Generation, propagation, and breaking of an internal wave beam

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We report upon an experimental study of internal gravity waves generated by the large-amplitude vertical oscillations of a circular cylinder in uniformly stratified fluid. Quantitative measurements are performed using a modified synthetic schlieren technique for strongly stratified solutions of NaCl or NaI. The oscillatory forcing leads to the development of turbulence in the region bounding the cylinder. This turbulence is found to be the primary source of the observed quasimonochromatic wave beams, whose characteristics at early times differ from theoretical predictions and experimental investigations of waves generated by small-amplitude cylinder oscillations. In particular, their wavelength is set by the Ozmidov scale rather than the size of the cylinder. The wave frequency is set by the buoyancy frequency $N$ if the cylinder frequency is larger or much less than $N$. Otherwise it is set by the cylinder oscillation frequency. Over long times the finite-amplitude waves that have propagated away from their source are observed to break down and the process is examined quantitatively through conductivity probe measurements and qualitatively through unprocessed synthetic schlieren images. From an analysis of the location of wave breakdown we determine that the likely mechanism for breakdown is through parametric subharmonic instability. This conclusion is supported by fully nonlinear numerical simulations of the evolution of a temporally, although not spatially, monochromatic internal wave beam. © 2010 American Institute of Physics. [doi:10.1063/1.3455432]

I. INTRODUCTION

Density stratified fluids support the propagation of internal gravity waves that arise from buoyancy restoring forces. Energy and momentum are transported in geophysical fluids by the internal waves radiating from localized sources. Observations, modeling, and experiments have been used to study in detail several generation mechanisms and the properties of the resulting waves. In particular, topographic forcing by tidal flow over features of the ocean floor is observed to be a major source of oceanic internal waves, which are subsequently responsible for significant diapycnal mixing. Similarly, flow over mountains may generate moderate- to large-amplitude atmospheric waves, the turbulent breakdown of which has been observed directly through in-flight measurements. Fritts and Alexander reviewed the generation of atmospheric internal waves by several primary sources, including topography, convection, shear, geostrophic adjustment, and wave-wave interactions. In general, the existence of such disturbances can have a significant non-local effect on the mean flow through the propagation and breaking of internal waves.

While turbulent flows are involved to varying extent in geophysical sources of internal waves, the turbulent generation process is currently not well understood. It has been observed in experiments and numerical simulations that turbulent sources generate waves in a narrow frequency range relative to the background buoyancy frequency. In a recent numerical study, Taylor and Sarkar found that the internal waves generated by oceanic bottom boundary layer turbulence propagated at angles between $35^\circ$ and $60^\circ$ from the vertical. It was shown that linear differential viscous decay could produce the observed spectral peak and decay in wave amplitudes. However, the proposed model may not be an adequate explanation in the case of larger amplitude waves. A key difference between their numerics and laboratory experiments is the use of large-eddy simulation for the numerical boundary layer and the resulting loss of resolution of the finescale turbulence. Although a viscous model was in agreement with the numerical results, experiments show the immediate generation of narrow-band internal waves on a timescale that is less than the viscous timescale required for differential decay.

In this paper we present the results of a study of internal waves generated by large-amplitude oscillations of a circular cylinder in strongly stratified fluids. We classify the forcing as large amplitude because in all cases the half peak-to-peak amplitude of the cylinder displacement was on the order of, but less than, the radius of the cylinder. Moderate-amplitude forcing in Boussinesq fluids has been investigated in laboratory experiments by Sutherland et al. and Sutherland and Linden, for circular and elliptical cylinders, respectively. In general, good qualitative agreement was found between the experiments and the linear, viscous, Boussinesq theory of Hurley and Keady. However, the beam width was consistently underpredicted because the theory neglects the formation of a viscous boundary layer around the cylinder. The large-amplitude forcing used in the current work results in boundary layer separation so that the internal waves are launched effectively by an oscillatory turbulent patch. In this sense, the extension of previous experimental work to in-
clude large-amplitude effects also alters the generation process.

As well as standard sodium chloride (NaCl) solutions, a new experimental technique that is applied in this work is the use of sodium iodide (NaI) to produce stronger stratifications than those in conventional tank experiments. Whereas typical stratifications have a density change on the order of 5% over the depth of the fluid, the density differences in this study are approximately 20% for solutions of NaCl and 50% for NaI. The use in this work of significant variations in the background density profile is motivated by the phenomenon of anelastic growth in amplitude for upward-propagating atmospheric waves. When the vertical distance traversed by a propagating wavepacket is a significant fraction of the local density scale height, non-Boussinesq effects result in an increase in wave amplitude with height. As a result, nonlinear effects may have a more pronounced influence on the evolution of such internal waves. Previous work has shown that weakly nonlinear effects modify the amplitude growth of internal wavepackets, thus affecting the location and intensity of wave breaking, should it occur. In the current experimental work the waves are generated at finite amplitude and the vertical scale of propagation is such that non-Boussinesq growth may be measurable. The use of both NaCl and NaI solutions allows for observation of the wave behavior over different fractions of a density scale height, since the vertical scale of propagation remains the same while the strength of the stratification is varied.

While the large-amplitude forcing acts to modify the source region by causing boundary layer separation, it also has the effect of generating moderate- to large-amplitude waves. Tabaei and Akylas showed through an asymptotic analysis that nonlinear effects are relatively insignificant for an isolated beam with slow along-beam modulations in a uniform, Boussinesq stratification. Dispersive and viscous effects were found to be the dominant factors in determining the propagation of isolated beams. A subsequent paper investigated the role of nonlinearity in situations where there exists a region of interaction, namely, the reflection of a wave beam from a slope or the collision of two beams. In such cases it was found that nonlinear effects result in the generation of higher-harmonic beams that propagate out of the interaction region and into the far field. In this study we find that nonlinear effects have a significant influence on the evolution of the finite-amplitude waves, resulting in wave breaking for a single beam in the absence of critical layers. Rather than the slow modulations considered by Tabaei and Akylas, we observe that the wave breaks down due to parametric subharmonic instability (PSI). This mechanism for wave breakdown has been studied analytically and in laboratory experiments for unbounded plane internal waves and wave modes in a rectangular domain. Ours is the first examination of PSI occurring for an internal wavebeam in continuously stratified fluid.

A brief summary of the analytic results of Hurley and Keady for small-amplitude cylinder oscillations is given in Sec. II. In Sec. III we describe the experimental apparatus and the implementation of the synthetic schlieren technique in strongly stratified solutions. We also explain in detail the analysis of wave frequency, wavenumber, and amplitude from the schlieren processed data. The final content of Sec. III is a description of the qualitative criterion for the onset of wave breakdown. In Sec. IV we present the results of the quantitative analysis of the wave field for coherent beam structures. A discussion of the characteristics of the observed instabilities and subsequent wave breakdown is the subject of Sec. V, where we also present our findings from numerical simulations of a localized wave beam. We summarize the results of the study in Sec. VI.

II. THEORY

The experiments in this work use strong stratifications in which the background density varies by up to 50% over the fluid depth. Therefore, we calculate the background buoyancy frequency $N$ using the non-Boussinesq relation

$$N^2(z) = -g d\bar{\rho}/\bar{\rho} dz,$$  \hspace{1cm} (1)

where $g$ is the acceleration due to gravity and $\bar{\rho}(z)$ is the background density profile as a function of height. Given the non-Boussinesq form of the equation as shown above, the construction of a near-exponential density profile in the experiments yields a uniform stratification with an approximately constant background buoyancy frequency denoted by $N_0$.

Although our experiments use large density gradients, we may treat the fluid as Boussinesq over our region of focus for the generation of the beams because density variations are relatively small over the vertical extent of the cylinder. Previous experimental work on internal wave generation by oscillating cylinders has been compared to the analytic solution of Hurley and Keady for the waves generated by small-amplitude oscillations of a cylinder in a viscous, uniform, Boussinesq stratification. For the purposes of comparing their predictions to our large-amplitude experiments, here we present their final equations recast in a modified set of variables as they relate to our analyses.

For a cylinder oscillating vertically with angular frequency $\omega_c$ in a uniformly stratified fluid with background buoyancy frequency $N_0$, such that $\omega_c < N_0$, four wave beams emanate from the source region at a fixed angle to the vertical, given by

$$\Theta = \cos^{-1}\left(\frac{\omega_c}{N_0}\right).$$  \hspace{1cm} (2)

The coordinate transformation from the $(x,z)$ plane to cross-beam and along-beam coordinates, $(\sigma,r)$, of the beam in the first quadrant is given by

$$\sigma = -x \cos \Theta + z \sin \Theta, \hspace{0.5cm} r = x \sin \Theta + z \cos \Theta.$$  \hspace{1cm} (3)

The coordinate transformations for other quadrants may be obtained using symmetry properties of the solution. Assuming time-periodic solutions and ignoring viscosity, the spatial dependence of the streamfunction describing the beam in the first quadrant is given by
To obtain disturbances that were uniform in the spanwise (y) direction, the length of the cylinder was 2 mm less than the inner separation of the tank walls. The tank of dimensions \( W_t=122.3 \) cm and \( L_t=15.5 \) cm was filled to a depth of \( H_t=55 \) cm. For both types of stratification, experiments were performed with the cylinder centered approximately 12 cm above the bottom of the tank or approximately 8 cm below the fluid surface. This allows for a comparison between the characteristics of upward- and downward-propagating waves. For all experiments the cylinder was located approximately 80 cm (=2\( W_t/3 \)) from the left side of the tank. The spatial region for quantitative analysis was restricted to the left side of the cylinder, as we assume symmetry of the wave properties about the vertical axis. As in the case of small-amplitude forcing, four wave beams were observed emanating from the source region. Hereafter we will refer to the “primary beam” and the “reflected beam” as they are shown in Fig. 1. The terms are used similarly for the case with the cylinder near the top of the tank, but the beam reflection occurs off of the fluid surface rather than the bottom of the tank.

Approximately exponential density stratifications were made from solutions of either NaCl or NaI, as described by Clark and Sutherland.\(^{16}\) For each subset of experiments the density profile was measured with a conductivity probe and the data were fitted with an exponential function of the form

\[
\bar{\rho}(z) = \rho(z_0)\exp\left[-\frac{(z-z_0)}{H}\right],
\]

where \( H \) is the density scale height and \( z_0 \) is the smallest measured vertical coordinate. \( H \approx 270 \) cm for NaCl stratifications, and \( H \approx 130 \) cm for experiments using NaI. These values yield background buoyancy (angular) frequencies of \( N_0\approx1.9 \) rad/s and \( N_0\approx2.7 \) rad/s for NaCl and NaI, respectively, according to Eq. (1).

### A. Synthetic schlieren technique

For the quantitative analysis of the internal wave field, we implemented the synthetic schlieren technique as depicted in Fig. 1(b). Due to the large variations in background density over the tank depth for both NaCl and NaI stratifications, we have used a modified form of the synthetic schlieren equations for non-Boussinesq fluids.\(^{16}\) Thus we take into account the full vertical dependence of the density profile and the corresponding index of refraction for each type of stratification.

From measurements of the apparent vertical displacement \( \Delta z \) of an image of horizontal black and white lines placed behind the tank [see Fig. 2(a)], we may calculate the perturbation to the squared buoyancy frequency \( \Delta N^2 \) due to waves within the tank.\(^{13}\) In this work we focus on the time derivative of this field, denoted by \( \dot{N}^2 \), given by

\[
\dot{N}^2 = - \frac{1}{\gamma} \left[ L_p^2 + L_d \bar{n} \left( \frac{L_p}{n_p} + \frac{L_d}{n_d} \right) \right]^{-1}.
\]

The lengths \( L_c=15.5 \) cm and \( L_p=1.7 \) cm are shown in Fig. 1(b). Here \( \bar{n} \) is the index of refraction profile for the background stratification, while \( n_p=1.49 \) and \( n_d=1.0 \) are the indices of refraction of the acrylic tank walls and air,
respectively. The parameter $\gamma(z)$ is evaluated in terms of the background density and index of refraction profiles as \(^{(9)}\)

$$\gamma = \frac{1}{g} \left[ a_1 + a_2 (\bar{\rho} - \rho_0) \right],$$

where $\rho_0$ = 0.99823 g/cm\(^3\) is the density of fresh water and

$$a_1 = 0.2458 \text{ cm}^3/\text{g},$$

$$a_2 = -0.1208 \text{ cm}^6/\text{g}^2 \text{ for NaCl},$$

$$a_1 = 0.1894 \text{ cm}^3/\text{g}, \quad a_2 = -0.0086 \text{ cm}^6/\text{g}^2 \text{ for NaI}.$$
become \( r \) and \( \sigma \), respectively, as plotted in Fig. 4(a) for a typical experiment with \( N_0 = 2.7 \) rad/s, \( \omega = 1.96 \) rad/s, \( R = 2.98 \) cm, and \( A_c = 2.0 \) cm. The snapshot shown corresponds to a time \( t = [4T_{igw}, 5T_{igw}] \). Note that the \( r \) axis as shown does not start from zero. For small values of the radial coordinate the schlieren processed data are unreliable due to the presence of the cylinder and the surrounding three-dimensional turbulence. We have restricted our analysis to \( \sigma \approx 0 \) because of potential interference between the primary beam and the reflected beam in the lower flank, for which \( \sigma < 0 \). Although interference is not predicted by theory, the turbulent generation process results in more diffuse boundaries for the experimental beams. The heavy dashed lines in Fig. 4(a) correspond to the edges of the original schlieren image that has undergone reflection and rotation. The amplitude is not necessarily zero in the upper corners of Fig. 4(a), but the region for data acquisition did not include the areas outside of the dashed curves. In some cases it was evident that a clockwise rotation by the angle \( \pi/2 - \theta = \pi/2 - \cos^{-1}(\omega_{igw}/N_0) \) resulted in beams that were not horizontal despite time series images confirming that the waves had reached quasisteady state. Although we have been unable to determine the cause of this observation, in these cases a correction of up to \( \sim 0.1 \) rad was applied to the rotation angle so that a vertical profile through the image would be close to perpendicular to the phase lines. A spatially averaged profile, plotted as the solid curve in Fig. 4(b), was computed from 11 evenly spaced profiles over \( r \in [5R_c, 6R_c] \), which are represented by the dashed vertical lines in Fig. 4(a). The dashed curves in Fig. 4(b) are the envelope of the \( r \)-averaged \( N^2 \) profiles over one wave period, with the lower curve being the reflection of the positive amplitudes obtained from the rms. The profile shown at a particular phase moderately overshoots the envelope because the experimental signal includes noise and is not perfectly sinusoidal in time. We then calculated the power spectrum, shown in Fig. 4(c), of the Fourier transformed data.

In all cases we observed a distribution of power over a range of wavenumbers, which is expected for a beam of internal waves. However, here we focus on the magnitude of

![Image](75x590 to 216x727)

FIG. 3. (a) A horizontal time series of the \( N^2 \) field is shown with vertical lines bounding the region of spatial averaging. A profile through the contour plot at the location of the leftmost solid dark line is shown in (b), with the corresponding power spectrum of the Fourier transform in time plotted in (c). The forcing and buoyancy frequencies are marked on the horizontal axis.

![Image](72x98 to 211x233)

FIG. 4. Contours of \( N^2 \) are shown in (a) after reflection and rotation about the position of the cylinder. The heavy dashed line demarcates the regions in the upper corners of the image where no schlieren information is available. The \( r \)-averaged profile is given by the solid curve in (b), with the envelope over one wave period given by the dashed curve. The power spectrum resulting from the Fourier transform of the solid curve is plotted in (c) as a function of cross-beam wavenumber.
the wavenumber with the maximum associated power in the Fourier spectrum and denote it as $k^*_{w}$. For each spectrum as shown in Fig. 4(c), the peak value was found from a quadratic fit to the three points with the maximum amplitudes. The resulting peak values from 16 evenly spaced snapshots were then averaged in time for $t \in [4T_{lw}, 5T_{lw}]$. The wave period was calculated from the measured wave frequency as $T_{lw} = 2\pi / \omega_{lw}$. This interval in time was chosen because the beams were well developed and significant distortions due to wave instabilities were not yet present. The uncertainty in the measurement of the cross-beam wavenumber $d \Delta_{w}^{a}$, was taken to be the standard deviation determined from the averaging process. This procedure provides a characteristic value for an analysis of the wavelengthscale and is used in further calculations that involve the polarization relations. We acknowledge that this treatment is a simplification, and in using this value we do not intend to imply that the wave field is entirely monochromatic in space. However, the characteristics of the Fourier wavenumber spectra are consistent with the selection of the dominant wavenumber to describe the dynamics.

**D. Measurements of wave amplitude**

For the analysis of wave amplitudes, we have focused on the $N^2$ field for times $t \in [4T_{lw}, 5T_{lw}]$ and we compute the envelope as described in Sec. III A. In order to capture the properties of the beam, we have computed a profile in the $\sigma$ direction, averaged over a radial coordinate of $r \in [10 \text{ cm}, 20 \text{ cm}]$. Although we work with the envelope, $\langle N^2 \rangle$, for this analysis, some “patchiness” is still evident in the final image that we use for further analysis. Averaging in the $r$-direction reduces some of the variability in the signal that is an artifact of processing rather than an indication of the wave structure. The lower bound of the spatial interval was chosen such that the measurements would be outside of the turbulent boundary layer around the cylinder. The characteristic amplitude, denoted by $A_{N^2}$, was taken as the maximum value extracted from the $r$-averaged profile, with its uncertainty given by the standard deviation.

**E. Wave breakdown: Qualitative observations**

While synthetic schlieren was used to obtain quantitative measurements of the wave field in the quasisteady regime, the technique was not applicable to the strongly disturbed flow that was evident at the onset of instability. Meaningful data are obtained from synthetic schlieren measurements only in the case of spanwise-uniform waves of moderate amplitude. This property precludes the use of quantitative synthetic schlieren when the background image becomes highly distorted due to the development of three-dimensional flow. In order to gain insight into the mechanism for the observed instabilities, a qualitative examination was performed of the raw video footage for a subset of experiments. The collection of experiments included both NaCl and NaI stratifications, as well as the full range of cylinder radius, amplitude, and oscillation frequency. A consistent, although qualitative, criterion was chosen to mark the onset of significant disturbances to the beam structure; we will refer to this phenomenon as wave breakdown. The horizontal black and white lines of the schlieren image were monitored visually for the first occurrence of the lines appearing to be oriented vertically at a location outside of the turbulent region bounding the cylinder. This is an indication that the displacement of isopycnals is so extreme that the image behind the tank is magnified and significantly distorted. Although our criterion is not an indication of wave overturning, we found that the image blurred around this location shortly after the lines becomes vertical which is an indication of the wave breakdown through the development of small-scale three-dimensional structures. Our criterion thus provides an objective measure of the location and onset time of the start of wave breakdown.

An example of the visual characteristics of the schlieren image is shown in Fig. 5 for an experiment in a NaCl stratification with $N_{0} = 1.9 \text{ rad/s}$, $\omega_{c} = 1.53 \text{ rad/s}$, $R_{c} = 2.98 \text{ cm}$, and $A_{c} = 2.0 \text{ cm}$. The spatial region is the same for each frame, with time increasing from Fig. 5(a) to Fig. 5(c). A typical image resulting from a coherent wave beam is shown in Fig. 5(a), which includes visible deflections of the lines.

**FIG. 5.** Closeup view of synthetic schlieren background showing evolution of image characteristics. (a) Typical distortions of the image due to waves. (b) The image when satisfying the criterion for breakdown. (c) Loss of resolution of the lines.
from their undisturbed orientation. The region is shown in Fig. 5(b); 4 s (= 1.7Tigw) later with clear qualitative changes occurring in the image. Our criterion for wave breakdown is satisfied at (x, z) = (−11 cm, 28 cm), where the lines become vertical and appear to be “overturning.” The image shown corresponds to a time of approximately 60 s (= 1.5Tigw) from the start of the oscillations of the cylinder. The image in Fig. 5(c), taken 7 s (= 1.7Tigw) after the time of Fig. 5(b), shows blurring and the inability of the camera to resolve each separate line due to the development of fully three-dimensional structures in the flow. We interpret this evolution of the raw image as evidence of the evolution of the beam instability, but these features alone do not provide significant insight into the instability mechanism. We provide a discussion of potential causes for wave breakdown in Sec. V B.

For each experiment, several frames separated by 1 s were captured from the video recording, starting at approximately the time of breakdown. In each case, the still images were used to estimate the time and location at which breakdown occurred, which could then be compared for different forcing parameters of the cylinder. We emphasize that this analysis focuses on the first occurrence of the image overturning shown in Fig. 5. Such overturning often was subsequently observed at different locations in the tank for later times in the experiment.

IV. WAVE STRUCTURE AND TRANSPORT

A. Wave frequencies

In Fig. 6 we compare the normalized wave frequency ωigw/N0 to the normalized forcing frequency ωf/N0. In Fig. 6(a) the average power spectrum for each experiment is shown with an offset on the horizontal axis corresponding to the value of ωf/N0. The spectra are plotted on a linear scale and have been rescaled such that the maximum is the same arbitrary value for all experiments. Multiple spectra shown at the same value of ωf/N0 represent experiments with the same forcing frequency but different cylinder radii and forcing amplitudes. Here we focus on the dependence of the wave frequency upon the forcing frequency only, since linear theory predicts a direct correspondence between these quantities. Thus, in the plot we do not make distinctions based on Re or A. Note that each spectrum displays similar content to that shown in Fig. 3, although the spectra for Fig. 6 are the average results and the ωigw axis has been normalized by the buoyancy frequency. We include the presentation of the frequency spectra in this form to provide additional information about the shape of the spectra for varying forcing frequency. For example, an increase in the width of the spectral peaks is evident for cylinder frequencies that are approaching or above the buoyancy frequency. In addition, at near-critical frequencies increased variation can be seen among multiple spectra at the same forcing frequency, as in the case of ωf/N0 ≈ 0.87, and the wave power is found at frequencies other than the forcing value. In Fig. 6(b) the value of ωigw with peak power is plotted with a solid circle. The open circles correspond to secondary, lower amplitude peaks in the frequency spectrum. The dashed line with a slope of 1 corresponds to the prediction of linear theory that the wave frequency is equal to the forcing frequency. Two additional dashed lines corresponding to frequency superharmonics are also shown in Fig. 6(b). In the low forcing frequency experiments, the majority of the power in the measured spectrum was at twice the forcing value with a peak also occurring at the forcing frequency. For the lowest forcing frequency, another small-amplitude peak can be seen in the spectrum at three times the forcing value, but frequency tripling is not permitted in other experiments due to the upper limit of the buoyancy frequency. Also according to linear theory, no waves can be generated directly by the cylinder for ωf above N0, which is indicated on the plot by the dotted vertical line. Although the spectra are broader for forcing frequencies above the buoyancy frequency, we nonetheless observed propagating internal waves with ωigw ≈ 0.5N0 in these experiments due to generation by the oscillatory turbulent patch.
The observed relative frequency corresponds to propagation at approximately 60° from the vertical. This result is near the largest angle of the ranges for turbulently generated waves found numerically by Taylor and Sarkar\textsuperscript{12} (35°–60°) and experimentally by Dohan and Sutherland\textsuperscript{9} (42°–55°).

In summary, we find that the dominant frequency of waves generated by the oscillatory turbulent patch surrounding the cylinder tends to lie between 0.5\(N_0\) and 0.8\(N_0\), whether the cylinder itself oscillates with very low or very high frequency. Agreement with the theoretical prediction is closest for the range \(\omega_c/N_0 \in [0.5, 0.8]\), where we see that the experimental data lie on the theoretical curve within the error bars in all cases. In the following sections, we focus on experiments in this frequency range because the waves exhibit the most coherent beam structure, thereby facilitating our analysis and interpretation of the data.

B. Cross-beam wavelengths

In the linear regime the length scale of the waves is completely determined by the size of the cylinder.\textsuperscript{15} In the current experiments, for which \(A_c\) is of the same order as \(R_c\), we anticipate that the large-amplitude forcing may influence the wavenumber. In Fig. 7(a) the inverse of the characteristic cross-beam wavenumber, \(1/k^*_{\sigma}\), is plotted as a function of the cylinder radius. From the plot we observe that the points corresponding to the smallest values of oscillation frequency are separated from the remaining experiments. We conclude that \(R_c\) alone is not an adequate predictor of the wavenumber, particularly for small oscillation frequencies.

Due to the turbulent nature of the wave generation process, we propose that the Ozmidov scale \(L_O\) should set the value of \(1/k^*_{\sigma}\). The Ozmidov scale is a measure of the vertical extent of the largest turbulent eddies that have sufficient kinetic energy to overturn in a given stratification. In terms of the turbulent dissipation rate \(\epsilon\) and the buoyancy frequency,\textsuperscript{26}

\[
L_O = \epsilon^{1/2} N^{-3/2}. \tag{10}
\]

To estimate \(\epsilon\), we assume that the energy of the transmitted waves is small in comparison with the total energy input by the cylinder, so that the turbulent dissipation rate may be approximated by the rate of energy input per unit mass.

From observations of the unprocessed experimental data, we find that when the cylinder begins to oscillate, fluid in the bounding region above or below the cylinder is displaced and flows around the cylinder to the opposite side. This appears to be the primary process occurring in the generation of the turbulent patch, and it is also consistent with our observation that in quasisteady state the oscillatory turbulence is out of phase with the cylinder. We consider the area in the \(x-z\) plane, \(A\), of fluid that is displaced by the motion of the cylinder with amplitude \(A_c\) during a quarter cycle. A direct calculation of this area through integration yields

\[
A = A_c R_c \sqrt{1 - \frac{A_c^2}{4 R_c^2}} + R_c^2 \left[ \pi - 2 \sin^{-1} \left( \sqrt{1 - \frac{A_c^2}{4 R_c^2}} \right) \right]. \tag{11}
\]

Assuming that \(A_c \approx R_c\), which is the case for our experiments, this expression may be approximated by \(A \sim A_c R_c\). To estimate the characteristic speed of the displaced fluid, we assume that it travels a vertical distance of \(2 R_c\) in a timescale of \(\omega_c^{-1}\), so that \(v \sim 2 \omega_c R_c\). From these calculations, we find that the rate of kinetic energy input per unit length along the cylinder for the displaced fluid is approximately \(\rho_0 (A_c R_c) \times 2 (2 \omega_c R_c)^2 \omega_c\). To obtain the input rate of energy per unit mass, we divide this quantity by the fluid density and the area of the region of energy input, which to leading order is \(\pi R_c^2\). Thus, we estimate

\[
\epsilon \sim \frac{(A_c R_c)(2 \omega_c R_c)^2 \omega_c}{\pi R_c^2} \sim A_c R_c \omega_c^3. \tag{12}
\]

and so from Eq. (10),

\[
L_O \sim \sqrt{A_c R_c \left( \frac{\omega_c}{N} \right)^{3/2}}. \tag{13}
\]

In Fig. 7(b), \((k^*_{\sigma})^{-1}\) is plotted as a function of \(L_O\). We do not expect perfect collapse of the data because of the complicated structure of the fluid motion surrounding the cylinder. However, Fig. 7(b) demonstrates that a clearer trend

![Figure 7](image-url)
emerges through a comparison between the wavelength scale and a length scale of turbulence. In particular, the separation of the data according to frequency, as mentioned above for Fig. 7(a), is reduced in Fig. 7(b). Here we note that the values of \( L_O \) that we calculate based on Eq. (13) are between approximately 1 and 2 cm. Although \( L_O \) is crudely estimated, the relationship between \( (k_x)^{-1} \) and \( L_O \) is of order of 1, which makes a direct correspondence between these quantities more physically plausible.

The estimated values of \( L_O \) are consistent with the vertical extent of turbulent patches surrounding the cylinder, as shown in Fig. 8. This shows an unprocessed synthetic schlieren image at a time approximately four wave periods from the start of the cylinder oscillation, which corresponds to the lower bound of the time interval for the wavenumber analysis, as described in Sec. III C. Fully three-dimensional disturbances in a stratified fluid scatter light passing through the tank from the background image of black and white lines. We identify the resulting region where the image is blurred as being turbulent. Figure 8 shows that the turbulent boundary layer above the cylinder is of comparable extent to the Ozmidov scale estimated using Eq. (13). This result is typical of all the large-amplitude cylinder oscillation experiments we have performed. Outside the turbulent boundary layer, the image of black and white lines is distorted but not blurred, indicating the presence of spanwise-coherent internal waves emanating from the oscillatory turbulent patch.

Thus it is reasonable to suppose that \( L_O \) gives a measure of the largest length scales of the turbulent patch that acts as a source for the internal waves. Some of the scatter in the data shown in Fig. 7 may be attributed to the value of \( k_x^\prime \).

Since we have retained only the peak wavenumber in our analysis, the results for each experiment may be affected differently by this simplification depending on the true wavenumber distribution.

C. Wave amplitudes

Although we measured the amplitude \( A_{N_x^2} \) directly, we may use the Boussinesq polarization relations to find the amplitude of the vertical displacement of the waves, given by

\[
A_x = \frac{A_{N_x^2}}{N_0^2 k_x \sin \Theta} = \frac{A_{N_x^2}}{N_0^2 k_x^2 \cos \Theta \sin \Theta}.
\]

We normalize \( A_x \) by the horizontal wavelength, \( \lambda_x = 2\pi/k_x \), to provide a more physically meaningful interpretation of the data. The final values shown in Fig. 9 were calculated as

\[
A_x = \frac{A_{N_x^2}}{2\pi N_0^2 \sin \Theta}.
\]

which have been plotted as a function of \( A_x/L_O \). As shown in Fig. 9, there is a clear separation in the ratio \( A_x/\lambda_x \) for NaCl and NaI experiments, with an approximately constant ratio of \( \approx 2.5\% \) for all NaI experiments. In previous work with NaCl stratifications it was observed across experiments that \( A_x/\lambda_x \) collapsed to a value in the range of 2%–4%, regardless of the forcing amplitude. We do not have a clear explanation for the observed increase in the ratio for NaCl experiments. The effect of the smaller value of \( N_0 \) for NaCl stratifications is magnified by the cubic power of \( N_0 \) in Eq. (14). However, there is no obvious physical reason to anticipate that this difference between the amplitude ratios should be based on stratification alone. We have estimated the Reynolds number based on the cylinder diameter, oscillation amplitude, frequency, and a characteristic kinematic viscosity of the fluid. Viscosity measurements were performed using an Anton Paar DMA 500 density meter for solutions of NaI with densities between fresh water and approximately 1.6 g/cm³. For the experiments of focus here, we estimate Reynolds num-
bers of approximately 1500–2000 for both types of stratification. Thus, viscosity differences between solutions of NaI and NaCl do not account for the results shown in Fig. 9. The separation of the results for NaCl and NaI may indicate that the wave behavior is outside the applicability of the polarization relations, which were derived under the assumptions of linear theory. There also exists the possibility that a quantity other than $A_0/\lambda_0$, with a different dependence on $N_0$, remains relatively constant despite changes in the forcing amplitude. We have presented the results as shown to allow for comparison within the context of previous studies.

For all calculations in the analysis of the wave generation, we have assumed that the fluid can be treated as Boussinesq because the extent of vertical propagation is small in comparison with the density scale height. As shown in Fig. 9, the vertical error bars are sufficiently large as to produce a region of overlap between the data for upward- and downward-propagating waves. If trends in the data are significant, the slightly reduced amplitudes for downward-propagating waves may indicate that the region of observation is at the threshold of non-Boussinesq asymmetry between the directions of vertical propagation. However, we cannot conclude that the data shown contain evidence of non-Boussinesq wave behavior.

D. Wave power

We use the measurements of the characteristic wavenumber and amplitude, as described in the previous subsections, to calculate the average energy flux of the primary beam, and hence the wave power. The present analysis is restricted to experiments for which $\delta k'/k' < 0.1$ and $\delta A_0^2/A_0^2 < 0.2$ simultaneously, so that the values used for the calculation of power are as unambiguous as possible.

From the Boussinesq polarization relations we obtain the following expression for the time-averaged vertical energy flux of a monochromatic plane wave with wavenumber $k_0^*:

$$\langle F_E \rangle = \frac{A_0^2}{L^2} \frac{\rho_0}{N_0^2} \cos \Theta \sin \Theta \frac{\sin \theta}{k_0^*}.$$  \hspace{1cm} (16)

The use of this expression in our analysis requires that modes other than $k_0$ do not contribute significantly to the $N_0^2$ field. Based on the breadth of the wavenumber spectra, we have modified the above expression to account for the contributions from all cross-beam modes, $k_n$, with non-negligible power. We replace the characteristic amplitude and wavenumber with a sum over modes, i.e.,

$$\langle F_E \rangle = \frac{1}{2} \rho_0 \frac{A^2}{N_0^2} \frac{1}{\cos \Theta \sin \Theta} \sum_n \frac{A_n^2}{k_n}.$$  \hspace{1cm} (17)

The images used in the present analysis were processed using the same procedure as for the wavenumber analysis, as described in Sec. III C. Signal attenuation with increasing $\sigma$ and a geometric effect caused by the image rotation resulted in a region of zero amplitude for the largest values of $\sigma$. Based on the properties of the FFT algorithm that we have employed, we account for this in our analysis by scaling the amplitude of a mode according to the ratio $L_n/L$, where $L_n$ is a measure of the beam width and $L$ is the length of the spatial domain for the Fourier transform. In this case the amplitude $A_n$ that one would obtain from the transform is related to the amplitude of the real signal, $A$, through

$$A_n = \left( \frac{L_n}{L} \right) A.$$  \hspace{1cm} (18)

We find that the squared amplitude is given in terms of the power by

$$A_n^2 = \left( \frac{2}{L_n^2} \right)^2 \mathcal{P}_n,$$  \hspace{1cm} (19)

in which $\mathcal{P}_n$ represents the squared magnitude of the Fourier coefficient of the $n$th mode of the $N_0^2$ field.

To obtain the total power of the primary beam, we multiply the vertical energy flux by the area of a horizontal cross section through the beam, $L_n L_n = L_n^2 \cos \Theta$, in which $L_n$ is the length of the cylinder. Through this step and the substitution of Eq. (19) into Eq. (17), we arrive at the expression for the total measured power of the experimental beam,

$$P_{\text{exp}} = \frac{2 \rho_0 L_n^2}{N_0^2 \cos^2 \Theta} \sin \Theta \frac{\sin \theta}{k_0^*} \sum_n \mathcal{P}_n.$$  \hspace{1cm} (20)

In order to use the above expression we also require a quantitative method of determining the beam width $L_n$. We express $L_n$ in terms of a multiple of a characteristic wavelength $\lambda^*$ as

$$L_n = \alpha \lambda^* = \frac{2 \pi \alpha}{k_0^*}.$$  \hspace{1cm} (21)

Substituting this expression into Eq. (18) with $A_n \rightarrow A_\lambda$ corresponding to the amplitude of the waves with $k_n \rightarrow k_n^*$, we obtain

$$\alpha = \frac{A_\lambda}{A} \left( \frac{k_n^* L}{\pi} \right) = \frac{1}{\alpha} \frac{A}{A} \left( \frac{k_n^* L}{\pi} \right)^2.$$  \hspace{1cm} (22)

Thus, we find

$$\alpha = \frac{\mathcal{P}_n^{3/4}}{\sqrt{A}} \left( \frac{k_n^* L}{\pi} \right).$$  \hspace{1cm} (23)

For our estimate of $\alpha$, we take the power of the mode for which $k_n$ is closest to $k_0^*$ and we use the characteristic amplitude $A_\lambda^2$ as described in Sec. III D. This yields values of $\alpha \in [1.6, 2.2]$ across all experiments. Comparison with the structure of the original $N_0^2$ profiles in the $\sigma$ direction shows that this range of $\alpha$ is reasonable when we consider that $\alpha$ characterizes the number of wavelengths contained in the primary beam.

In order to gain insight into the effects of the large-amplitude forcing on the resulting energy transport of the waves, we compare Eq. (20) with the theoretically predicted time-averaged power of the primary beam using Eq. (6). Denoting the prediction as $P_{\text{the}}$ and recasting expression (6) in terms of our experimental parameters, we obtain
The use of both $\omega_{gw}$ and $\omega_c$ in the calculation of $P_{thy}$ arises from the conversion of different parameters in Eq. (6) to the variables in our notation. The product of $A_c\omega_c$ is the magnitude of the maximum velocity of the oscillating cylinder, whereas the single power of $\omega_{gw}$ is a result of our distinction between the properties of the waves and the cylinder. We have expressed $P_{thy}$ in terms of the wave frequency rather than the angle from the vertical because frequency was the directly measured quantity. For calculations of both $P_{exp}$ and $P_{thy}$ we have used characteristic densities of $\rho_0=1.25$ g/cm$^3$ for NaI stratifications and $\rho_0=1.1$ g/cm$^3$ for NaCl stratifications. These values were estimated from the experimental measurements of the density at the vertical level of our analysis of wavenumber and amplitude.

Figure 10 shows a plot of the experimentally measured power versus the theoretical prediction. The large vertical error bars on the data points are a result of adding significant contributions from several of the variables in Eq. (20). Namely, the largest contributions to the final error were due to the uncertainties in $\Theta$, $L_c$, and $P_{thy}$, the last of which was estimated from the standard deviation of the time-averaged spectrum.

The experimental measurements and theoretical predictions are similar for small values of $P_{thy}$ but the two values deviate more with increasing forcing intensity. In general, we expect the experimental values to be less than the theoretically predicted power because the coupling of the cylinder to the internal waves is affected by the development of the turbulent boundary layer. This behavior is observed for all experiments, but the low values of $P_{exp}$ for large $P_{thy}$ also suggest that we may be observing a saturation of the wave field. Where the theory predicts an increasing rate of energy transport by waves, we hypothesize that much of the forcing energy is lost to turbulent kinetic energy in the bounding region of the cylinder. Here we may use the expression for $\varepsilon$, Eq. (12), to estimate the total turbulent dissipation rate. The product of $\rho_0\varepsilon$ with the characteristic volume of displaced fluid, $\sim A_c R_c L_c$, yields an approximate dissipation rate of $\rho_0(A_c R_c)^2 L_c \omega_c^2$. Using characteristic experimental values, we obtain an estimate of $\sim 2000$ erg/s. Therefore, it is reasonable to observe smaller wave powers than predicted by theory.

In particular, the estimated dissipation rate is comparable to the discrepancy between the values of $P_{thy}$ and $P_{exp}$ as the forcing intensity is increased.

V. WAVE INSTABILITIES AND BREAKING

A. In situ probe measurements

Synthetic schlieren provides a means of measuring quantitatively the structure and amplitude of internal waves in space and time. In the previous sections we have demonstrated the application of synthetic schlieren to experiments in strongly stratified fluids. We have used a separate quantitative technique for one characteristic experiment, in which measurements were made of the waves at a fixed location as they evolved in time. This provides an independent means through which we may observe the establishment of the wave field and its subsequent breakdown. Additional information is gained through this technique since quantitative synthetic schlieren measurements are not possible as the transition to instability occurs, as discussed in Sec. III E.

A conductivity probe was used to perform a vertical traverse of the background stratification in the region of interest, with approximately 42 measurements of voltage $V$ per vertical centimeter. The probe was then placed at fixed locations in space for a series of three experiments in which a cylinder with $R_c=4.43$ cm and $A_c=2.0$ cm was oscillating with frequency $\omega_c=1.96$ rad/s approximately 10 cm above the bottom of the tank. The probe provided measurements of the voltage with a resolution in time of $\Delta t=0.05$ s. The horizontal location of the probe was approximately 30 cm from the center of the cylinder while the vertical coordinate was set at 45, 40, and 35 cm successively above the bottom of the tank.

The time series measurements of voltage were translated into densities using the function $\tilde{\rho}(V)$ obtained from a linear fit to four discrete measurements of voltage for densities in the range of [0.998, 1.30] g/cm$^3$. With waves in the fluid, the perturbation density at each vertical level was computed by subtracting the initial background density. The wave vertical displacement $\xi$ was then calculated according to

$$\rho = -\frac{d\tilde{\rho}}{dz}\xi,$$

where $\rho$ is the perturbation density and the background density gradient at each of the three vertical levels was found through

$$\frac{d\tilde{\rho}}{dz} = \frac{dV}{dt}\frac{d\tilde{\rho}}{dV}dt.$$

Time series of the vertical displacement are shown in Fig. 11 for each vertical level. The motor driving the oscillations of the cylinder was turned on at $t=30$ s for Fig. 11(a) and $t=20$ s for Figs. 11(b) and 11(c) and was turned off at $t\approx 150$ s for each experiment. In all cases, we observe at
B. Qualitative analysis of synthetic schlieren

As discussed in previous sections, we observed the breakdown of propagating beams through both quantitative and qualitative techniques. Here we provide the results of further investigations of the wave breakdown and the associated instability mechanism. First we note that the measured normalized vertical displacement amplitudes $A_e / \lambda_v$, as presented in Sec. IV C, were well below the magnitude required for overturning instability. Non-Boussinesq growth of the wave amplitudes to overturning values does not account for the observations because the density scale height was large in comparison with the distance of vertical propagation.

The qualitative data, obtained from unprocessed schlieren images as described in Sec. III E, yielded several significant results. Instability was not observed in every experiment; in some cases, the breakdown criterion was not satisfied at any time that the cylinder was oscillating. It was common to these experiments that the forcing frequency was very near to or above the background buoyancy frequency, or the amplitude of oscillation was the smallest of our parameter range. This result for high-frequency forcing is consistent with the quantitative synthetic schlieren measurements, in which the wave signal was weaker and less coherent than for midrange forcing frequencies. Thus, we should not necessarily expect significant growth and transition to instability for the waves generated by high-frequency forcing. The occurrence of wave breakdown for both upward- and downward-propagating beams confirms that non-Boussinesq growth of upward-propagating waves is not responsible for the behavior. Another trend in the observations is that the time of wave breakdown varied significantly across experiments. For the same cylinder radius and amplitude, a change in the forcing frequency yielded an opposite change in the observed time of breakdown, i.e., a decrease (increase) in frequency resulted in a later (earlier) breakdown. This is a physically reasonable consequence of the change in the timescale for beam development. The effects of cylinder radius and amplitude on the breakdown time and location appear to be dominated by the forcing frequency in this qualitative analysis.

There are several possible scenarios, which are represented schematically in Fig. 12, that could lead to the breakdown of waves as observed. We expect that nonlinear effects are the most significant in regions of beam self-interaction, such as when the primary beam reflects off of the surface, or in a beam-beam interaction that could arise through multiple reflections off of the tank walls and the fluid surface. These situations are depicted in Figs. 12(a) and 12(b), respectively. We also consider the possibility of the breakdown of a freely propagating, noninterfering primary beam, as shown in Fig. 12(c). Breakdown in a region of beam reflection is the most straightforward to identify from observations because of the proximity to the surface or the bottom of the tank. In the case of beam-beam interactions at mid-depth, we expect that the location of the breakdown would vary significantly according to the wave frequency and the corresponding angle of the beam.
propagation. Through geometrical considerations, the region of interference would move farther from the position of the cylinder with decreasing forcing frequency.

The greatest insight into the underlying mechanism for the instability has been obtained from a comparison of the breakdown location across experiments, for which the data are plotted in Fig. 13. The experiments have been separated into upward- and downward-propagating beams in Figs. 13(a) and 13(b), respectively, with further distinctions made according to the forcing frequency relative to the background buoyancy frequency, as shown by the legend. For reference, we also include lines with slopes predicted by the value of \( \omega_c/N_0 \) according to Eq. (2), by which we can place approximate bounds on the expected location of the primary beam at a given frequency. Note that in this qualitative analysis of video footage, the experiments were not restricted to the frequency range that was used for quantitative analyses. In Fig. 13, a marker indicates the location of wave breakdown for each experiment. We observe that as a group, the markers are displaced somewhat upward in Fig. 13(a) and downward in Fig. 13(b) relative to a line through the center of the cylinder. This is consistent with our observations from quantitative synthetic schlieren analysis that the primary beam was not centered about the equilibrium position of the cylinder. Such an effect is evident in Fig. 4(a), in which the rotated beam is displaced from a line through the center of the cylinder. Therefore, the locations of the markers agree well with our expectation that the path of the primary beam emanates from a turbulent patch above and below the cylinder. The horizontal coordinates of breakdown were clustered within approximately 20 cm from the center of the cylinder, and there were no apparent trends in this location based on forcing parameters. The initial wave breakdown occurred at a vertical coordinate that does not correspond to a region of surface or bottom wave reflection, nor does the horizontal distance change significantly with frequency as it would in the case of interacting beams illustrated in Fig. 12. As an additional qualitative demonstration, a larger view of the synthetic schlieren background image during a breakdown event is shown in Fig. 14. The experiment is the same as that shown in Fig. 5, with the expanded view corresponding to the image in Fig. 5(b). The image in Fig. 14 is typical of the observations across experiments with different forcing parameters and stratifications. Significant distortions and blurring of the image occur close to the cylinder, as discussed in previous sections, and the wave breakdown is evident at \((x,z) \approx (11 \, \text{cm}, 28 \, \text{cm})\). The vertical lines at \(x = -15, -36 \, \text{cm} \) are due to junctions in the background image and they do not influence the analysis or the interpretation of results. From Fig. 14 we note that visible distortions of the background image outside of the turbulent patch are confined to the locations of the primary and reflected wave beams as determined from quantitative synthetic schlieren analysis at earlier times in the experiment. If present, reflected beams from the right side of the tank would result in disturbances in the upper right-hand corner of the image. Although the absence of a visually detectable signal does not imply the absence of small-amplitude waves, this example demonstrates that reflected waves from the side-walls or free surface were not apparent at the time and location of breakdown. Considering all of the observations discussed above, we conclude that beam-beam interactions are not responsible for the breakdown. Rather, the waves contained in a single beam undergo a transition to instability independently of interactions with boundaries or other beams. A candidate mechanism for the breakdown of an iso-

[FIG. 12. Schematic illustration of potential causes for wave breakdown: (a) beam superposition due to surface reflection, (b) beam-beam interference, and (c) breakdown of a freely propagating beam due to instability. The domain and the cylinder size and position are shown to scale.]

[FIG. 13. Schematic of experimental apparatus (to scale) with a marker denoting the location of wave breakdown for each experiment as indicated in the legend. Results for upward- and downward-propagating primary beams are shown in (a) and (b), respectively.]
lated beam is PSI, whereby energy is transferred to waves of lower frequency and higher wavenumber than the primary disturbance. \(^{27,28,24}\) This hypothesis provides the motivation for the numerical simulations described in Sec. V C.

C. Numerical simulations

A two-dimensional, fully nonlinear, Boussinesq code\(^ {18}\) was used to simulate the evolution of an internal wave beam. Previous work\(^ {18,29}\) has focused on plane waves and spatially localized wavepackets, often with uniform structure in the along-stream direction. The simulations presented here are not an attempt to model accurately all of the characteristics of the oscillating cylinder experiments. Our objective in performing simulations was to investigate potential instabilities in an established beam of finite width. For this study the spatial domain was horizontally and vertically periodic with a resolution of 128 by 512 points in the \(x\) and \(z\) directions, respectively. The code uses finite differencing in the vertical with periodic upper and lower boundary conditions and was run with 64 spectral modes in the horizontal direction. While a horizontally plane wave structure can be resolved with far fewer modes, the finite beam width in the current study means that increased horizontal resolution was required for adequate sampling across the signal.

We have initialized the simulations with a perturbation in the form of plane wave structure in the cross-beam (\(\sigma\)) direction with a Gaussian envelope to determine the beam width. In order to satisfy the doubly periodic boundary conditions, the full disturbance consisted of a superposition of three identical beams separated by a fixed distance. The beams decayed sufficiently rapidly with \(\sigma\) to prevent an increase in amplitude of the neighboring beams due to superposition. Given a domain \(x \in [0, L], z \in [0, H]\), the beam separation was given by

\[
\sigma_i = \frac{LH}{\sqrt{L^2 + H^2}}. \tag{27}
\]

This distance guaranteed the periodicity of the structure by positioning the center line of the two secondary beams at the appropriate corners of the domain. The center line of the primary beam was from corner to corner of the domain, regardless of the dimensions \(L\) and \(H\). These parameters determined the angle of the beam to the vertical direction, and hence the frequency \(\omega_{\text{gw}}\), through the relations

\[
\Theta = \tan^{-1}\left(\frac{L}{H}\right) = \cos^{-1}\left(\frac{\omega_{\text{gw}}}{N_0}\right). \tag{28}
\]

The maximum amplitude of the perturbation, cross-beam wavenumber, and standard deviation of the Gaussian envelope, denoted by \(a_0\), \(k_{\sigma r}\), and \(\sigma_0\), respectively, are free parameters. The initial structure of the streamfunction is then given by

\[
\psi(\sigma, t = 0) = a_0 \left\{ \exp\left[\frac{-\sigma^2}{2\sigma_0^2}\right] \cos(k_{\sigma r} \sigma) + \exp\left[\frac{-(\sigma - \sigma_r)^2}{2\sigma_0^2}\right] \cos[k_{\sigma r}(\sigma - \sigma_r)] + \exp\left[\frac{-(\sigma + \sigma_r)^2}{2\sigma_0^2}\right] \cos[k_{\sigma r}(\sigma + \sigma_r)] \right\}. \tag{29}
\]

For each spatial coordinate pair \((x, z)\) in the domain, a transformation to the \(\sigma\) coordinate was performed using Eq. (3). The value of \(\psi\) was calculated for the resulting value of \(\sigma\) according to Eq. (29) and was then assigned at the original grid point. Small-amplitude randomly generated noise was also superimposed on the field over the entire domain to seed any physical instabilities evenly.

For a beam in the first quadrant, the vertical component of the group velocity is positive. Therefore, in order to obtain the correct signs of the horizontal and vertical wavenumbers \(k_x\) and \(k_z\), they were calculated as

\[
k_x = |k_{\sigma r}| \cos \Theta, \quad k_z = -|k_{\sigma r}| \sin \Theta, \tag{30}\]

where \(k_{\sigma r} < 0\).

The parameters that determine the flow were chosen to model the experimental conditions. In all cases, the background velocity was zero and \(N_0\) was the value determined from the density profile, as described in Sec. III. For a given experiment, the characteristic cross-beam wavenumber \(k_{\sigma r}\), frequency \(\omega_{\text{gw}}\), and vertical displacement amplitude \(A_{\xi}\) were known. The initial streamfunction amplitude was determined through polarization relations as

\[
a_0 = \frac{\omega_{\text{gw}}}{k_x} A_{\xi} = \frac{\omega_{\text{gw}}}{|k_{\sigma r}^*| \cos \Theta} A_{\xi}, \tag{31}\]

such that \(a_0 > 0\). The value of \(A_{\xi}\) was determined similarly using a polarization relation, so the final value of \(a_0\) should be considered an estimate due to the propagation of uncertainties in our experimental measurements. We also have an
approximate measure of the beam width from experiments in terms of the parameter $\alpha$, given by Eq. (23). For the numerical beam initialization, we have attempted to obtain approximately two wavelengths across the width of the beam for consistency with observations and the measurements of $\alpha$. Contours of the vorticity field, $\zeta$, are shown at initialization in Fig. 15(a). Note that the extrema of the contour range are the same for each panel.

Although experimental parameters are used to initialize the simulations, the plots in Fig. 15 are cast in nondimensional form, emphasizing, for example, that the results should scale in time as $N^{-1}$.

For all simulations that were initialized using parameters comparable to experimental conditions, an instability developed along the central beam after an initial period of regular propagation of phase lines through the beam at a constant angle. The onset of the instability occurred along the center line of the beam, where the initial amplitude was largest, and the transition appeared visually to occur along the entire length of the beam simultaneously, as shown in Fig. 15(b). Therefore, we have confidence that the instability is physical and is not caused by boundary effects in the numerical formulation. Within the beam structure, waves began to develop at a larger angle to the vertical direction, and hence a lower frequency, than the initial disturbance. A cascade of energy to smaller scales was also observed. As in other studies of PSI for internal waves, we find that the disturbance which grows fastest from the background noise is that with frequency half that of the wavebeam, as illustrated in Fig. 15(c).

The phase lines of the waves including the developed instability align well with the expected direction for the subharmonic, thereby supporting the conclusion that PSI was the primary mechanism for the breakdown of the wave beams in the numerical context. In general, the instability grew in amplitude until overturning began to occur, after which the simulations broke down rapidly. For waves with smaller initial amplitudes, the simulations ran for the full time of the corresponding experiment. However, the development of PSI was responsible for a complete loss of the coherent beam structure. This effect may explain our observations in raw experimental footage of the sudden spread at late times of large disturbances in the tank that did not correspond with the expected location of beams. A similar effect was observed in experiments by McEwan, who noted that density microstructure became evident surprisingly rapidly in regions outside of breaking due to PSI.

The use of experimentally realistic parameters to initialize the simulations facilitates a comparison between the observed time of PSI onset in the simulations and the experimentally observed time of significant visual distortions of the schlieren image, as described in Sec. V B. Although we cannot specify what physical effect was occurring at the time of observation, the hypothesis that PSI arose in the experiments may be supported or refuted through an order-of-magnitude comparison with the results of the simulations. Runs were performed with parameters modeling five characteristic experiments in two different stratifications with a range of forcing frequencies. We have found that the estimated time of the onset of PSI in the simulations differs from the experimental breakdown time by a maximum of a factor of 2. This is reasonable agreement if one considers that the simulations serve to model approximately some of the characteristics of the experimentally measured waves. Using the numerics, we have verified that instabilities became evident in a physically reasonable timescale given physically realistic input parameters. There were cases in which the experimental time was approximately 30% less than the time from simulations and vice versa, so no systematic pattern emerged for these particular runs. The similarity of the experimental and numerical timescales for the development of instabilities serves as support for the hypothesis that PSI was the cause for breakdown of the beams in the experiments.

VI. CONCLUSIONS

We have studied the generation, propagation, and eventual breakdown of internal wave beams generated by the large-amplitude vertical oscillations of a cylinder in strong stratifications of NaCl or NaI. Quantitative measurements of wave frequency, wavenumber, and amplitude were made using a generalized form of synthetic schlieren that takes into account the full vertical profile of the density and index of refraction for fluids with significant density variations with height. The large-amplitude forcing produced a turbulent boundary layer around the cylinder that modified the genera-
tion region and the resulting characteristics of the wave beams.

It was found that the wave frequency was equal to the forcing frequency only for the interval $\omega_0 / N_0 = 0.5 - 0.8$. For small forcing frequencies, beams were generated at higher harmonics and with larger associated power than the primary beam. With forcing frequencies above $N_0$, the waves generated by the localized turbulent patch had frequencies of approximately $0.5 N_0$. In all cases, the waves with the maximum associated power were observed in the range $\omega_{gw} / N_0 = 0.5 - 0.8$ regardless of the forcing frequency. These results indicate a preferred frequency range that is in agreement with previous work on the turbulent generation of internal waves.

Wave frequency selection in the typical range occurred despite a dominant frequency component due to forcing by the cylinder, of the turbulent source. Coherent quasimonochromatic beam structures were observed emanating directly from the source without an intermediate spatial region of waves with a broad frequency distribution. Thus, differential viscous decay does not account for the observations of frequency selection in this study.

The Ozmidov scale, which characterizes the vertical scale of the eddies in stratified turbulence, was found to be more predictive of the length scale of the waves than the cylinder radius alone. Also as a consequence of the turbulent generation mechanism, the wave amplitudes were found to be an approximately constant fraction of the horizontal wavelength, which has been noted in previous experimental studies.

However, the magnitude of this ratio differed for experiments in NaCl and NaI stratifications. This observation is currently unexplained and requires further investigation.

Qualitative observations from unprocessed videos were used to characterize the time and location of wave breakdown in the experiments. With the motivation of examining potential instabilities for a temporally monochromatic wave beam, fully nonlinear numerical simulations were performed with experimentally realistic input parameters. The results of the simulations showed the development of PSI at times comparable to the observed values for the experiments. Based on this outcome, we conclude that PSI of the isolated primary beam was responsible for the breakdown of the waves in the experiments at relatively late times.

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