real. Taking real parts of the fields we have

$$\begin{aligned} \xi(\vec{x},t) &= A_\xi e^{\gamma z} \cos(k_x x - \omega t) \\ u(\vec{x},t) &= -\frac{\gamma}{k_x} A_\xi e^{\gamma z} \cos(k_x x - \omega t) \\ w(\vec{x},t) &= \omega A_\xi e^{\gamma z} \sin(k_x x - \omega t), \end{aligned} \tag{3.74}$$

in which it is assumed that k_x and ω are prescribed. The value of γ is given by (3.73), in which the sign is set to ensure bounded solutions as $z \to \pm\infty$.

So, as with the case of interfacial waves, the components of the velocity field are now 90° out of phase and the fluid follows elliptical rather than straight-line paths. In the limit $N_0 \to 0$, $\gamma \to \pm k_x$, so (3.72) is consistent with the structure for interfacial waves given by (2.81) and (2.82), in which the velocity potential is a product of an exponential function of z with periodic functions of x and t. Indeed, if the fluid surrounding an interface is weakly stratified, we expect the motion nonetheless to behave approximately as if it were irrotational.

Disturbances with exponentially decreasing amplitude in stratified fluids are said to be 'evanescent', meaning vanishing. This is the same terminology used in electromagnetic theory to describe light waves. When light is incident upon a lens at a glancing angle, its amplitude decreases exponentially towards the lens interior. Sometimes evanescent disturbances are called 'evanescent waves'. But this ambiguous terminology will be avoided here.

3.4 Transport by Boussinesq internal waves

As internal waves propagate vertically through a continuously stratified fluid they carry momentum with them. Momentum is conserved and so where the waves are generated they exert drag and accelerate the fluid where they break. Likewise, the waves transport energy, extracting it where they are created and depositing it where they dissipate. As such, the waves act as a conduit through which the middle and upper atmosphere can be influenced by processes near the ground and through which the deep ocean can respond to the surface mixed region.

In the atmosphere, the transport of momentum by internal waves is most significant dynamically. This is because the large-scale flows of the atmosphere are primarily zonal, travelling predominantly eastwards at mid-latitudes and westwards near the equator in the troposphere. Thus waves generated by unidirectional flow over topography, for example, carry momentum upwards and accelerate the flow in a direction opposing the ambient wind where they break (see Section 5.4). Energy transport is less significant because, although energy deposition results in turbulence, the absorption of solar radiation in the atmosphere acts efficiently to restratify the mixed region.

Conversely, the transport of energy by internal waves in the ocean is usually the most significant dynamically. Internal waves generated by oscillatory tidal flows over topography (see Section 5.5) excite waves travelling in opposing directions so that the net momentum transport is zero. Even beneath strong unidirectional currents, the flow has embedded within it eddies that act to redirect momentum. On the other hand, where the waves break the energy they transport acts to mix the fluid. Because sunlight does not penetrate significantly below about 100 m of the surface, solar radiation does not act to restratify this mixed fluid. Therefore energy transport by waves and consequent mixing is crucial for understanding the salinity and temperature structure of the oceans.

Here we consider the transport of momentum and energy by small-amplitude, two-dimensional plane waves in uniformly stratified, stationary fluid with no background rotation. For now we will suppose the waves are Boussinesq, though this will be relaxed in Section 3.7.

3.4.1 Zero mass/internal energy transport

For an incompressible fluid, the fully nonlinear flux-form of the internal energy equation is given by (1.67). In the x–z plane, this becomes

$$\frac{\partial \rho}{\partial t} = -\frac{\partial u\rho}{\partial x} - \frac{\partial w\rho}{\partial z} - w\frac{d\bar{\rho}}{dz}, \tag{3.75}$$

in which the total density ϱ has explicitly been written in terms of the background and fluctuation components: $\bar{\rho}(z) + \rho(\vec{x},t)$.

Averaging (3.75) over one horizontal wavelength gives the mass conservation equation

$$\frac{\partial \langle \rho \rangle}{\partial t} = -\frac{\partial \langle w\rho \rangle}{\partial z}. \tag{3.76}$$

The averaging in x has eliminated the first and last terms on the right-hand side of (3.75). The left-hand side of (3.76) symbolically represents the mean change in density at a point in space. The equation thereby states that, on average, the density changes in time due to the divergence of the vertical mass flux $\langle w\rho \rangle$.

The polarization relations for small-amplitude, Boussinesq internal waves show that the vertical velocity and perturbation density are 90° out of phase, as illustrated in Figure 3.6. In Section 1.15.8 we found that the correlation is zero between two periodic disturbances that are 90° out of phase. And so we have

$$\langle w\rho \rangle = 0. \tag{3.77}$$

Therefore the average mass flux due to internal waves is identically zero. The waves may periodically change the density at a point, but on average the density remains the same. Even if the waves break, there is no net density change, only a redistribution of the background density due to mixing. Here the statement of zero mass transport is equivalent to the statement that internal waves do not transport heat or salinity.

3.4.2 Mechanical energy transport

The flux-form of the kinetic energy equation for an incompressible fluid is derived by taking the dot-product of $\vec{u}$ with the vector form of the momentum equations (1.64) and then bringing each component of $\vec{u}$ inside the derivatives using the incompressibility relation (1.30).

If there is no background horizontal flow, for a Boussinesq fluid we find

$$\frac{\partial E_K}{\partial t} = -\nabla \cdot [(E_K + p)\vec{u}] - g w \rho, \tag{3.78}$$

in which

$$E_K = \frac{1}{2}\varrho_0 |\vec{u}|^2 \tag{3.79}$$

is the kinetic energy density, with units of energy per unit volume.

From (1.94), the total potential energy is classically defined as the volume integral of $\varrho g z$. But this is not a particularly useful quantity for studying internal waves. Under this definition, potential energy is converted entirely into kinetic energy only if the fluid as a whole is brought down to a reference level, $z = 0$ say. Clearly incompressibility does not allow this to occur. It is the potential energy difference between the perturbed and unperturbed states that is dynamically relevant for internal waves. This is the available potential energy.

Although the formula for the available potential energy density can be derived by computing the difference between the potential energy of the disturbed and undisturbed states, here we will derive it heuristically by a fluid parcel argument. This is done for a liquid, but a similar argument can be used for a gas. Suppose the fluid has approximately constant buoyancy frequency about some arbitrary level, which we denote by $z = 0$. The background density near $z = 0$ can be represented approximately by $\bar{\rho} \simeq \varrho_0(1 - z/H)$, in which H is the local value of the density scale height. We wish to exchange two parcels located at $z = \pm\delta$. This costs potential energy given by the difference between the potential energy after and before the exchange. That is, the energy required is

$$\begin{aligned}\Delta E &= g\varrho_0[(1-\delta/H)(-\delta)+(1+\delta/H)(\delta)]\\ &\quad - g\varrho_0[(1+\delta/H)(-\delta)+(1-\delta/H)(\delta)]\\ &= 4\varrho_0\delta^2 g/H.\end{aligned} \tag{3.80}$$

The available potential energy density associated with one of the two fluid parcels is half this amount, $\Delta E/2$.

An explicit definition of the available potential energy density is derived from (3.80) by replacing g/H with ${N_0}^2$ and setting the total displacement 2δ to be ξ. Thus $\Delta E/2$ becomes

$$E_{\mathrm{P}} = \frac{1}{2}\varrho_0 {N_0}^2 \xi^2. \tag{3.81}$$

This form is analogous to the depth-integrated available potential energy (2.36) associated with surface waves. Equation (3.81) shows that the available potential energy is larger if the waves vertically displace fluid to a greater extent against the background stratification, or if the background stratification itself is stronger.

Sometimes it is more useful to represent E_{P} in terms of the fluctuation density, ρ, rather than ξ. Using (3.66) to relate ξ to ρ for small-amplitude disturbances we have

$$E_{\mathrm{P}} = \frac{1}{2}\frac{1}{\varrho_0}\frac{g^2}{{N_0}^2}\rho^2. \tag{3.82}$$

Despite possible computational advantages, this form of the available potential energy density is not all that intuitive. The presence of ${N_0}^2$ in the denominator of (3.82) seems to imply that E_{P} becomes large as ${N_0}^2$ becomes small. However, for a wave with a given vertical displacement amplitude, A_ξ, the fluctuation density is proportional to ${N_0}^2$ and so E_{P} in fact decreases as ${N_0}^2$ decreases, a fact made obvious by (3.81).

The evolution of available potential energy is given by multiplying $g\rho/(-d\bar{\rho}/dz)$ on both sides of the internal energy equation (1.67) and converting the result to flux-form. In uniformly stratified fluid, the background density gradient is constant and may be brought inside the partial z-derivative terms. Using (3.82), we thereby find

$$\frac{\partial E_{\mathrm{P}}}{\partial t} = -\nabla\cdot(\vec{u}E_{\mathrm{P}}) + g w \rho. \tag{3.83}$$

Comparing (3.78) and (3.83), we realize that $g w \rho$ represents the conversion from kinetic to available potential energy when vertical motions carry dense fluid upwards or light fluid downwards against buoyancy forces. Conversely, kinetic energy increases at the expense of available potential energy when relatively heavy fluid descends or light fluid rises. As such, the term $g w \rho$ is sometimes called the 'buoyancy flux'.

Adding together the kinetic and available potential energy equations gives the total energy conservation law

$$\frac{\partial E}{\partial t} = -\nabla \cdot (\vec{u}(E+p)), \tag{3.84}$$

in which

$$E \equiv E_K + E_{\mathrm{P}}. \tag{3.85}$$

Equation (3.84) shows that energy in an incompressible fluid changes solely due to the divergence of the energy flux, $\vec{u}(E+p)$.

Now suppose that perturbations are due to small-amplitude periodic internal waves. Horizontally averaging over one wavelength and using the polarization relations in Table 3.1, the mean kinetic energy represented in terms of the vertical displacement amplitude is

$$\langle E_K \rangle = \frac{1}{4} \varrho_0 {N_0}^2 {A_\xi}^2. \tag{3.86}$$

Likewise, from the formula for the available potential energy density given by (3.81), the mean available potential energy is identical to the mean kinetic energy. So the total mean energy is

$$\langle E \rangle = 2 \langle E_K \rangle = 2 \langle E_{\mathrm{P}} \rangle = \frac{1}{2} \varrho_0 {N_0}^2 {A_\xi}^2. \tag{3.87}$$

This equipartitioning of kinetic and available potential energy, which was also seen for interfacial waves, is a general result for waves in a non-inertial (non-rotating) reference frame.

Horizontally averaging (3.78) and (3.83) and keeping only the leading-order terms, which are quadratic in amplitude, gives the conservation laws

$$\begin{aligned} \frac{\partial}{\partial t} \langle E_K \rangle &= -\frac{\partial}{\partial z} \langle w p \rangle \\ \frac{\partial}{\partial t} \langle E_{\mathrm{P}} \rangle &= 0. \end{aligned} \tag{3.88}$$

In formulating these equations the buoyancy flux term has vanished as a result of (3.77).

Horizontally averaging the total energy equation (3.84), or equivalently summing (3.88), gives the energy conservation law

$$\frac{\partial \langle E \rangle}{\partial t} = -\frac{\partial \langle \mathcal{F}_E \rangle}{\partial z}, \tag{3.89}$$

in which the vertical energy flux is given by

$$\langle \mathcal{F}_E \rangle \equiv \langle wp \rangle = \frac{1}{2} \varrho_0 \frac{{N_0}^3}{k_x} \sin\Theta \cos^2\Theta \, {A_\xi}^2. \tag{3.90}$$

Comparing this with the vertical component of the group velocity in (3.62) and with (3.87), we arrive at the expected result

$$\langle \mathcal{F}_E \rangle = c_{gz} \langle E \rangle \,. \tag{3.91}$$

Horizontally averaged energy is vertically transported at the vertical group speed of the waves. In particular, for waves with fixed amplitude and horizontal extent, the vertical energy transport is greatest for waves with frequency $\omega = \sqrt{2/3}N_0 \simeq 0.82N_0$ ($\Theta \simeq 35^o$), corresponding to waves with the maximum vertical group velocity.

Some care must be taken in the interpretation of (3.89). For vertically and horizontally periodic internal waves, (3.90) shows that the energy flux is independent of z. Hence (3.89) predicts that the energy is unchanging with time. This does not imply that the mean energy is zero, only that as much energy is transported into a vertical slab from below as is transported out of it from above.

A clearer illustration of what determines $\langle E \rangle$ is shown in Figure 3.7a. Here the internal waves are manifest as a quasi-monochromatic wavepacket whose amplitude envelope moves upwards at speed c_{gz}. The energy flux is given approximately

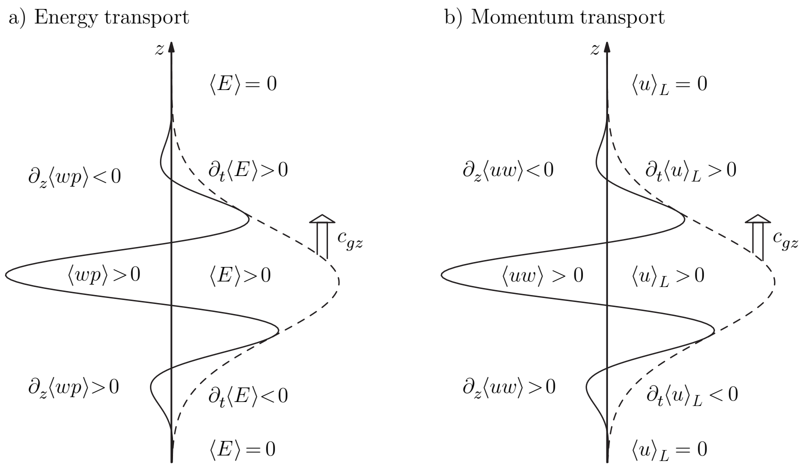

Fig. 3.7. a) Energy transport associated with an upward-propagating, vertically localized wavepacket and b) the corresponding transport of momentum per unit mass. The latter results in a wave-induced mean flow $\langle u \rangle_L$ that moves upwards with the wavepacket.

by (3.90), where now it is understood that $A_\xi \equiv A_\xi(z)$ is the amplitude envelope of the vertical displacement field. At the leading edge of the wavepacket, the amplitude envelope decreases with height meaning that the divergence of the flux is negative. Hence, according to (3.89), the mean energy at a fixed vertical position increases as the leading edge moves past it. Similarly the mean energy decreases at the trailing edge where the flux divergence is positive. The energy change due to a passing wavepacket that does not break is reversible in the sense that the change in energy after the wavepacket has passed is zero. When wave breaking occurs, the energy due to the wave's flux goes to zero and (3.89) predicts the resulting irreversible deposition of energy to the ambient.

3.4.3 Wave action and pseudoenergy

In stationary fluid, energy as defined by (3.87) is conserved in the sense that its time rate of change at a point is given in terms of the divergence of the energy flux. But energy, so defined, is not conserved for waves that propagate in a background horizontal flow whose speed changes with height. To see this, we represent the total horizontal velocity field by the sum of the background flow $\bar{U}(z)$ and the fluctuation horizontal velocity due to waves $u(\vec{x},t)$. The energy equation is found by substituting $\bar{U}+u$ into the momentum equations (as done explicitly in Section 3.6.1), taking the dot-product of these equations with $\vec{u}=(u,w)$, horizontally averaging, and keeping only the leading-order quadratic terms. Further assuming the average buoyancy flux is zero gives

$$\frac{\partial \langle E\rangle}{\partial t} = -\varrho_0 \langle uw\rangle \frac{d\bar{U}}{dz} - \frac{\partial \langle wp\rangle}{\partial z}. \qquad (3.92)$$

If $\bar{U}=U_0$ is constant, (3.92) reduces to the energy conservation law (3.89).

The first term on the right-hand side of (3.92) indicates that energy changes in a shear flow not only because the energy flux diverges, but also because momentum is transported vertically across the shear flow. For this reason, $\varrho_0 \langle uw\rangle \frac{d\bar{U}}{dz}$ is called the 'energy production term'.

The energy changes because of the way in which a shear flow effectively changes the wave frequency. As will be shown in Section 6.3, the intrinsic frequency, ω, (that seen from a stationary observer) does not change as the waves pass through a steady background flow. Likewise, that section shows that k_x is constant if the ambient is horizontally uniform. However, because the background flow changes in z the vertical wavenumber can change and does so according to the dispersion relation for internal waves. Explicitly,

$$k_z = \pm k_x \sqrt{\frac{{N_0}^2}{(\omega-\bar{U}k_x)^2}-1} = \pm k_x \sqrt{\frac{{N_0}^2}{\Omega^2}-1}, \qquad (3.93)$$

in which the sign of the square root depends upon the direction of wave propagation. Here we have introduced the extrinsic frequency, $\Omega = \omega - \bar{U} k_x$, which does change with height. By analogy to the apparent change in the frequency of sound that originates from a moving object, Ω is sometimes called the Doppler-shifted frequency.

We have already seen that the group velocity depends upon the wavenumber and frequency of the waves. So as the background flow Doppler-shifts the waves, the speed at which they vertically transport energy changes. If the resulting vertical group velocity of upward-propagating waves decreases with height, the energy piles up, as predicted by the energy production term.

We would like to define a conserved quantity analogous to energy for internal waves in a shear flow. It should be conserved in the sense that it remains constant for waves propagating through shear and it changes only when the waves are generated or dissipated. This quantity is called the 'wave action'.

For small-amplitude waves, it is given by

$$\langle \mathcal{A} \rangle = \frac{\langle E \rangle}{\Omega} = \frac{\langle E \rangle}{\omega - k_x \bar{U}}, \tag{3.94}$$

the ratio of the mean energy and extrinsic frequency.

Wave action satisfies the conservation law

$$\frac{\partial \langle \mathcal{A} \rangle}{\partial t} = -\frac{\partial \langle \mathcal{F}_A \rangle}{\partial z}, \tag{3.95}$$

in which the flux of wave action is given by the product of the vertical group velocity and the wave action:

$$\langle \mathcal{F}_A \rangle = c_{gz} \langle \mathcal{A} \rangle = \frac{\langle \mathcal{F}_E \rangle}{\omega - k_x \bar{U}}. \tag{3.96}$$

Traditionally, wave action has been used as a proxy for energy conservation for waves in the presence of shear. A closely related conserved quantity has also been proposed in its stead. This is called the 'pseudoenergy', which is the appropriate conserved quantity for waves in a time-invariant system.

For small-amplitude waves, the pseudoenergy is simply the product of the wave action and the intrinsic frequency:

$$\langle \mathcal{E} \rangle = \omega \langle \mathcal{A} \rangle = \langle E \rangle \frac{\omega}{\Omega}. \tag{3.97}$$

Because wave action is conserved and the intrinsic frequency ω does not change with time, the product must also be conserved. Unlike the definition for wave action, pseudoenergy has the same units as energy and indeed equals the wave energy in the absence of any background flow. As such, pseudoenergy is more intuitive as a conserved quantity than wave action.

Multiplying both sides of (3.95) by ω gives the small-amplitude pseudoenergy conservation law

$$\frac{\partial \langle \mathcal{E} \rangle}{\partial t} = -\frac{\partial \langle \mathcal{F}_{\mathcal{E}} \rangle}{\partial z}, \tag{3.98}$$

in which

$$\langle \mathcal{F}_{\mathcal{E}} \rangle = c_{gz} \langle \mathcal{E} \rangle . \tag{3.99}$$

Pseudoenergy and its flux can be written generally for large-amplitude waves. However, the addition of terms of the order of the amplitude-cubed and higher do not significantly change $\langle \mathcal{E} \rangle$, even for waves close to breaking amplitude. We therefore expect (3.97) and (3.99) sufficiently characterize energy transport in shear for most practical applications.

3.4.4 Momentum transport

The flux-form of the horizontal momentum equation restricted to the x–z plane is given for an incompressible fluid by

$$\varrho \left(\frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uw}{\partial z} \right) = -\frac{\partial p}{\partial x}, \tag{3.100}$$

in which $\varrho = \bar{\rho} + \rho$ is the total density, and we have assumed the fluid is incompressible.

In the Boussinesq approximation, we assume that ϱ can be treated as a constant, ϱ_0, on the left-hand side of (3.100). Horizontally averaging eliminates the x-derivatives and so gives the momentum conservation law

$$\frac{\partial \langle M \rangle}{\partial t} = -\frac{\partial \langle \mathcal{F}_M \rangle}{\partial z}, \tag{3.101}$$

in which $M \equiv \varrho_0 u$ and

$$\mathcal{F}_M \equiv \varrho_0 \langle uw \rangle . \tag{3.102}$$

The momentum flux $\mathcal{F}_M$, sometimes denoted by the symbol τ, is one component of the Reynolds stress tensor.

Dividing by ϱ_0 on both sides of (3.101) gives

$$\frac{\partial \langle u \rangle}{\partial t} = -\frac{\partial \langle uw \rangle}{\partial z}. \tag{3.103}$$

This states that the horizontally averaged flow accelerates if the mean vertical transport (w) of horizontal momentum per unit mass (u) diverges.

Because the horizontal and vertical velocity fields of propagating waves are directly correlated, they do indeed transport momentum. Explicitly, in terms of the amplitude of the vertical displacement field for Boussinesq waves, the momentum flux per unit mass is

$$\langle uw \rangle = -\frac{1}{2}\text{sign}(k_x k_z)\,\omega^2 \sqrt{\frac{N_0{}^2}{\omega^2} - 1}\,|A_\xi|^2 = -\frac{1}{4} N_0{}^2 \sin 2\Theta\,|A_\xi|^2. \qquad (3.104)$$

The last expression has used the definition (3.56) for Θ, the sign of which equals the sign of $k_z k_x$. The negative sign in both expressions in (3.104) indicates that the momentum transport is upwards, for example, if k_x is positive (the wave crests move from left to right) and k_z is negative (the crests move downwards and the vertical component of the group velocity is upwards). A surprising result of (3.104) is that for waves of fixed amplitude, their momentum transport is largest if $|\Theta| = 45°$ ($\omega = N_0/\sqrt{2}$), independent of their wavenumber.

Our choice of writing the momentum flux in terms of the vertical displacement amplitude is somewhat arbitrary. Expressions similar to those in (3.104) can be written in terms of the amplitude of the vertical velocity or any other field for that matter. One reason the expressions are written here in terms of the vertical displacement is because A_ξ is related directly to hill height if the waves are generated by flow over topography.

Equation (3.104) is appropriate only for propagating internal waves. If the disturbance fields are evanescent as a result of being forced at a frequency larger than the buoyancy frequency, then their associated momentum flux is zero. Mathematically this follows from the fact that k_z is imaginary if $\omega > N_0$. Hence, as shown in (3.74), the horizontal velocity field is 90° out of phase with the vertical velocity field.

For vertically propagating internal waves, the usual interpretation of (3.103) is that the flux $\langle uw \rangle$ diverges when internal waves break and this acts irreversibly to accelerate the background flow. However, as shown in Figure 3.7b, an alternative interpretation is that a wavepacket accelerates the background at its leading edge and decelerates it at its trailing edge, so that there is no change in the flow speed before and after the passage of the waves. The horizontally averaged, reversible flow associated with a wavepacket is referred to as the wave-induced mean flow.

3.4.5 Wave-induced mean flow

In our study of deep water surface waves, we found that the mean horizontal displacement of a fluid parcel due to the passage of waves was zero at leading order. But accounting for the variation in velocity with depth at next order, the mean horizontal parcel displacement was found to move in the same direction as the phase velocity of the waves. This time-averaged mean flow, known as the Stokes drift, is the appropriate measure of momentum per unit mass associated with a deep surface wave.

Likewise, there is a wave-induced mean flow associated with internal waves. An explicit formula for this in terms of the properties of the waves can be derived using Stokes' theorem, which connects the vorticity in a domain to the circulation surrounding it. Generally, Stokes' theorem is written $\iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$, in which $\mathcal{S}$ is an open surface and C is a closed curve circumscribing the opening in the surface. Taking the vector function $\vec{F}$ to be the velocity and collapsing the surface to the x–z plane we have

$$\iint_{\mathcal{D}} (\nabla \times \vec{u}) \cdot \hat{y}\, dA = \oint_C \vec{u} \cdot d\vec{r}, \tag{3.105}$$

in which we take $\mathcal{D}$ to be the area between upward-displaced fluid and its equilibrium position in the first half-period of the wave's oscillation. C is the counter-clockwise closed curve oriented along the equilibrium position for the first half-period and then tracing back along the displaced isopycnal, as illustrated in Figure 3.8.

Assuming that the vertical displacement is small so that the vorticity is independent of z, the area integral on the left-hand side of (3.105) becomes $-\int_0^{\lambda_x/2} \xi \zeta\, dx$. In a right-handed co-ordinate system, the x-axis points leftwards when looking at the x–z plane down the y-axis. So we have introduced a negative sign in front of the integral as a result of reversing the direction of x-integration.

On the right-hand side of (3.105), the circulation over one quarter-period along the isopycnal cancels that over the other quarter-period. The only part of the line integral that remains is the flow in the x-direction along the equilibrium position. Defining the average flow to be $\langle u \rangle_L$, the circulation integral is simply $\langle u \rangle_L \lambda_x/2$.

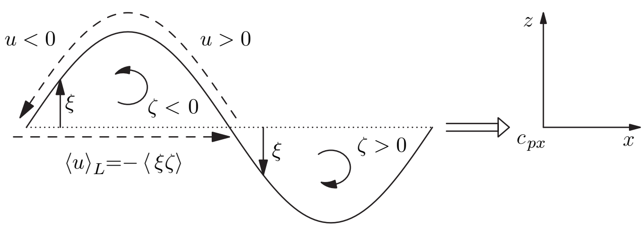

Fig. 3.8. Schematic illustrating the derivation of the wave-induced mean flow through the correlation of the vertical displacement and vorticity fields: $\langle u \rangle_L = -\langle \xi \zeta \rangle$. The dashed curves indicate the closed path C enclosing an area $\mathcal{D}$ of the fluid between the upward-displaced fluid parcels and the equilibrium position (dotted line). Stokes' theorem relates the circulation around the path to the area integral of the vorticity ζ, which is negative below the crest for rightward-moving waves.

Comparing the right- and left-hand integrals we therefore have

$$\langle u \rangle_L = -\frac{1}{\lambda_x/2}\int_0^{\lambda_x/2} \xi\zeta\, dx = -\langle \xi\zeta \rangle\,. \tag{3.106}$$

Repeating the procedure over the second half-period gives the same result. The sign of $\langle u \rangle_L$ does not change because the vorticity changes sign as the displacement changes sign.

Using the polarization relations for Boussinesq internal waves of vertical displacement amplitude A_ξ, (3.106) becomes

$$\langle u \rangle_L = \frac{1}{2}\frac{{N_0}^2}{\omega}k_x|A_\xi|^2 = \frac{1}{2}N_0 k_x \sec\Theta |A_\xi|^2, \tag{3.107}$$

in which Θ is given by (3.56). This is the Stokes drift for internal waves.

In (3.106) the subscript L has been introduced as a reminder that this is the horizontally averaged Lagrangian motion of fluid parcels due to the passage of waves. From a Lagrangian perspective, fluid parcels do not simply zig-zag back and forth at an angle Θ to the vertical, but over each cycle they drift in the same direction as the horizontal phase speed of the waves. Thus we should interpret $\langle u \rangle$ in (3.109), not as the average of a sinusoidal motion in an Eulerian frame, but as the Lagrangian mean motion, $\langle u \rangle_L$.

In the language of Hamiltonian fluid dynamics, $\mathcal{M} \equiv \varrho_0 \langle u \rangle_L$ is called the 'pseudomomentum', which is interpreted as the appropriate conserved quantity for waves in a horizontally invariant system. Hamiltonian fluid dynamics predicts that the vertical flux of horizontal pseudomomentum is

$$\langle \mathcal{F}_M \rangle = \varrho_0 \langle uw \rangle\,. \tag{3.108}$$

This is identical to the Reynolds stress (3.102) and it is consistent with the momentum conservation formula (3.101).

The wave-induced mean flow for internal waves is alternately and, perhaps, more intuitively derived if one considers the motion of a wavepacket. The approach is similar to that used in the derivation of (2.63) to get the formula (2.58) for the Stokes drift of deep surface waves. We consider horizontally periodic internal waves whose amplitude envelope varies slowly in the vertical and in time and which translates upwards at the vertical group velocity, c_{gz}. Explicitly, we write

$$u = \Re\{A_u(Z,T)\exp[i(\vec{k}\cdot\vec{x} - \omega t)]\} \;\text{ and }\; w = \Re\{A_w(Z,T)\exp[i(\vec{k}\cdot\vec{x} - \omega t)]\},$$

in which $Z = \epsilon(z - c_{gz}t)$ describes the translation of the wavepacket and $T = \epsilon^2 t$ describes the relatively slow change of the amplitude envelope due to dispersion, as represented by Schrödinger's equation (1.136). The quantity $\epsilon \equiv (\sigma_z k_z)^{-1} \ll 1$ measures the vertical wavelength relative to the vertical extent of the wavepacket.

Putting these expressions into the flux-form of the horizontal momentum equations (3.100) and averaging over one horizontal wavelength gives the momentum conservation law (3.103), but now written in terms of the vertically varying amplitude envelopes of the u and w fields:

$$\frac{\partial}{\partial t}\langle u\rangle = -\frac{\partial}{\partial z}\langle uw\rangle = -\frac{\partial}{\partial z}\left(\frac{1}{2}\Re\{A_u(Z,T)\, A_w{}^{\star}(Z,T)\}\right), \tag{3.109}$$

in which the star denotes the complex conjugate.

Because the wave-induced mean flow $\langle u\rangle$ moves upwards with the wavepacket, it should depend upon the translating co-ordinate Z. Thus on the left-hand side of (3.109) we have $\partial_t = -\epsilon c_{gz}\partial_Z + \epsilon^2\partial_T$ and on the right-hand side we have $\partial_z = \epsilon\,\partial_Z$. Neglecting the ϵ^2 term and vertically integrating the resulting equation in Z gives an implicit expression for the wave-induced mean flow due to a horizontally periodic internal wave of finite vertical extent:

$$\langle uw\rangle = \frac{1}{2}\Re\{A_u A_w{}^{\star}\} = c_{gz}\,\langle u\rangle_L. \tag{3.110}$$

Rearranging (3.110) to isolate $\langle u\rangle_L$, using the polarization relations and the explicit expression for c_{gz} in (3.62), we once again find the explicit formula (3.107) for the wave-induced mean flow of Boussinesq internal waves having vertical displacement amplitude A_ξ.

Thus we have found that the mean momentum flux per unit mass $\langle uw\rangle$ equals the wave-induced mean flow times the vertical group velocity. The formula is analogous to (3.91), which states that the mean energy flux equals the mean energy times the vertical group velocity. In this sense, the wave-induced mean flow can be identified with the momentum per unit mass of horizontally periodic internal waves.

This consideration of the wave-induced mean flow provides a different interpretation of the momentum conservation law (3.103), as illustrated schematically in Figure 3.9. The usual interpretation is that internal waves transport momentum, having no effect upon the mean flow until they break, at which point the background flow accelerates where momentum is deposited. This makes sense only if the incident waves are plane waves. In more realistic circumstances, internal waves propagate as wavepackets that change the mean flow as they propagate without breaking.

Like the Stokes drift, the wave-induced mean flow should not be confused with the background mean flow, $\bar{U}(z)$, which exists both in the presence and in the absence of waves. $\bar{U}(z)$ can be prescribed arbitrarily, but $\langle u\rangle_L$ is a function of the

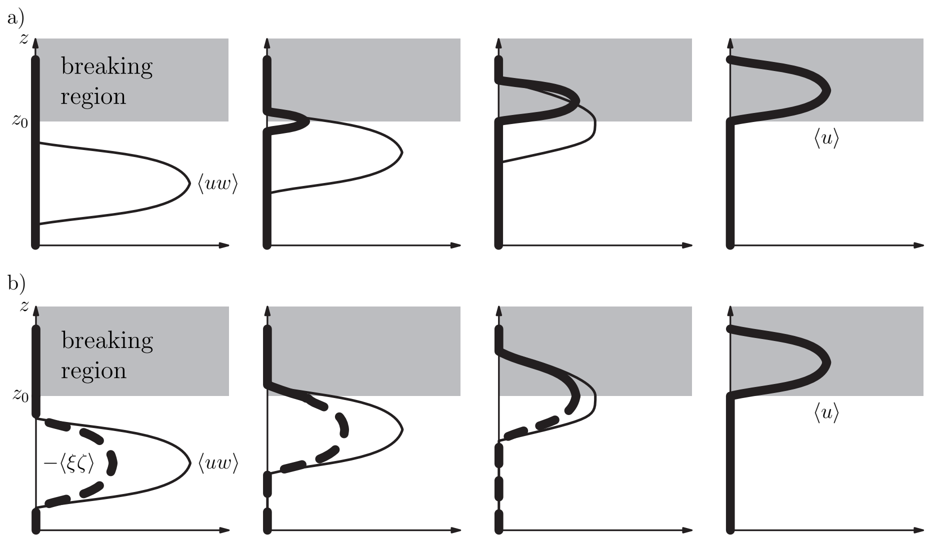

Fig. 3.9. a) Schematic illustrating the position of an upward-propagating wavepacket encountering a region where it breaks (shaded grey). The thin solid line represents the momentum flux per unit mass, $\langle uw\rangle$. The thick line represents the horizontally averaged flow. b) Before the waves break (dashed line) this is the wave-induced mean flow, $\langle u\rangle_L = -\langle \xi\zeta\rangle = \langle uw\rangle / c_{gz}$. After the waves break (solid line) momentum is permanently deposited to the background horizontal flow.

structure and amplitude of the internal waves. Thus the total horizontally averaged flow at any level z is the superposition $\langle u\rangle = \bar{U}(z) + \langle u\rangle_L$.

When internal waves break, their wave-induced mean flow is irreversibly transformed into a steady background mean flow. Momentum conservation requires that breaking internal waves accelerate the background flow so that the vertically integrated wave-induced mean flow, $\int \langle u\rangle_L \, dz$, equals the vertically integrated change to the background flow, $\int \Delta\bar{U}\, dz$, after they break.

3.5 Bounded internal waves

As for interfacial waves in domains with vertical side-walls, the effect of the side boundaries is to limit the allowable horizontal wavenumbers of internal waves to a discrete spectrum. Similarly, internal waves bounded above and below by horizontal boundaries are confined to a discrete vertical wavenumber spectrum. They form a discrete set of modes.

If the basin is not rectangular, the waves behave in a remarkably different fashion. They develop into beams rather than modes through a process called geometric