1. An automobile tire is inflated to a pressure of 270 kPa at the beginning of a trip. After 3 hours of high-speed driving the pressure is 300 kPa but the volume remains constant at 0.057 m^3.

   a) What is the change of internal energy (in Joules) of air in the tire between the two pressure measurements?

   b) What is the change of enthalpy (in Joules)?

a) USE $PV = nRT$ AND $\Delta U = C_V \Delta T$

$$\Rightarrow T_f - T_i = \frac{1}{nR}(P_f - P_i)V$$

$$C_V = \frac{f}{2}(nR),\ f = 5$$

$$\Rightarrow U_f - U_i = \frac{C_V}{nR}(P_f - P_i)V = \frac{5}{2}(P_f - P_i)V$$

$$= \frac{5}{2}\left[(300 - 270)\times 10^3\ \mathrm{Pa}\right](0.057\,\mathrm{m}^3)$$

$$\Rightarrow \boxed{\Delta U \simeq 4.3\times 10^3\ \mathrm{J}}$$

b) USE $dH = d(U + PV)$

$$\Rightarrow \Delta H = \Delta U + \Delta(PV) = \Delta U + V\Delta P$$

$$\simeq 4.3\times 10^3\ \mathrm{J} + (0.057\,\mathrm{m}^3)\left[(300 - 270)\times 10^3\ \mathrm{Pa}\right]$$

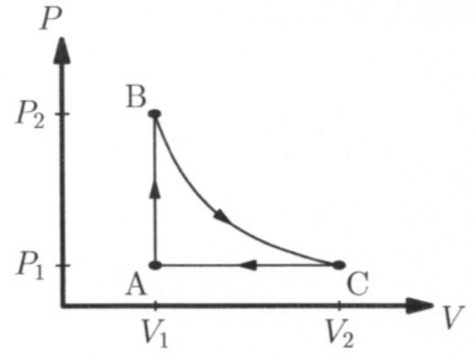
$$\simeq 4.3\times 10^3\ \mathrm{J} + 1.7\times 10^3\ \mathrm{J}$$

$$\Rightarrow \boxed{\Delta H = 6.0\times 10^3\ \mathrm{J}}$$

2. Suppose an ideal diatomic gas operates in a heat engine whose PV cycle, shown to the right, consists of an isochoric process ($A \to B$), an adiabatic process ($B \to C$) and an isobaric process ($C \to A$). Suppose $V_2 = 2V_1$.

   a) Write an expression for P_2 in terms of P_1.

   b) What is the efficiency of this engine?



a) ALONG BC $\quad P \propto V^{-\gamma}$

$$\Rightarrow P = P_1\left(\frac{V}{V_2}\right)^{-\gamma}$$

$$\Rightarrow P_2 = P_1\left(\frac{V_1}{V_2}\right)^{-\gamma} \underset{\gamma=7/5}{\overset{V_2=2V_1}{=}} P_1\left(\frac{1}{2}\right)^{-7/5} = \boxed{P_1\left(2^{7/5}\right)} \simeq 2.64P_1$$

b) WORK IS AREA OF PV DIAGRAM

$$W = \int_{V_1}^{V_2} P_1\left(\frac{V}{V_2}\right)^{-\gamma} dV + \int_{V_2}^{V_1} P_1\, dV$$

$$= P_1\left(\frac{1}{V_2}\right)^{-\gamma}\left[\frac{1}{1-\gamma} V^{1-\gamma}\Big|_{V_1}^{V_2}\right] + P_1(V_1 - V_2)$$

$$= \frac{1}{1-\gamma} P_1\left(\frac{1}{V_2}\right)^{-\gamma}\left[V_2^{1-\gamma} - V_1^{1-\gamma}\right] - P_1(V_2 - V_1)$$

$$= \frac{1}{1-\gamma} P_1 V_2 \left[1 - \left(\frac{V_1}{V_2}\right)^{1-\gamma}\right] - P_1(V_2 - V_1)$$

USE $\gamma = 7/5$, $V_2 = 2V_1$

$$\Rightarrow \quad W = -\frac{5}{2}(2P_1V_1)\left[1 - \left(\frac{1}{2}\right)^{-2/5}\right] - P_1V_1$$

$$= P_1V_1\left[5\left(2^{2/5} - 1\right) - 1\right] = P_1V_1\left[5\cdot 2^{2/5} - 6\right] \quad (\simeq 0.598P_1V_1$$

HEAT IN ALONG A→B (ISOCHORIC PROCESS)

$$\Delta Q = \Delta U = C_V(T_B - T_A) = C_V\left(\frac{P_2V_1}{nR} - \frac{P_1V_1}{nR}\right) = \frac{C_V}{nR}(P_2 - P_1)V_1$$

$$= \frac{5}{2}\left(2^{7/5} - 1\right)P_1V_1 \quad (\simeq 4.098\,P_1V_1)$$

$$\text{EFFICIENCY} = \frac{W}{\Delta Q} = \frac{5\cdot 2^{2/5} - 6}{\frac{5}{2}\left(2^{7/5} - 1\right)} \simeq \frac{0.598}{4.098} \simeq 0{,}146$$

$$\boxed{= 14.6\%}$$

3. This problem demonstrates the importance of the ocean as a heat reservoir.

a) Consider a vertical column of a dry atmosphere having a horizontal cross-sectional area $A = 1\ \text{m}^2$ and assume it is isothermal with temperature $T = 20°\text{C}$ extending from the Earth's surface (where the pressure is 1 atm) to space. Estimate how much energy (in Joules) is required to raise the temperature of this column by 1°C.

b) Now consider a vertical column of the ocean with the same horizontal area A and isothermal temperature T extending down to some depth D. Give an estimate, as a function of D, for the energy required to raise the temperature of this column by 1°C. (For simplicity, suppose the column consists of fresh water and assume D is sufficiently small that you can ignore the thermodynamics of pressure acting upon water.)

c) Equating the energy found in (b) to your result in (a), find the corresponding depth D (in meters) of the ocean which contains as much energy as the entire atmosphere above it.

a) $\Delta Q = C_v \Delta T = c_v m \Delta T$, m IS MASS OF AIR COLUMN

$$m = \int_0^\infty \rho A\, dz = \rho_0 A \int_0^\infty e^{-z/H_\rho} dz , \quad H_\rho = R_a T/g$$

$$= \rho_0 A \left[-H_\rho e^{-z/H_\rho} \Big|_0^\infty \right] = \rho_0 A H_\rho$$

$$\text{So} \quad \Delta Q = c_v (\rho_0 A H_\rho) \Delta T \overset{P=\rho R_a T}{=} c_v \left(\frac{P_0}{R_a T} \cdot A \cdot \frac{R_a T}{g} \right) \Delta T$$

$$= c_v \frac{P_0 A}{g} \Delta T \simeq \left(718 \frac{\text{J}}{\text{kg K}}\right) \left(\frac{1.013 \times 10^5 \text{Pa} \cdot 1\text{m}^2}{9.81\ \text{m/s}^2} \right) (1\text{K})$$

$$\simeq 7.41 \times 10^6\ \text{J}$$

b) $\Delta Q = \overset{\swarrow \text{WATER}}{c_p} m \Delta T = c_p (D A \rho_w) \Delta T$

$$\simeq \left(4186 \frac{\text{J}}{\text{kgK}}\right) \left(1\text{m}^2 \cdot 998 \frac{\text{kg}}{\text{m}^3}\right) (1\text{K})\ D$$

$$\simeq (4.18 \times 10^6\ \text{J/m})\ D$$

c) $(4.18 \times 10^6 \frac{\text{J}}{\text{m}}) D = 7.41 \times 10^6\ \text{J}$

$\Rightarrow D = 1.8\text{m}$

4. A parcel of moist air situated at the base of a cloud has temperature 10°C, pressure 900 mbar and 100% humidity.

   a) What would be the temperature of the air if brought adiabatically directly to the ground where the pressure is 1000 mbar? [Give your answer in °C.]

   b) What would be the temperature of the air if it was lifted adiabatically until all the water vapour in it condensed, and then the resulting dry air was brought adiabatically to the ground where the pressure is 1000 mbar? [Give your answer in °C.]

a) $\theta = T\left(\frac{P}{P_{00}}\right)^{-x} \simeq (283.15\,K)\left(\frac{900}{1000}\right)^{-2/7} \simeq (283.15K)(1{,}03056)$

$\simeq 291{,}80\,K \boxed{= 18.7^\circ C}$

THIS IS TEMPERATURE AT "GROUND"

b) $\theta_e = \theta \, EXP\left[\frac{L_V W_S}{C_p T}\right]$

WHERE $W_S = \frac{\varepsilon_W}{P} e_s = \frac{\varepsilon_W}{P} e_0 \, EXP\left[\frac{L_V}{R_V}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$

$\simeq \frac{0{,}622}{9{,}00 \times 10^4\,Pa}(611\,Pa)\,EXP\left[\frac{2.50\times10^6}{462}\left(\frac{1}{273.15} - \frac{1}{283.15}\right)\right]$

$\simeq 4.2\times10^{-3}\,EXP\left[5.4\times10^3\left(1{,}29\times10^{-4}\right)\right] = 4.2\times10^{-3}\,EXP[0.698]$

$\simeq 8.4\times10^{-3}$

So $\theta_e \simeq (291{,}80)\,EXP\left[\frac{2.50\times10^6}{1005} \cdot \frac{0{,}0084}{283.15}\right]$

$\simeq (291.80)\,EXP[0{,}0738]$

$\simeq 314{,}15\,K$

$\boxed{\simeq 41.0^\circ C}$

5. A simple model of a polymer (a long-chained molecule as in rubber), assumes it forms a one dimensional chain made up of N identical elements, each of length ℓ. The angle between successive elements can be either 0° or 180°, but there is no difference in internal energy between these two possibilities. For the sake of counting, one can think of each element as either pointing to the right (+, if oriented at 0° from the previous element) or to the left (−, if oriented at 180° from the previous element). Then one has $N = n_+ + n_-$ for the total number of elements and $L = \ell(n_+ - n_-)$ for the length of the chain to the right of the first element. Assume N, n_+ and n_- are all *very* large.

   a) Find the entropy, S, as a function of N, n_+ and k.

   b) The relationship between entropy and the tension, F, on the chain is given by the first law of thermodynamics: $0 = TdS + FdL$. Use this to find an explicit expression for the tension as a function of N, n_+, T, k and ℓ.

   c) Using your result in b), show that in the high temperature limit (for which $L \ll N\ell$) the chain satisfies Hookes Law: $F \propto L$. Explicitly find the proportionality constant as it depends upon N, T, k and ℓ.

a) $S = k \operatorname{Ln} \Omega$

WHERE $\Omega = \binom{N}{n_+} = \dfrac{N!}{n_+!(N-n_+)!}$

$$\Rightarrow S = k[\operatorname{Ln} N! - \operatorname{Ln} n_+! - \operatorname{Ln}(N-n_+)!]$$

$$\Rightarrow \boxed{S \simeq k[N \operatorname{Ln} N - n_+ \operatorname{Ln} n_+ - (N-n_+)\operatorname{Ln}(N-n_+)]} \quad (*)$$

b) FROM $L = \ell(n_+ - n_-) = \ell(n_+ - (N-n_+)) = \ell(2n_+ - N)$

$$\Rightarrow n_+ = \tfrac{1}{2}\left(\tfrac{L}{\ell} + N\right)$$

PUT IN (*)

$$\Rightarrow S = k\left[N \operatorname{Ln} N - \left(\tfrac{1}{2}\left(\tfrac{L}{\ell}+N\right)\right)\operatorname{Ln}\left(\tfrac{1}{2}\left(\tfrac{L}{\ell}+N\right)\right) - \left(\tfrac{1}{2}\left(N-\tfrac{L}{\ell}\right)\right)\operatorname{Ln}\left(\tfrac{1}{2}\left(N-\tfrac{L}{\ell}\right)\right)\right] \quad (**)$$

$$0 = T dS + F dL \Rightarrow \frac{dS}{dL} = -\frac{F}{T}$$

$$(**) \Rightarrow -\frac{F}{T} = k\left[-\frac{1}{2\ell}\operatorname{Ln}\left(\tfrac{1}{2}\left(\tfrac{L}{\ell}+N\right)\right) - \frac{1}{2\ell} + \frac{1}{2\ell}\operatorname{Ln}\left(\tfrac{1}{2}\left(N-\tfrac{L}{\ell}\right)\right) + \frac{1}{2\ell}\right]$$

$$= k\frac{1}{2\ell}\operatorname{Ln}\left[\frac{\frac{1}{2}\left(N-\frac{L}{\ell}\right)}{\frac{1}{2}\left(\frac{L}{\ell}+N\right)}\right] = k\frac{1}{2\ell}\operatorname{Ln}\left[\frac{N-L/\ell}{N+L/\ell}\right]$$

$$\Rightarrow \boxed{F = kT\frac{1}{2\ell}\operatorname{Ln}\left[\frac{N+L/\ell}{N-L/\ell}\right]}$$

c) $$F = kT\frac{1}{2\ell}\operatorname{Ln}\left[\frac{1+\frac{L}{N\ell}}{1-\frac{1}{NL}}\right] = kT\frac{1}{2\ell}\left[\operatorname{Ln}\left(1+\frac{L}{N\ell}\right) - \operatorname{Ln}\left(1-\frac{L}{N\ell}\right)\right]$$

$$\simeq kT\frac{1}{2\ell}\left[\frac{L}{N\ell} - \left(-\frac{L}{N\ell}\right)\right] = \left(kT\frac{1}{N\ell^2}\right)L$$

So $F \propto L$ WITH PROPORTIONALITY CONSTANT $\boxed{\dfrac{kT}{N\ell^2}}$

6. The energy density of photons in a box (a "photon gas") at temperature T is

$$u = \frac{4}{c}\sigma T^4$$

in which c is the speed of light and σ is the Stefan-Boltzmann constant.

a) What is the heat capacity of photons in a box of volume V? (Give your answer in terms of T, V, c and σ.)

b) Assuming this formula is accurate all the way down to absolute zero, what is the entropy of the photon gas, with $S(T)$ defined so that $S(0) = 0$?

c) Use $P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$ to find the pressure of a photon gas as a function of the energy density, u.

a) $C_V = \left(\frac{\partial U}{\partial T}\right)_V$ WITH $U = uV$

$$\Rightarrow C_V = V\left(\frac{\partial}{\partial T}\left(\frac{4}{c}\sigma T^4\right)\right) = \boxed{V\frac{16}{c}\sigma T^3}$$

b) $dU = T\,dS$

WRITE $dU = \left(\frac{\partial U}{\partial T}\right)_V dT$

$$\Rightarrow dS = \frac{1}{T}\left[V\frac{16}{c}\sigma T^3\right]dT$$

$$\Rightarrow S = \frac{16}{3}\frac{V\sigma}{c}T^3 \qquad (*)$$

c) $P = -\left(\frac{\partial}{\partial V}(U)\right)_S$

ENSURE S IS FIXED AS V CHANGES

$$(*) \Rightarrow T = \left[\frac{3c}{16V\sigma}S\right]^{1/3}$$

$$\Rightarrow U = uV = \frac{4}{c}\sigma\left[\frac{3c}{16\sigma}\frac{S}{V}\right]^{4/3}V = \frac{4}{c}\sigma\left[\frac{3c}{16\sigma}S\right]^{4/3}V^{-1/3}$$

$$\Rightarrow P = -\left[-\frac{1}{3}V^{-4/3}\frac{4}{c}\sigma\left[\frac{3c}{16\sigma}S\right]^{4/3}\right]$$

$$= \frac{1}{3}\frac{4}{c}\sigma\left[\frac{3c}{16\sigma}\frac{S}{V}\right]^{4/3} = \frac{1}{3}\frac{4}{c}\sigma T^4$$

$$\Rightarrow \boxed{P = \frac{1}{3}u}$$