1.15.1 Representation of plane waves

Any small-amplitude disturbance can be represented by a superposition of perfectly sinusoidal waves and its evolution can be described by tracking the individual progress of each sinusoid. If the disturbance is a single sinusoidal wave that extends to infinity, it is called a 'plane wave' or sometimes a 'monochromatic wave', a metaphor referring to light waves of one frequency and, hence, one colour.

Mathematically a plane wave can be prescribed, for example, by

$$\eta(\vec{x},t) = A_0 \cos(\vec{k}\cdot\vec{x} - \omega t), \tag{1.104}$$

$$\eta(\vec{x},t) = A_0 \sin(\vec{k}\cdot\vec{x} - \omega t), \tag{1.105}$$

$$\eta(\vec{x},t) = \mathcal{A}_0 \exp[\imath(\vec{k}\cdot\vec{x} - \omega t)], \tag{1.106}$$

$$\eta(\vec{x},t) = \frac{1}{2}\mathcal{A}_0 \exp[\imath(\vec{k}\cdot\vec{x} - \omega t)] + \text{cc}. \tag{1.107}$$

Here η could represent displacement or fluctuations of velocity components, density, pressure, etc.

In all these forms, $\vec{x}$ can be a vector in one, two or three dimensions depending upon the geometry of the waves under consideration. The quantities ω and $\vec{k}$ represent the frequency and wavenumber vector, respectively. In the first two expressions A_0 is the wave amplitude. In the last two expressions, $\mathcal{A}_0$ is a complex number which is a measure of both amplitude and phase. The abbreviation cc in the last expression denotes the 'complex conjugate' of the preceding term.

Frequency is measured in units of radians per unit time (e.g., s^{-1} for radians per second). In terms of the wave period T, the (angular) frequency is $\omega = 2\pi/T$. For a frequency, ν, measured in cycles per unit time (e.g., cps for cycles per second), we would replace ωt with $2\pi\nu t$ to describe the time-variation of η. To avoid writing 2π explicitly in the arguments to cos and sin, it is typical to write frequency in terms of radians per time.

In three dimensions $\vec{k} = (k_x, k_y, k_z)$ in which k_x, k_y and k_z are the components of the wavenumber vector in the x-, y- and z-directions, respectively. The wavenumber, with units of radians per distance, measures the spatial extent of a periodic wave. For example, $k_x \equiv 2\pi/\lambda_x$, in which λ_x is the wavelength (the distance between two successive crests) in the x-direction. Generally, the wavelength $\lambda = 2\pi/|\vec{k}|$ is the shortest distance between successive crests. Hence the magnitude of the wavenumber $|\vec{k}|$ is small for long waves and is large for short waves.

Typically, a positive frequency $\omega > 0$ indicates that the waves advance forward in time and the orientation of the wavenumber vector indicates the direction in which the waves propagate through space. In particular, for $\omega > 0$ waves move rightwards in the x-direction if $k_x > 0$ and move leftwards if $k_x < 0$. However,

there is some ambiguity in the description of waves: one could instead describe leftward-moving waves by setting $\omega < 0$ and $k_x > 0$. Effectively this corresponds to observing rightward-propagating waves moving backwards in time. By convention, we will assume $\omega \geq 0$ when possible and use the wavenumber vector to denote the propagation direction. We will see that in some cases we cannot do this. For example, with deep interfacial waves in a two-layer fluid (see Section 2.3) we must fix the sign of k_x to be positive and describe rightward- and leftward-propagating waves by $\omega > 0$ and $\omega < 0$, respectively.

The term 'amplitude' is sometimes used ambiguously to mean either the value of the field at some position $\vec{x}$ and time t or, specifically, the maximum value of the field. Here we use 'amplitude' to mean the latter. In (1.104) and (1.105), A_0 is the real-valued amplitude of η which, for example, could represent maximum displacement or maximum fluctuation pressure. The two representations differ in the phase of the waves they describe. In (1.104) the wave peaks at the origin at time $t = 0$, whereas the wave given by (1.105) has zero value at that position, reaching a peak a quarter-cycle later. This is illustrated in Figure 1.24a.

In (1.106) $\mathcal{A}_0$ is complex-valued and it is understood that the actual amplitude (the value of η at any position $\vec{x}$ and time t) is the real part of η. Thus if $\mathcal{A}_0 = A_{0r} + \imath A_{0i}$, the actual structure of η is $A_{0r}\cos(\vec{k}\cdot\vec{x} - \omega t) - A_{0i}\sin(\vec{k}\cdot\vec{x} - \omega t)$. It is more illuminating, however, to write $\mathcal{A}_0$ in polar form as

$$\mathcal{A}_0 = A_0 \exp(\imath \phi_0), \tag{1.108}$$

in which the magnitude $A_0 \equiv |\mathcal{A}_0| = ({A_{0r}}^2 + {A_{0i}}^2)^{1/2}$ is the amplitude, and the argument

$$\phi_0 \equiv \tan^{-1}(A_{0i}/A_{0r}) \tag{1.109}$$

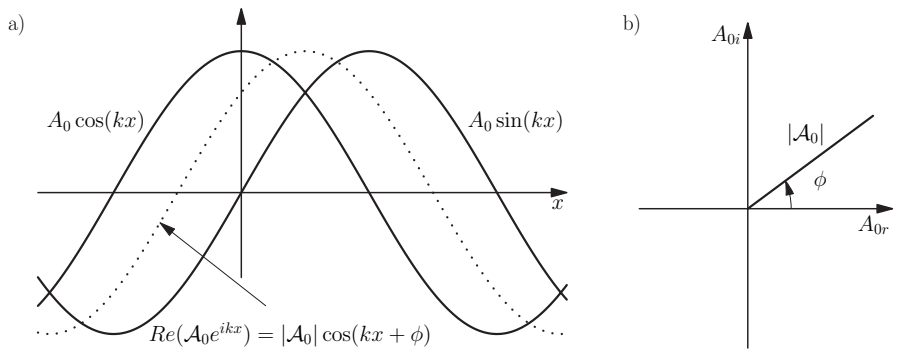

Fig. 1.24. a) Sketch of waves at time $t = 0$ represented by cosine, sine and complex exponential functions. b) Representation of complex amplitude $\mathcal{A}_0 = A_{0r} + \imath A_{0i}$ in complex Cartesian and polar co-ordinates.

is the phase. Strictly speaking, if we treat (A_{0r}, A_{0i}) as a vector in the complex plane, then ϕ_0 is the angle formed between this vector and the positive x-axis, as shown in Figure 1.24b. Using (1.108), the structure of η given by the real-part of the right-hand side of (1.106) is

$$\eta = A_0 \cos(\vec{k} \cdot \vec{x} - \omega t + \phi_0). \tag{1.110}$$

In particular, if $\mathcal{A}_0 = A_0$ is real and positive, then (1.109) shows that $\phi_0 = 0$ (or an integer multiple of 2π). If $\mathcal{A}_0 = \imath A_0$ with A_0 real and positive, then $\phi_0 = \pi/2$ and (1.110) becomes (1.105).

Although using (1.106) to describe waves seems unintuitive at first, its form is preferable to (1.104) and (1.105) because it makes no explicit assumption about the phase of the wave. The form (1.107) is more cumbersome but provides a useful compromise between describing the wave field explicitly as a real function, while still using a notation that does not explicitly prescribe the phase of the waves. Summing a complex number with its complex conjugate gives twice the real part of the number. Thus the definition of (1.107) is identical to taking the real part of (1.106).

Instead of (1.107), some texts write $\eta = \mathcal{A}_0 \exp[\imath(\vec{k} \cdot \vec{x} - \omega t)] + \text{cc}$. In this notation $|\mathcal{A}_0|$ is the 'quarter peak-to-peak amplitude'. Here, we consistently define η by either (1.106) or (1.107), in which case $|\mathcal{A}_0|$ is the more conventional 'half peak-to-peak amplitude'.

The complex representation also has the advantage that linear operators commute with the process of taking real and imaginary parts while not changing the form of the function upon which $\vec{x}$ and t depend. For example, the x-derivative of the real part of $\exp(\imath kx)$ is $d\cos(kx)/dx = -k\sin(kx)$. This equals the real part of the x-derivative of $\exp(\imath kx)$: $\Re\{\imath k \exp(\imath kx)\} = -k\sin(kx)$.

We will see that this is useful in solving coupled linear partial differential equations in which the relative phases of the different fields in the equations are unknown. This is particularly useful when computing the polarization relations, which interrelate fields associated with waves. For example, the vertical velocity field, w, is related to the vertical displacement field, ξ, by $w = \partial_t \xi$. Assuming $\xi = \mathcal{A}_\xi \exp[\imath(kx - \omega t)]$ and $w = \mathcal{A}_w \exp[\imath(kx - \omega t)]$, the differential relation immediately gives $\mathcal{A}_w = -\imath\omega\mathcal{A}_\xi$. Because $-\imath = \exp(-\imath\pi/2)$, we see that w lags in phase by 90° from ξ and the amplitude of the vertical velocity field is $|\mathcal{A}_\xi|\omega$.

The operation of taking the product of wave fields (as would be done, for example, to compute cross-correlations or to derive equations for moderately large amplitude waves) is nonlinear and so the representation of waves by (1.106) is inappropriate and can lead to incorrect computations (see Section 1.15.8). Instead, it is useful to represent their structure by (1.107).

1.15.2 The dispersion relation

For all waves, the frequency depends upon the wavenumber through what is called a dispersion relation. This describes how a wavepacket spreads, and hence disperses, if waves of different wavelength travel at different speeds. If the waves are sufficiently large amplitude that nonlinear effects cannot be neglected, then ω is additionally a function of amplitude. In most of the discussion below we will consider only small-amplitude waves.

Waves that travel at the same speed for all wavelengths are 'nondispersive'. Examples of such waves include light waves, sound waves and long waves on shallow water. The evolution of these waves is prescribed by the well-known 'wave equation':

$$\frac{\partial^2 \eta}{\partial t^2} = c^2 \nabla^2 \eta. \tag{1.111}$$

Here c is a constant and $\eta(\vec{x}, t)$ represents the amplitude of the wave at position $\vec{x}$ and time t. The symbol $\nabla^2 \equiv \nabla \cdot \nabla$ is the Laplacian operator, sometimes denoted by the symbol Δ. In three-dimensional Cartesian co-ordinates $\nabla^2 \equiv {\partial_x}^2 + {\partial_y}^2 + {\partial_z}^2$. Straightforward substitution of (1.106) into (1.111), which amounts to taking a Fourier transform, gives the dispersion relation for nondispersive waves

$$\omega^2 = c^2 |\vec{k}|^2. \tag{1.112}$$

We will see that short interfacial waves and internal waves in continuously stratified fluid are dispersive. Their dispersion relation is not given by (1.112), but more generally by

$$\omega = \omega(\vec{k}). \tag{1.113}$$

For example, surface waves in deep water have the dispersion relation

$$\omega^2 = g|\vec{k}|, \tag{1.114}$$

in which $\vec{k} = (k_x, k_y)$.

As in (1.112) and (1.114), the dispersion relation for waves in otherwise stationary fluid is often given in terms of the squared frequency ω^2. This indicates that two types of waves are captured by the dispersion relations: those that propagate both forwards and backwards in time.

1.15.3 Phase velocity

By definition, the speed at which wave crests move is called the phase speed. Of course, there is nothing special about the wave crests: the motion of the trough or of any point of constant phase suffices in the definition of phase speed. The phase velocity describes the direction as well as speed of propagation.

For a one-dimensional wave having structure in the x-direction alone, the phase speed is $c_p \equiv \omega/k = \lambda/T$, meaning the crest moves one wavelength λ in the time of one wave period T.

For waves having structure in two or three dimensions, the phase velocity can be represented in one of two ways. In the first definition, which is standard, we imagine we are sitting on a wave crest moving with the wave in a direction perpendicular to the along-crest direction. In this perspective, the phase velocity is

$$\vec{c}_p \equiv \frac{\omega}{|\vec{k}|}\hat{k} = \frac{\omega}{|\vec{k}|^2}\vec{k}, \tag{1.115}$$

in which $\hat{k} \equiv \vec{k}/|\vec{k}|$ is the unit vector pointing in the direction of $\vec{k}$, as illustrated in Figure 1.25a. In particular, the x-component of the phase speed for a two-dimensional wave is

$$c_{px} = \omega \frac{k_x}{{k_x}^2 + {k_y}^2}.$$

The phase speed is just the magnitude $c_p \equiv |\vec{c}_p| = \omega/|\vec{k}|$.

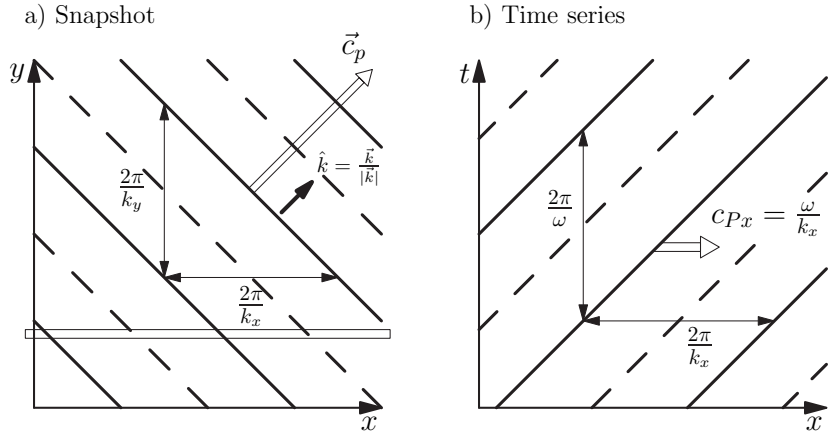

Fig. 1.25. a) Snapshot of plane waves with crests illustrated by solid lines and troughs by dashed lines. The wavelengths in the x and y directions are indicated by double-headed arrows; the unit wavenumber vector $\hat{k}$, indicated by the single arrow, is perpendicular to the crests; the phase velocity $\vec{c}_p$, indicated by the double-tailed arrow, is parallel to $\vec{k}$. b) A time series constructed by examining how the flow evolves within the horizontal window indicated by the thin box in a). The x-component of the phase speed, $\vec{c}_{Px} = \omega/k_x$, is not necessarily equal to the x-component of $\vec{c}_p$.

If instead we imagine we are sitting at a fixed position with wave crests moving past, the speeds of the crests in the x, y, and z directions are, respectively

$$c_{Px} = \frac{\omega}{k_x},\ c_{Py} = \frac{\omega}{k_y},\ \text{and}\ c_{Pz} = \frac{\omega}{k_z}. \tag{1.116}$$

Here we have used an upper-case 'P' in the subscript to distinguish this definition from that in (1.115). The definitions in (1.116) are useful when analysing time series data either from observations or numerics, as illustrated in Figure 1.25b. It does not make sense to compose the vector $\vec{c}_P$ from the components in (1.116) because $|\vec{c}_P|$ does not equal the appropriate value for the phase speed, $c_p = \omega/|\vec{k}|$. This is only the case for one-dimensional waves.

Using the dispersion relation, the phase speed can be expressed explicitly in terms of the wavenumber alone. For example, using (1.112), one-dimensional shallow water waves have phase speed $c_p = \pm c$, a constant for all wavenumbers. The plus and minus signs correspond to rightward and leftward propagating waves, respectively. For one-dimensional deep water waves with a dispersion relation given by (1.114), the phase speed is $c_p = \pm(g/k)^{1/2}$. So long waves (small k) travel at faster speeds.

1.15.4 Group velocity

More dynamically important yet more difficult to perceive than the phase velocity is the group velocity, $\vec{c}_g$.This is the velocity at which energy is transported by small-amplitude waves. The magnitude and direction of the group velocity are not necessarily the same as those of the phase velocity, as illustrated in Figure 1.26. So one should not look at the direction of propagation of wave crests to infer where the energy is being transported. In particular, Section 3.3.4 shows that the group and phase velocities of internal waves have different directions as well as magnitudes.

In Section 1.15.6 we will develop the mathematics used to describe the evolution of the amplitude envelope of a spatially localized packet of waves. There we will show that $c_g = |\vec{c}_g|$ is the speed of the group of waves.

Elementary texts derive the group speed of one-dimensional waves by considering a superposition of two waves having wavenumbers k_0 and $k_0 + k_\Delta$ and respective frequencies $\omega_0 \equiv \omega(k_0)$ and $\omega_0 + \omega_\Delta$. Assuming $|k_\Delta| \ll |k|$, the difference in frequencies is found from the second term of the Taylor-series expansion of ω about k_0:

$$\omega_\Delta \simeq \omega'(k_0)\, k_\Delta, \tag{1.117}$$

in which the prime denotes the k derivative of ω.

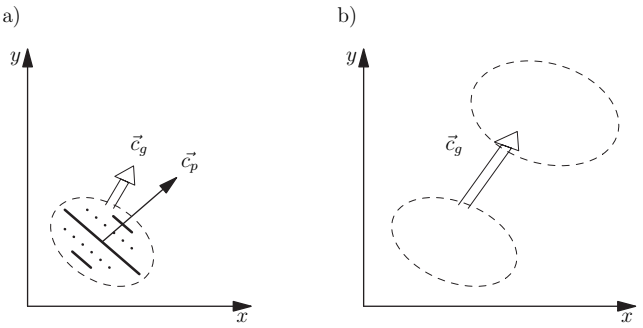

Fig. 1.26. a) The vector $\vec{c}_p$ indicates the speed and direction at which crests (solid lines) and troughs (dotted lines) propagate within a wavepacket (enclosed by dashed lines). This is not necessarily the same as the group velocity, $\vec{c}_g$, which indicates the speed and direction at which the wavepacket as a whole moves, as illustrated in b).

Taking the amplitudes to be the same and arbitrarily matching their initial phase at $x = 0$, the disturbance amplitude of the two waves is given by

$$\begin{aligned}\eta &= A_0 \cos(k_0 x - \omega_0 t) + A_0 \cos[(k_0 + k_\Delta)x - (\omega_0 + \omega_\Delta)t] \\ &\simeq A_0 \left\{1 + \cos[k_\Delta (x - \omega'(k_0)t)]\right\} \cos(k_0 x - \omega_0 t). \end{aligned} \tag{1.118}$$

In deriving the second expression, we have used (1.117) and a double-angle trigonometric identity to write the cosine of two angles as the sum of a product of cosines and sines. The term involving a product of sines has been neglected under the assumption that k_Δ is small.

Equation 1.118 shows that the superposition of waves acts like a plane wave of wavenumber k_0 and frequency ω_0, but whose amplitude changes in space and time as

$$A(x,t) = A_0 \left\{1 + \cos[k_\Delta (x - \omega'(k_0)t)]\right\}. \tag{1.119}$$

This is the so-called 'amplitude envelope' of the wave. Thus we have shown that the superposition of the two waves gives a disturbance in which crests move at the phase speed $c_p = \omega_0/k_0$, but for which the peak of the amplitude envelope moves at speed $c_g = \omega'(k_0)$, the group speed.

More generally, the group velocity determines how a wavepacket travels in one, two or three dimensions. Mathematically, it is given by

$$\vec{c}_g \equiv \nabla_{\vec{k}}\, \omega, \tag{1.120}$$

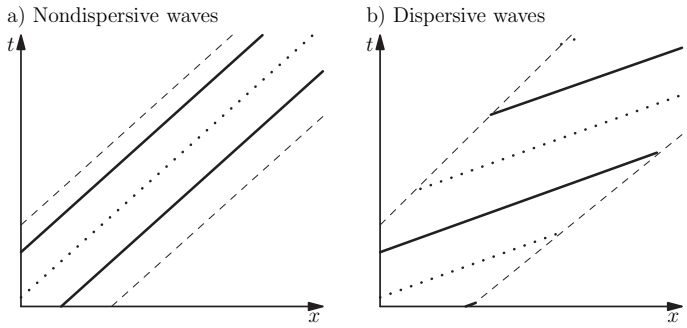

Fig. 1.27. Time series showing the phase lines (solid and dotted) moving within a wavepacket (dashed lines) for waves that are a) nondispersive and b) dispersive with $c_g < c_p$.

in which $\nabla_{\vec{k}}$ is the gradient operator that takes derivatives with respect to each component of the wavenumber vector, $\vec{k}$. For example, the x-component of the group velocity is $c_{gx} = \partial\omega/\partial k_x$.

For nondispersive waves, the group and phase velocity are identical: the wavepackets move in the same direction and at the same speed as the wave crests, as shown in Figure 1.27a. This is not the case for dispersive waves. For example, the group velocity of one-dimensional, rightward-propagating deep water waves is $c_g = (g/k)^{1/2}/2 = c_p/2$. In this case, the group moves in the same direction as the wave crests but at half the speed. Crests advance from the back to the front of a wavepacket during its propagation, as shown in Figure 1.27b.

1.15.5 Representation of wavepackets

A 'wavepacket' is a localized wavy disturbance. Typically this is a superposition of waves having frequencies in a range about a central value, and phases are set so that the amplitude of the wavepacket drops to negligibly small values away from its centre. Such wavepackets are said to be 'quasi-monochromatic' because they behave similarly to (monochromatic) plane waves with single frequency.

For example, a one-dimensional, small-amplitude wavepacket as it evolves over time from a known initial condition can be written as the superposition of plane waves by

$$\eta(x,t) = \int_{-\infty}^{\infty} \hat{\eta}(k) e^{\imath(kx-\omega t)}\, dk. \tag{1.121}$$

Here, $\hat{\eta}(k)$ is the amplitude (per unit wavenumber) of waves with wavenumber k. This can be determined explicitly from the initial conditions by the inverse Fourier

transform

$$\hat{\eta}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(x,0)e^{-\imath kx}\,dx. \tag{1.122}$$

Alternately, the spatial structure of the waves as established from known variations of the amplitude at a boundary ($x = 0$, say) may be represented by

$$\eta(x,t) = \int_{-\infty}^{\infty} \hat{\eta}(\omega)e^{\imath(kx-\omega t)}\,d\omega, \tag{1.123}$$

in which

$$\hat{\eta}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(0,t)e^{\imath\omega t}\,dt. \tag{1.124}$$

In these equations, we have defined the transforms so that a constant factor, in this case $1/2\pi$, appears only in the formulae for the inverse transforms (1.122) and (1.124). To exploit symmetry, mathematicians sometimes define the transform pairs with constant factors $(2\pi)^{-1/2}$ leading each integral. We do not do this here so that we can associate $\hat{\eta}$ more directly with the half peak-to-peak amplitude of the waves.

In some idealized studies, a wavepacket is conveniently represented by the product of a plane wave with a smooth, non-negative function which is the 'amplitude envelope'. For example, a one-dimensional 'Gaussian wavepacket' with peak value at the origin is represented by

$$\eta(x,0) = \mathcal{A}_0 \exp\left[-\frac{x^2}{2\sigma^2}\right] e^{\imath k_0 x}. \tag{1.125}$$

This is sketched in Figure 1.28. Here σ measures the width of the wavepacket envelope consisting of waves with wavenumber k_0. If the amplitude $\mathcal{A}_0 = A_0$ is a

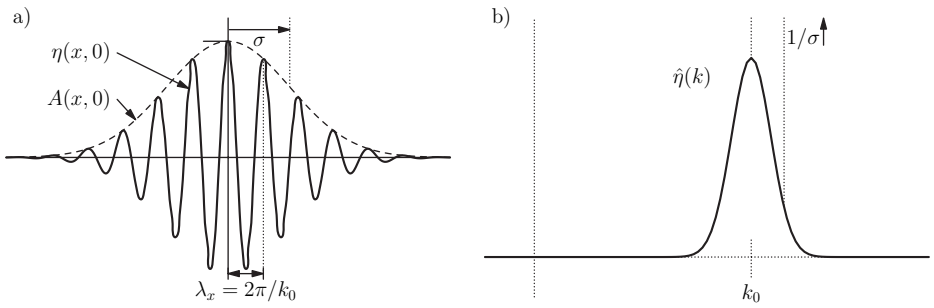

Fig. 1.28. a) A Gaussian wavepacket with initial amplitude envelope $A(x,0) = |\mathcal{A}|\exp(-x^2/2\sigma^2)$ containing waves with wavenumber k_0. b) The Fourier transform of this wavepacket is peaked about wavenumber $k = k_0$. The width of the peak narrows as σ increases but the area under the curve remains constant.

real number, then the actual structure of η, given by the real part of the right-hand side of (1.125), is a Gaussian times a cosine function.

This wavepacket can be thought of as a superposition of waves having non-negligible amplitude only for a range of wavenumbers near k_0. This is revealed by the inverse Fourier transform of $\eta(x,0)$ through (1.122):

$$\hat{\eta}(k) = (\sqrt{2\pi}\sigma)A_0 \exp\left[-\frac{1}{2}\sigma^2(k-k_0)^2\right]. \tag{1.126}$$

The range of wavenumbers with significant amplitude is measured by $1/\sigma$, as shown in Figure 1.28b. In the limit $\sigma k_0 \to \infty$, the wavepacket is a plane wave in which only waves with wavenumber equal to k_0 have significant amplitude. The corresponding Fourier transform (1.126) becomes a Dirac delta function: $\hat{\eta}(k) = \mathcal{A}_0\,\delta(k-k_0)$. Substituting this into (1.121) gives $\eta = \mathcal{A}_0 \exp[\imath\,(k_0 x - \omega(k_0)t)]$, as expected.

For $\sigma k_0 \ll 1$, the wavepacket more closely resembles a 'wave-pulse' than a group of waves. Such waves are not quasi-monochromatic and are usually not considered in the theory of wavepacket propagation.

1.15.6 Plane wave and wavepacket evolution equations

Knowing the dispersion relation, we can formulate a differential equation that describes the evolution of the waves. More generally, one can determine how the amplitude envelope of a quasi-monochromatic wavepacket translates and disperses in time. The equation that describes the latter process for dispersive waves is known as 'Schrödinger's equation'. Although best known for its application in quantum mechanics, generally it is a formula that describes the evolution of dispersive wavepackets.

We begin by showing how to derive a differential equation for the evolution of plane waves given the dispersion relation. This amounts to transforming the equations from frequency space (in ω and $\vec{k}$) to real space (in t and $\vec{x}$). For waves in the form (1.106), we see that time derivatives give $\partial_t \eta = -\imath\omega\eta$ and spatial derivatives give $\nabla\eta = \imath\vec{k}\eta$. So, in going from real space to wavenumber space, we can simply replace derivatives with algebraic quantities: $\partial_t \to -\imath\omega$ and $\nabla \to \imath\vec{k}$. We follow the same procedure going backwards to convert a dispersion relation into a differential equation: $\omega \to \imath\,\partial_t$ and $\vec{k} \to -\imath\nabla$. Thus for one-dimensional waves with dispersion relation $\omega = \omega(k_x)$, the differential equation describing the time evolution of the waves is given by

$$\left[\imath\frac{\partial}{\partial t}\right]\eta = \left[\omega\left(-\imath\frac{\partial}{\partial x}\right)\right]\eta, \tag{1.127}$$

in which $\eta(x,t)$ represents the structure of the waves in real space.

For example, the one-dimensional analogue of (1.112) gives the dispersion relation of nondispersive waves: $\omega = \pm ck_x$. The upper sign corresponds to rightward-propagating waves and the lower sign to leftward-propagating waves. Then (1.127) gives the differential equation describing their evolution: $\imath\,\partial_t \eta = \pm c(-\imath\,\partial_x)\eta$. This is simplified to give

$$\frac{\partial \eta}{\partial t} \pm c\frac{\partial \eta}{\partial x} = 0. \tag{1.128}$$

This advection equation is the one-dimensional analogue of (1.111), separately describing waves propagating rightwards with speed $+c$ and leftwards with speed $-c$.

Next we derive an equation for the evolution of a wavepacket in an unbounded domain. We suppose the structure in terms of η can be written as the Fourier transform

$$\eta(x,t) = \int_{-\infty}^{\infty} \hat{\eta}(k) e^{\imath(kx-\omega t)}\, dk. \tag{1.129}$$

To ensure η is real, we insist that $\hat{\eta}(-k) = \hat{\eta}^\star(k)$, in which the star denotes the complex conjugate.

Initially, the structure of η can be represented by

$$\eta(x,0) = \mathcal{A}(x,0)\, e^{\imath k_0 x}, \tag{1.130}$$

in which $\mathcal{A}$ is the possibly complex-valued amplitude envelope containing waves with wavenumber k_0. For example, $\mathcal{A}(x,0) = \mathcal{A}_0 \exp[-x^2/2\sigma^2]$ for the Gaussian wavepacket in (1.125). Here it is understood that η is given by the real part of the expression on the right-hand side of (1.130).

The initial condition (1.130) together with (1.129) implicitly defines $\hat{\eta}$ through

$$\mathcal{A}(x,0) = \int_{-\infty}^{\infty} \hat{\eta}(k) e^{\imath(k-k_0)x}\, dk. \tag{1.131}$$

As before, $\hat{\eta}$ can be written explicitly in terms of $\mathcal{A}(x,0)$ by the inverse Fourier transform except with k replaced by $k - k_0$.

For example, Figure 1.28 shows a Gaussian wavepacket and its Fourier transform. The initial amplitude is $A_0 = |\mathcal{A}(0,0)|$ and its width is σ. For quasi-monochromatic waves, we assume $\sigma k_0 \gg 1$. In this case the Fourier transform of (1.130) is sharply peaked about wavenumbers with $k \simeq k_0$.

If the waves are quasi-monochromatic initially, we may assume they remain quasi-monochromatic as the wavepacket evolves. Only the amplitude envelope, $\mathcal{A}(x,t)$, will change, albeit slowly compared with the wave period. We therefore wish to develop an equation describing the evolution of $\mathcal{A}(x,t)$ from which the structure $\eta(x,t)$ of the waves can then be determined from

$$\eta(x,t) = \mathcal{A}(x,t) e^{\imath[k_0 x - \omega(k_0)t]}. \tag{1.132}$$

Substituting (1.132) into (1.129) and multiplying both sides by $\exp\{-\imath[k_0 x - \omega(k_0)t]\}$ gives an integral equation for $\mathcal{A}$:

$$\mathcal{A}(x,t) = \int_{-\infty}^{\infty} \hat{\eta}(k) e^{\imath[(k-k_0)x - (\omega(k)-\omega(k_0))t]}\, dk. \tag{1.133}$$

Because the Fourier transform, $\hat{\eta}(k)$, of the wavepacket is sharply peaked about wavenumbers $k \simeq k_0$, the integrand of (1.133) is non-negligible only in a small range about k_0. Thus we need only be concerned with the behaviour of the dispersion relation near $k \simeq k_0$, which can be found through a Taylor-series expansion:

$$\omega(k) \simeq \omega(k_0) + \omega'(k_0)(k-k_0) + \frac{1}{2}\omega''(k_0)(k-k_0)^2. \tag{1.134}$$

Here, the primes denote k-derivatives of ω, and we have chosen to truncate the Taylor series in ω at second order in $(k-k_0)$.

Substituting (1.134) into (1.133), taking x- and t-derivatives, and comparing terms we arrive at the following approximate equation for $\mathcal{A}$:

$$\frac{\partial \mathcal{A}}{\partial t} \simeq -\omega' \frac{\partial \mathcal{A}}{\partial x} + \imath \frac{1}{2}\omega'' \frac{\partial^2 \mathcal{A}}{\partial x^2}. \tag{1.135}$$

The first term on the right-hand side of (1.135) indicates that the wavepacket translates at the group speed $c_g = \omega'(k_0)$. Typically, (1.135) is further simplified by a change of co-ordinates to a frame translating at speed c_g, which has the effect of eliminating this term. Explicitly defining $X = x - c_g t$, (1.135) becomes

$$\frac{\partial \mathcal{A}}{\partial t} \simeq +\imath \frac{1}{2}\omega'' \frac{\partial^2 \mathcal{A}}{\partial X^2}. \tag{1.136}$$

This is Schrödinger's equation.

The right-hand side of (1.136), or equivalently the last term on the right-hand side of (1.135), represents the leading-order effect of dispersion. This describes how the wavepacket may change shape as it propagates because waves with different wavenumbers propagate at different speeds if they are dispersive. Typically, dispersion has the effect of widening the wavepacket while its maximum amplitude gradually decreases.

The discussion in this section has been confined to small-amplitude waves. Consequently, equation (1.136) is sometimes called the linear Schrödinger equation, emphasizing that the equation is linear and also that it captures only the linear dispersion of small-amplitude waves. The extension of this equation to include

moderately large-amplitude effects results in the nonlinear Schrödinger equation. Its derivation and interpretation are described in Section 4.2.3.

1.15.7 Wave modes

The complex-exponential representation of waves given by (1.106) is appropriate particularly for propagating waves in an unbounded medium. If physical boundaries or ambient conditions (as in a wave duct) confine the waves to a finite-sized region, it is often more useful to describe the waves in terms of modes. Wave modes are sometimes called 'standing waves', meaning that as the crests move up and down the positions of the nodes between them remain stationary.

One-dimensional, horizontally propagating wave modes can be thought of as the superposition of two propagating waves both having the same frequency and spatial structure, but moving in opposite directions with locked-in phase at the boundaries. Thus, for example, the superposition of rightward- and leftward-propagating waves represented by (1.104) in one dimension is the wave mode

$$A\cos(kx-\omega t)+A\cos(-kx-\omega t)=2A\cos(\omega t)\cos(kx). \tag{1.137}$$

This represents a stationary wave with spatial structure given by $\cos(kx)$ but whose amplitude changes in time according to $2A\cos(\omega t)$. The variations in time and space have explicitly been separated.

A one-dimensional wave mode is generally represented by

$$\eta(x,t)=\eta(x)\cos(\omega t+\phi_0), \tag{1.138}$$

in which η is a real function describing the spatial structure and amplitude of the wave and ϕ_0 is an arbitrary but constant phase factor. One could describe the displacement of waves by $A\cos(kx)$, for example, in which A represents the maximum vertical displacement of the surface. The wavenumber k varies continuously if the domain is infinitely large, but it can hold only a discrete set of values if the domain is constrained by requiring $0\le x\le L$. In this case, the allowable values of k are $k_n=n\pi/L$ for $n=1,2,\ldots$ The zeroth mode of cosine-shaped waves is constant and so is neglected.

The dispersion relation $\omega\equiv\omega(k)$ is the same whether or not the domain is bounded. Because $k=k_n$ can hold only a discrete set of values in a bounded domain the frequency $\omega_n\equiv\omega(k_n)$ is likewise discrete-valued.

With the co-ordinate system set up with the origin at one of the boundaries, cosine functions are appropriate if the slope of the field is required to be zero at the boundary. These are called Neumann boundary conditions, and are used, for

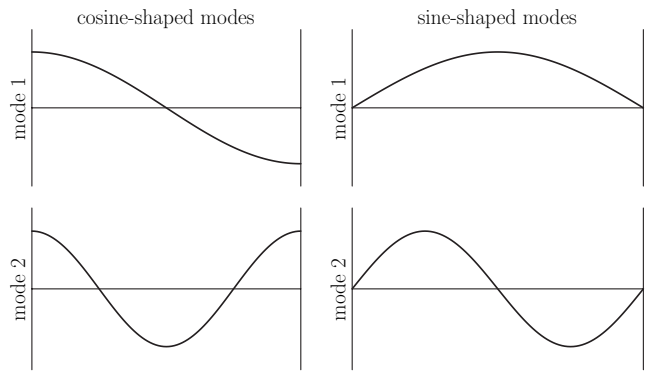

Fig. 1.29. Lowest wave-like modes of one-dimensional waves in a finite-sized domain. Their structure, $\eta(x)$, is represented by cosine functions if the side-wall boundary condition requires zero slope (zero Neumann boundary conditions). It is represented by sine functions if the boundary condition requires zero value (zero Dirichlet boundary conditions).

example, to describe the vertical displacement of small-amplitude waves in a box. Sine functions are appropriate for Dirichlet boundary conditions, which require the value of the field to be zero at the boundary. These would be used, for example, to describe the horizontal velocity field of laterally bounded waves.

The left-hand column of Figure 1.29 shows mode 1 and 2 for laterally bounded cosine-shaped waves and the right-hand column shows these modes for sine-shaped waves. The lowest mode has a wavelength that is twice the extent of the domain while exactly one wavelength of mode-2 waves fills the domain.

Any disturbance in a bounded domain can be written as a sum of modes. If the disturbance must have zero slope conditions at the boundaries, it is written as a superposition of cosine functions using the Fourier cosine series. If the disturbance is necessarily zero at the boundaries, then the Fourier sine series is used.

In our discussion of internal waves in Section 3.5, we will see that in a domain with sloping sides the description of waves as a superposition of modes is not so practical and the use of attractors gives a more useful description.

1.15.8 Cross-correlations

Cross-correlations help to diagnose the transport properties of waves. For example, the vertical transport of mass by waves is characterized by computing the cross-correlation between the density field, ρ, and the vertical velocity field, w: if this correlation is zero, the waves do not transport mass; if positive, mass is transported upwards; if negative, mass is transported downwards.

For spatially periodic one-dimensional waves, the cross-correlation between fields η and ξ is denoted by $\langle \eta\xi \rangle$ in which

$$\langle \eta\xi \rangle \equiv \frac{1}{\lambda}\int_0^\lambda \eta(x)\xi(x)\,dx. \tag{1.139}$$

Here $\lambda = 2\pi/k$ is the horizontal wavelength corresponding to wavenumber k. For example, if $\eta = A_\eta \cos(kx)$ and $\xi = A_\xi \cos(kx)$, the correlation of the two fields is $\langle \eta\xi \rangle = A_\eta A_\xi/2$.

Although (1.139) is given as a spatial average, the cross-correlation can be defined as an average over one period rather than one wavelength. For plane waves defined in space and time, it turns out that it does not matter whether the averaging is done over a wavelength or over a period; the result is the same. Putting $\eta = A_\eta \cos(kx - \omega t)$ and $\xi = A_\xi \cos(kx - \omega t)$ in (1.139) still gives the correlation $\langle \eta\xi \rangle = A_\eta A_\xi/2$.

If η and ξ represent the same field, (1.139) is called the autocorrelation. Its square root is the root-mean-square average from which one can determine the amplitude of the field by multiplying by $2^{1/2}$.

Some shortcuts are possible in the computation of (1.139). For example, suppose η and ξ are the real parts of $\mathcal{A}_\eta \exp(\imath kx)$ and $\mathcal{A}_\xi \exp(\imath kx)$, respectively. Then

$$\begin{aligned}\langle \eta\xi \rangle &= \frac{1}{\lambda}\int_0^\lambda \left[\frac{1}{2}\left(\mathcal{A}_\eta e^{\imath kx} + \text{ cc}\right)\right]\left[\frac{1}{2}\left(\mathcal{A}_\xi e^{\imath kx} + \text{ cc}\right)\right] dx \\ &= \frac{1}{4\lambda}\int_0^\lambda \mathcal{A}_\eta\mathcal{A}_\xi e^{2\imath kx} + \mathcal{A}_\eta\mathcal{A}_\xi{}^\star + \mathcal{A}_\eta{}^\star\mathcal{A}_\xi + \mathcal{A}_\eta{}^\star\mathcal{A}_\xi{}^\star e^{-2\imath kx}\,dx. \end{aligned} \tag{1.140}$$

Here cc and the star superscript denote the complex conjugate. Note that we need to represent the fields explicitly as real quantities through (1.107) because taking the product of fields is a nonlinear operation: the real part of their product is not the product of the real parts.

The first and last expressions in the integrand are periodic in x with period λ and so their integral is zero. The middle two terms are constant (in fact, one is the complex conjugate of the other) and so these can be pulled outside the integral. Thus, without resorting to applying trigonometric identities in the integrand, we immediately arrive at a succinct formula for the correlation of two fields associated with a plane wave:

$$\langle \eta\xi \rangle = \frac{1}{4}\left(\mathcal{A}_\eta\mathcal{A}_\xi{}^\star + \text{ cc}\right) = \frac{1}{2}\Re\left(\mathcal{A}_\eta\mathcal{A}_\xi{}^\star\right) = \frac{1}{2}\Re\left(\mathcal{A}_\eta{}^\star\mathcal{A}_\xi\right), \tag{1.141}$$

in which $\Re(\mathcal{Z})$ takes the real part of $\mathcal{Z}$.

One immediate consequence of (1.141) is that the cross-correlation is zero between two fields that are out of phase by 90°. In particular, if $\eta = \Re\{A_\eta \exp(\imath kx)\} \propto \cos(kx)$ and $\xi = \Re\{-\imath A_\xi \exp(\imath kx)\} \propto \sin(kx)$, then $\langle \eta\xi \rangle = 0$.