(5) APPLICATIONS OF THE FIRST LAW (SPECIFIC)

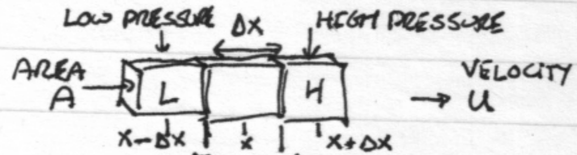
1] THE SPEED OF SOUND

THE CORRECT PREDICTION FOR THE SPEED OF SOUND REQUIRES THE THERMODYNAMICS OF ADIABATIC PROCESSES.

FIRST A DERIVATION OF THE EQUATION FOR SOUND PROPAGATION

i) FORCE LAW : $ma = \Sigma F$

$$\Rightarrow (\rho \Delta x A)\frac{\partial u}{\partial t} = AP\Big|_{x-\frac{\Delta x}{2}} - AP\Big|_{x+\frac{\Delta x}{2}}$$



$$\Delta x \to dx \Rightarrow \rho \frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} \qquad (1)$$

ii) MASS CONSERVATION : $\frac{\partial M}{\partial t} = \Delta(\text{AREA} \times \text{MASS FLUX})$

$$\Rightarrow (\Delta x A)\frac{\partial \rho}{\partial t} = A\rho u\big|_{x-\frac{\Delta x}{2}} - A\rho u\big|_{x+\frac{\Delta x}{2}}$$

$$\Delta x \to dx \Rightarrow \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) \qquad (2)$$

SIMPLIFYING ASSUMPTIONS

i) SUPPOSE DENSITY CHANGE IS SMALL COMPARED TO DENSITY ITSELF

$\Rightarrow \rho \frac{\partial u}{\partial t} \approx \rho_0 \frac{\partial u}{\partial t}$ (ρ_0 IS "CHARACTERISTIC" (UNPERTURBED) DENSITY)

AND $\frac{\partial}{\partial x}(\rho u) = \rho \frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} \simeq \rho_0 \frac{\partial u}{\partial x}$

ii) SUPPOSE PRESSURE IS ONLY A FUNCTION OF DENSITY (AS IN (8) ON p.25)

$\Rightarrow P = P(\rho) \Rightarrow \frac{\partial P}{\partial x} = \frac{dP}{d\rho}\frac{\partial \rho}{\partial x}$

SO

$$(1) \Rightarrow \rho_0 \frac{\partial u}{\partial t} \simeq -\left(\frac{dP}{d\rho}\right)\frac{\partial \rho}{\partial x} \qquad (1)'$$

$$(2) \Rightarrow \frac{\partial \rho}{\partial t} \simeq -\rho_0 \frac{\partial u}{\partial x} \qquad (2)'$$

DEFINE $C_S^2 \equiv \frac{dP}{d\rho}$ AND ASSUME APPROXIMATELY CONSTANT

THEN, ELIMINATING u FROM (1)' & (2)' GIVES

$$\frac{\partial^2 \rho}{\partial t^2} = C_S^2 \frac{\partial^2 \rho}{\partial x^2} \quad : \text{ THE WAVE EQUATION} \qquad (3)$$

SOLUTIONS GIVE $\rho(x,t) = f(x - C_S t) + g(x + C_S t)$ FOR ARBITRARY f & g.

SO $C_S = \sqrt{dP/d\rho}$ DESCRIBES SPEED OF RIGHT- AND LEFT-PROPAGATING DISTURB.

NEWTON USED $P = \rho R_0 T$ WITH T-CONSTANT TO PREDICT $C_S = \sqrt{R_0 T} \overset{\text{AT } T=300K}{\simeq} \sqrt{(287)(300K)} \simeq 290\,\text{m/s}$

NOW WE KNOW $P = P_0(\rho/\rho_0)^\gamma \Rightarrow \frac{dP}{d\rho} = (P_0/\rho_0^\gamma)\gamma\rho^{\gamma-1} = \gamma P/\rho = \gamma R_0 T$.

SO AT $T = 300K$ $C_S = \sqrt{\gamma R_0 T} \simeq \sqrt{(7/5)(287)(300)} \simeq 350\,\text{m/s}$ ✓

2] DIFFUSION OF HEAT

HEAT PASSES THROUGH SOLIDS BY CONDUCTION. THE EVOLUTION OF TEMPERATURE THROUGH A SOLID IS FOUND THROUGH RELATING (HEAT) ENERGY FLUX TO TEMPERATURE CHANGES AND THROUGH ENERGY CONSERVATION.

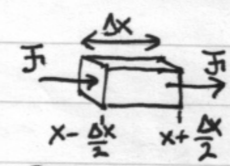
i) FICK'S LAW: ENERGY FLOWS FROM HOT TO COLD AT A RATE PROPORTIONAL TO THE TEMPERATURE DIFFERENCE:

IN 1D $$\mathcal{F} = -k\frac{\partial T}{\partial x}, \qquad (1)$$

IN WHICH k ~~IS~~ THE "THERMAL CONDUCTIVITY".

(eg FOR WATER $k \simeq 0.58\ \mathrm{W/m\cdot K}$ ... HIGHEST OF ALL BUT LIQUID METALS)

ii) ENERGY CONSERVATION: $\frac{\partial E}{\partial t} = \Delta(\text{AREA} \times \text{ENERGY FLUX})$



BUT $\Delta E \approx c_p(\rho\,\Delta x A)\,\Delta T$

$$\Rightarrow \frac{\partial E}{\partial t} \approx c_p(\rho\,\Delta x A)\frac{\partial T}{\partial t} = -Ak\frac{\partial T}{\partial x}\Big|_{x-\frac{\Delta x}{2}} - \left(-Ak\frac{\partial T}{\partial x}\right)\Big|_{x+\frac{\Delta x}{2}}$$

$$\Rightarrow \frac{\partial T}{\partial t} = \frac{k}{c_p\rho}\left[\frac{\partial T}{\partial x}\Big|_{x+\frac{\Delta x}{2}} - \frac{\partial T}{\partial x}\Big|_{x-\frac{\Delta x}{2}}\right]/\Delta x$$

$$\Delta x \to dx \quad \Rightarrow \quad \frac{\partial T}{\partial t} = \kappa_T\frac{\partial^2 T}{\partial x^2} \qquad (2)$$

WITH $\kappa_T \equiv \frac{k}{c_p\rho}$, THE "THERMAL DIFFUSIVITY".

(eg FOR WATER $\kappa_T = (0.58\ \frac{W}{m\cdot K})/[(4.2\times10^3\ \frac{J}{kg\,K})(1000\ \frac{kg}{m^3})]$
$\simeq 1.42\times10^{-7}\ m^2/s$)

(2) IS CALLED THE "HEAT EQUATION" (AKA "DIFFUSION EQUATION")

EXAMPLE: FOR A POINT OF HIGH TEMPERATURE AT ORIGIN AT $t=0$ IN AN UNBOUNDED MEDIUM, THE SOLUTION OF (1) IS

$$T(x,t) = \mathcal{C}\, t^{-1/2}\, \mathrm{EXP}\left[-x^2/(4\kappa_T t)\right], \qquad (3)$$

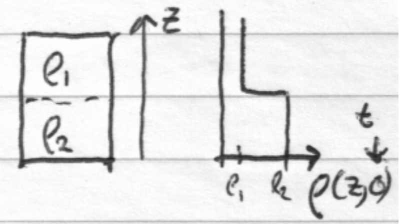
FOR A CONSTANT $\mathcal{C}$ THAT DEPENDS ON THE INITIAL ENERGY.

So HEAT SPREADS OUT WITH DISTANCE AS $\sigma = \sqrt{\kappa_T t}$

THIS CAN BE USED TO ESTIMATE THE TIME TO HEAT A SOLID: THE TIME TO HEAT A 5mm THICK IRON PAN ($\kappa_T = 2.3\times10^{-5}\ m^2/s$) IS ON THE ORDER $t \sim L^2/\kappa_T = (0.005m)^2/(2.3\times10^{-5}\ m^2/s) \simeq 1.1\,s$.

2] (CONT'D)

SALT DIFFUSION EXAMPLE



$$\frac{\partial \rho}{\partial t} = \kappa_S \frac{\partial^2 \rho}{\partial z^2} \quad (*) \text{ WITH } \kappa_S \text{ THE SALT DIFFUSIVITY}$$

$$\text{AND } \rho = \rho_0 [1 + \beta_S (S)]$$

EXPECT $\rho(z,t)$ IS "SELF-SIMILAR" SO THAT $\rho = f(\eta)$ WITH $\eta = \frac{z}{\sqrt{\kappa_S t}}$

$$\text{So } \frac{\partial \rho}{\partial t} = \frac{df}{d\eta}\frac{\partial \eta}{\partial t} = -\frac{1}{2}\frac{z}{\sqrt{\kappa_S}} t^{-3/2} f' = -\frac{1}{2}\frac{1}{t}\eta f'$$

$$\frac{\partial \rho}{\partial z} = \frac{df}{d\eta}\frac{\partial \eta}{\partial z} = \frac{1}{\sqrt{\kappa_S t}} f'$$

$$\frac{\partial^2 \rho}{\partial z^2} = \frac{1}{\sqrt{\kappa_S t}}\left[\frac{1}{\sqrt{\kappa_S t}} f''\right]$$

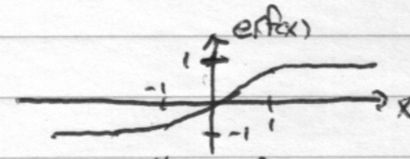
$$\text{HENCE } (*) \Rightarrow -\frac{1}{2}\frac{1}{t}\eta f' = \kappa_S \left[\frac{1}{\kappa_S t} f''\right]$$

$$\Rightarrow f'' = -\frac{1}{2}\eta f'$$

$$\text{LET } g = f' \Rightarrow g' = -\frac{1}{2}\eta g$$

$$\Rightarrow g = \mathcal{C} e^{-\eta^2/4} \qquad \mathcal{C} \text{ - INTEGRATION CONSTANT}$$

$$\text{So } f = \int_0^{\eta} \mathcal{C} e^{-\tilde{\eta}^2/4}\, d\tilde{\eta} + \mathcal{C}_2$$



THIS LOOKS LIKE AN ERROR FUNCTION; $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\tilde{x}^2} d\tilde{x}$

LET $\hat{\eta} = \tilde{\eta}/2 \Rightarrow f = 2\mathcal{C}\int_0^{\eta/2} e^{-\hat{\eta}^2} d\hat{\eta} + \mathcal{C}_2 = 2\mathcal{C}\frac{\sqrt{\pi}}{2}\mathrm{erf}(\eta/2) + \mathcal{C}_2$

$\Rightarrow f(\eta) = C_1 \mathrm{erf}(\eta/2) + \mathcal{C}_2$ ($C_1 = 2\mathcal{C}\frac{\sqrt{\pi}}{2}$ ANOTHER CONSTANT)

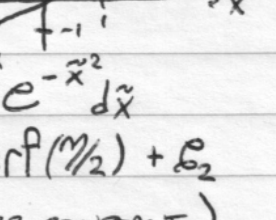
FIND CONSTANTS SUPPOSING $\rho(z \to \infty, t) = \rho_1$, $\rho(z \to -\infty, t) = \rho_2$

$\Rightarrow f(\eta \to \infty) = \rho_1$, $f(\eta \to -\infty) = \rho_2 \Rightarrow C_1 + C_2 = \rho_1$, $-C_1 + C_2 = \rho_2$

$\Rightarrow C_1 = \frac{\rho_1 - \rho_2}{2}$, $C_2 = \frac{\rho_1 + \rho_2}{2}$

$$\text{So } f(\eta) = \tfrac{1}{2}(\rho_1 - \rho_2)\mathrm{erf}(\eta/2) + \tfrac{1}{2}(\rho_1 + \rho_2)$$

$$\Rightarrow \rho(z,t) = \tfrac{1}{2}(\rho_1 + \rho_2) - \tfrac{1}{2}(\rho_2 - \rho_1)\mathrm{erf}\left(\frac{1}{2}\frac{z}{\sqrt{\kappa_S t}}\right)$$



FOR NaCl SOLUTION, $\kappa_S \simeq 2 \times 10^{-5}\ \mathrm{cm^2/s}$

SO FOR A THIN INTERFACE TO THICKEN TO $d = 8\,\mathrm{cm}$ TAKES $t = \frac{1}{\kappa_S}\left(\frac{d}{2}\right)^2 = \cdots = 10$ DAYS

3] POTENTIAL TEMPERATURE

WE HAVE ALREADY FOUND THAT IN AN ADIABATIC PROCESS
$T = T_0 (P/P_0)^{\varkappa}$, (WITH $\varkappa = 2/7$ FOR AIR).

DEFINE "<u>POTENTIAL TEMPERATURE</u>" : $\theta \equiv T(P/P_0)^{-\varkappa} \quad (= T_0)$. ①

THIS IS THE TEMPERATURE THAT GAS WITH TEMPERATURE T AND PRESSURE P WOULD HAVE IF BROUGHT ADIABATICALLY TO A HEIGHT WHERE THE PRESSURE IS P_0.

IN ATMOSPHERIC SCIENCE, IT IS CONVENTIONAL TO SET $P_0 = 1$ BAR ($= 1000$ mBAR)
SO θ IS TEMPERATURE AIR WOULD HAVE IF BROUGHT TO GROUND.

EXAMPLE: 1) DURING A CHINOOK WIND EVENT, AIR WITH TEMPERATURE $-20°C$ AT 500 mBAR (ABOUT 5 km UP) IS BROUGHT TO GROUND WHERE PRESSURE IS $P_0 = 1000$ mBAR. WHAT IS ITS TEMPERATURE AT GROUND?

SOL^{N}: POTENTIAL TEMPERATURE IS $\theta \simeq (273-20)\left(\frac{500\text{ mBAR}}{1000\text{ mBAR}}\right)^{-2/7}$

$$\Rightarrow \theta \simeq (253)(0.5)^{-2/7} \simeq 308\,K \simeq 35°C$$

BY CONSTRUCTION THIS IS TEMPERATURE AT GROUND//

2) THE SAME PARCEL DESCENDS ONLY TO 950 mBAR (ABOUT 400 m UP). WHAT IS TEMPERATURE?

SOL^{N}: WE ALREADY FOUND $\theta = 308\,K$, WHICH DOESN'T CHANGE DURING ENTIRE DESCENT. SO

$$308 = T\left(950\text{ mBAR}/1000\text{ mBAR}\right)^{-2/7}$$

$$\Rightarrow T \simeq 308\left(\frac{950}{1000}\right)^{2/7} \simeq 304\,K \simeq 31°C //$$

BECAUSE θ DOESN'T CHANGE DURING THE ADIABATIC MOTION OF AIR, IT IS A FUNDAMENTAL QUANTITY IN ATMOSPHERIC MODELLING. BECAUSE POSITION CHANGES IN TIME, ONE WRITES

$$\frac{d}{dt}\theta(t, x(t), y(t), z(t)) = \frac{\partial\theta}{\partial t} + \frac{dx}{dt}\frac{\partial\theta}{\partial x} + \frac{dy}{dt}\frac{\partial\theta}{\partial y} + \frac{dz}{dt}\frac{\partial\theta}{\partial z} = \frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial x} + \frac{\partial\theta}{\partial z} = 0$$

SUCCINTLY, DEFINE $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$. $\Rightarrow \boxed{\frac{D\theta}{Dt} = 0}$ ②

THE "MATERIAL DERIVATIVE".

4] POTENTIAL DENSITY AND STABILITY

AS WITH POTENTIAL TEMPERATURE, WE CAN USE $P = P_0 (\rho/\rho_0)^{\gamma}$ TO DEFINE THE POTENTIAL DENSITY $\rho_{POT} \equiv \rho (P/P_0)^{-1/\gamma}$ $(= \rho_0)$ IN WHICH eg $\gamma = 7/5$ FOR AIR.

THIS CAN BE USED TO ASSESS THE STABILITY OF THE ATMOSPHERE TO BUOYANT CONVECTION.

EXAMPLE: AN AIR PARCEL AT THE GROUND ($P_0 = 1000$ mBAR) HAS DENSITY $1.30\,kg/m^3$. IT IS CARRIED ADIABATICALLY UPWARDS 1km WHERE THE PRESSURE IS 900mBAR AND THE SURROUNDING AIR HAS DENSITY $1.25\,kg/m^3$. IS THE PARCEL BUOYANT?

SOL$^{\underline{N}}$: THE POTENTIAL DENSITY OF THE PARCEL IS $\rho_{POT} = 1.30 \left(\frac{P_0}{P_0}\right)^{-1/\gamma} = 1.30\,kg/m^3$

TO ASSESS STABILITY WE CAN TAKE ONE OF 2 APPROACHES

i) CALCULATE DENSITY OF AIR PARCEL AT 900 mBAR:

$$1.30 = \rho\,(900/1000)^{-1/(7/5)}$$

$$\Rightarrow \rho = 1.30\,(0.900)^{5/7} \simeq 1.21\,kg/m^3$$

THIS IS LIGHTER THAN SURROUNDING AIR, SO PARCEL IS BUOYANT AND WILL CONTINUE TO RISE//

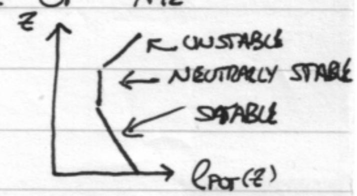
ii) CALCULATE POTENTIAL DENSITY OF AIR AT 900mBAR:

$$\rho_{POT,900} = 1.25\,(900/1000)^{-5/7} \simeq 1.35\,kg/m^3 > \rho_{POT}$$

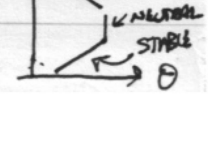
BECAUSE THE POTENTIAL DENSITY OF AIR ALOFT IS LARGER THAN POTENTIAL DENSITY OF AIR PARCEL, THE PARCEL IS BUOYANT//

FROM A SNAPSHOT OF THE ATMOSPHERE, CAN MEASURE DENSITY AND PRESSURE AS A FUNCTION OF HEIGHT. AND SO CAN COMPUTE A PROFILE OF THE POTENTIAL DENSITY: $\rho_{POT}(z) = \rho(z)\left[P(z)/P_0\right]^{-1/\gamma}$.

THE ATMOSPHERE IS STABLE IF $\frac{d\rho_{POT}}{dz} \leq 0$

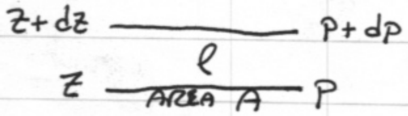


IN PRACTISE, STABILITY IS USUALLY ASSESSED BY POTENTIAL TEMPERATURE. SINCE $\rho_{POT}/\rho_0 = T_0/\theta$ [EXERCISE] HAVE STABILITY IF $\frac{d\theta}{dz} \geq 0$.



5] THE ISOTHERMAL ATMOSPHERE (GOOD FOR STRATOSPHERE)

SUPPOSE $T(z) = T_0$ AND ASSUME NO MOTION SO THAT PRESSURE CHANGES ONLY BECAUSE OF THE WEIGHT OF AIR ABOVE IT:



WEIGHT OF FLUID IN SLAB IS $(\rho A dz)g$

SO FORCE OVER AREA A AT z IS GREATER THAN FORCE AT $z+dz$ BY WEIGHT $(\rho A dz)g$. $\Rightarrow Ap - A(p+dp) = (\rho A dz)g$

$$\Rightarrow dp = -\rho g \, dz$$

I.e $$\boxed{\frac{dP}{dz} = -\rho g} \quad \text{"HYDROSTATIC BALANCE"} \qquad (1)$$

BEING ISOTHERMAL, AT EACH HEIGHT $P = \rho R_a T_0$

SO $(1) \Rightarrow \frac{dP}{dz} = -\left(\frac{P}{R_a T_0}\right) g = -\left(\frac{g}{R_a T_0}\right) P = -\frac{1}{H_p} P \qquad (2)$

IN WHICH $H_p \equiv R_a T_0 / g$ (A CONSTANT) IS THE "PRESSURE SCALE HEIGHT"

Eg. FOR $T_0 = 250K$, $H_p \approx (287 \frac{J}{kg K})(250K)/(9.8 m/s^2) \simeq 7.3 km$.

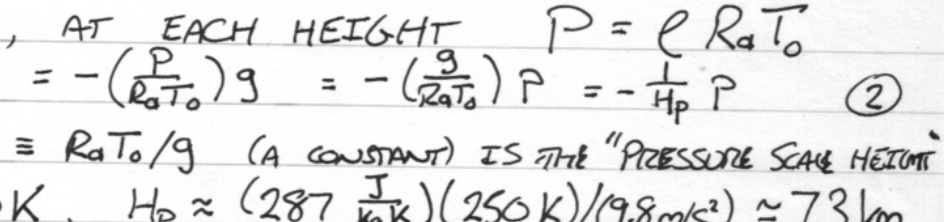
SOLVING (2) GIVES $P = P_0 e^{-z/H_p}$, WHERE P_0 IS PRESSURE AT $z=0$. (THIS COULD BE THE GROUND OR ARBITRARILY HIGHER UP)

USING $P = \rho R_a T_0 \Rightarrow \rho = \rho_0 e^{-z/H_p}$

ALSO USING $\theta = T(P/P_0)^{-\kappa} = T_0 (e^{-z/H_p})^{-\kappa} = T_0 e^{z/H_\theta}$

WHERE $H_\theta = H_p/\kappa$ IS POTENTIAL TEMPERATURE SCALE HEIGHT (AT 250K, $H_\theta \simeq \frac{7.3km}{2/7} \simeq 26 km$)

TYPICAL PROFILES LOOK LIKE



SO THE AIR GETS "THINNER" DIRECTLY AS A CONSEQUENCE OF HYDROSTATIC BALANCE AND EQUATION OF STATE.

NOTE $\theta(z)$ IS AN INCREASING FUNCTION ($\frac{d\theta}{dz} > 0$ FOR ALL z).

SO AN ISOTHERMAL ATMOSPHERE IS <u>VERY</u> STABLE. VERTICAL MOTION IS INHIBITED AN FLUID FLOWS IN HORIZONTAL LAYERS ... HENCE <u>STRATO</u>SPHERE

6] THE NEUTRALLY STABLE ATMOSPHERE (GOOD FOR TROPOSPHERE)

THE ATMOSPHERE IS NEUTRALLY STABLE IF $\theta(z)$ IS CONSTANT [THE EQUIVALENT FOR WATER IS HAVING UNIFORM DENSITY. CONVECTION OCCURS IF HEATED BELOW OF COOLED ABOVE].

USING $\theta(z) = T(z)\left(P(z)/P_0\right)^{-\varkappa}$, THE CONDITION FOR NEUTRAL STABILITY IS

$$0 = \frac{d\theta}{dz} = \frac{dT}{dz}\left(\frac{P}{P_0}\right)^{-\varkappa} + T\left[-\varkappa\left(\frac{P}{P_0}\right)^{-\varkappa-1}\frac{1}{P_0}\frac{dP}{dz}\right]$$

$$\times\left(\frac{P}{P_0}\right)^{\varkappa} \Rightarrow \quad 0 = \frac{dT}{dz} - \varkappa T\frac{1}{P}\frac{dP}{dz}$$

NOW INVOKE HYDROSTATIC BALANCE : $\frac{dP}{dz} = -\rho g$

$$\Rightarrow \quad 0 = \frac{dT}{dz} - \varkappa\frac{T}{P}(-\rho g)$$

AND USE EQUATION OF STATE $P = \rho R_a T$

$$\Rightarrow \quad 0 = \frac{dT}{dz} + \varkappa g/R_a$$

FINALLY, RECALL $\varkappa \equiv 1 - 1/\gamma$, $\gamma \equiv C_P/C_V$, $C_P - C_V = R_a$

$$\text{SO} \quad \frac{dT}{dz} = -g\frac{\varkappa}{R_a} = -g\left(1-\frac{C_V}{C_P}\right)/(C_P - C_V) = -g/C_P \qquad (1)$$

DEFINE $\boxed{\Gamma \equiv g/C_P}$ TO BE THE "(DRY) ADIABATIC LAPSE RATE"

THUS WE HAVE FOUND THAT IN A NEUTRALLY STABLE ATMOSPHERE THE TEMPERATURE DECREASES LINEARLY WITH HEIGHT

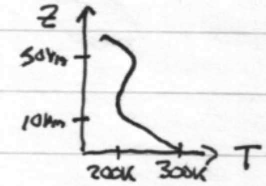
$$\boxed{T(z) = T_0 - \Gamma z} \qquad (2)$$

FOR AIR, $\Gamma \simeq (9.8\ m/s^2)/(1005\ J/(kg\,K)) \simeq 9.8\times10^{-3}\ K/m$
$= 9.8\ K/km.$

THIS IS WHY, EVEN THOUGH HOT AIR RISES, IT IS COLD ON MOUNTAIN TOPS. IN A 'WELL-STIRRED' TROPOSPHERE, RISING AIR EXPANDS AND COOLS BY ~ 10°C FOR EVERY KILOMETER. EVEN IN SUMMER AT 20°C ON GROUND CAN HAVE SNOW AT TOP OF 2000m MOUNTAIN

6A] SCALE HEIGHTS IN GENERAL

SUPPOSE $T(z)$ IS THE OBSERVED TEMPERATURE PROFILE OF THE ATMOSPHERE.



HYDROSTATIC BALANCE GIVES $\frac{dp}{dz} = -\rho g = -\frac{p}{R_a T} g$

SOLVING GIVES $p(z) = p_0 \,\mathrm{EXP}\left[-\int_0^z \frac{g}{R_a T(\tilde z)} d\tilde z\right] = p_0 \,\mathrm{EXP}\left[-\int_0^z \frac{1}{H_p(\tilde z)} d\tilde z\right]$

WHERE $\boxed{H_p(z) = \frac{R_a}{g} T(z)}$ (1) IS PRESSURE SCALE HEIGHT

(EXPLICIT DEFINITION IS $H_p \equiv -\left(\frac{1}{p}\frac{dp}{dz}\right)^{-1}$)

NOW FIND DENSITY SCALE HEIGHT $H_\rho \equiv -\left(\frac{1}{\rho}\frac{d\rho}{dz}\right)^{-1}$ IN TERMS OF T USING EQ$^{\underline{N}}$ OF STATE $\rho = \frac{p}{R_a T}$ AND HYDROSTATIC BALANCE $\frac{dp}{dz} = -\rho g$

$$\frac{d\rho}{dz} = \frac{1}{R_a}\left[-\frac{p}{T^2}\frac{dT}{dz} + \frac{1}{T}\frac{dp}{dz}\right] = -\frac{\rho}{T}\frac{dT}{dz} + \frac{1}{R_a T}(-\rho g)$$

$$= -\frac{\rho}{T}\left[\frac{dT}{dz} + \frac{g}{R_a}\right]$$

$\Gamma = \frac{\kappa g}{R_a} = g/c_p$ IS ADIABATIC LAPSE RATE

SO $\boxed{H_\rho = \left[\frac{1}{T}\left(\frac{dT}{dz} + \frac{1}{\kappa}\Gamma\right)\right]^{-1}}$ (2)

NOW FIND POTENTIAL TEMPERATURE SCALE HEIGHT $H_\theta \equiv +\left(\frac{1}{\theta}\frac{d\theta}{dz}\right)^{-1}$ IN TERMS OF T USING DEFINITION $\theta = T(p/p_0)^{-\kappa}$

$$\frac{d\theta}{dz} = \left(\frac{p}{p_0}\right)^{-\kappa}\frac{dT}{dz} + T(-\kappa)\left(\frac{p}{p_0}\right)^{-\kappa-1}\frac{1}{p_0}\frac{dp}{dz}$$

$$= \frac{1}{T}\theta\frac{dT}{dz} - \kappa\,\theta\frac{1}{p}\frac{dp}{dz} = \theta\left[\frac{1}{T}\frac{dT}{dz} - \kappa\frac{1}{p}(-\rho g)\right]$$

$$= \frac{\theta}{T}\left[\frac{dT}{dz} + \frac{\kappa g}{R_a}\right]$$

SO $\boxed{H_\theta = \left[\frac{1}{T}\left(\frac{dT}{dz} + \Gamma\right)\right]^{-1}}$ (3)

NOTE THAT THE THREE SCALE HEIGHTS ARE INTER-RELATED:

$$\frac{1}{H_\theta} = \frac{1}{T}\frac{dT}{dz} + \frac{1}{T}\Gamma = \frac{1}{T}\frac{dT}{dz} + \frac{1}{\kappa}\frac{\Gamma}{T} + \frac{\Gamma}{T}\left(1 - \frac{1}{\kappa}\right)$$

$$= \frac{1}{H_\rho} + \frac{g}{R_a T}(\kappa - 1)$$ (USING $\Gamma = \frac{\kappa g}{R_a}$)

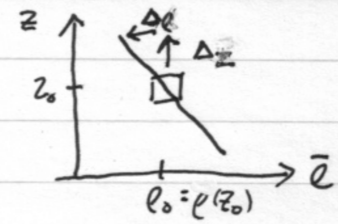
$\Rightarrow \boxed{\frac{1}{H_\theta} = \frac{1}{H_\rho} - \frac{1}{\gamma}\frac{1}{H_p}}$ (4) (USING $\gamma = \frac{1}{1-\kappa}$)

IN THE TROPOSPHERE $\frac{dT}{dz} \approx -\Gamma$ SO (3) $\Rightarrow H_\theta \to \infty$

IN THE STRATOSPHERE (NEARLY ISOTHERMAL) $H_p \approx H_\rho \Rightarrow H_\theta \approx H_p \frac{\gamma}{\gamma - 1} = \frac{1}{\kappa} H_p \approx \frac{7}{2} H_p \approx 20\text{km}$

6 B] THE BUOYANCY FREQUENCY

IN A "STRATIFIED" LIQUID ITS DENSITY DECREASE WITH HEIGHT DUE TO DECREASING SALINITY OR INCREASING TEMPERATURE. IF A FLUID PARCEL IS VERTICALLY DISPLACED IT EXPERIENCES AN OPPOSING BUOYANCY FORCE. THE RESULTING MOTION CAN BE EXPRESSED BY NEWTON'S LAW:



$$\rho_0 \frac{d^2 \Delta_z}{dt^2} = -g \Delta_\rho \qquad (1)$$

WHERE Δ_ρ IS THE DENSITY DIFFERENCE BETWEEN THE PARCEL AND THE AMBIENT AT $z = z_0 + \Delta_z$: $\Delta_\rho = \rho_0 - \rho(z_0 + \Delta_z)$

TAYLOR SERIES $\Rightarrow \simeq \rho_0 - \left[\rho(z_0) + \frac{d\rho}{dz}\Big|_{z_0} \Delta_z \right]$

$= - \frac{d\rho}{dz}\Big|_{z_0} \Delta_z$

So (1) $\Rightarrow \rho_0 \frac{d^2 \Delta_z}{dt^2} = \left(g \rho'(z_0) \right) \Delta_z$

$\Rightarrow \frac{d^2 \Delta_z}{dt^2} + \left(-\frac{g}{\rho_0} \rho'(z_0) \right) \Delta_z = 0$

$\Rightarrow \frac{d^2 \Delta_z}{dt^2} + N^2 \Delta_z = 0 \Leftarrow$ THE SPRING EQUATION

WITH $N^2 = -\frac{g}{\rho} \frac{d\bar{\rho}}{dz}\Big|_{z=z_0}$

SOLUTIONS ARE $\Delta_z(t) = A \cos Nt + B \sin Nt$, A,B CONSTANTS

SO THE PARCEL OSCILLATES VERTICALLY WITH THE "BUOYANCY FREQUENCY": $\boxed{N = \sqrt{\frac{-g}{\rho} \frac{d\rho}{dz}}} \simeq \sqrt{\frac{-g}{\rho_{00}} \frac{d\rho}{dz}}$ WHERE ρ_{00} IS CHARACTERISTIC DENSITY

THE BUOYANCY PERIOD IS

$$T_B = \frac{2\pi}{N}$$

IN STABLE STRATIFICATION $\frac{d\bar{\rho}}{dz} < 0$ SO N IS REAL-VALUED.

EXAMPLE: A GLASS OF SALT WATER HAS DENSITY DECREASE LINEARLY FROM $\rho = 1.03\, g/cm^3$ TO $1.00\, g/cm^3$ OVER 15 cm.

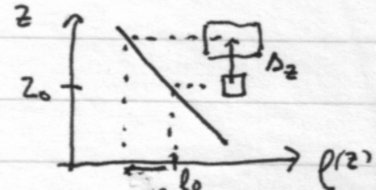
So $N \simeq \left[-(g/\rho_{00}) \frac{d\rho}{dz} \right] = \left[-(980\, cm/s^2)/(1.00\, g/cm^3) \left(\frac{1.00 - 1.03}{15} \frac{g/cm^3}{cm} \right) \right]^{1/2}$

$\simeq \left[1.96 \frac{1}{s^2} \right]^{1/2} \simeq 1.40\, s^{-1}$

So $T_B = 4.49\, s$ //

6B] (CONT'D)

NOW ADAPT THIS TO A GAS ACCOUNTING FOR DENSITY DECREASE OF A FLUID PARCEL AS IT MOVES UPWARD INTO LOWER PRESSURE.



DURING ADIABATIC ASCENT, DENSITY CHANGES WITH PRESSURE ACCORDING TO $\rho = \rho_0 (P/P_0)^{1/\gamma}$

SO THE DENSITY OF THE DISPLACED PARCEL IS

$$\rho_{PARCEL} = \rho_0 \left[\frac{P(z_0 + \Delta z)}{P(z_0)} \right]^{1/\gamma} \simeq \rho_0 \left[1 + \frac{1}{\gamma} \frac{1}{P_0} P'(z_0) \Delta z \right]$$

i.e. $(\rho_{PARCEL} - \rho_0)/\rho_0 \simeq \frac{1}{\gamma} \frac{P'(z_0)}{P_0} \Delta z = -\frac{1}{\gamma} \frac{1}{H_P} \Delta z$

WHERE $H_P \equiv -\left(\frac{1}{P}\frac{dP}{dz}\right)^{-1}$ IS THE PRESSURE SCALE HEIGHT

THE DENSITY OF THE SURROUNDING AIR AT $z_0 + \Delta z$ IS

$$\rho_{BKGD} = \rho(z_0 + \Delta z) \simeq \rho(z_0) + \rho'(z_0)\Delta z$$

i.e $(\rho_{BKGD} - \rho_0)/\rho_0 \simeq \frac{1}{\rho_0} \rho'(z_0) \Delta z = -\frac{1}{H_\rho} \Delta z$

WHERE $H_\rho \equiv -\left(\frac{1}{\rho}\frac{d\rho}{dz}\right)^{-1}$ IS THE DENSITY SCALE HEIGHT

SO THE RELATIVE DENSITY DIFFERENCE BETWEEN THE PARCEL & BKGD IS

$$\Delta_\rho/\rho_0 = (\rho_{PARCEL} - \rho_{BKGD})/\rho_0 \simeq \left(\frac{1}{H_\rho} - \frac{1}{\gamma}\frac{1}{H_P}\right)\Delta z$$

USING THE RELATIONSHIP BETWEEN SCALE HEIGHTS (eqn 4 on p. 31a)

$$\Rightarrow \quad \Delta_\rho/\rho_0 = \Delta z / H_\theta$$

IN WHICH $H_\theta \equiv \left(\frac{1}{\theta}\frac{d\theta}{dz}\right)^{-1}$ IS THE POTENTIAL TEMPERATURE SCALE HEIGHT

SO NEWTON'S LAW GIVES $\rho_0 \frac{d^2\Delta z}{dt^2} = -g\Delta_\rho = -\rho_0 g \frac{1}{H_\theta} \Delta z$

$$\Rightarrow \quad \frac{d^2\Delta z}{dt^2} + N^2 \Delta z = 0$$

IN WHICH THE BUOYANCY FREQUENCY IS NOW DEFINED BY

$$N = \sqrt{g/H_\theta} = \left[\frac{g}{\theta}\frac{d\theta}{dz}\right]^{1/2}$$

IN THE STRATOSPHERE A TYPICAL VALUE IS $N \simeq \sqrt{\frac{9.8\,m/s^2}{20{,}000\,m}} \simeq 0.02\,s^{-1}$

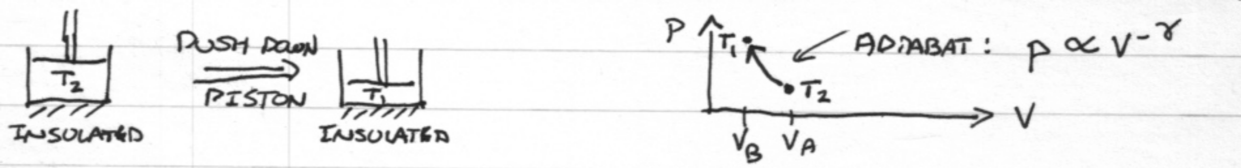
THE CORRESPONDING BUOYANCY PERIOD IS $T_B = \frac{2\pi}{N} \simeq 300s \sim 5$ MINUTES

7] HEAT ENGINES: THE CARNOT CYCLE

A HEAT ENGINE USES HEAT TO CREATE WORK. THERE ARE MANY THEORETICAL AND PRACTICAL MANIFESTATIONS OF SUCH ENGINES. BUT THE MOST EFFICIENT IS THE IDEALIZATION DUE TO CARNOT (1796-1832) WHO CONCEIVED OF A REVERSIBLE CYCLE ALTERNATING BETWEEN ADIABATIC AND ISOTHERMAL CHANGES TO A CLOSED SYSTEM CONTAINED BY A PISTON AND IN CONTACT WITH DIFFERENT THERMAL BATHS.

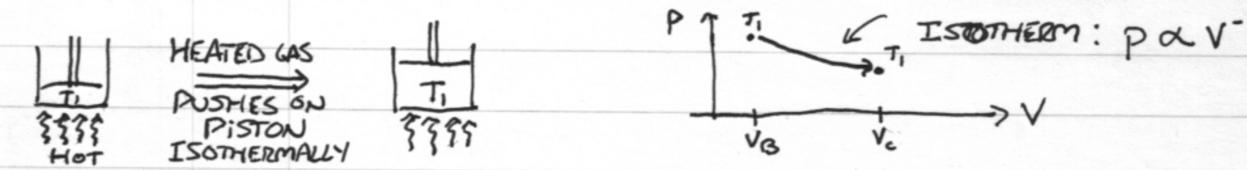
THE 4 STEPS OF THE CYCLE ARE REPRESENTED ON A 'PV'-DIAGRAM:

i)



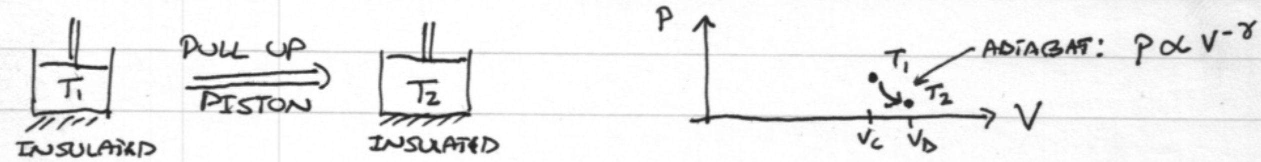
GAS HEATS AS IT IS ADIABATICALLY COMPRESSED

ii)



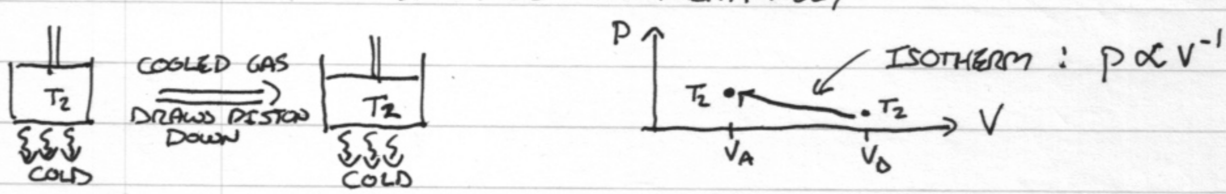
GAS PUSHES ON PISTON MAINTAINING CONSTANT T BY ABSORBING HEAT

iii)



GAS COOLS AS IT EXPANDS ADIABATICALLY

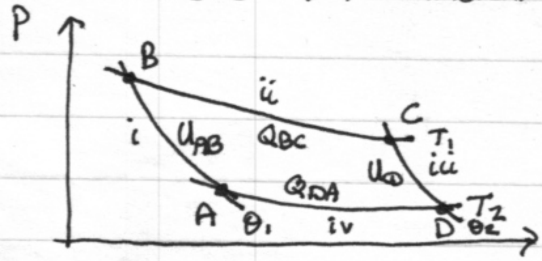
iv)



GAS PULLS ON PISTON MAINTAINING CONSTANT T WHILE LOSING HEAT

7] (cont'd)

THE CARNOT CYCLE IS SUMMARIZED BY SHOWING ALL 4 STEPS ON ON PV-DIAGRAM



$$\left.\begin{aligned} &\text{ALONG AB: } \theta=\theta_1 \Rightarrow P = P_A\left(\frac{V}{V_A}\right)^{-\gamma} \\ &\text{ALONG BC: } T=T_1 \Rightarrow P = P_B\left(\frac{V}{V_B}\right)^{-1} \\ &\text{ALONG CD: } \theta=\theta_2 \Rightarrow P = P_C\left(\frac{V}{V_C}\right)^{-\gamma} \\ &\text{ALONG DA: } T=T_2 \Rightarrow P = P_D\left(\frac{V}{V_D}\right)^{-1} \end{aligned}\right\} \quad (1)$$

THE WORK DONE IN EACH STEP IS $\int p\,dV$. THE TOTAL WORK IS THE AREA OF $\overline{ABCD}$. WE WILL CALCULATE THIS EXPLICITLY AND, IN PROCESS, FIND THE RESULT CAN BE FOUND INTUITIVELY.

FIRST NOTE HOW VOLUMES V_A, V_B, V_C AND V_D ARE RELATED USING (1)

$$P_A\left(\frac{V_A}{V_B}\right)^{\gamma} = P_B = \left(\frac{V_C}{V_B}\right)P_C = \left(\frac{V_C}{V_B}\right)\left[\left(\frac{V_D}{V_C}\right)^{\gamma}P_D\right] = \left(\frac{V_C}{V_B}\right)\left(\frac{V_D}{V_C}\right)^{\gamma}\left[\left(\frac{V_A}{V_D}\right)P_A\right]$$

$$\Rightarrow \left(\frac{V_A}{V_B}\right)^{\gamma} = \left(\frac{V_C}{V_B}\right)\left(\frac{V_D}{V_C}\right)^{\gamma}\left(\frac{V_A}{V_D}\right) \Rightarrow \left(\frac{V_C V_A}{V_B V_D}\right)\left(\frac{V_D V_B}{V_C V_A}\right)^{\gamma} = 1 \Rightarrow \left(\frac{V_D V_B}{V_C V_A}\right) = 1$$

So $\boxed{V_C/V_B = V_D/V_A}$ (2)

1) CHANGE IN INTERNAL ENERGY (STEPS i & iii)

ON $\overline{AB}$ AND $\overline{CD}$ NO HEAT ENTERS OR LEAVES THE SYSTEM ($\delta Q = 0$)

$\Rightarrow dU = -P\,dV$ (i.e WORK CHANGES INTERNAL ENERGY)

$$U_{AB} = -\int_{V_A}^{V_B} P\,dV = -\int_{V_A}^{V_B} P_A\left(\frac{V}{V_A}\right)^{-\gamma}dV = -P_A V_A^{\gamma}\frac{1}{1-\gamma}\left[V_B^{1-\gamma} - V_A^{1-\gamma}\right] = P_A V_A \frac{1}{\gamma-1}\left[\left(\frac{V_A}{V_B}\right)^{\gamma-1} - 1\right]$$

SIMILARLY $U_{CD} = P_C V_C \frac{1}{\gamma-1}\left[\left(\frac{V_C}{V_D}\right)^{\gamma-1} - 1\right]$

BUT, USING (1) & (2)

$$\Rightarrow U_{CD} = P_B V_B \frac{1}{\gamma-1}\left[\left(\frac{V_B}{V_A}\right)^{\gamma-1} - 1\right] = \left[P_A\left(\frac{V_A}{V_B}\right)^{\gamma}\right]V_B\frac{1}{\gamma-1}\left[\left(\frac{V_B}{V_A}\right)^{\gamma-1} - 1\right] = P_A V_A \frac{1}{\gamma-1}\left[1 - \left(\frac{V_A}{V_B}\right)^{\gamma-1}\right]$$

$\Rightarrow \boxed{U_{CD} = -U_{AB}}$ (3)

WE COULD HAVE GUESSED THIS BECAUSE U DOES NOT CHANGE ON $\overline{BC}$ OR $\overline{DA}$ AND THE FINAL STATE IS THE SAME $\Rightarrow \Delta U = 0$

7] (CONT'D)

2) CHANGE IN HEAT (STEPS ii AND iv)

ON $\overline{BC}$ AND $\overline{DA}$ INTERNAL ENERGY DOES NOT CHANGE ON ISOTHERMS

$\Rightarrow 0 = \delta Q - P\,dV$ (i.e WORK COMES AND GOES AS HEAT)

$$Q_{BC} = \int_{V_B}^{V_C} P\,dV = \int_{V_B}^{V_C} P_B V_B \frac{1}{V}\,dV = P_B V_B \ln\left(\frac{V_C}{V_B}\right) \overset{\text{IDEAL GAS}}{=} nRT_1 \ln\left(\frac{V_C}{V_B}\right)$$

AND $Q_{DA} = nRT_2 \ln(V_A/V_D)$

[NOTE $Q_{BC} > 0$ BECAUSE $V_C > V_B$, $Q_{DA} < 0$ BECAUSE $V_A < V_D$]

USING (2), $Q_{DA} = nRT_2 \ln(V_B/V_C) = -\frac{T_2}{T_1} Q_{BC}$

$$\Rightarrow \boxed{\frac{1}{T_2} Q_{DA} = -\frac{1}{T_1} Q_{BC}} \qquad (4)$$

LATER WE WILL SEE THAT THIS IS A STATEMENT THAT ENTROPY CHANGES ALONG ISOTHERMS, NOT ADIABATS.

3) TOTAL WORK

WE HAVE SEEN THAT THE WORK IN STEP iii CANCELS THAT IN STEP i. SO TOTAL WORK RESULTS FROM THE HEAT ENTERING AND LEAVING THE SYSTEM THROUGH ISOTHERMAL PROCESSES:

$$\Delta W = Q_{BC} + Q_{DA} = Q_{BC} + \left(-\frac{T_2}{T_1} Q_{BC}\right) = Q_{BC}\left(1 - \frac{T_2}{T_1}\right)$$

FROM THIS RESULT WE FIND A SIMPLE FORMULA FOR THE "EFFICIENCY" OF THE CARNOT HEAT ENGINE, DEFINED BY

$$\eta \equiv \frac{\text{BENEFIT}}{\text{COST}} = \frac{\Delta W}{Q_{BC}} \Rightarrow \boxed{\eta = 1 - \frac{T_2}{T_1}} \qquad (5)$$

EFFICIENCY IS BETTER IF $T_2 \ll T_1$. BUT CAN NEVER BE 100% EFFICIENT

EXAMPLE: IF HOT STEP HAS $T_1 = 100°C$ AND COLD STEP AS $T_2 = 20°C$ THEN

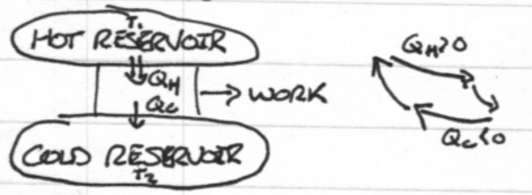
$$\eta \simeq 1 - \frac{273+20}{273+100} \simeq 0.21$$

REAL ENGINES HAVE ADDITIONAL LOSSES DUE TO FRICTION, etc

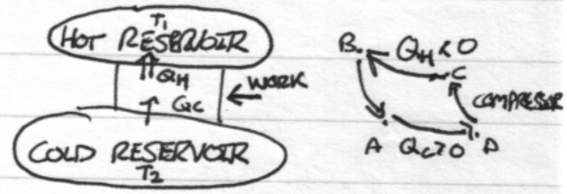
8] REFRIDGERATORS

IF WE CONSIDER THE CARNOT CYCLE IN REVERSE, THE WORK IS INPUT TO EXTRACT HEAT. THIS IS THE ESSENCE OF REFRIDGERATION AND AIR CONDITIONING.

ENGINE



REFRIDGERATOR



IN THE REFRIDGERATOR THE WORK IS DONE IN CONTACT WITH THE COLD RESERVOIR. I.e. AFTER A PISTON IS PULLED UP TO EXPAND & COOL A GAS, THE CHAMBER DRAWS HEAT FROM THE FRIDGE; THE PISTON THEN COMPRESSES THE GAS AND THIS EXCESS HEAT IS THEN EXPELLED INTO THE ROOM.

THE WORK ALONG $\overline{AD}$ IS $Q_{AD} = \int_{V_A}^{V_D} P\,dV = nRT_2 \ln\left(\frac{V_D}{V_A}\right) > 0$

HEAT IS EXPELLED ALONG $\overline{CB}$: $Q_{CB} = nRT_1 \ln\left(\frac{V_B}{V_C}\right) = -\frac{T_1}{T_2} Q_{AD} < 0$

THE TOTAL WORK IS $\Delta W = |Q_{CB}| - Q_{AD} = Q_{AD}\left(\frac{T_1}{T_2} - 1\right)$

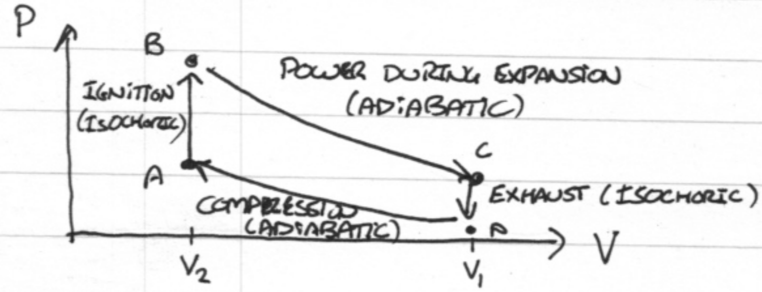
THE EFFICIENCY OF A REFRIDGERATOR (ITS "COEFFICIENT OF PERFORMANCE") IS

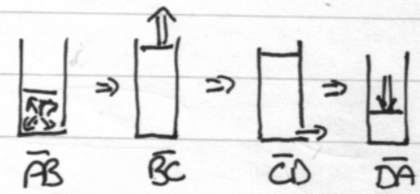
$$C.O.P = \frac{\text{BENEFIT}}{\text{COST}} = \frac{Q_{AD}}{\Delta W} = \frac{Q_{AD}}{Q_{AD}(T_1/T_2 - 1)} = \frac{T_2}{T_1 - T_2}$$

NOTE THAT PERFORMANCE IMPROVES IF T_1 IS NOT MUCH LARGER THAN T_2. BUT AGAIN FRICTION, etc DEGRADES PERFORMANCE.

9] A GASOLINE ENGINE

AN IDEALIZATION OF A COMBUSTION ENGINE THAT EXPLODES GAS INSIDE A CYLINDER TO DRIVE A PISTON IS SHOWN ON A PV-DIAGRAM:





TO CALCULATE WORK, IGNORE CHEMISTRY OF COMBUSTION AND MASS GAIN AN LOSS DURING FUEL INJECTION AND EXHAUST.

$W_{AB} = W_{CD} = 0$ BECAUSE PROCESSES ARE ISOCHORIC $(dV=0)$

$$W_{BC} = \int_{V_2}^{V_1} P\,dV = \int_{V_2}^{V_1} P_B \left(\frac{V}{V_2}\right)^{-\gamma} dV = \cdots = P_B V_2 \frac{1}{\gamma-1}\left[1-\left(\frac{V_2}{V_1}\right)^{\gamma-1}\right]$$

$$W_{DA} = \int_{V_1}^{V_2} P\,dV = \cdots = -P_D V_1 \frac{1}{\gamma-1}\left[\left(\frac{V_1}{V_2}\right)^{\gamma-1}-1\right]$$

$$= -\left[P_A\left(\frac{V_2}{V_1}\right)^{\gamma}\right] V_1 \frac{1}{\gamma-1}\left[\left(\frac{V_1}{V_2}\right)^{\gamma-1}-1\right] = -P_A V_2 \frac{1}{\gamma-1}\left[1-\left(\frac{V_2}{V_1}\right)^{\gamma-1}\right]$$

SO THE TOTAL WORK IS $\Delta W = W_{BC} + W_{DA} = (P_B - P_A) V_2 \frac{1}{\gamma-1}\left[1-\left(\frac{V_2}{V_1}\right)^{\gamma-1}\right]$ ①

BECAUSE $\overline{BC}$ & $\overline{DA}$ ARE ADIABATS (NO HEAT LOSS/GAIN) $-\Delta W = \Delta U$, THE CHANGE IN INTERNAL ENERGY DURING THE EXPANSION-COMPRESSION STEPS.

THE INTERNAL ENERGY GAIN DURING IGNITION STEP, $\overline{AB}$, IS GIVEN BY

$$dU = C_V\,dT = \delta Q$$

$$\Rightarrow Q_{AB} = C_V \Delta T_{AB} = C_V \Delta\left(\frac{PV_2}{nR}\right) = C_V \frac{V_2}{nR}\Delta P = C_V \frac{V_2}{nR}(P_B - P_A)$$

BUT $nR = C_P - C_V$ AND $\gamma \equiv C_P/C_V$

$$\Rightarrow Q_{AB} = \frac{1}{\gamma-1} V_2 (P_B - P_A) \qquad ②$$

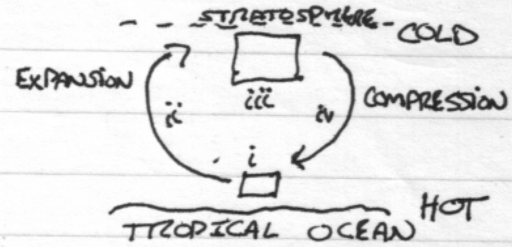
THUS, FROM ① & ② THE EFFICIENCY IS

$$\eta = \frac{\Delta W}{Q_{AB}} = 1 - (V_2/V_1)^{\gamma-1}$$

NOTE: THE COMPRESSION RATIO (NOT TEMPERATURE RATIO) IS MOST IMPORTANT FOR EFFICIENCY.

10] THE ATMOSPHERE AS A HEAT ENGINE

THE REASON WE EXPERIENCE WINDS IS BECAUSE THE EARTH CONVERTS HEAT ENERGY FROM THE SUN INTO MECHANICAL ENERGY. THIS IS DONE EFFICIENTLY THROUGH COMPRESSION AN EXPANSION PROCESSES FOR AIR DESCENDING FROM A COLD RESERVOIR (THE STRATOSPHERE) AND RISING FROM A HOT RESERVOIR (THE GROUND)



FOR EXAMPLE, CONSIDER AN IDEALIZATION OF A HURRICANE:

i) AIR TAKES UP HEAT (AND MOISTURE) FROM THE ~ 300K (27°C) OCEAN WHILE COMPRESSED AT ~ 1000 mBAR.

ii) THE

ii) THE RELATIVELY HOT AIR RISES BUOYANTLY (IN FACT, AS WATER VAPOUR CONDENSES WHILE EXPANDING AIR COOLS, MORE HEAT IS RELEASED INTO THE AIR PARCEL, DRIVING ITS MOTION ALL THE WAY TO THE STRATOSPHERE. BUT WE'LL IGNORE MOISTURE HERE.)

iii) THE EXPANDED AIR STOPS JUST BELOW THE STRATOSPHERE AT ~ 100 mBAR (ABOUT 16km UP IN TROPICS) WHERE THE TEMPERATURE IS ~ 150K (-123°C)

iv) THE COOLED (AND DRY) AIR DESCENDS AND THE PROCESS REPEATS.

CRUDELY MODELLING THIS AS A CARNOT CYCLE, THE EFFICIENCY IS

$$\eta = 1 - \frac{T_{COLD}}{T_{HOT}} \simeq 1 - \frac{150K}{300K} = 50\%$$

A VERY EFFICIENT ENGINE INDEED!

NO WONDER HURRICANES PACK A WHOLLUP.