ASSIGNMENT 2: SOLUTIONS

1 a) FOR AN ISOTHERMAL PROCESS, WORK IS $W = nRT \ln(V_f/V_i)$
WHERE $nRT = P_i V_i$

$$\Rightarrow W = P_i V_i \ln(V_f/V_i)$$
$$\simeq (10^5 \text{ Pa})(10^{-4} \text{ m}^3) \ln(0.5)$$
$$\Rightarrow \boxed{W \simeq -6.9 \text{ J}}$$

b) FOR AN ADIABATIC PROCESS, $W = \frac{1}{1-\gamma}[P_f V_f - P_i V_i]$
WHERE $P_f = P_i (V_f/V_i)^{-\gamma}$ AND $\gamma \simeq 7/5$ FOR AIR
$$\simeq (10^5 \text{ Pa})(0.5)^{-7/5}$$
$$\simeq 2.64 \times 10^5 \text{ Pa}$$

So $W \simeq \frac{1}{1-7/5}[(2.64 \times 10^5 \text{ Pa})(0.5 \times 10^{-4} \text{ m}^3) - (10^5 \text{ Pa})(10^{-4} \text{ m}^3)]$
$$\simeq -\tfrac{5}{2}[3.2 \text{ J}]$$
$$\Rightarrow \boxed{W \simeq -8.0 \text{ J}}$$

c) $T_f = T_i (P_f/P_i)^x$
$$\simeq (293 \text{ K})[(2.64 \times 10^5 \text{ Pa})/(10^5 \text{ Pa})]^{2/7}$$
$$\simeq 387 \text{ K}$$
$$\Rightarrow \boxed{T_f \simeq 114\,^\circ\text{C}}$$

[ALTERNATELY USE $PV = nRT \Rightarrow nR = P_i V_i / T_i$
So $T_f = P_f V_f/(nR) = T_i (P_f V_f)/(P_i V_i) = T_i (P_f/P_i)(V_f/V_i)$
$$\simeq (293 \text{ K})(2.64)(0.5)$$
$\simeq 387$ K, AS ABOVE]

2 a) INTERFACE THICKNESS GROWS DIFFUSIVELY AS $\sigma = \sqrt{\kappa t}$ (BUT SEE BELOW*)

INITIALLY. (AT SOME NON-ZERO TIME t_0) $0.5\,cm = \sqrt{\kappa t_0}$

$\Rightarrow \kappa t_0 = 0.25\,cm^2$

AFTER AN HOUR $2.0\,cm = \sqrt{\kappa(t_0 + 3600s)} = \sqrt{\kappa t_0 + \kappa(3600s)}$

$= \sqrt{(0.25\,cm^2) + \kappa(3600s)}$

$\Rightarrow \kappa = [(2.0\,cm)^2 - (0.25\,cm^2)]/(3600s)$

$\Rightarrow \boxed{\kappa \simeq 0.0010\ cm^2/s}$

b) AFTER ANOTHER HOUR

$\sigma = \sqrt{\kappa(t_0 + 7200s)} = \sqrt{(0.25\,cm)^2 + \kappa(7200s)}$

$\simeq \sqrt{(0.25\,cm^2) + (0.0010\,cm^2/s)(7200s)}$

$\Rightarrow \boxed{\sigma \simeq 2.8\,cm}$

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*) SOME MAY CHOOSE TO DEFINE THE THICKNESS AS $\sigma = 2\sqrt{\kappa t}$, WHICH IS ALSO ACCEPTABLE.

SO IN 2a) $2\sqrt{\kappa t_0} = 0.5\,cm \Rightarrow \kappa t_0 = (0.25\,cm)^2$

AFTER AN HOUR $2.0\,cm = 2\sqrt{\kappa t_0 + \kappa(3600s)}$

$\Rightarrow \kappa = [(\frac{2.0}{2}\,cm)^2 - (\frac{0.5}{2}\,cm)^2]/(3600s)$

$= \frac{1}{4}(0.0010\,cm^2/s) = \underline{0.00025\,cm^2/s}$

IN 2b) $\sigma = 2\sqrt{\kappa(t_0 + 7200s)} = 2\sqrt{(0.25\,cm)^2 + (0.00025\,cm^2/s)(7200s)}$

$= \underline{2.8\,cm}$

3a) SPEED OF SOUND IS $c_s = \sqrt{\gamma R_g T}$

CO_2 IS NOT DIATOMIC. SO USE $\gamma = \frac{c_p}{c_v} = \frac{844}{655} \simeq 1.29$

GAS CONSTANT IS FOUND FROM $R_g = c_p - c_v = (844-655)\frac{J}{kg K}$

$\Rightarrow R_g \simeq 189\ J/(kg K)$

[ALTERNATELY USE MOLAR MASS $M_{CO_2} \simeq 44 g/mol$ AND USE $R_g = \frac{R}{M_{CO_2}}$]

So $c_s \simeq \sqrt{(1.29)(189 \frac{J}{kg K})(\underbrace{(467+273)}_{740K} K)}$

$\Rightarrow \boxed{c_s \simeq 425\ m/s}$

b) $\rho_0 = P/(R_g T) \simeq (9.00 \times 10^5\ Pa)/[(189 \frac{J}{kg K})(740 K)]$

$\Rightarrow \boxed{\rho_0 \simeq 6.4\ kg/m^3}$

c) IN AN ISOTHERMAL ATMOSPHERE $p = p_0 e^{-z/H_\rho}$ AND $\rho = \rho_0 e^{-z/H_\rho}$

WHERE $H_\rho = R_g T_0 / g$ IS THE DENSITY SCALE HEIGHT

$\Rightarrow H_\rho \simeq (189\ J/kg K)(740 K)/(8.9 m/s^2)$

$\simeq 1.6 \times 10^4\ m$ $\boxed{\simeq 16\ km}$

So $\rho = \rho_0 e^{-z/H_\rho} = (6.4 \frac{kg}{m^3}) \mathrm{EXP}[-z/(16\ km)]$

d) MASS $=$ AREA $\times \int_0^{z_{TOP}} \rho\, dz$ WHERE $z_{TOP} = 70\ km$, AREA $= 1 m^2$

$= (1 m^2)\, \rho_0 [-H_\rho e^{-z/H_\rho} |_0^{z_{TOP}}] = (1 m^2)\, \rho_0 H_\rho (1 - e^{-z_{TOP}/H_\rho})$

$\simeq (6.4\ kg/m^3)(1 m^2)(\underbrace{16 km}_{16 \times 10^3 m})(1 - \underbrace{e^{-70/16}}_{\simeq 0.013})$

$\Rightarrow$ MASS $\simeq 1.0 \times 10^5\ kg$

4a) FROM HYDROSTATIC BALANCE, $\frac{dp}{dz} = -\rho g$, AND THE EQUATION OF STATE, $p = \rho R_a T$, ELIMINATE ρ TO GET

$$\frac{dp}{dz} = -\frac{g}{R_a}\frac{1}{T(z)}p \qquad \text{WITH } p(0) = p_0$$

GENERAL SOLUTION IS

$$p(z) = p_0 \,\mathrm{EXP}\left[-\int_0^z \frac{g}{R_a}\frac{1}{T(\tilde z)}d\tilde z\right]$$
$$= p_0 \,\mathrm{EXP}\left[-\frac{g}{R_a}\int_0^z \frac{1}{T_0 - \Gamma\tilde z}d\tilde z\right]$$
$$= p_0 \,\mathrm{EXP}\left[+\frac{g}{R_a}\frac{1}{\Gamma}\underbrace{\left(\ln(T_0-\Gamma z) - \ln T_0\right)}_{=\ln(1-\Gamma z/T_0)}\right]$$

SO $p = p_0\left(1 - \Gamma z/T_0\right)^{g/(R_a\Gamma)}$

CAN SIMPLIFY USING $\Gamma = g/c_p \Rightarrow \frac{g}{R_a\Gamma} = \frac{c_p}{R_a} = \frac{c_p}{c_p - c_v} = \frac{\gamma}{\gamma-1} = \frac{1}{\kappa}$

SO $\boxed{p(z) = p_0\left(1 - \frac{\Gamma}{T_0}z\right)^{1/\kappa}}$ WITH $\kappa = 2/7$ FOR AIR

b) FROM $\frac{dp}{dz} = -\rho g$ AND SOLUTION IN a)

$$\rho = -\frac{1}{g}\left[p_0\frac{1}{\kappa}\left(1-\frac{\Gamma}{T_0}z\right)^{1/\kappa - 1}\left(-\frac{\Gamma}{T_0}\right)\right]$$
$$= \frac{1}{\kappa}\frac{p_0\Gamma}{gT_0}\left(1 - \frac{\Gamma}{T_0}z\right)^{1/\kappa - 1}$$

SIMPLIFY COEFFICIENT: $\frac{p_0\Gamma}{gT_0} = \frac{(\rho_0 R_a T_0)(g/c_p)}{gT_0} = \rho_0\frac{R_a}{c_p} = \rho_0\kappa$

$$\Rightarrow \boxed{\rho = \rho_0\left(1 - \frac{\Gamma}{T_0}z\right)^{\frac{1}{\kappa}-1}}$$

c) $\theta(z) = T(z)\left(p(z)/p_0\right)^{-\kappa}$

$$= (T_0 - \Gamma z)\left[\left(1 - \frac{\Gamma}{T_0}z\right)^{1/\kappa}\right]^{-\kappa}$$
$$= (T_0 - \Gamma z)\left(1 - \frac{\Gamma}{T_0}z\right)^{-1}$$

$\Rightarrow \boxed{\theta(z) = T_0}$ CONSTANT; AS EXPECTED FOR WELL-MIXED ATMOSPHERE.