(1D) STATISTICAL THERMODYNAMICS

1] STATISTICS REVIEW

A] BINOMIAL THEOREM

CONSIDER A SYSTEM OF N PARTICLES (e.g. COINS) EACH OF WHICH CAN EXIST IN ONE OF TWO STATES (e.g. HEADS OR TAILS)

THE SYSTEM CAN EXIST IN ONE OF 2^N STATES. BUT MOST ARE NOT UNIQUE. (e.g. THERE ARE N STATES WITH ONE HEAD AND THE REST TAILS).

FROM THE BINOMIAL THEOREM: $(H+T)^N = \sum_{n=0}^{N} \binom{N}{n} H^n T^{N-n}$
WE SEE THAT THE STATE WITH n HEADS EXISTS $\binom{N}{n}$ TIMES
WHERE

$$\binom{N}{n} = \frac{N!}{n!(N-n)!} = \frac{N(N-1)\cdots(2)(1)}{[n(n-1)\cdots(2)(1)][(N-n)(N-n-1)\cdots(2)(1)]}$$

B] STIRLINGS FORMULA (TREATMENT OF $N!$ FOR LARGE N)

DEFINE GAMMA FUNCTION TO BE $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$.
EASILY SEE THAT $\Gamma(1) = \Gamma(2) = 1$ AND $\Gamma(x+1) = x\Gamma(x)$.
So $\Gamma(N+1) = N\Gamma(N) = N(N-1)\Gamma(N-1) = \cdots = N(N-1)\cdots(2)\Gamma(2)$
$\Rightarrow \Gamma(N+1) = N!$ FOR INTEGERS N.

THE ASYMPTOTIC APPROXIMATION TO $\Gamma(x)$ FOR LARGE x GIVES STIRLINGS FORMULA: $\boxed{N! = \Gamma(N+1) \simeq \sqrt{2\pi N}\,(N/e)^N}$ N LARGE

OFTEN CARE MORE ABOUT $\ln N! \simeq N\ln N - N$, N LARGE

N	$N!$	$\sqrt{2\pi N}\left(\frac{N}{e}\right)^N$	ERROR	$\ln N!$	$N\ln N - N$	ERROR
10	3628800	3598696	0.83%	15.1	13.0	13.8%
100	9×10^{157}	9×10^{157}	0.083%	364	360	0.89%

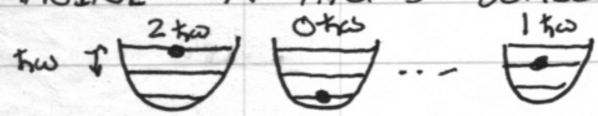
*VERY GOOD APPROXIMATION FOR $N \sim 10^{23}$.

2] THE EINSTEIN SOLID

IN A SOLID, ATOMS SIT IN POTENTIAL WELLS RESULTING FROM MUTUAL ATTRACTION/REPULSION OF ELECTRONS & NUCLEI. IN 1D, THE POTENTIAL WELL IS LIKE THAT OF A SPRING: $\frac{1}{2}K_s x^2$

THANKS TO QUANTUM MECHANICS, WE CAN SUPPOSE THE ENERGY OF ANY ONE ATOM OCCURS IN DISCRETE STEPS OF SIZE $\hbar\omega$.

IMAGINE N ATOMS WHICH HAVE TOTAL ENERGY $q(\hbar\omega)$



THIS CAN OCCUR IN $\binom{q+N-1}{q}$ WAYS.

e.g. FOR $N=3$, $q=3$, USE q '•' TO REPRESENT EACH ENERGY, AND USE $N-1$ '|' TO DENOTE $N-1$ PARTITIONS BETWEEN ATOMS

$\vert\,\vert\,\bullet\,\bullet\,\bullet \leftarrow$  0 0 3

$\vert\,\bullet\,\vert\,\bullet\,\bullet \leftarrow$ 0 1 2

$\vert\,\bullet\,\bullet\,\vert\,\bullet$ $\qquad$ $\bullet\,\vert\,\bullet\,\bullet\,\vert$

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TOTAL NUMBER OF COMBINATIONS IS $\binom{q+(N-1)}{q} = \binom{3+2}{3} = \binom{5}{3} = 10$ ✓

IN GENERAL, ATOMS IN A LATTICE ARE FREE TO MOVE IN 3 INDEPENDENT DIRECTIONS. SO N REFERS TO THE # DEGREES OF FREEDOM, AND THE NUMBER OF ATOMS IS $N/3$.

3] INTERACTING EINSTEIN SOLIDS.

CONSIDER 2 EINSTEIN SOLIDS, ONE WITH ENERGY $q_{A0}(\hbar\omega)$ AND N_A DEGREES OF FREEDOM, THE OTHER WITH $q_{B0}(\hbar\omega)$ & N_B.

WHEN PUT IN THERMAL CONTACT, EXPECT THE TOTAL ENERGY

$$q(\hbar\omega) = (q_{A0} + q_{B0})\hbar\omega = (q_A + q_B)\hbar\omega$$

TO BE REDISTRIBUTED AMONGST THE $N = N_A + N_B$ DEGREES OF FREEDOM

TOTAL # STATES IS $\Omega = \binom{q+N-1}{q}$ AS BEFORE.

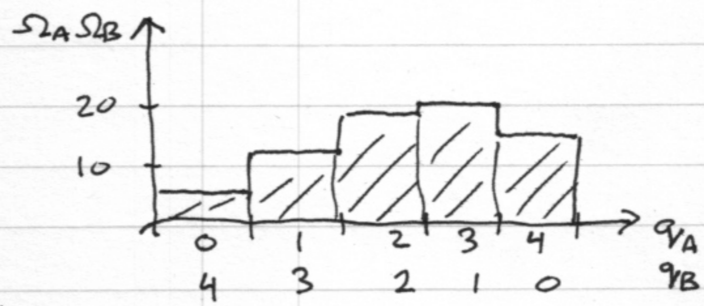
OF THESE, COUNT # STATES IN WHICH A HAS ENERGY $q_A(\hbar\omega)$ AND B HAS ENERGY $q_B(\hbar\omega) = (q - q_A)(\hbar\omega)$:

$$= \underbrace{\binom{q_A + N_A - 1}{q_A}}_{\Omega_A} \underbrace{\binom{q_B + N_B - 1}{q_B}}_{\Omega_B}$$

EXAMPLE: FOR $N_A = 3$, $N_B = 2$, $q = 4$ HAVE

q_A	Ω_A	q_B	Ω_B	$\Omega_A \Omega_B \equiv \Omega_{TOTAL}$
0	1	4	5	5
1	3	3	4	12
2	6	2	3	18
3	10	1	2	20
4	15	0	1	15

$$\Omega = 70 = \binom{4+5-1}{4} = \binom{8}{4}$$

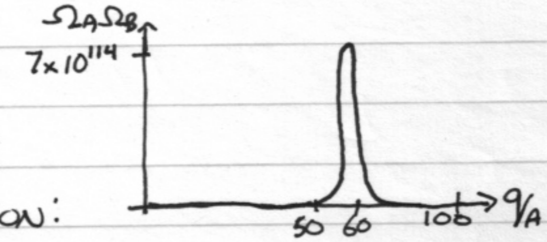


NOTE BIAS FOR MORE ENERGY IN A ON AVERAGE SINCE $N_A > N_B$

3] (CONT'D)

THIS BRINGS US TO THE <u>FUNDAMENTAL ASSUMPTION OF STATISTICAL MECHANICS</u>: ASSUME ENERGY IS DISTRIBUTED RANDOMLY SO THAT ALL STATES ARE POSSIBLE WITH EQUAL PROBABILITY.

THEN STATISTICS DICTATE THE MOST LIKELY STATE IS THE ONE WITH THE LARGEST NUMBER OF MULTIPLICITIES, $\Omega_{TOTAL} = \Omega_A \Omega_B$. FOR LARGE SYSTEMS THE MOST LIKELY STATE HAS A <u>VERY</u> SMALL DEVIATION ABOUT ITS MEAN.



EXAMPLE: FOR $N_A = 300$, $N_B = 200$ AND $q = 100$, $\Omega_{TOTAL}(q_A)$ HAS THE FOLLOWING DISTRIBUTION:

LETS COMPUTE THE MEAN AND STANDARD DEVIATION IN GENERAL ASSUMING N_A, N_B AND q LARGE ($q \gg N$)

FIRST ESTIMATE THE TOTAL # STATES × MULTIPLICITIES: $\Omega = \binom{q+N-1}{q}$

NOTE $\ln\Omega = \ln\left[\frac{(q+N-1)!}{q!\,(N-1)!}\right] \simeq \ln(q+N)! - \ln q! - \ln N!$

STIRLING: $\simeq [(q+N)\ln(q+N) - (q+N)] - [q\ln q - q] - [N\ln N - N]$

$= (q+N)\ln(q+N) - q\ln q - N\ln N$

SINCE $q \gg N \Rightarrow \ln(q+N) = \ln q\left(1+\frac{N}{q}\right) = \ln q + \ln\left(1+\frac{N}{q}\right)$

$\simeq \ln q + \frac{N}{q}$

SO $\ln\Omega \simeq (q+N)\ln q + (q+N)\left(\frac{N}{q}\right) - q\ln q - N\ln N$

$= N\ln(q/N) + N + N^2/q \simeq N\ln(q/N) + N$

SO $\Omega \approx \mathrm{EXP}\left[N\ln\frac{q}{N} + N\right] = e^N\left(\frac{q}{N}\right)^N = \underline{\underline{\left(\frac{eq}{N}\right)^N}}$.

3] (CONT'D)

LIKEWISE WE CAN ESTIMATE THE MULTIPLICITY OF EACH STATE

$$\Omega_{TOTAL} = \Omega_A \Omega_B \approx \left(\frac{e q_A}{N_A}\right)^{N_A} \left(\frac{e(q-q_A)}{N_B}\right)^{N_B} = \frac{e^N q^N}{N_A^{N_A} N_B^{N_B}} \tilde{q}^{N_A}(1-\tilde{q})^{N_B}$$

IN WHICH $\tilde{q} \equiv q_A/q$ IS THE FRACTION OF ENERGY IN A.

SUPPOSE THE MEAN IS CLOSE TO THE PEAK OF THE DISTRIBUTION (WILL CONFIRM THIS LATER). TO FIND PEAK OF $\mathcal{E} x^a (1-x)^b$, SET DERIVATIVE TO ZERO:

$$0 = \mathcal{E} a x^{a-1}(1-x)^b - \mathcal{E} b x^a (1-x)^{b-1}$$

$$= \mathcal{E} x^{a-1}(1-x)^{b-1}\left[a(1-x) - bx\right]$$

$$\Rightarrow x = a/(a+b)$$

SO $\bar{\tilde{q}} \simeq \frac{N_A}{N_A+N_B} = N_A/N$

$\Rightarrow \bar{q}_A \simeq q N_A/N$ (i.e. FRACTION OF ENERGY IN A IS FRACTION OF ATOMS IN A)

NOTE, THE VALUE OF THE PEAK IS $\frac{e^N q^N}{N_A^{N_A} N_B^{N_B}}\left(\frac{N_A}{N}\right)^{N_A}\left(1-\frac{N_A}{N}\right)^{N_B} = \left(\frac{eq}{N}\right)^N \approx \Omega$

THAT THE MULTIPLICITY OF THIS ONE STATE IS ALMOST THE TOTAL NUMBER OF STATES x MULTIPLICITIES. IN AN INDICATION OF HOW NARROW THE PEAK MUST BE.

LETS FIND THE WIDTH OF THE PEAK IN THE SIMPLE CASE $N_A = N_B = \frac{1}{2}N$

WRITE $\tilde{q} = \bar{\tilde{q}} + \delta$ WITH $\delta \ll 1$ THE 'DISTANCE" FROM PEAK.

SO $\Omega_{TOTAL} = \frac{e^N q^N}{N_A^{N_A} N_B^{N_B}}\left(\frac{N_A}{N} + \delta\right)^{N_A}\left(\frac{N_B}{N} - \delta\right)^{N_B} \approx \Omega\left(1 + \delta\frac{N}{N_A}\right)^{N_A}\left(1 - \delta\frac{N}{N_B}\right)^{N_B}$

$N_A = N_B = \frac{N}{2} \Rightarrow \quad = \Omega(1+2\delta)^{N/2}(1-2\delta)^{N/2} = \Omega(1-4\delta^2)^{N/2}$

BUT $\ln\left[(1-4\delta^2)^{N/2}\right] = \frac{N}{2}\ln(1-4\delta^2) \simeq \frac{N}{2}(-4\delta^2) = -2N\delta^2$

$\Rightarrow (1-4\delta^2)^{N/2} \simeq \mathrm{EXP}\left[-2N\delta^2\right]$

SO $\Omega_{TOTAL} \simeq \Omega e^{-2N\delta^2} = \Omega e^{-2N(\tilde{q}-\bar{\tilde{q}})^2} = \Omega e^{-2N(q_A - \bar{q}_A)^2/q^2}$

WHICH IS TO SAY THE DISTRIBUTION IS GAUSSIAN WITH STANDARD DEVIATION $\sigma = \frac{1}{2\sqrt{N}} q$ ABOUT MEAN $\bar{q}_A = q\frac{N_A}{N} = \frac{1}{2}q$

e.g. IF $N = 10^{20} \Rightarrow \sigma \approx 5\times10^{-11} q \lll \bar{q}_A$!

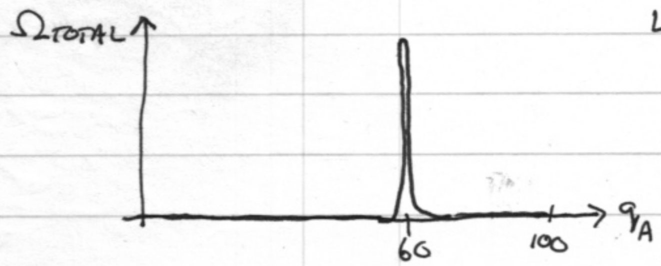
4] ENTROPY OF AN EINSTEIN SOLID

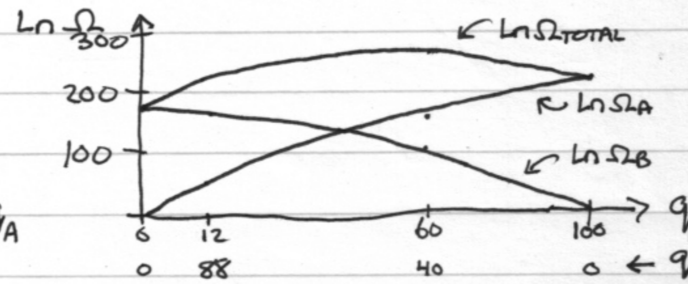
WE WISH TO CONSIDER APPROACH TO EQUILIBRIUM OF 2 EINSTEIN SOLIDS TO DRAW A CONNECTION BETWEEN INTERNAL ENERGY $(U = q(\hbar\omega))$, ENTROPY AND TEMPERATURE.

FOR ILLUSTRATION PURPOSES SUPPOSE $N_A = 300$, $N_B = 200$, $q = 100$ (AS IN P. 81 OF NOTES) (DON'T WORRY THAT $q < N$)

HERE ARE SOME VALUES OF MULTIPLICITIES FOR DIFFERENT STATES q_A

q_A	Ω_A	$\ln\Omega_A$	q_B	Ω_B	$\ln\Omega_B$	Ω_{TOTAL}	$\ln\Omega_{TOTAL}$
0	1	0	100	2.8×10^{81}	187.5	2.8×10^{81}	187.5
12	1.4×10^{21}	48.7	88	3.4×10^{75}	173.9	4.7×10^{96}	222.6
60	1.3×10^{69}	159.1	40	5.3×10^{45}	105.5	6.9×10^{114}	264.4
100	1.7×10^{96}	221.6	0	1	0	1.7×10^{96}	221.6





NOTE THE FOLLOWING:

A] AT EQUILIBRIUM $\frac{d}{dq_A}\ln\Omega_{TOTAL} = 0 \Rightarrow \frac{d}{dq_A}(\ln\Omega_A + \ln\Omega_B) = 0$

$$\Rightarrow \frac{d}{dq_A}\ln\Omega_A = -\frac{d}{dq_A}\ln\Omega_B \overset{q_B = q - q_A}{=} +\frac{d}{dq_B}\ln\Omega_B \quad \text{AT } q_A = \bar{q}_A,\ q_B = \bar{q}_B$$

BUT $U_A = q_A(\hbar\omega)$ & $U_B = q_B(\hbar\omega)$ ARE INTERNAL ENERGIES OF A & B

SO IN EQUILIBRIUM $\frac{d\ln\Omega_A}{dU_A} = \frac{d\ln\Omega_B}{dU_B}$

B] IF INITIALLY $q_{A0} < \bar{q}_A$ THEN $\frac{d\ln\Omega_A}{dq_A} > \frac{d\ln\Omega_B}{dq_B}$

i.e. ENERGY FLOWS FROM $B \to A$ IN APPROACH TO EQ^M WHILE $\frac{d\ln\Omega_A}{dU_A} > \frac{d\ln\Omega_B}{dU_B}$

LIKEWISE, IF $q_{A0} > \bar{q}_A$ INITIALLY $\Rightarrow \frac{d\ln\Omega_A}{dq_A} < \frac{d\ln\Omega_B}{dq_B}$

SO ENERGY FLOWS FROM $A \to B$ WHILE $\frac{d\ln\Omega_A}{dU_A} < \frac{d\ln\Omega_B}{dU_B}$

4] (CONT'D)

COMPARE THESE OBSERVATIONS WITH WHAT WE KNOW ABOUT PUTTING TWO OBJECTS IN THERMAL CONTACT

A] IN EQUILIBRIUM THE TEMPERATURES ARE THE SAME

B] AWAY FROM EQUILIBRIUM HEAT FLOWS FROM HIGH TO LOW TEMPERATURE.

LEADINGLY, WE ALSO HAVE THE DEFINITION OF TEMPERATURE IN TERMS OF INTERNAL ENERGY AND ENTROPY:

$$T = \left(\frac{\partial U}{\partial S}\right)_V \qquad \text{(SEE p.40 OF NOTES)}$$

$$\Rightarrow \left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T} \qquad \text{(RECIPROCAL RELATION).}$$

SO FOR SOLID A IN CONTACT WITH SOLID B

A] $\left(\frac{\partial S_A}{\partial U_A}\right)_V = \frac{1}{T} = \left(\frac{\partial S_A}{\partial U_B}\right)_V$ IN EQUILIBRIUM

B] $\left(\frac{\partial S_A}{\partial U_A}\right)_V > \left(\frac{\partial S_B}{\partial U_B}\right)_V$ IF $T_A < T_B$ (SO HEAT FLOWS INTO A)

AND $\left(\frac{\partial S_A}{\partial U_A}\right)_V < \left(\frac{\partial S_B}{\partial U_B}\right)_V$ IF $T_A > T_B$ (SO HEAT FLOWS OUT OF A)

SO WE EXPECT $S = \mathcal{E} \ln \Omega$ FOR SOME CONSTANT $\mathcal{E} > 0$.

IN WHICH Ω IS THE MULTIPLICITY OF THE STATE. WHY NOT $S \propto \Omega$?

BECAUSE S IS EXTENSIVE (i.e. $S(2N) = 2S(N)$) AND $\Omega \sim (eq/N)^N \Rightarrow \ln\Omega(2N) \sim 2\ln\Omega(N)$

E.g. FOR AN EINSTEIN SOLID IN EQUILIBRIUM WITH $q \gg N$, WE FOUND $\Omega \simeq (eq/N)^N$

$$\Rightarrow S = \mathcal{E}\ln\Omega = \mathcal{E}N[\ln q - \ln N + 1] = \mathcal{E}N\Big[\ln\underbrace{\left(\frac{U}{\hbar\omega}\right)}_{q} - \ln N + 1\Big]$$

$$= \mathcal{E}N\ln U + \mathcal{E}[-N\ln(\hbar\omega) - \ln N + 1]$$

$$\Rightarrow \frac{\partial S}{\partial U} = \mathcal{E}N/U$$

$$\text{So } \left(\frac{\partial S}{\partial U}\right)_V = \frac{1}{T} \Rightarrow U = \mathcal{E}NT.$$

ONE EACH FOR INTERACTION TO LEFT AND RIGHT

BUT EACH OSCILLATOR HAS ENERGY $\frac{1}{2}kT$ AND THERE ARE $2N$ OSCILLATORS

$\Rightarrow U = 2N(\frac{1}{2}kT) = NkT$. So $\boxed{\mathcal{E} = k}$, THE BOLTZMAN CONSTANT

5] STATISTICAL MECHANICS DEFINITION OF ENTROPY

FOR THE EINSTEIN SOLID, WE FOUND THAT ENTROPY WAS GIVEN BY $\boxed{S = k \ln \Omega}$ WITH $k = 1.381 \times 10^{-23}$ J/K BOLTZMAN'S CONST. AND Ω = MULTIPLICITY OF STATE.

IT TURNS OUT THAT THIS DEFINITION IN STATISTICAL MECHANICS WORKS FOR ANY SYSTEM. IN PRACTISE, HOWEVER, Ω AND HENCE S, CAN ONLY BE CALCULATED FOR A SMALL NUMBER OF IDEALIZED SYSTEMS.

ITS DEFINITION HELPS US RECONSIDER THE CLASSICAL DEFINITION OF S (THROUGH $dS = \delta Q/T$) IN TERMS OF STATISTICAL MECHANICS:

A) $dS = 0$ (HENCE $\delta Q = 0$) MEANS $d\Omega = 0$.
I.e. THE MULTIPLICITY OF STATES IS UNCHANGED IF NO HEAT ENTERS THE SYSTEM.

B) $dS \geq 0$ (SECOND LAW) MEANS SYSTEM ALWAYS EVOLVES TO STATES WITH LARGER MULTIPLICITY: $d\Omega \geq 0$

C) $S = 0$ OCCURS IF $\Omega = 1$: ONLY ONE STATE EXISTS
THIS OCCURS ONLY IF THE SYSTEM IS REQUIRED TO BE AT ITS MINIMUM ENERGY STATE. AND THIS IS THE CASE ONLY AT ABSOLUTE ZERO TEMPERATURE (THIRD LAW)

6] ENTROPY OF AN IDEAL GAS

1) SINGLE ATOM IN VOLUME V WITH KINETIC ENERGY E

STATES DETERMINED BY # LOCATIONS AND # VALUES OF MOMENTUM

DISCRETIZE V BY RESOLUTION Δ_s^3 (UNCERTAINTY IN POSITION)

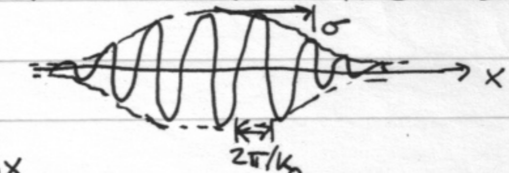
DISCRETIZE MOMENTUM $|\underline{p}| = \sqrt{2mE}$ BY RESOLUTION Δ_p^3 (UNCERTAINTY IN MOM$^{\underline{M}}$)

HEISENBERG'S UNCERTAINTY PRINCIPLE STATES $\Delta_s \Delta_p \sim \hbar$

[THIS FOLLOWS FROM FOURIER TRANSFORM OF A GAUSSIAN WAVEPACKET

$$f(x) = A e^{-x^2/2\sigma^2} e^{ik_0 x} \quad \text{(ACTUALLY REAL PART)}$$

$$= \int_{-\infty}^{\infty} F(k)\, e^{ikx}\, dk$$



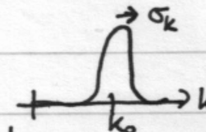
IN WHICH

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} A e^{-x^2/2\sigma^2} e^{-i(k-k_0)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} A\, \sigma\, e^{-\sigma^2 (k-k_0)^2/2}$$

$$= \frac{1}{\sqrt{2\pi}} A\, \sigma\, e^{-(k-k_0)^2/2\sigma_k^2}, \text{ WITH } \sigma_k = \frac{1}{\sigma}$$



SO, UNCERTAINTY IN LOCATION IS $\Delta_s = \sigma$

THIS CORRESPONDS TO UNCERTAINTY IN WAVENUMBER BY $\sigma_k = \frac{1}{\sigma}$.

AND THIS CORRESPONDS TO UNCERTAINTY IN MOMENTUM BY $\hbar\sigma_k = \hbar/\sigma$

SO $\Delta_s \Delta_p \simeq \sigma\,(\hbar/\sigma) = \hbar = \frac{h}{2\pi}$. (A MORE ACCURATE APPROACH GIVES $\Delta_s \Delta_p = h$)

→ HEISENBERG IS JUST A RESULT OF THE PROPERTIES OF WAVES ANALYSED BY FOURIER TRANSFORMS TO CONNECT SPATIAL TO WAVENUMBER STRUCTURE. QUANTUM MECHANICS IS INVOKED ONLY TO RELATE WAVENUMBER TO MOMENTUM THROUGH $p = \hbar k$.]

SO # STATES IS $\left(\frac{V}{\Delta_s^3}\right)\left(\frac{V_{p1}}{\Delta_p^3}\right) = \frac{V\, V_{p1}}{h^3} \Rightarrow \Omega_1 = \frac{1}{h^3} V V_{p1}$

IN WHICH V_{p1} IS THE VOLUME OF THE SPHERICAL SHELL OF RADIUS $\sqrt{2mE}$ AND THICKNESS $\Delta_p \Rightarrow V_{p1} \propto S_3 = 4\pi\left(\sqrt{2mE}\right)^2$

IN WHICH S_3 IS SURFACE AREA OF 3D SPHERE

6] (cont'd)

2) N atoms in volume V with total kinetic energy E.

For 2 atoms, # states is $\Omega_2 = \frac{1}{2}\frac{1}{h^6}V^2 V_{P2}$

in which $\frac{1}{2}$ accounts for duplication of atoms switching places

V_{P2} counts all values of $\underline{P}_1$ & $\underline{P}_2$ satisfying $|\underline{P}_1|^2 + |\underline{P}_2|^2 = 2mE$,

So $V_{P2} \propto S_6$ the surface area of a 6 dimensional hypersphere.

For N atoms, # states is $\Omega_N = \frac{1}{N!}\frac{1}{h^{3N}}V^N V_{PN}$

in which $V_{PN} \propto S_{3N}$ is surface area of $3N$ dimensional hypersphere

Explicitly $S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} r^{d-1}$ $\left(\Gamma(\frac{d}{2}) = (\frac{d}{2}-1)!\ ;\ \Gamma(\frac{3}{2}) = (\frac{1}{2}!) = \frac{\sqrt{\pi}}{2}\right)$

[So $S_2 = \frac{2\pi}{\Gamma(1)} r = 2\pi r$; $S_3 = \frac{2\pi^{3/2}}{\Gamma(3/2)} r^2 = 4\pi r^2$; etc]

Hence $\Omega_N \propto \frac{1}{N!}\frac{1}{h^{3N}}V^N\left[\frac{2\pi^{3N/2}}{\Gamma(3N/2)}(2mE)^{\frac{3N-1}{2}}\right]$

$N \gg 1 \Rightarrow \quad \simeq \frac{1}{N!}\frac{1}{h^{3N}}V^N \frac{2\pi^{3N/2}}{(3N/2)!}(2mE)^{3N/2}$

$\Rightarrow \boxed{\Omega \simeq f(N)\, V^N E^{3N/2}}$ (*) in which $f(N) = \frac{1}{N!}\frac{1}{h^{3N}}\frac{2\pi^{3N/2}}{(3N/2)!}(2m)^{3N/2}$

3) Entropy

So $S \equiv k \ln \Omega = k \ln\left[\frac{1}{N!}\frac{1}{h^{3N}}V^N\frac{2\pi^{3N/2}}{(3N/2)!}(2mE)^{3N/2}\right]$

Stirling approx $\Rightarrow \quad \simeq k \ln\left[\frac{1}{\sqrt{2\pi N}}\left(\frac{N}{e}\right)^{-N}\frac{1}{h^{3N}}V^N\frac{\pi^{3N/2}}{\sqrt{2\pi(3N/2)}}\left(\frac{3N}{2e}\right)^{-3N/2}(2mE)^{3N/2}\right]$

$\Rightarrow \boxed{S \simeq kN\left[\ln\left(\frac{V}{N}\left(\frac{4\pi m E}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right]}$

The Sackur – Tetrode equation

Letting $E = U$, the internal energy,

(*) $\Rightarrow S = k \ln\left[f(N)\, V^N U^{3N/2}\right]$

$= kN \ln V + k\frac{3N}{2}\ln U + k \ln f(N)$

7] PROPERTIES OF IDEAL GAS, REVISITED

WE HAVE FOUND FOR A MONATOMIC GAS: $S = kN\ln V + k\frac{3N}{2}\ln U + k\ln f(N)$ (*)

1) TEMPERATURE: $\left(\frac{\partial S}{\partial U}\right)_{N,V} = \frac{1}{T}$ ① (P.40 OF NOTES)

$(*) \Rightarrow \left(\frac{\partial S}{\partial U}\right)_{N,V} = k\frac{3N}{2}\left(\frac{1}{U}\right)$

TOGETHER WITH ① $\Rightarrow$ $U = k\frac{3N}{2}T = \underline{\frac{3}{2}NkT} = N\times 3\times\left(\frac{1}{2}kT\right)$

THIS IS EXPECTED RESULT FOR GAS OF ATOMS IN WHICH EACH ATOM HAS ENERGY $\frac{1}{2}kT$ ASSOCIATED WITH EACH OF 3 COMPONENTS OF TRANSLATIONAL KINETIC ENERGY.

2) PRESSURE: $P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$ ② (P.40 OF NOTES)

TAKE $\frac{\partial}{\partial V}(*)\big|_{S,N}$

$$\Rightarrow \quad 0 = kN\frac{1}{V} + k\frac{3N}{2}\frac{1}{U}\overbrace{\left(\frac{\partial U}{\partial V}\right)_{S,N}}^{-P} + 0$$

$$\Rightarrow \quad P = \left(kN\frac{1}{V}\right)/\left(k\frac{3N}{2}\frac{1}{U}\right)$$

$$= \frac{2}{3}\frac{U}{V}$$

BUT $U = \frac{3}{2}NkT \quad \Rightarrow \quad P = \frac{2}{3}\frac{1}{V}\left(\frac{3}{2}NkT\right) = \underline{NkT/V}$

THIS IS EXPECTED FROM IDEAL GAS LAW.

(EQUIVALENTLY, WITH U FIXED $dU = TdS - PdV \Rightarrow P = T\left(\frac{\partial S}{\partial V}\right)_{U,N}$

SO: $(*) \Rightarrow P = T\left(kN\frac{1}{V}\right) = NkT/V$, AS ABOVE)

3) CHEMICAL POTENTIAL: $\mu = \left(\frac{\partial U}{\partial N}\right)_{S,V}$

EQUIVALENTLY, WITH U (AND V) FIXED $dU = TdS - PdV + \mu dN$ (P.62 OF NOTES)

$$\Rightarrow \quad \mu = -T\left(\frac{\partial S}{\partial N}\right)_{U,V}$$

S-TETR* P.87 $\Rightarrow$ $= -Tk\left[\left\{\ln\left(\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right\} + N\left\{-\frac{5}{2}\frac{1}{N}\right\}\right]$

$$= -kT\ln\left[\frac{V}{N}\left(\frac{4\pi m}{3h^2}\frac{U}{N}\right)^{3/2}\right]$$

$$= -kT\ln\left[\frac{V}{N}\left(\frac{2\pi m}{h^2}kT\right)^{3/2}\right]$$

8] PARAMAGNET

UNLIKE A FERROMAGNET, A PARAMAGNET BECOMES MAGNETIZED ONLY IN THE PRESENCE OF A MAGNETIC FIELD (LIKE YOUR FRIDGE REACTING TO A FRIDGE MAGNET).

IN AN IDEALIZED 2-STATE PARAMAGNET, IT IS COMPOSED OF MAGNETIC DIPOLES, WHICH ARE RANDOMLY ORIENTED UP OR DOWN IN THE ABSENCE OF AN EXTERNAL MAGNETIC FIELD.
SCHEMATICALLY : UP $\uparrow\uparrow\downarrow\uparrow\downarrow\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\uparrow\downarrow$ DOWN

IF THERE ARE N DIPOLES, THE NUMBER OF STATES WITH $N_\uparrow$ 'UP' DIPOLES IS $\binom{N}{N_\uparrow} = N!/[N_\uparrow!\,N_\downarrow!]$
(JUST LIKE COUNTING HEADS + TAILS AMONG N COINS)

NOW CONSIDER MAGNETIC POTENTIAL ENERGY ASSOCIATED WITH DIPOLES IN A UNIFORM MAGNETIC FIELD $\underline{B}$. THE ENERGY REQUIRED TO FLIP A DIPOLE FROM PARALLEL TO ANTIPARALLEL IS $2\mu B$, IN WHICH μ IS THE (CONSTANT) DIPOLE MOMENT. (FOR ELECTRON $\mu = \frac{1}{2} e^- \hbar/m_e = 9.274\times10^{-24}\,J/T = 5.788\times10^{-5}\,eV/T$)

INVOKING SYMMETRY, SAY THAT THE ENERGY OF AN 'UP' DIPOLE (i.e PARALLEL) IS $-\mu B$ AND OF 'DOWN' (ANTIPARALLEL) IS $+\mu B$

$$\underline{B}\ \Uparrow \qquad \uparrow^{UP}_{-\mu B} \xrightarrow{2\mu B} \downarrow^{DOWN}_{+\mu B}$$

SO, IN AN ARRAY OF $N_\uparrow$ 'UP' AND $N_\downarrow$ 'DOWN' DIPOLES, THE TOTAL ENERGY RELATIVE TO A WELL-MIXED (EQUAL UP + DOWN) STATE IS

$$U = \mu B(N_\downarrow - N_\uparrow)$$

$$N = N_\uparrow + N_\downarrow \Rightarrow U = \mu B(N - 2N_\uparrow)$$

8] (CONT'd)

NOW FIND ENTROPY IN TERMS OF MULTIPLICITY OF STATES AND RECAST THIS IN TERMS OF U.

$$S = k \ln \Omega = k \ln \binom{N}{N_\uparrow} = k \ln \left[N! / (N_\uparrow! (N - N_\uparrow)!) \right]$$

STIRLING APPROX $\Rightarrow \simeq k \left[N \ln N - N_\uparrow \ln N_\uparrow - (N - N_\uparrow) \ln (N - N_\uparrow) \right]$

$U = \mu B (N - 2N_\uparrow) \Rightarrow \simeq k \left[N \ln N - \frac{1}{2}\left(N - \frac{U}{\mu B}\right) \ln \left[\frac{1}{2}\left(N - \frac{U}{\mu B}\right)\right] - \frac{1}{2}\left(N + \frac{U}{\mu B}\right) \ln \left[\frac{1}{2}\left(N + \frac{U}{\mu B}\right)\right] \right]$

SO TEMPERATURE IS FOUND USING $\frac{1}{T} \equiv \left(\frac{\partial S}{\partial U}\right)_{N,B}$

$$\Rightarrow \frac{1}{T} = k \left[\frac{1}{2} \frac{1}{\mu B} \ln \left[\frac{1}{2}\left(N - \frac{U}{\mu B}\right)\right] + \frac{1}{\mu B} - \frac{1}{2} \frac{1}{\mu B} \ln \left[\frac{1}{2}\left(N + \frac{U}{\mu B}\right)\right] - \frac{1}{\mu B} \right]$$

$$= k \frac{1}{2\mu B} \ln \left[\left(N - \frac{U}{\mu B}\right) / \left(N + \frac{U}{\mu B}\right) \right]$$

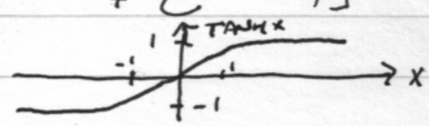
REWRITE THIS TO GET $U(T)$:

$$\left(1 - \frac{U}{N\mu B}\right) / \left(1 + \frac{U}{N \mu B}\right) = \mathrm{EXP}\left[\frac{2\mu B}{kT}\right]$$

$$\Rightarrow U = N \mu B \left[\left(1 - \mathrm{EXP}\left(\frac{2\mu B}{kT}\right)\right) / \left(1 + \mathrm{EXP}\left(\frac{2\mu B}{kT}\right)\right) \right]$$

$$= - N \mu B \left[\left(e^{\mu B/kT} - e^{-\mu B/kT}\right) / \left(e^{\mu B/kT} + e^{-\mu B/kT}\right) \right]$$

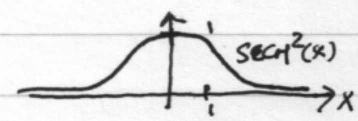
$$\Rightarrow \boxed{U = - N \mu B \tanh(\mu B / kT)}$$



HEAT CAPACITY (AT CONSTANT MAGNETIC FIELD)

$$C_B \equiv \left(\frac{\partial U}{\partial T}\right)_{B,N} = - N \mu B \,\mathrm{sech}^2\left(\frac{\mu B}{kT}\right) \left(-\frac{\mu B}{k} \frac{1}{T^2}\right)$$

$$\Rightarrow \boxed{C_B = N k \left(\frac{\mu B}{kT}\right)^2 \mathrm{sech}^2\left(\frac{\mu B}{kT}\right)}$$



MAGNETIZATION

$$M = \mu (N_\uparrow - N_\downarrow) = \frac{U}{B} = N \mu \tanh\left(\frac{\mu B}{kT}\right)$$

NOTE: AT ROOM TEMPERATURE IN A 1T FIELD $\frac{\mu B}{kT} \simeq \frac{5.788 \times 10^{-5} eV}{0.025 eV} \approx 2.3 \times 10^{-3} \ll 1$

$$\text{So } C_B \approx N \left(\frac{(\mu B)^2}{k}\right) \frac{1}{T^2}$$

$$\text{AND } M \approx N \frac{\mu^2 B}{k} \frac{1}{T}$$

THAT $M \propto 1/T$ WAS FIRST DISCOVERED EXPERIMENTALLY BY PIERRE CURIE. IT IS NOW KNOWN AS "CURIES LAW."

9] Maxwell-Boltzman velocity distribution: Revisited

Suppose N particles are partitioned amongst a discrete set of m energy levels: $[0, \delta E)$, $[\delta E, 2\delta E)$, ..., $[(m-1)\delta E, m\delta E)$
Let n_i be # particles in range $[(i-1)\delta E, i\delta E)$.
So $\quad N = \sum_{i=1}^{m} n_i$, where maximum energy is $< m\delta E$.

Now in each energy level, i, a particle can be in any of g_i states. (g_i's are the "Density of states")

ways to distribute n_i particles in g_i states is $g_i^{n_i}$
ways to choose n_i particles from N and distribute is $\binom{N}{n_i} g_i^{n_i}$
Doing this, successively populating levels $i = 1, 2, \ldots, m$
gives total # arrangements

$$\Omega = \binom{N}{n_1} g_1^{n_1} \cdot \binom{N-n_1}{n_2} g_2^{n_2} \cdot \binom{N-n_1-n_2}{n_3} g_3^{n_3} \cdots$$
$$= \left[\binom{N}{n_1}\binom{N-n_1}{n_2}\binom{N-n_1-n_2}{n_3}\cdots\right] g_1^{n_1} g_2^{n_2} g_3^{n_3} \cdots$$
$$\Rightarrow \Omega = N!/[n_1! n_2! n_3! \cdots n_m!] \; g_1^{n_1} g_2^{n_2} g_3^{n_3} \cdots g_m^{n_m}$$

The underlying principle of Statistical Mechanics is that the state with greatest multiplicity will be manifest.
So we want to maximize Ω (or, equivalently, maximize $\ln \Omega$) subject to constraints

$$N = \sum n_i$$
$$E = \sum n_i E_i \qquad (\text{where } E_i = i\,\delta E)$$

First note $\quad \ln \Omega = \ln N! - \sum \ln n_i! + \sum n_i \ln g_i$

Stirling approx $\Rightarrow$
$$\simeq (N \ln N - N) - \left(\sum n_i \ln n_i - n_i\right) + \sum n_i \ln g_i$$
$$\simeq (N \ln N - N) + \sum (n_i \ln g_i - n_i \ln n_i + n_i)$$

9] (CONT'D)

TO MAXIMIZE A FUNCTION SUBJECT TO CONSTRAINTS, USE THE METHOD OF LAGRANGE MULTIPLIERS.

$$\text{SEEK TO MAXIMIZE } F \equiv \ln \Omega + \alpha \overbrace{(N - \Sigma n_i)}^{\text{CONSTRAINTS}} + \beta \overbrace{(E - \Sigma n_i E_i)}^{}$$

LAGRANGE MULTIPLIERS

WITH RESPECT TO $n_1, n_2, \ldots, n_m$, α AND β.

COMPUTING TOTAL CHANGE WRT n_i'S : $dF = \frac{\partial F}{\partial n_1} dn_1 + \cdots + \frac{\partial F}{\partial n_m} dn_m$

AT MAXIMUM HAVE

$$\begin{aligned} 0 &= \Sigma(\ln g_i \, dn_i - \ln n_i \, dn_i - dn_i + dn_i) - \Sigma \alpha \, dn_i - \Sigma \beta E_i \, dn_i \\ &= \Sigma(\ln(g_i/n_i) - \alpha - \beta E_i) \, dn_i \end{aligned}$$

SO, MUST HAVE $\ln(g_i/n_i) - \alpha - \beta E_i = 0$ FOR EACH i

$$\Rightarrow n_i = g_i e^{-\alpha} e^{-\beta E_i}$$

GOING FROM DISCRETE TO CONTINUOUS ENERGY LEVELS, WE GET

$$dN = g(E) e^{-\alpha} e^{-\beta E} dE = \mathcal{C} g(E) e^{-\beta E} dE$$

IN WHICH $E = \frac{1}{2} m u^2$

IT REMAINS TO FIND THE DENSITY OF STATES, $g(E)$.
DO THIS BY CONVERTING FROM u-SPACE TO ENERGY.

IN 3D: $dN = 4\pi u^2 du$ ← 'VOLUME' OF SPHERICAL SHELL OF RADIUS u

$E = \frac{1}{2} m u^2 \Rightarrow \quad = 4\pi \left(\frac{2E}{m}\right) d\left(\frac{2E}{m}\right)^{1/2} = 2\pi \left(\frac{2E}{m}\right)^{1/2} \frac{2}{m} dE \propto E^{1/2} dE$

SO $g(E) \propto E^{1/2}$

THUS WE HAVE $dN \propto E^{1/2} e^{-\beta E} dE = \left(\frac{1}{2} m u^2\right)^{1/2} e^{-\beta(\frac{1}{2} m u^2)} d\left(\frac{1}{2} m u^2\right)$

$= \mathcal{C} u^2 e^{-\beta(\frac{1}{2} m u^2)} du$, FOR SOME CONSTANT $\mathcal{C}$

FINALLY, USE CONSTRAINTS $N = \int dN$, $E_{TOT} = \frac{3}{2} NkT = \int \frac{1}{2} m u^2 dN$
TO FIND $\mathcal{C}$ AND β (COMPARE WITH CALCULATIONS ON p. 68)

$$\Rightarrow dN = 4\pi u^2 N \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mu^2/2kT} du = N(u) \, du \quad \text{(SEE p. 68)}$$

10] ENTROPY AND TEMPERATURE OF A BLACK HOLE

FIRST ESTIMATE RADIUS, R, OF BLACK HOLE BASED ON GRAVITATIONAL CONSTANT G [UNITS $(M\frac{L}{T^2})\frac{L^2}{M^2} = L^3T^{-2}M^{-1}$], MASS [$M$] AND SPEED OF LIGHT c [L/T]

FROM DIMENSIONAL ANALYSIS $R \sim GM/c^2$

ASSUME ENERGY IN BLACK HOLE IS ASSOCIATED WITH TRAPPED PHOTONS OF WAVELENGTH $\lambda \sim R$, i.e. WAVENUMBER $K = \frac{2\pi}{\lambda} \sim R^{-1}$

FREQUENCY IS $\omega = cK \sim c/R$

PHOTON ENERGY IS $\hbar\omega \sim \hbar c/R$

IF TOTAL ENERGY IS $Mc^2 \Rightarrow$ # PHOTONS IS $N \approx \frac{Mc^2}{\hbar c/R} = \frac{McR}{\hbar}$

. BUT $R = GM/c^2$

$\Rightarrow N \sim GM^2/(c\hbar)$

FOR LARGE N, $S = k\ln\Omega = k\ln[f(N)^N]$ FOR SOME $f(N)$)

$\Rightarrow S \approx kN$ TO WITHIN AN ORDER OF MAGNITUDE

SO ENTROPY IS $S \simeq GM^2k/(c\hbar)$

[HAWKING MORE RIGOROUSLY FOUND $S = 4\pi\, GM^2k/(c\hbar)$

$\Rightarrow S \simeq 4\pi\,(6.7\times10^{-11}\ m^3s^{-2}kg^{-1})(2\times10^{30}\ kg)^2(1.38\times10^{-23}\ J/K)/(3\times10^8\ m/s \times 1.05\times10^{-34}\ Js)$

$\simeq 1.5\times10^{54}\ J/K$]

TO FIND TEMPERATURE, RECAST S IN TERMS OF ENERGY $U = Mc^2$

$\Rightarrow S = 4\pi\, G\,(U/c^2)^2\, k/(c\hbar)$

$= 4\pi[Gk/(c^5\hbar)]\,U^2$

SO $\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right) = 8\pi\,[Gk/(c^5\hbar)]\,U \overset{U=Mc^2}{=} 8\pi GMk/(c^3\hbar)$

$\Rightarrow T = \frac{1}{8\pi}c^3\hbar/(GMk)$

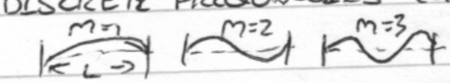
$\simeq 6\times10^{-8}\ K$

BLACKBODY RADIATION

CONSIDER DISTRIBUTION OF ENERGY OF PHOTONS IN A BOX OF TEMPERATURE T

IN 1D, FOR BOX OF LENGTH L, MODES HAVE DISCRETE FREQUENCIES (CYCLES/SECOND)

$$\nu = \frac{c}{\lambda} = \left\{ \frac{c}{2L} m \;\middle|\; m = 1, 2, \ldots \right\}$$



CORRESPONDING ENERGY IS $h\nu$, h - PLANCK'S CONSTANT

IN 3D, FOR CUBICAL BOX $V = L^3$, FREQUENCIES ARE $\left\{ \frac{c}{2L}(m^2+n^2+l^2)^{1/2} \;\middle|\; m, n, l = 1, 2, 3, \ldots \right\}$

FOR LARGE # OF MODES CAN ESTIMATE THE NUMBER OF MODES WITH FREQUENCY BETWEEN $\nu = q\left(\frac{c}{2L}\right)$ AND $\nu + d\nu = (q + dq)\frac{c}{2L}$, WITH $q = \sqrt{m^2+n^2+l^2}$

$$dN = \frac{1}{8}(4\pi q^2)dq = \frac{1}{2}\pi\left(\frac{2L}{c}\right)^3 \nu^2 d\nu = 4\pi V \frac{\nu^2}{c^3} d\nu$$

POSITIVE OCTANT — DENSITY OF STATES

BUT FOR EACH MODE WITH FREQUENCY ν, CAN HAVE 2 POLARIZATIONS OF LIGHT

SO $$dN = 8\pi V \frac{\nu^2}{c^3} d\nu$$

FROM BOLTZMANN DISTRIBUTION, THE PROBABILITY OF HAVING ENERGY E_n IS $\propto \exp(-E_n/kT)$ WHERE $E_n = nh\nu$, $n = 0, 1, 2, \ldots$

SO $$p(n) = \exp(-E_n/kT) \Big/ \sum_{n=0}^{\infty} \exp(-E_n/kT)$$

IS PROBABILITY THE STATE CONTAINS n PHOTONS OF FREQUENCY ν

MEAN ENERGY IS $$\overline{E_\nu} = \sum_{n=0}^{\infty} E_n p(n) = \sum_{n=0}^{\infty} nh\nu \exp[-nh\nu/kT] \Big/ \sum \exp[-nh\nu/kT]$$

$$= \cdots = h\nu \Big/ \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]$$

SO THE PROPORTION OF ENERGY IN MODES WITH FREQUENCY BETWEEN ν AND $\nu + d\nu$ IS $\overline{E_\nu}\, dN = 8\pi V \frac{\nu^2}{c^3} \overline{E_\nu}\, d\nu = U(\nu)\, d\nu$.

WRITE IN TERMS OF THE ENERGY DENSITY $u_\nu = U/V$

$$\Rightarrow u(\nu) = 8\pi \frac{1}{c^3} h\nu^3 \Big/ \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]$$ THE PLANCK DISTRIBUTION

TOTAL ENERGY DENSITY $$u = \int_0^\infty u_\nu\, d\nu = \frac{8\pi h}{c^3} \int_0^\infty \nu^3 \left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1} d\nu$$

LET $x = \frac{h\nu}{kT}$ $\Rightarrow$ $$u = \frac{8\pi h}{c^3}\left(\frac{kT}{h}\right)^4 \underbrace{\int_0^\infty x^3 [e^x - 1]^{-1} dx}_{= \frac{1}{15}\pi^4}$$

$$\Rightarrow u = aT^4, \quad a = \left(\frac{8\pi^5 k^4}{15 c^3 h^3}\right) \simeq 7.57 \times 10^{-16}\ \mathrm{J/(m^3 K^4)}$$

FROM KINETIC THEORY, # PHOTON COLLISIONS/AREA/TIME $= \frac{1}{4} n\bar{u} = \frac{1}{4} nc$

SO ENERGY FLUX IS $$F = \frac{1}{4} n(h\bar{\nu})c = \frac{1}{4} uc = \sigma T^4, \quad \sigma = \frac{1}{4} ca = \left(\frac{2\pi^5 k^4}{15 c^2 h^3}\right) = 5.67 \times 10^{-8}\ \mathrm{W/(m^2 K^4)}$$

GLOBAL WARMING

FROM PLANCK DISTRIBUTION, $u_\nu = 8\pi h \frac{\nu^3}{c^3}\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]^{-1}$; CAN ESTIMATE TEMPERATURE OF SUN'S SURFACE GIVEN SPECTRUM OF LIGHT REACHING EARTH

TYPICALLY $T_{SUN} \sim 5800K$.

SO THE POWER ASSOCIATED WITH LIGHT RADIATION LEAVING SURFACE IS $(\sigma T_{SUN}^4)(4\pi R_S^2)$

BY ENERGY CONSERVATION, THE SAME POWER CROSSES THE SURFACE AREA OF A SPHERE WITH THE RADIUS OF THE EARTH'S ORBIT.

SO THE FLUX INCIDENT UPON THE EARTH IS $F_S = (\sigma T_{SUN}^4)(4\pi R_S^2)/(\pi R_E^2)$

$$\Rightarrow F_S \simeq (5.67\times10^{-8}\,\tfrac{W}{m^2K^4})(5800\,K)^4\,(4\cdot(6.96\times10^5\,km)^2)/(6.38\times10^3\,km)^2$$

$$\simeq 1361\ W/m^2$$ [MEASURED BY SATELLITE]

WE MEASURE $\alpha = 0.3$ OF THIS INCIDENT RADIATION IS REFLECTED,

THE REST $(1-\alpha)F_S$ IS ABSORBED OVER THE EARTH'S SURFACE AREA.

FOR BALANCE, THE EARTH MUST RADIATE THIS BACK TO SPACE.

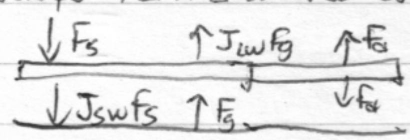
IGNORING ATMOSPHERE, AND TREATING EARTH AS BLACKBODY FOR NON-REFLECTED LIGHT,

$$(\sigma T_E^4)(4\pi R_E^2) = (1-\alpha)F_S(\pi R_E^2)$$

$$\Rightarrow T_E = \left[\frac{1}{4}\frac{1}{\sigma}(1-\alpha)F_S\right]^{1/4} \simeq \left[\frac{1}{4}\frac{1}{5.67\times10^{-8}\,\frac{W}{m^2K^4}}(1-0.3)(1361\,\tfrac{W}{m^2})\right]^{1/4}$$

$$\simeq 255\,K \quad (\simeq -28^\circ C)$$

THIS MUCH LOWER THAN OBSERVED AVERAGE TEMPERATURE OF $\sim 15^\circ C \sim 288K$



ACCOUNT FOR ABSORPTION BY ATMOSPHERE

FOR EQUILIBRIUM $\quad F_S = J_{LW}F_g + F_a \quad$ AT TOP OF ATMOSPHERE

$J_{SW}F_S + F_a = F_g \quad$ AT SURFACE

ELIMINATE $F_a \Rightarrow F_g = F_S\left(\frac{1+J_{SW}}{1+J_{LW}}\right)$

$$\Rightarrow T = \left(\frac{1}{\sigma}F_g\right)^{1/4} = \left(\frac{1}{\sigma}F_S\right)^{1/4}\left(\frac{1+J_{SW}}{1+J_{LW}}\right)^{1/4} = 255K\left(\frac{1+J_{SW}}{1+J_{LW}}\right)^{1/4}$$

ATMOSPHERE MOSTLY TRANSPARENT TO SHORT WAVE RADIATION (EXCEPT UV) $\Rightarrow J_{SW} \approx 0.9$

ATMOSPHERE ABSORBS LONGWAVE RADIATION (CO_2, H_2O, CH_4) $\Rightarrow J_{LW} \simeq 0.2$

SO $T = 255\left(\frac{1+0.9}{1+0.2}\right)^{1/4} \simeq 286K$ CLOSE TO OBSERVED 288K.