ASSIGNMENT 4 SOLUTIONS

1a) THE CLAUSIUS-CLAPEYRON EQUATION IS $\frac{dP}{dT} = \frac{LM}{T\Delta V}$

WHERE, HERE, L IS THE LATENT HEAT OF FUSION AND $\Delta V = V_L - V_S$ IS THE DIFFERENCE IN VOLUMES BETWEEN LIQUID WATER AND ICE. DIVIDING TOP AND BOTTOM BY MASS, M, GIVES

$$\frac{dP}{dT} = \frac{L}{T}\frac{1}{\frac{V_L}{M} - \frac{V_S}{M}} = \frac{L}{T}\left(\rho_L^{-1} - \rho_S^{-1}\right)^{-1}, \text{ WITH } \rho_L = \frac{M}{V_L},\ \rho_S = \frac{M}{V_S}$$

BUT $\rho_L \simeq 1000\ kg/m^3$ AND $\rho_S = 917\ kg/m^3 \Rightarrow \frac{1}{\rho_L} < \frac{1}{\rho_S}$

SO $\frac{dP}{dT} < 0$

b) TAKING $L = L_F$, ρ_L AND ρ_S AS CONSTANT, CAN SOLVE ODE:

$$\frac{dP}{dT} = \frac{L}{T}\frac{1}{\rho_L^{-1} - \rho_S^{-1}} \Rightarrow P = \frac{L}{(\rho_L^{-1} - \rho_S^{-1})}\ln T + C = \frac{L}{\rho_L^{-1} - \rho_S^{-1}}\ln\left(\frac{T}{T_0}\right) + P_0$$

WHERE $P(T_0) = P_0$ WITH $T_0 = 273.15\,K$ AND $P_0 = 100\ kPa$

SO PRESSURE CHANGE FOR MELTING AT $-1°C = 272.15\,K$ IS

$$P - P_0 = \left(3.33\times10^5\ \tfrac{J}{kg}\right)\left(\frac{1}{1000\ kg/m^3} - \frac{1}{917\ kg/m^3}\right)^{-1}\ln(272.15/273.15)$$

$$\Rightarrow \Delta P \simeq 1.35\times10^7\ Pa = 135\ \text{BAR}$$

c) THE WEIGHT OF A COLUMN OF ICE OF HEIGHT H AND AREA A IS $W = (\rho_S A H)g$.

SO THE PRESSURE AT THE BOTTOM DUE TO WEIGHT ABOVE IS

$$P = \frac{W}{A} = \rho_S H g$$

EQUATE THIS TO SOLUTION IN b):

$$\rho_S H g = \Delta P \Rightarrow H = \frac{\Delta P}{\rho_S g} = \frac{1.35\times10^7\ Pa}{(917\ kg/m^3)(9.81\ m/s^2)}$$

$$\Rightarrow H = 1.50\times10^3\ m = 1.5\ km$$

SO GLACIER WOULD NEED TO BE 1.5 KM THICK TO ALLOW MELTING AT $-1°C$. [THIS IS THE CASE FOR MANY GLACIERS e.g. IN GREENLAND AND ANTARCTICA.]

2) RELATIVE HUMIDITY IS $R.H. = e/e_s$, WHERE e_s IS THE SATURATION VAPOUR PRESSURE, GIVEN BY

$$e_s(T) = e_0 \,\mathrm{EXP}\left[\frac{L_v}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right] \qquad (*)$$

$$= (6.11 \text{ mBAR})\,\mathrm{EXP}\left[\frac{2.50\times10^6 \frac{J}{kg}}{462 \text{ J/(kg}\cdot\text{K)}}\left(\frac{1}{273.15} - \frac{1}{278.15}\right)\right]$$

$$\simeq 8.72 \text{ mBAR}$$

SO $e = (R.H.)\, e_s = (0.85)(8.72 \text{ mBAR}) \simeq \boxed{7.42 \text{ mBAR}}$

AT DEW POINT, T_d, $e_s(T_d) = 7.42$ mBAR

INVERTING $(*)$, GIVES

$$T_d = \left[\frac{1}{T_0} - \frac{R_v}{L_v}\ln\left(\frac{e_s(T_d)}{e_0}\right)\right]^{-1}$$

$$= \left[\frac{1}{273.15} - \frac{462}{2.50\times10^6}\ln\left(\frac{7.42}{6.11}\right)\right]^{-1}$$

$$\simeq 275.85\,K = 2.7^\circ C$$

3) USE FORMULA FOR THE EQUIVALENT POTENTIAL TEMPERATURE:

$$\theta_e = \theta \, \mathrm{EXP}\left[\frac{L_v}{C_p} \frac{W_s}{T}\right]$$

POTENTIAL TEMPERATURE AT GROUND IS THE SAME AS TEMPERATURE:

$$\theta = 273.15 + 5 = 278.15\,K$$

THE MIXING RATIO FOR A SATURATED AIR PARCEL IS

RESULT IN QUESTION 2

$$W_s = \frac{\varepsilon}{p} e_s = \frac{0.622}{1000\,\mathrm{mBAR}} (8.72\,\mathrm{mBAR})$$

$$\simeq 5.42 \times 10^{-3} \qquad (5.42\ g/Kg)$$

$$\text{So } \theta_e \simeq (278.15\,K)\,\mathrm{EXP}\left[\frac{2.50 \times 10^6}{1005} \cdot \frac{0.00542}{278.15}\right]$$

$$\simeq 291.96\,K$$

THIS IS POTENTIAL TEMPERATURE AIR WILL HAVE WHEN ALL MOISTURE HAS CONDENSED, RELEASING HEAT.

WHEN THIS DRY AIR IS BROUGHT TO THE GROUND ITS TEMPERATURE EQUALS THE POTENTIAL TEMPERATURE:

$$T = 291.96\,K \simeq 18.8^\circ C$$

4a) THE SPEED DISTRIBUTION IS $\mathcal{N}(u) = N\, 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} u^2 \, EXP\left[-\frac{mu^2}{2kT}\right]$

RECAST IN TERMS OF NONDIMENSIONAL $\tilde{u} \equiv u/u_m$, WITH $u_m = \sqrt{2kT/m}$

$$u = \tilde{u}\, u_m \Rightarrow \mathcal{N}(\tilde{u}) = N\, 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} \left(\tilde{u}^2 \frac{2kT}{m}\right) EXP\left[-\frac{m}{2kT}\left(\tilde{u}^2 \frac{2kT}{m}\right)\right]$$

$$\Rightarrow \mathcal{N}(\tilde{u}) = N \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT}\right)^{1/2} \tilde{u}^2 e^{-\tilde{u}^2}$$

THE PROBABILITY THE SPEED IS BETWEEN u AND $u + du$ IS $\frac{1}{N}\mathcal{N}(u)du$

SO FOR NONDIMENSIONAL SPEED BETWEEN $\tilde{u}$ AND $\tilde{u} + d\tilde{u}$, THE PROBABILITY IS

$$\frac{1}{N}\mathcal{N}(\tilde{u})\, d(\tilde{u}\, u_m) = \frac{u_m}{N}\mathcal{N}(\tilde{u})\, d\tilde{u} = \boxed{\frac{4}{\sqrt{\pi}} \tilde{u}^2 e^{-\tilde{u}^2} d\tilde{u}} = D(\tilde{u})\, d\tilde{u}$$

(NOTE, THIS IS WHAT IS INTEGRATED IN THE TABLE)

b) FOR N_2 AT 1000K

$$u_m \simeq \left[2(1.38 \times 10^{-23} \tfrac{J}{K})(1000K)/(28 \times 1.67 \times 10^{-27} kg)\right]^{1/2}$$
$$\simeq 768 \text{ m/s}$$

FOR ESCAPE VELOCITY $u_e = 11 \times 10^3$ m/s, $\tilde{u}_e = \frac{u_e}{u_m} \simeq \boxed{14.3}$

c) $$P(u > u_e) = P(\tilde{u} > \hat{u}_e) = \int_{\tilde{u}_e}^{\infty} D(\tilde{u})\, d\tilde{u}$$
$$= \int_{14.3}^{\infty} \frac{4}{\sqrt{\pi}} \tilde{u}^2 e^{-\tilde{u}^2} d\tilde{u}$$

FROM THE GIVEN TABLE THE PROBABILITY IS 2.5×10^{-88}

SINCE THERE ARE FAR LESS THAN 10^{88} MOLECULES ($\simeq 10^{64}$ MOLES) IN THE THERMOSPHERE, EXPECT VERY FEW TO ESCAPE IN EARTH'S LIFETIME

d) FOR THE MOON'S ESCAPE VELOCITY $u_{MOON} = 2.4 \times 10^3$ m/s,

$\tilde{u}_{MOON} = (2.4 \times 10^3)/(768) \simeq 3.1$

$$P(\tilde{u} > \hat{u}_{MOON}) = \int_{3.1}^{\infty} \frac{4}{\sqrt{\pi}} \tilde{u}^2 e^{-\tilde{u}^2} d\tilde{u} \simeq 2.5 \times 10^{-4}.$$

THOUGH SMALL, A SIGNIFICANT FRACTION OF 1 MOLE (6×10^{23} MOLECULES) WOULD MEET THIS CRITERION. IN TIME ALMOST ALL WOULD ESCAPE.