence of $C'_b$ upon the incident bolus speed $C_{bo}$ and the along-slope distance between where deceleration begins and the surface. Figure 13b shows that these quantities are related approximately by

$$C'_b \approx 1.2(\pm 0.1)C_{bo}^2/d_b.$$

(15)

[76] There is significant scatter in the data, which is to be expected because detailed physical processes have been explicitly neglected. Implicit in equation (15) are the influences of buoyancy (determined through $C_{bo}$) and the upper layer return flow and gravity (determined through $d_b$ and not $H_b$). Arguably, friction does not play a significant role at least during the early stages of deceleration.

5. Discussion and Conclusions

[77] In light of recent ocean observations revealing internal solitary waves of great amplitude (many times the depth of the surface layer) at the continental shelf, we have performed laboratory experiments examining shoaling of large-amplitude solitary waves on slopes ranging from shallow to steep. The study and analyses have been performed in a way to assist in the interpretation of ocean observations, particularly time series constructed from moored thermistor chains or CTD profiles [Klymak and Moum, 2003; Bourgault et al., 2007; Scotti et al., 2008; Nash et al., 2012].

[78] Being generated by lock release in an approximately two-layer fluid, solitary waves of depression were created with amplitudes up to 3.5 times the shallow-layer depth. Consistent with Stamp and Jacka [1995] and Grie et al. [1999], we confirmed that KdV theory should not be used to infer the width of the waves from their amplitude, if $A_{sw}$ is comparable to or larger than the characteristic depth $H_f$. Instead, the width is best determined as the half width, $L_{sw}$, from measurements of the horizontal wave speed (observed at the surface or by comparing a wave passing through two closely spaced mooring sites) and the time for the interface to descend from half to maximum vertical displacement. This measure of horizontal wave extent is more practical than its definition in terms of area over amplitude, which requires longer time series and which assumes the wave is indeed solitary with no breaking in the lee, a process observed in the lab for very large amplitude waves [Carr et al., 2008].

[79] Our analysis of breaking regimes classified in terms of the Iribarren number was consistent with earlier experiments [Boegman et al., 2005a] and numerical simulations [Aghsae et al., 2010], although the critical Iribarren numbers distinguishing between collapsing, plunging, and surging differed because we defined $Ir$ in terms of the half width $L_{sw}$.

[80] We developed a simple theory predicting the maximum interface descent at breaking. Although qualitatively

Figure 13. (a) Height of rise upslope of the bolus while it moves at constant speed compared with the maximum interface descent. (b) Magnitude of along-slope deceleration of the bolus after its constant-speed phase compared with the characteristic deceleration based upon its upslope speed and along-slope distance to the surface. In both plots, the symbols indicate the magnitude of slope as in Figure 10.
different in form than the empirical equations of Boegman et al. [2005a] and Aghaee et al. [2010], its predictions were consistent with experiments.

[81] Significantly, whereas the empirical predictions rely on a constant slope from base to interface, the prediction (10) requires only a comparison between the area of the solitary wave to the area of the lower-layer fluid drawn down by the incoming wave. For the former, we approximated the area by the product of amplitude and twice the half width. For the latter, because our lower boundary had constant slope, we took the area to be that of a right-angled triangle bounded by the original interface position, the slope, and the height of maximum drawdown. This formula may readily be applied to observations of solitary waves shoaling on the continental shelf. For example, Table 1 computes the maximum descent and runout of the thermocline for shoaling solitary waves in the Sulu Sea, the Australian North West Shelf, and the Scotian Shelf. The results show that the interface can descend from tens to hundreds of meters with runout lengths on the order of kilometers.

[82] These calculations assumed constant topographic slope. More generally, if the topographic height above the abyss is given by \( h(x) \), then one can estimate the horizontal position, \( x_b \), where breaking occurs through the implicit relation

\[
2A_{sw}L_{sw} = \int_{x_b}^{10} (H_1 - h) dx. \tag{16}
\]

[83] Here \( H_1 \) is the depth below the thermocline in the open ocean and \( x_0 \) is the position where the undisturbed thermocline intersects the topography (i.e., where \( h(x_0) = H_1 \)). Using equation (16) to find \( x_b \), the maximum interface descent is simply \( H_1 - h(x_b) \).

[84] The stress and turbulence associated with solitary wave breaking can resuspend sediments and nutrients into the fluid column. As such, to manage marine ecosystems and to assess potential hazards for the offshore-oil industry, it is important to predict the location of breaking sites and the along-slope range and stresses associated with boluses that run upslope following a breaking event. The diagnostics presented here give crude order of magnitude estimates of breaking locations and the evolution of the leading bolus. Examining the impact of these processes upon bedload transport and sediment resuspension is the subject of work in preparation.

[85] Acknowledgments. The authors are deeply grateful to Marco Ghisabalberti and Nicole Jones for their assistance in providing equipment for the experiments in the School of Environmental Systems Engineering at the University of Western Australia. Part of this research was conducted by Sutherland as a Gledden Senior Visiting Fellow at the University of Western Australia.

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