ASSIGNMENT 3: SOLUTIONS

1a) TEMPERATURE IS A PIECEWISE FUNCTION GIVEN BY

$$T(z) = \begin{cases} T_1 & 0 \le z \le h_1 \\ T_1 + \frac{\Delta T}{\Delta h}(z-h_1) & h_1 < z \le h_2 \\ T_2 - \Gamma(z-h_2) & z > h_2 \end{cases} \quad ; \Delta T = T_2 - T_1, \ \Delta h = h_2 - h_1$$

TO GET $\theta(z) \equiv T(z)\left(p(z)/p_0\right)^{-\kappa}$, WITH $\kappa \equiv \frac{R_a}{c_p}$ ($= 2/7$ FOR AIR), NEED $p(z)$.

GENERALLY, GIVEN $T(z)$, $\quad p(z) = p_0 \, \mathrm{EXP}\left[-\frac{g}{R_a}\int_0^z \frac{1}{T(\tilde{z})}d\tilde{z}\right]$

FOR $0 \le z \le h_1$, $\quad T(z) = T_1 \Rightarrow p(z) = p_0 e^{-z/H_1}$ WITH $H_1 = R_a T_1/g$

SO $\theta(z) = T_1\left[e^{-z/H_1}\right]^{-\kappa} = T_1 e^{\kappa z/H_1}$

FOR $h_1 < z \le h_2$, $\quad T(z) = T_1 + \frac{\Delta T}{\Delta h}(z-h_1)$

$$\begin{aligned}
\Rightarrow p(z) &= p_0 \, \mathrm{EXP}\left[-\frac{g}{R_a}\left(\int_0^{h_1}\frac{1}{T_1}d\tilde{z} + \int_{h_1}^{z}\frac{1}{T(\tilde{z})}d\tilde{z}\right)\right] \\
&= p_1 \, \mathrm{EXP}\left[-\frac{g}{R_a}\int_{h_1}^{z}\left(T_1 + \frac{\Delta T}{\Delta h}(\tilde{z}-h_1)\right)^{-1}d\tilde{z}\right] \quad \text{WITH } p_1 = p(h_1) = p_0 e^{-h_1/H_1} \\
&= p_1 \, \mathrm{EXP}\left[-\frac{g}{R_a}\frac{\Delta h}{\Delta T}\ln\left[\left(T_1 + \frac{\Delta T}{\Delta h}(z-h_1)\right)/T_1\right]\right] \\
&= p_1\left[T(z)/T_1\right]^{-\delta} \quad \text{WITH } \delta = \frac{g}{R_a}\frac{\Delta h}{\Delta T}
\end{aligned}$$

$$\begin{aligned}
\text{SO } \theta(z) &= T(z)\left(\frac{p_1}{p_0}\left(\frac{T(z)}{T_1}\right)^{-\delta}\right)^{-\kappa} \\
&= T_1\left(\frac{p_1}{p_0}\right)^{-\kappa}\left(\frac{T(z)}{T_1}\right)^{\kappa\delta+1} = T_1 e^{\kappa h_1/H_1}\left[1 + \frac{\Delta T}{\Delta h}\frac{1}{T_1}(z-h_1)\right]^{\kappa\delta+1}
\end{aligned}$$

FOR $z > h_2$, $\quad T(z) = T_2 - \Gamma(z-h_2)$

$$\begin{aligned}
\Rightarrow p(z) &= p_2 \, \mathrm{EXP}\left[-\frac{g}{R_a}\int_{h_2}^{z}\left(T_2 - \Gamma(\tilde{z}-h_2)\right)^{-1}d\tilde{z}\right] \quad \text{WITH } p_2 = p(h_2) = p_1\left(\frac{T_2}{T_1}\right)^{-\delta} \\
&= p_2 \, \mathrm{EXP}\left[-\frac{g}{R_a}\frac{1}{\Gamma}\ln\left[\left(T_2 - \Gamma(z-h_2)\right)/T_2\right]\right] \\
&= p_2\left[T(z)/T_2\right]^{-1/\kappa} \quad \text{USING } \kappa = \frac{R_a}{c_p} = \frac{R_a}{g}\frac{g}{c_p} = \frac{R_a}{g}\frac{1}{\Gamma}
\end{aligned}$$

$$\begin{aligned}
\text{SO } \theta(z) &= T(z)\left(\frac{p_2}{p_0}\left(\frac{T(z)}{T_2}\right)^{-1/\kappa}\right)^{-\kappa} \\
&= T_2\left(\frac{p_2}{p_0}\right)^{-\kappa} = T_2\left[\frac{p_1}{p_0}\left(\frac{T_2}{T_1}\right)^{-\delta}\right]^{-\kappa} = T_2 e^{\kappa h_1/H_1}\left(\frac{T_2}{T_1}\right)^{\kappa\delta}
\end{aligned}$$

IN SUMMARY

$$\theta(z) = \begin{cases} T_1 e^{\kappa z/H_1} \\ T_1 e^{\kappa h_1/H_1}\left[1 + \frac{\Delta T}{\Delta h}\frac{1}{T_1}(z-h_1)\right]^{\kappa\delta+1} \\ T_2 e^{\kappa h_1/H_1}\left[T_2/T_1\right]^{\kappa\delta} \end{cases}$$

WITH $H_1 \equiv R_a T_1/g$, $\Delta T = T_2 - T_1$, $\Delta h = h_2 - h_1$, $\delta = \frac{g}{R_a}\frac{\Delta h}{\Delta T}$, $\kappa = \frac{R_a}{c_p}$

1b) THE SQUARED BUOYANCY FREQUENCY IS $N^2 = \frac{g}{\bar{\theta}}\frac{d\bar{\theta}}{dz}$

AT $z = 90m < h_1$, $\theta = T_1 e^{\varkappa z/H_1}$

$\Rightarrow d\theta/dz = T_1(\varkappa/H_1)e^{\varkappa z/H_1} = (\varkappa/H_1)\theta(z)$

$\Rightarrow N^2 = g\varkappa/H_1$ (FOR ALL $z < H_1$)

$$H_1 = \frac{R_d T_1}{g} \simeq (287 \tfrac{J}{kg K})(273+10)K)/(9.81 m/s^2) \simeq 8.3\times10^3 m$$

$\Rightarrow N^2 \simeq (9.81 m/s^2)(2/7)/(8.3\times10^3 m) \simeq 3.4\times10^{-4} s^{-2}$

SO BUOYANCY FREQUENCY IS $N \simeq 0.018 s^{-1}$

AND BUOYANCY PERIOD IS $T_B = 2\pi/N \simeq 3.4\times10^2 s$

AT $z = 110m$ $(h_1 < z < h_2)$, $\theta = T_1 e^{\varkappa h_1/H_1}\left[1 + \frac{\Delta T}{\Delta h}\frac{1}{T_1}(z-h_1)\right]^{\varkappa\delta+1}$

$\Rightarrow d\theta/dz = T_1 e^{\varkappa h_1/H_1}(\varkappa\delta+1)\frac{\Delta T}{\Delta h}\frac{1}{T_1}\left[1+\frac{\Delta T}{\Delta h}\frac{1}{T_1}(z-h_1)\right]^{\varkappa\delta}$

$= (\varkappa\delta+1)\frac{\Delta T}{\Delta h}\frac{1}{T_1}\theta(z)/\left[1+\frac{\Delta T}{\Delta h}\frac{1}{T_1}(z-h_1)\right]$

$$\Rightarrow N^2_{(z)} = g\,(\underbrace{\varkappa\delta}_{=\frac{R_d}{c_p}\cdot\frac{g}{R_d}\frac{\Delta h}{\Delta T} = \Gamma\frac{\Delta h}{\Delta T} \simeq (10K/km)(20m/5K) = 0.04}+1)\frac{\Delta T}{\Delta h}\frac{1}{T_1}\Big/\left[1+\frac{\Delta T}{\Delta h}\frac{1}{T_1}(z-h_1)\right]$$

$\Rightarrow N^2(110m) \simeq (9.81 m/s^2)(1.04)\frac{5K}{20m}\cdot\frac{1}{283K}/\left[1+\frac{5K}{20m}\frac{1}{283K}(10m)\right]$

$\simeq 0.0089 s^{-2}$

SO BUOYANCY FREQUENCY IS $N \simeq 0.095 s^{-1}$

AND BUOYANCY PERIOD IS $T_B = 2\pi/N \simeq 66 s$

AT $z = 130m$ $(z > h_2)$, $\theta = T_2 e^{\varkappa h_1/H_1}[T_2/T_1]^{\varkappa\delta}$

THIS IS INDEPENDENT OF z

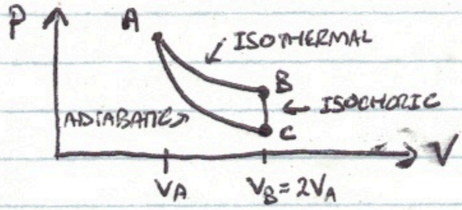
$\Rightarrow d\theta/dz = 0$

SO BUOYANCY FREQUENCY $N = 0 s^{-1}$

AND BUOYANCY PERIOD IS EFFECTIVELY INFINITE

(NO OSCILLATORY MOTION)

2 a)



b) ON $\overline{AB}$, $T = T_A$ IS CONSTANT

SO $dU = C_V dT = 0$

SO, FROM FIRST LAW, $dU = \delta Q - PdV \Rightarrow \delta Q = PdV$

INTEGRATE FROM $A \to B$ USING $P = nRT_A/V$ ①

$\Rightarrow \Delta Q_{AB} = nRT_A \int_{V_A}^{V_B} \frac{1}{V} dV = nRT_A \ln(V_B/V_A)$ $(= \Delta W_{AB})$ ②

$V_B = 2V_A \Rightarrow \Delta Q_{AB} = nRT_A \ln(2)$

THIS IS POSITIVE. SO HEAT <u>ENTERS</u> THE SYSTEM.

c) ON $\overline{BC}$, $V = V_B$ IS CONSTANT. SO $pdV = 0$

$\Rightarrow \delta Q = dU = C_V dT = C_V d(PV_B/nR) = \frac{1}{nR} C_V V_B \, dP$

$\Rightarrow \Delta Q_{BC} = C_V \frac{1}{nR} V_B \int_{P_B}^{P_C} dP = C_V \frac{1}{nR} V_B (P_C - P_B)$

TO FIND P_C & P_B, NOTE $P_A = \frac{1}{V_A}(nRT_A)$

FROM ① $P_B = nRT_A/V_B$

ALONG $\overline{AC}$, $P = P_A (V/V_A)^{-\gamma} \Rightarrow P_C = P_A (V_B/V_A)^{-\gamma} = \frac{1}{V_A}(nRT_A)\left(\frac{V_B}{V_A}\right)^{-\gamma}$

USE $V_B = 2V_A \Rightarrow \Delta Q_{BC} = C_V \frac{1}{nR}(2V_A)\left[\frac{1}{V_A}(nRT_A) 2^{-\gamma} - \frac{1}{2}\frac{1}{V_A} nRT_A\right]$

$= C_V T_A \, 2[2^{-\gamma} - \frac{1}{2}]$

USE $\gamma = 7/5 \Rightarrow \Delta Q_{BC} = C_V T_A [2^{-2/5} - 1] = -\overbrace{C_V T_A (1 - 2^{-2/5})}^{\geq 0} \simeq -0.24\, C_V T_A$

THIS IS NEGATIVE. SO HEAT <u>LEAVES</u> THE SYSTEM.

d) EFFICIENCY = TOTAL WORK OVER HEAT INPUT: $\eta = \frac{\Sigma \Delta W}{\Sigma \Delta Q_{IN}}$

$\Delta W_{AB} = nRT_A \ln(V_B/V_A)$ (FROM ②

$\Delta W_{BC} = 0$

$\Delta W_{CA} = \int_{V_B}^{V_A} P_A (V/V_A)^{-\gamma} dV = P_A V_A \left[\frac{1}{1-\gamma}\left(\frac{V}{V_A}\right)^{1-\gamma}\right]_{V_B}^{V_A} = \frac{1}{1-\gamma}\overbrace{P_A V_A}^{= nRT_A}\left[1 - \left(\frac{V_B}{V_A}\right)^{1-\gamma}\right]$

USE $V_B = 2V_A \Rightarrow \Sigma \Delta W = nRT_A \ln(2) + \frac{1}{1-\gamma}(nRT_A)[1 - 2^{1-\gamma}]$

HEAT GOES OUT ALONG $\overline{BC}$ AND $\Delta Q = 0$ ALONG $\overline{CA}$ (ADIABATIC)

SO $\Sigma \Delta Q_{IN} = \Delta Q_{AB} = nRT_A \ln(2)$

SO $\eta = 1 + \frac{1}{1-\gamma}(1 - 2^{1-\gamma})/\ln(2)$ WITH $\gamma = 7/5$

$\simeq 1 - \frac{5}{2}(1 - 2^{-2/5})/\ln(2) \simeq 1 - 0.873$

$\Rightarrow \boxed{\eta \simeq 12.7\%}$

2e) $dS = \delta Q/T$

ON $\overline{AB}$ WE FOUND $\delta Q = PdV$

SO $dS = \frac{1}{T}(PdV)$ WITH $T = T_A$ CONSTANT

$\Rightarrow \Delta S_{AB} = \frac{1}{T_A}\Delta Q_{AB} = nR\,Ln(2)$

ON $\overline{BC}$ WE FOUND $\delta Q = dU = C_V dT$

SO $dS = \frac{1}{T}(C_V dT)$

$\Rightarrow \Delta S_{BC} = C_V \int_{T_B}^{T_C} \frac{1}{T}dT = C_V\,Ln(T_C/T_B)$

BECAUSE $\overline{AB}$ IS ISOTHERMAL $\Rightarrow T_B = T_A$

FOR T_C USE $T_C = \frac{1}{nR}P_C V_C$

$= \frac{1}{nR}\left[P_A\left(V_B/V_A\right)^{-\gamma}\right]\left(\overset{V_C}{\overset{\shortparallel}{V_B}}\right)$

$P_A = \frac{1}{V_A}nRT_A \Rightarrow T_C = T_A\left(V_B/V_A\right)^{1-\gamma}$

$\Rightarrow \Delta S_{BC} = C_V\,Ln\left[\left(V_B/V_A\right)^{1-\gamma}\right] = (1-\gamma)C_V\,Ln(V_B/V_A)$

$= (1-\gamma)C_V\,Ln(2)$

SIMPLIFY FURTHER USING $\gamma = \frac{C_P}{C_V} = \frac{C_P}{C_V}$ AND $nR = C_P - C_V$

$\Rightarrow \Delta S_{BC} = \left(1 - \frac{C_P}{C_V}\right)C_V\,Ln(2) = -nR\,Ln(2)$

ON $\overline{CA}$, $\delta Q = 0$ (ADIABATIC)

$\Rightarrow \Delta S_{CA} = 0$

COMBINING RESULTS, THE TOTAL CHANGE OF ENTROPY AROUND THE CLOSED PATH IS

$\sum\Delta S = \Delta S_{AB} + \Delta S_{BC} + \Delta S_{CA} = nR\,Ln(2) - nR\,Ln(2) + 0$

$= 0$, AS EXPECTED.

3a) IN TIME Δt THE HEAT ENTERING THE ROOM FROM OUTSIDE IS

$$\Delta Q = \Delta t \; K(T_o - T_i) \qquad (1)$$

TO KEEP THE ROOM AT CONSTANT TEMPERATURE, THE AIR CONDITIONER MUST REMOVE THE SAME HEAT IN TIME Δt.

WE KNOW THE "EFFICIENCY" OF A CARNOT REFRIDGERATOR IS $\Delta Q/\Delta W = T_i/(T_o - T_i)$

SO WORK REQUIRED IN TIME Δt IS

$$\Delta W = \Delta Q \,(T_o - T_i)/T_i$$

USE (1)

$$= \Delta t \; K\,(T_o - T_i)^2/T_i$$

SO POWER IS $P = \frac{\Delta W}{\Delta t} = K \frac{1}{T_i}(T_o - T_i)^2$

b) FOR $T_i = 21°C$ ($\sim 294K$) AND $T_o = 27°C$ ($\sim 300K$)

POWER IS

$$P_1 = \frac{K}{T_i}(27-21)^2$$

(OK TO WORK IN °C SINCE TAKING DIFFERENCE)

IF INSTEAD $T_o = 30°C$, BUT STILL HAVE $T_i = 21°C$,

POWER IS

$$P_2 = \frac{K}{T_i}(30-21)^2$$

SO RELATIVE INCREASE IN POWER IS

$$\frac{P_2 - P_1}{P_1} = \frac{(30-21)^2}{(27-21)^2} - 1 = \frac{9^2}{6^2} - 1 \simeq 1.25$$

SO PERCENT INCREASE IN POWER IS 125%