the waves do not veer significantly towards the shore. Only as the lower-layer depth becomes comparable to H_1 do the waves feel the influence of the bottom slope. At this point, just as in the case for surface waves, the crests of the interfacial waves rotate so as to become more parallel to the beach. Of course, the waves never reach the shore. In reality they would grow in amplitude and develop into nonlinear waves. These would eventually break, partially reflect or would otherwise be affected by near-slope currents where the interface intersects the bottom slope.

6.5 Ray theory for internal waves

Here we consider the vertical and horizontal propagation of internal waves in non-uniformly stratified fluid in which we also include the effects of vertically varying background winds. For mathematical simplicity, we focus upon Boussinesq internal waves and we ignore the effects of background rotation.

We begin by looking at internal waves restricted to the x–z plane, in which case the ray theory equations can be reduced to a simple integral expression analogous to (6.19). Two cases arise that are of particular interest. In one, internal waves asymptotically approach a vertical level where their extrinsic frequency is zero or, equivalently, where the horizontal crest speed of the waves matches the speed of the background flow. This is known as a critical level. In the other case, the background wind and stratification are prescribed so that at some height the extrinsic frequency equals the background buoyancy frequency. Waves reflect from this level.

6.5.1 Internal waves in two dimensions

Here we consider the general circumstance in which internal waves move in non-uniformly stratified Boussinesq fluid, with buoyancy frequency $N(z)$, and in vertically varying background flow $\bar{U}(z)$ that moves in the x-direction. For the assumptions of ray theory to remain valid, the vertical wavelength of waves must be short compared with the scale of variations of N and $\bar{U}$. The horizontal component of the wavenumber vector is taken parallel to the flow in the x-direction. This circumstance is not unrealistic: in Section 5.4 we found that internal waves generated by flow over two-dimensional obstacles have their horizontal wavenumber aligned with the flow.

It follows from (6.13) that the horizontal components of the wavenumber are constant following the motion of the wavepacket. Because the horizontal wavenumber is parallel to the flow direction we set $k_y = 0$ and $k_x = k_{x0}$. Likewise, for steady motion the intrinsic frequency $\omega = \omega_0$ is constant although the extrinsic frequency $\Omega = \omega_0 - k_{x0}\bar{U}$ can change following the wave motion because the background flow Doppler-shifts the waves.

From (6.12), the x- and z-positions of the wavepacket vary in time according to

$$\frac{dx}{dt} = c_{gx} + \bar{U} \tag{6.29}$$

and

$$\frac{dz}{dt} = c_{gz}. \tag{6.30}$$

Combining these equations, the path of the waves is given by solving the differential equation

$$\frac{dz}{dx} = \frac{c_{gz}}{c_{gx} + \bar{U}}, \tag{6.31}$$

in which the components of the group velocity are given by (3.62). Explicitly,

$$\frac{dz}{dx} = \frac{-N \sin\Theta \cos^2\Theta}{N \sin^2\Theta \cos\Theta + k_{x0}\bar{U}}, \tag{6.32}$$

in which, from (3.56), $\Theta(z) = \tan^{-1}(k_z/k_{x0})$ represents the angle formed between lines of constant phase and the vertical.

The solution to (6.32) is nontrivial in that N and $\bar{U}$ are functions of z, and Θ itself is a function of z through its dependence upon the vertically varying wavenumber k_z. Rearranging the dispersion relation (3.54), k_z is given explicitly in terms of N and $\bar{U}$ by

$$k_z(z) = -k_{x0}\sqrt{\frac{N^2}{(\omega_0 - k_{x0}\bar{U})^2} - 1}. \tag{6.33}$$

Here we have chosen the sign of the square root to correspond to upward- and rightward-propagating waves with $\Omega = \omega_0 - k_{x0}\bar{U} < N$. Thus $|k_z|$ decreases and the vertical wavelength increases as the waves move to heights where either the buoyancy frequency decreases or the extrinsic frequency Ω increases through Doppler-shifting by the background winds.

If $\bar{U} = 0$, the path of internal waves is given simply by

$$\frac{dz}{dx} = -\cot\Theta = -\frac{k_{x0}}{k_z}. \tag{6.34}$$

This predicts that the slope of the path increases to infinity as the extrinsic frequency of upward-propagating waves increases to N, in which case $|\Theta| \to 0$ and $|k_z| \to 0$.

In Section 6.4.1 we found that conservation of energy requires the amplitude of surface waves to increase as their horizontal group velocity decreases. A similar principle holds for internal waves. For waves propagating in a background shear flow, wave action, not energy, is conserved. Wave action, $\mathcal{A}$, is the ratio of energy to the extrinsic frequency, as defined by (3.94). For internal waves with vertical displacement amplitude A_ξ, (3.87) predicts the average energy per unit mass is $\langle E\rangle = (NA_\xi)^2/2$, which is independent of wavenumber and frequency. Assuming that the waves do not spread substantially in the horizontal as they move vertically (as is the case for horizontally periodic waves), the requirement that the wave action flux is non-divergent means that

$$c_{gz}\frac{N^2{A_\xi}^2}{\Omega} = c_{gz}\frac{N^2{A_\xi}^2}{\omega_0 - k_{x0}\bar{U}} = \text{ constant.} \tag{6.35}$$

So the amplitude increases as N or c_{gz} decreases or as the extrinsic frequency Ω increases. Whether nonlinear effects become important so that ray theory predictions become unreliable is assessed by the breaking conditions described in Section 4.6.

Because the vertical group velocity goes to zero as $|\Theta|$ approaches either 0 or $\pi/2$ the question arises as to where the energy ends up. Next we examine these two circumstances in detail beginning with a study of waves approaching a critical level, where $|\Theta| \to \pi/2$, followed by a study of waves approaching a reflection level, where $|\Theta| \to 0$.

6.5.2 Critical levels

In continuously stratified fluid a critical level is the height at which the extrinsic frequency, Ω, of internal waves is zero. Equivalently, it is where the horizontal speed ω/k_x of wave crests measured by a stationary observer matches the ambient flow speed, $\bar{U}$.

There is a subtle difference between this definition of critical levels and that which arises in the study of shear flow instability. In the latter case, horizontally periodic perturbations are determined as stable or unstable modes having the same phase speed as a point in the background flow. Where this matching occurs is called a critical level. In stability theory, the waves originate about the critical level itself. Conversely, here we are concerned with internal waves whose properties are set independently of the background flow and which move towards, rather than being situated at, a critical level.

In a well-studied case, one assumes that $N = N_0$ is constant and the ambient flow $\bar{U}$ increases linearly with height as $\bar{U}(z) = s_0 z$, in which the constant shear s_0 is positive. The two-dimensional internal waves are situated initially at the origin with horizontal wavenumber $k_{x0} > 0$ (which does not change in time) and initial vertical wavenumber $k_{z0} < 0$. Thus the waves are set to move upwards and to the right.

The initial intrinsic frequency of the waves situated at $z = 0$ is $\omega_0 = N_0 \cos\Theta_0$, in which $\Theta_0 = \tan^{-1}(k_{z0}/k_{x0})$. Because the flow is steady, ω_0 is constant for all time. Likewise, the horizontal phase speed measured by a stationary observer, ω_0/k_{x0}, is constant for all time. Thus the height of the critical level z_c can be determined immediately from the solution of $\bar{U}(z_c) = \omega_0/k_{x0}$. Explicitly,

$$z_c = \frac{N_0}{s_0 k_{x0}} \cos\Theta_0. \tag{6.36}$$

The actual path of the wavepacket as it approaches a critical level is found by solving (6.32) with Θ given by (3.56) and (6.33). Explicitly, the initial value problem is

$$\begin{aligned} &\frac{dx}{dz} = \tan|\Theta| + \frac{k_{x0} s_0 z}{N_0 \sin|\Theta| \cos^2|\Theta|}, \quad x(0) = 0 \\ &\text{with } \ |\Theta| = \tan^{-1}\left(\frac{{N_0}^2}{(\omega_0 - k_{x0} s_0 z)^2} - 1\right)^{1/2}. \end{aligned} \tag{6.37}$$

This is solved numerically through straightforward integration of both sides of the differential equation with respect to z.

The result is shown in Figure 6.4 for waves moving at the fastest vertical group velocity in a flow with shear strength $s_0 = 0.01 N_0$. The path shown in Figure 6.4a asymptotically approaches the critical level at $z_c \simeq 81.6 {k_{x0}}^{-1}$. At early stages during the propagation of the waves, where $\bar{U} \simeq 0$, lines of constant phase are nearly tangent to the path. This is consistent with the fact that the group velocity is oriented perpendicular to the wavenumber vector. At later times, the Doppler-shifting background wind results in a steeper angle of the phase lines compared with the slope of the path. In this calculation, after 200 buoyancy periods (in which one buoyancy period is $T_B = 2\pi/N_0$) the waves have travelled 91% of the vertical distance towards the critical level.

As the waves approach a critical level, (6.35) predicts that the amplitude will change in proportion to $[(\omega_0 - k_{x0} s_0 z)/c_{gz}]^{1/2}$. Near the critical level the waves become increasingly hydrostatic so that $c_{gz} \simeq (N_0/k_{x0}) \cot^2|\Theta|$. So, using the overturning condition (4.117), the level where internal waves break can be estimated numerically by finding the value of z where

$$\cot|\Theta(z)| = C(\omega_0 - k_{x0} s_0 z)^{1/4}. \tag{6.38}$$

Here C is a constant with respect to z that depends upon the initial wavenumber, frequency and amplitude of the waves.

This prediction does not guarantee the waves will break in reality: over sufficiently long distances the ambient flow may veer with height, in which case the

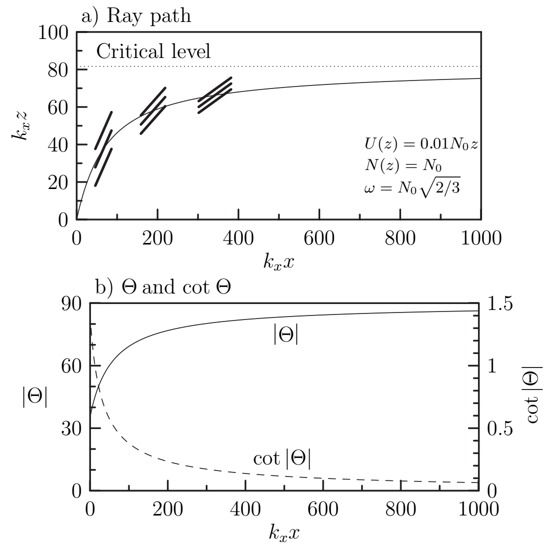

Fig. 6.4. a) The path followed by internal waves approaching a critical level in uniformly stratified fluid and in uniform shear of strength $s_0 = 0.01N_0$. The intrinsic wave frequency is set to be $\omega_0 = (2/3)^{1/2}N_0$, corresponding to waves with the fastest vertical group velocity in stationary fluid. The orientations of constant-phase lines are superimposed on the path at three positions. b) The angle Θ formed between phase lines and the vertical (solid line) and values of cot Θ (dashed line) as the waves move along the path. The value of cot Θ can be used to assess the stability of the waves to overturning depending upon their amplitude.

assumption of two-dimensional flow is no longer valid; over sufficiently long times the flow may no longer be steady; as the waves grow to large amplitude, weakly nonlinear effects will change their structure and may lead to instability through wave–wave interactions; if the fluid is sufficiently viscous, as may occur in laboratory experiments, the waves may broaden due to diffusion and so deposit energy to the mean flow without overturning and mixing.

The approach of internal waves to a critical level has been observed in several laboratory experiments. For example, Figure 6.5 shows internal waves launched by stratified flow over model topography (see Section 5.4) which then encounter a level in the flow above which the mean flow speed is zero, the same as that of the stationary hills. Consistent with the prediction of ray theory, the upward-propagating waves evolve to have decreasing vertical wavelength as they approach the critical level.

In Figure 6.5a the isopycnal surfaces, indicated by the dashed lines, become gradually less distorted as the waves approach the critical level. The incident waves have sufficiently small amplitude that viscosity damps the waves before they

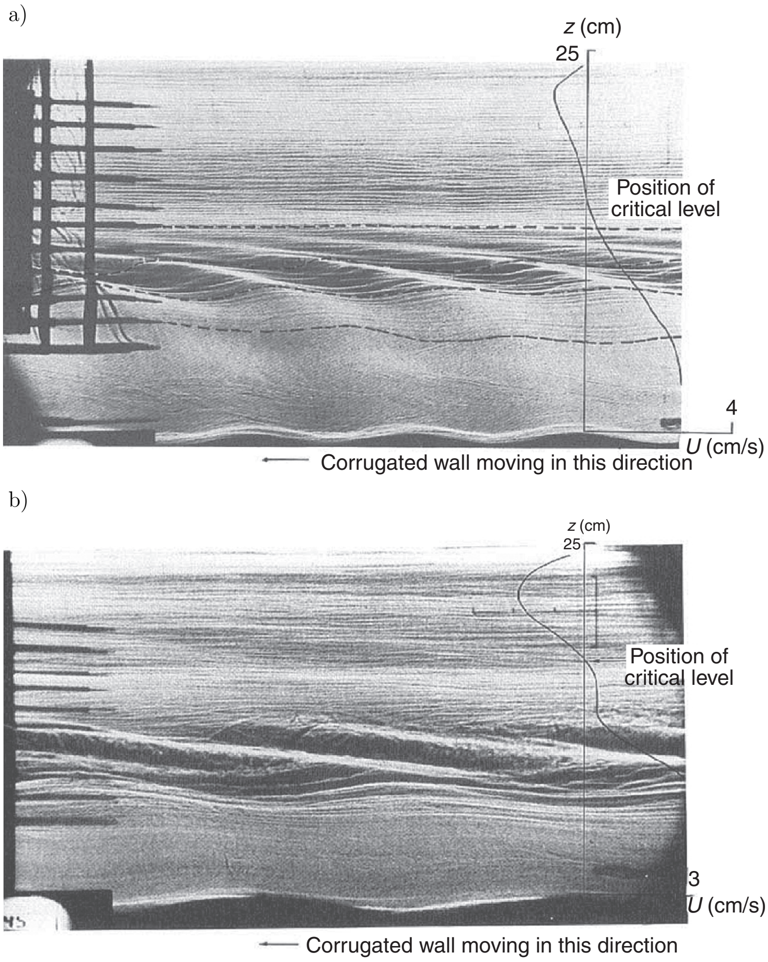

Fig. 6.5. Shadowgraph images taken from experiments in which a) small-amplitude and b) large-amplitude internal waves are launched by flow over a leftward-moving set of sinusoidal hills in a stratified shear flow. The waves approach a critical level where the velocity profile, indicated, crosses the vertical line. The dashed lines illustrate the distortion of isopycnal surfaces at different vertical levels. [Adapted, by permission of Cambridge University Press, from Figures 5 and 7 of Koop and McGee, J. Fluid Mech., **172**, 453–480 (1986).]

become overturning. In comparison, the large-amplitude incident waves shown in Figure 6.5b break turbulently below the critical level.

The second experiment demonstrates an important aspect of critical-level interactions. When waves break, they deposit momentum to the mean flow and this changes the background flow profile. Indeed, the velocity profile shown to the right of Figure 6.5b exhibits a nearly constant velocity between the critical level and the height at which the waves break. Over time the level at which the waves break occurs at progressively smaller heights above the source of the waves.

This interaction between waves and the mean flow has been used to explain the essential dynamics governing the Quasi-Biennial Oscillation (or, more succinctly, the 'QBO'). This refers to the observed zonal winds in the equatorial stratosphere that alternately flow eastwards and westwards with a period of about two years, as shown in Figure 6.6.

It is believed that the flow is driven by upward-propagating waves originating in the troposphere and which deposit momentum where they break in the stratosphere, as illustrated in Figure 6.7. Incident waves have both eastward and westward phase

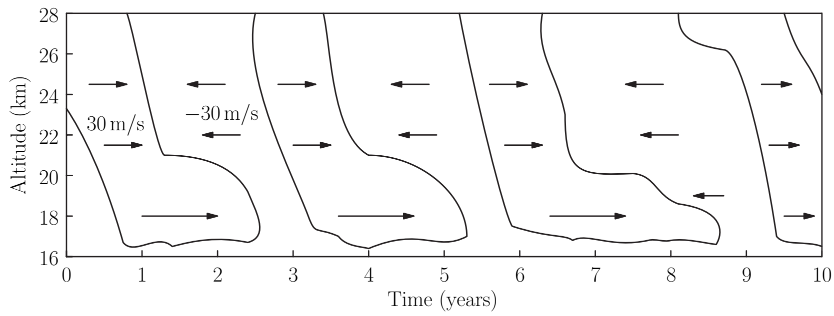

Fig. 6.6. Schematic of the alternating westward and eastward winds in the lower equatorial stratosphere associated with the Quasi-Biennial Oscillation.

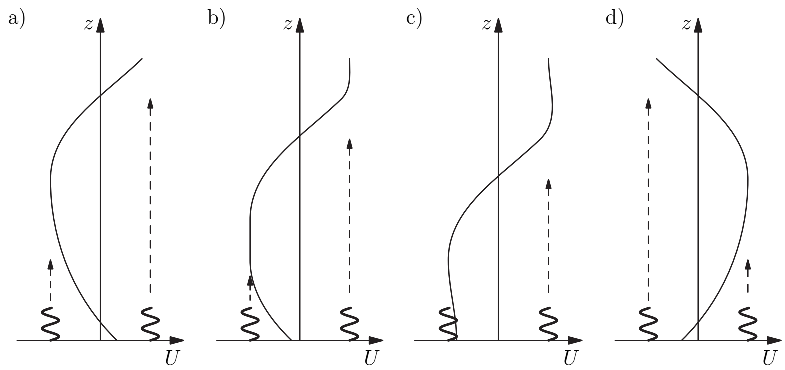

Fig. 6.7. Schematic illustrating how the absorption of internal waves at critical levels results in the alternating westward and eastward zonal flows associated with the Quasi-Biennial Oscillation. The solid line in each plot represents the zonal wind profile and the left and right sinusoidal curves represent upward-propagating waves respectively with westward and eastward zonal phase speeds. a) Westward waves deposit momentum at a critical level which b) reduces the altitude of the level until c) the critical level reaches the level of the source. d) The process then repeats for eastward-propagating waves.

speeds. During the phase of the QBO when the stratospheric winds are westward, the westward-propagating waves encounter a critical level and, by depositing their momentum, they lower the altitude of the critical level. The eastward-propagating waves do not encounter a critical level and so propagate high up into the stratosphere where they dissipate or break due to other processes and so accelerate an upper-level eastward flow.

Eventually the critical level for westward waves is so low in the stratosphere that it reaches the source of the waves. Afterwards these waves can pass freely upwards through the stratosphere. Meanwhile the eastward-propagating waves encounter a critical level high in the stratosphere and, through momentum deposition, progressively lower the altitude at which this critical level is situated. The process repeats itself so that the wind in the lower stratosphere alternately flows westwards, then eastwards, then westwards again.

6.5.3 Reflection levels

Ray theory predicts that internal waves reflect from a vertical level where the extrinsic frequency matches the background buoyancy frequency. This is called a reflection level. There are two distinct idealized circumstances in which this may occur, both of which are considered here for upward-propagating incident waves. In one, the ambient flow is stationary and the stratification decreases with height. In the other, the stratification is uniform and the ambient flow decreases with height, Doppler-shifting the waves to higher extrinsic frequencies.

First we consider rightward- and upward-propagating waves in decreasing stratification with $N(z) = N_0(1 - \sigma_0 z)$, in which σ_0 is a constant. The background flow is taken to be stationary so that the intrinsic and extrinsic frequencies are equal. For waves with intrinsic frequency ω_0, the reflection level occurs where

$$z = z_r = \frac{1 - \omega_0/N_0}{\sigma_0}. \tag{6.39}$$

The path followed by these waves is found by solving (6.32) with $\bar{U} = 0$. The solution is shown in Figure 6.8a for a case with $\sigma_0 = 0.003 k_{x0}$ and $\omega_0 = (2/3)^{1/2} N_0$.

As the waves approach the reflection level, the slope of the path increases with height and becomes infinite at $z = z_r$. The waves then reflect, still moving rightwards but now moving downwards along a path whose slope is negative and decreasing in magnitude.

Because there is no background flow the slope is given by (6.34): $dz/dx = -\cot\Theta$, in which Θ is the angle formed between constant-phase lines and the

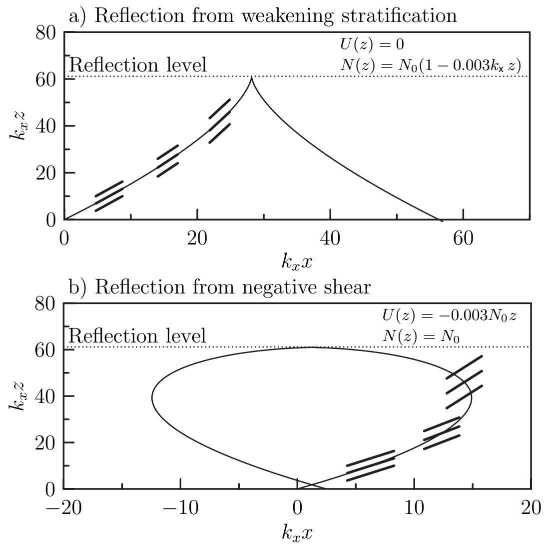

Fig. 6.8. The path followed by internal waves approaching a reflection level in a) stationary fluid whose stratification decreases according to $N = N_0(1 - 0.003\,k_x z)$ and b) uniformly stratified fluid in constant negative shear with $s_0 = -0.003\,N_0$. In both cases the initial intrinsic frequency is taken to be $\omega_0 = N_0\sqrt{2/3}$, corresponding to internal waves that move upwards at the fastest vertical group velocity in stationary fluid. In both plots, the orientations of constant-phase lines are superimposed on the path at three positions.

vertical and Θ is negative (positive) for upward- (downward-) propagating waves. Thus the slope of the phase lines matches the slope of the ray path everywhere along the path. In particular, at the reflection level phase lines are vertically oriented and the corresponding vertical wavelength is infinite.

In the second example, we consider the circumstance in which a uniformly stratified fluid has uniform but negative shear $\bar{U}(z) = -s_0 z$ with $s_0 > 0$. In this case, waves with intrinsic frequency ω_0 and horizontal wavenumber $k_{x0} > 0$ that move upwards from $z = 0$ are Doppler-shifted to increasing extrinsic frequencies $\Omega = \omega - k_x \bar{U} = \omega_0 + k_{x0} s_0 z$. Eventually, the waves reach a level z_r at which the extrinsic frequency equals the buoyancy frequency. Explicitly,

$$z_r = \frac{N_0 - \omega_0}{k_{x0}\, s_0}. \tag{6.40}$$

The path followed by an internal wavepacket reflecting from a negative shear flow is shown in Figure 6.8b. As expected, the wavepacket moves upwards from the origin and reflects from the reflection level predicted by (6.40). Unlike the previous case, however, the waves follow a counter-clockwise path and approach the reflection level tangentially rather than at a cusp. During the motion the phase

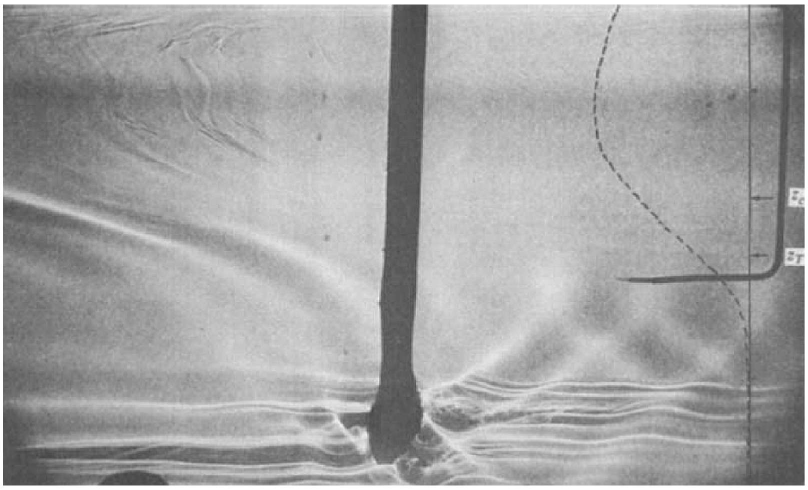

Fig. 6.9. Shadowgraph images taken from experiments in which internal waves generated by an oscillating cylinder move upwards into a leftward-moving shear flow. The leftward-propagating waves encounter a critical level; the rightward-propagating waves encounter a reflection level. [Reproduced, by permission of Cambridge University Press, from Figure 10 of Koop, J. Fluid Mech., **113**, 347–386 (1981).]

lines are not parallel with the path except at $z = 0$ where $\bar{U} = 0$. As in the previous example, the phase lines become more vertically oriented as the waves approach the reflection level, meaning that the vertical wavelength becomes infinite.

These two examples show that the energy and momentum transported by internal waves are reflected back towards the source of the waves if the waves encounter a level where their extrinsic frequency equals the background buoyancy frequency. The different behaviour of waves at a critical and reflection level is beautifully illustrated by the laboratory experiment shown in Figure 6.9. Here internal waves are generated in a uniformly stratified fluid by a vertically oscillating cylinder (see Section 5.2) and the left and right beams propagate upwards into a leftward-moving shear flow. The leftward-propagating waves encounter a critical level, their phase lines tilting towards the horizontal as they move upwards. The rightward-propagating waves encounter a reflection level resulting from their extrinsic frequency being Doppler-shifted to match the background buoyancy frequency. The shear is so strong in this case, that the looped path shown in Figure 6.8b occurs within a short vertical distance from the reflection level and so appears to be more cusp-like, as in Figure 6.8a.

In this section we have noted that as internal waves approach a reflection level their vertical wavelength becomes infinitely large. This poses a problem for ray theory, which is valid only in the limit of background variations being long compared to the vertical wavelength. Ray theory predicts its own demise for waves approaching a reflection level.

Nonetheless it is possible to model the behaviour of waves within the context of the WKB approximation by performing an asymptotic expansion of the wave equations about the reflection level. This is an example of the treatment of waves near so-called caustics.

6.5.4 Caustics

Caustics refer to singularities in the equations of ray theory which result when ray paths intersect. This occurs, for example, when internal waves encounter a reflection level as shown in Figure 6.8: the meeting at a cusp of the incident and reflected waves forms a caustic.

This class of caustics, resulting from wave reflection, is treated by solving the linearized equations of motion in a neighbourhood about the reflection level. The solutions can then be spliced together with the ray theory prediction far from the caustic to give an approximate solution for the evolution of incident and reflected waves.

To demonstrate the treatment of such caustics, we consider the evolution of incident internal waves in the lower-half plane that propagate upwards in a non-uniform shear flow with non-uniform stratification and which encounter a reflection level at $z_r = 0$. Suppose that $N(z)$ and $\bar{U}(z)$ vary continuously about $z = 0$, so that near this level we can approximate the background buoyancy frequency and horizontal velocity by linearized functions

$$N(z) = N_0(1 - \sigma_0 z) \text{ and } \bar{U}(z) = U_0 - s_0 z. \tag{6.41}$$

We assume that $\omega_0 > 0$ and $k_{x0} > 0$ and that the constants σ_0, s_0 and U_0 are positive so that the stratification weakens with height and the background shear Doppler-shifts upward-propagating waves to higher extrinsic frequencies $\Omega(z)$. For $z = 0$ to be a reflection level, we must have

$$\Omega(z = 0) = \omega_0 - U_0 k_{x0} = N_0. \tag{6.42}$$

The vertical structure of small-amplitude horizontally periodic disturbances in a parallel shear flow is prescribed by the Taylor–Goldstein equation (3.134). For the profiles defined by (6.41), this becomes

$$\frac{d^2\hat{\xi}}{dz^2} + k_{x0}{}^2 \left\{ \frac{[N_0(1 - \sigma_0 z)]2}{[N_0 + s_0 k_{x0} z]^2} - 1 \right\} \hat{\xi} = 0, \tag{6.43}$$

in which we have used the reflection level condition (6.42). For conceptual convenience, the equation has been recast as a formula for the vertical displacement amplitude, $\hat{\xi}(z)$, instead of the streamfunction amplitude, $\hat{\psi}(z)$.

In (6.43) the term in curly braces is zero at $z = 0$, and the signs of s_0 and σ_0 have been chosen so that the term is positive if $z < 0$ and negative if $z > 0$. Thus solutions have an oscillatory form in the lower-half plane, consistent with propagating waves, and have a monotonically decreasing form in the upper-half plane, consistent with the structure of evanescent disturbances.

We can examine the detailed structure near the reflection level by performing a Taylor-series expansion about $z = 0$ of the term in curly braces in (6.43) and keeping only the leading-order term in z. This yields the approximate differential equation

$$\frac{d^2\hat{\xi}}{dz^2} - 2{k_{x0}}^2\sigma_0\left(1 + \frac{s_0 k_{x0}}{N_0\sigma_0}\right) z\,\hat{\xi} \simeq 0. \tag{6.44}$$

Through a straightforward change of variables this can be converted into the canonical form of Airy's equation for the function $\xi(Z)$:

$$\xi'' + Z\xi = 0, \tag{6.45}$$

in which $Z = -\{2{k_{x0}}^2\sigma_0[1 + (s_0 k_{x0})/(N_0\sigma_0)]\}^{1/3} z$.

Generally, the solution of (6.45) is a superposition of the Airy functions Ai and Bi. However, the latter function is unbounded as $Z \to \infty$ and so is neglected. The plot of Ai(Z) is shown as the solid line in Figure 6.10a. The dashed lines represent asymptotic approximations to the Airy function:

$$\mathrm{Ai}(Z) \simeq \begin{cases} \dfrac{1}{2\sqrt{\pi} Z^{1/4}} \exp\left(-\dfrac{2}{3} Z^{3/2}\right) & Z \gg 0, \\ \dfrac{1}{\sqrt{\pi}(-Z)^{1/4}} \sin\left(\dfrac{2}{3}(-Z)^{3/2} + \dfrac{\pi}{4}\right) & Z \ll 0. \end{cases} \tag{6.46}$$

Below $z = 0$ the vertical wavelength and amplitude of the waves increases as Z increases. However, the full treatment of the Airy function shows that the vertical wavelength and amplitude do not approach infinity as $Z \to 0$. Above the reflection level the amplitude decreases exponentially as $\exp(-(2/3)Z^{3/2})/Z^{1/4}$.

The structure of the wavefield at a snapshot in time is shown in Figure 6.10b. Explicitly, the greyscale indicates values of $\mathrm{Ai}(Z)\cos(k_{x0}x)$ with light greys indicating positive values up to 0.5 and dark greys indicating negative values as low as −0.5. As a consequence of the superposition of the incident and reflected waves,

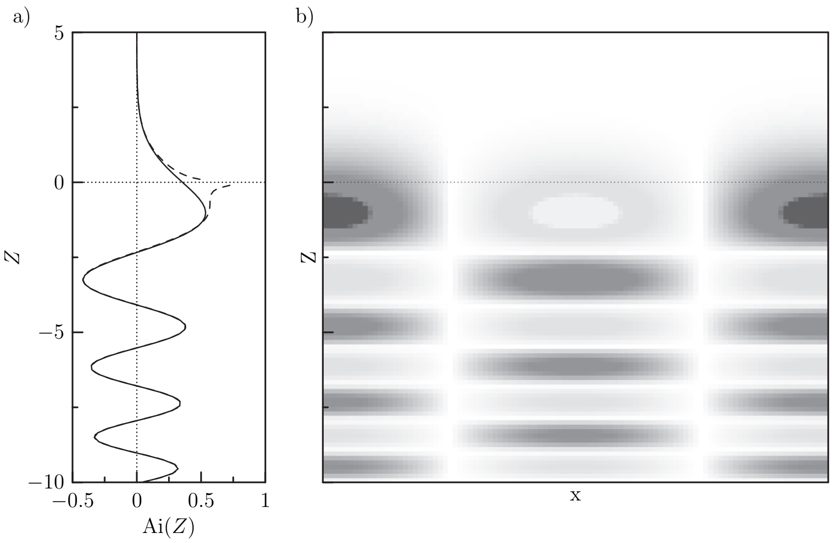

Fig. 6.10. a) Plot of Airy function (solid line) and asymptotic approximations to this function (dashed lines) for large and small Z as given by (6.46) and b) the structure of internal waves incident from below upon a reflection level where the buoyancy frequency decreases linearly with height. The greyscale indicates crests (light grey) and troughs (dark grey) of the vertical displacement field.

the disturbance field adopts a checkerboard pattern below the reflection level. The associated flux of wave action is everywhere zero, with no energy transport by the evanescent waves above $Z = 0$ and with the upward flux by incident waves cancelled by the downward flux by reflected waves below $Z = 0$.

6.6 Eckart resonance and tunnelling

Both the ocean and atmosphere are characterized by layers of strong and weak stratification. In the ocean, for example, the seasonal and main thermoclines are separated by a relatively weakly stratified region. In some circumstances internal waves can be generated in the seasonal thermocline with frequency less than the local buoyancy frequency in the region but greater than the buoyancy frequency immediately underneath. Such waves are said to be ducted, meaning that they are trapped within the stratified layer. However, depending upon their spatial scale it may be possible for the waves to transmit energy through to the main thermocline. In this sense, the seasonal thermocline acts as a leaky duct and, more generally, the waves are said to tunnel from one strongly stratified region to another through a region in which they are evanescent.