1. A (non-insulated) cylinder with a movable piston initially contains air with density, ρ_1, and pressure, P_1. The piston is slowly raised simultaneously decreasing the density, ρ, and increasing the pressure, P, with heat passing through the walls of the cylinder so that at every instant P and ρ are related by the equation

$$P = k/\rho,$$

in which k is a constant.

a) Write an expression for the initial temperature of the gas, T_1.
[Give your answer in terms of P_1, ρ_1, and R_a.]

b) Express the constant k in P_1, T_1, and R_a.

c) Find an expression for the temperature when the density has halved to $\rho_1/2$.
[Give your answer in terms of T_1]

a) $P = \rho R_a T \Rightarrow \boxed{T_1 = \frac{P_1}{\rho_1 R_a}}$

b) $k = P_1 \rho_1 = P_1 \left(\frac{P_1}{R_a T_1} \right) = \boxed{\frac{P_1^2}{R_a T_1}}$

c) IF $\rho_2 = \frac{\rho_1}{2} \Rightarrow P_2 = k/\rho_2 = k/(\rho_1/2) = 2\,k/\rho_1$

USE b) $\Rightarrow P_2 = 2 \left(\frac{P_1^2}{R_a T_1} \right) \frac{1}{\rho_1} = 2P_1$

So $T_2 = \frac{P_2}{R_a \rho_2} = \frac{2P_1}{R_a(\rho_1/2)} = 4 \left(\frac{P_1}{R_a \rho_1} \right) = \boxed{4T_1}$

2. As a radiosonde balloon rises, it finds the temperature is constant (T_0) as the pressure changes from P_1 to $P_2 < P_1$. What is the vertical distance between the two pressure levels, P_2 and P_1?
[Give your answer in terms of T_0, P_1, P_2, R_a and g.]

$P = \rho R_a T_0$

USE HYDROSTATIC BALANCE $\quad \frac{dP}{dz} = -\rho g = -\frac{g}{R_a T_0} P$

$$\Rightarrow \frac{1}{P} dP = -\frac{g}{R_a T_0} dz$$

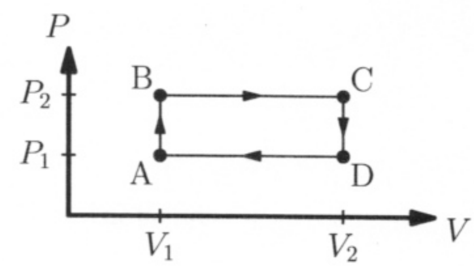
INTEGRATE BOTH SIDES FROM P_1 TO P_2 WHERE HEIGHT GOES FROM Z_1 TO Z_2

$$\Rightarrow \int_{P_1}^{P_2} \frac{1}{P} dP = -\underbrace{\frac{g}{R_a T_0}}_{\text{CONSTANT}} \int_{z_1}^{z_2} dz \Rightarrow \ln P \Big|_{P_1}^{P_2} = -\frac{g}{R_a T_0}(z_2 - z_1)$$

SO DISTANCE BETWEEN P_2 AND P_1 IS

$$Z_2 - Z_1 = -\frac{R_a T_0}{g}\left[\ln(P_2) - \ln(P_1)\right] = \boxed{\frac{R_a T_0}{g} \ln\left(P_1/P_2\right)}$$

3. Suppose an ideal gas operates in a heat engine with a 4-step cycle: isochoric for A→B and C→D; isobaric for B→C and D→A. Its PV diagram is shown to the right. The pressure along BC is twice the pressure along DA: $P_2 = 2P_1$. The volume along CD is three times the volume along AB: $V_2 = 3V_1$.



   a) What is the total work performed by this engine?

   b) What is the total heat input to the engine?

   c) Evaluate the efficiency of this engine.

[Give your answers in terms of any or all of P_1, V_1 and the adiabatic constant γ.]

a) WORK IS AREA OF PV DIAGRAM

$$W = (P_2 - P_1)(V_2 - V_1) = (2P_1 - P_1)(3V_1 - V_1) = \boxed{2P_1V_1}$$

b) HEAT IN AN ISOCHORIC PROCESS IS $\Delta Q = C_V \Delta T$

" " " ISOBARIC " IS $\Delta Q = C_P \Delta T$

$$Q_{AB} = C_V(T_B - T_A) \overset{PV=nRT}{=} C_V\left(\frac{P_2V_1}{nR} - \frac{P_1V_1}{nR}\right) = \frac{C_V}{nR} V_1 (P_2 - P_1)$$

$P_2 > P_1$, SO THIS IS HEAT IN

$$Q_{BC} = C_P(T_C - T_B) = C_P\left(\frac{P_2V_2}{nR} - \frac{P_2V_1}{nR}\right) = \frac{C_P}{nR} P_2 (V_2 - V_1)$$

$V_2 > V_1$, SO THIS IS ALSO HEAT IN

WHEN PRESSURE OR VOLUME DROPS HEAT GOES OUT ($Q_{CD} < 0$, $Q_{DA} < 0$)

SO TOTAL HEAT IN IS

$$Q_{IN} = \frac{C_V}{nR} V_1 (P_2 - P_1) + \frac{C_P}{nR} P_2 (V_2 - V_1)$$

USE $C_P - C_V = nR$ & $\gamma = \frac{C_P}{C_V}$ $\Rightarrow$ $\frac{C_V}{nR} = \frac{C_V}{C_P - C_V} = \frac{1}{\gamma - 1}$, $\frac{C_P}{nR} = \frac{C_P}{C_P - C_V} = \frac{\gamma}{\gamma - 1}$

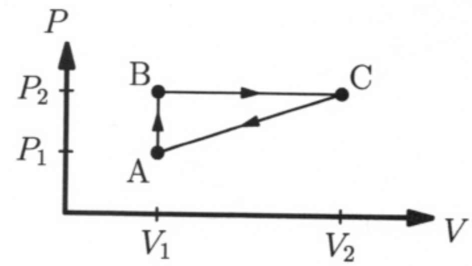
$$\Rightarrow Q_{IN} = \frac{1}{\gamma-1} V_1 (P_2 - P_1) + \frac{\gamma}{\gamma - 1} P_2 (V_2 - V_1) = \frac{1}{\gamma - 1} V_1 (2P_1 - P_1) + \frac{\gamma}{\gamma-1} 2P_1 (3V_1 - V_1)$$

$$= \frac{1}{\gamma - 1} V_1 P_1 + \frac{\gamma}{\gamma - 1} 4 V_1 P_1 = \boxed{\frac{1 + 4\gamma}{\gamma - 1} V_1 P_1}$$

c) EFFICIENCY $= \frac{W}{Q_{IN}} = \frac{2P_1V_1}{\left(\frac{1+4\gamma}{\gamma-1} P_1 V_1\right)} = \boxed{\frac{2(\gamma - 1)}{1 + 4\gamma}}$

4. Consider the triangular PV cycle shown to the right. Here, $P_2 = 2P_1$ and $V_2 = 3V_1$.

Find the change in entropy over each part of the cycle.
I.e. find ΔS_{AB} for A→B, ΔS_{BC} for B→C, and ΔS_{CA} for C→A.
[Give your answers in terms of C_v and the adiabatic constant γ.]



IN AN ISOCHORIC PROCESS $C_V dT = \delta Q = T dS \Rightarrow dS = C_V \frac{1}{T} dT$

So $\Delta S_{AB} = C_V \int_{T_A}^{T_B} \frac{1}{T} dT = C_V \ln T \Big|_{T_A}^{T_B} = C_V \ln(T_B/T_A)$

$$= C_V \ln\left[\frac{P_2V_1/nR}{P_1V_1/nR}\right] = C_V \ln\left(\frac{P_2}{P_1}\right) = \boxed{C_V \ln(2)}$$

IN AN ISOBARIC PROCESS $C_P dT = \delta Q = T dS \Rightarrow dS = C_P \frac{1}{T} dT$

So $\Delta S_{BC} = C_P \int_{T_B}^{T_C} \frac{1}{T} dT = C_P \ln(T_C/T_B) = C_P \ln\left(\frac{P_2V_2/nR}{P_2V_1/nR}\right)$

$$= C_P \ln\left(\frac{V_2}{V_1}\right) = \boxed{\gamma C_V \ln(3)}$$

CAN FIND ΔS_{CA} ONE OF 2 WAYS:

1) S IS A STATE VARIABLE $\Rightarrow \oint dS = 0 \Rightarrow \Delta S_{AB} + \Delta S_{BC} + \Delta S_{CA} = 0$

$\Rightarrow \Delta S_{CA} = -\Delta S_{AB} - \Delta S_{BC} = -C_V \ln(2) - \gamma C_V \ln(3)$

$$\boxed{\Delta S_{CA} = -C_V[\ln(2) + \gamma \ln(3)]}$$

2) $dU = T dS - P dV \Rightarrow dS = \frac{C_V dT}{T} + \frac{P}{T} dV$

BUT $\frac{P}{T} = \frac{nR}{V} \Rightarrow dS = C_V \frac{1}{T} dT + nR \frac{1}{V} dV$

$\Rightarrow \Delta S_{CA} = C_V \int_{T_C}^{T_A} \frac{1}{T} dT + nR \int_{V_C}^{V_A} \frac{1}{V} dV$

$$= C_V \ln\left(\frac{T_A}{T_C}\right) + nR \ln\left(\frac{V_A}{V_C}\right)$$

$$= C_V \ln\left(\frac{P_1V_1/nR}{P_2V_2/nR}\right) + (C_P - C_V) \ln\left(\frac{V_1}{V_2}\right)$$

$$= C_V\left[\ln\left(\tfrac{1}{6}\right) + \gamma \ln\left(\tfrac{1}{3}\right) - \ln\left(\tfrac{1}{3}\right)\right]$$

$$= C_V\left[\ln\left(\tfrac{1}{2}\right) + \gamma \ln\left(\tfrac{1}{3}\right)\right] = \boxed{-C_V[\ln(2) + \gamma \ln(3)]}$$