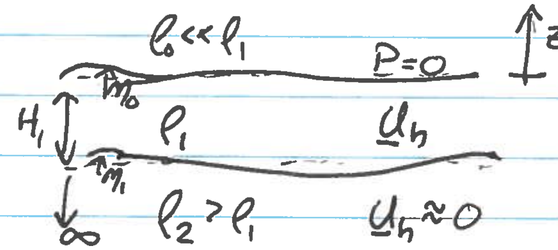
5] Waves in Stratified Fluid

So far have considered waves in uniform-density fluid characterized by surface displacement, η.

Now consider waves that move <u>within</u> a fluid whose density changes with height.

(1) $1\frac{1}{2}$ Layer Fluid

This describes a 2-layer fluid in which one layer is effectively infinitely deep.



The approximation is often good in describing undulations of the oceanic thermocline or atmospheric inversions.
Here we consider motions that are long compared to shallow layer

- In upper layer

Hydrostatic balance $\Rightarrow$ $P_1(z) = g\rho_1(\eta_0 - z)$

So horizontal Mom$^{\underline{n}}$ Eq^{n} is $\frac{D}{Dt}\underline{u}_h + \underline{f}\times\underline{u}_h = -\frac{1}{\rho_1}\nabla_h p = -g\nabla_h\eta_0$ (1)

- In lower layer

$$P_2(z) = P_1(-H_1+\eta_1) + \rho_2 g(\eta_1 - z) = g\rho_1(\eta_0+H_1-\eta_1) + \rho_2 g(\eta_1 - z)$$
$$= g\rho_1(\eta_0+H_1) + g(\rho_2-\rho_1)\eta_1 - g\rho_2 z$$
$$= \rho_1\left[g(\eta_0+H_1) + g'\eta_1\right] - g\rho_2 z \qquad (*)$$

where we have defined $\boxed{g' = g(\rho_2-\rho_1)/\rho_1}$ the "reduced gravity"

Because lower layer is ∞ deep, $\underline{u}_h \approx 0$ below $z = -H + \eta_1$
$\Rightarrow \nabla_h P_2 \approx 0$

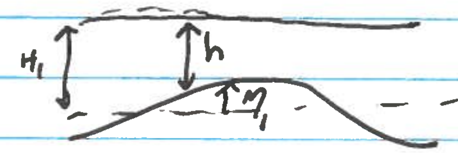
So (*) $\Rightarrow$ $g\nabla_h\eta_0 + g'\nabla_h\eta_1 \approx 0$ $\Rightarrow$ $g\eta_0 + g'\eta_1 \approx$ constant

But $g' \ll g$ since typically $\rho_2 \gtrsim \rho_1$ $\Rightarrow \|\eta_0\| \ll \|\eta_1\|$: surface $\approx$ flat

So rewrite (1) in terms of η_1: $\frac{D\underline{u}_h}{Dt} + \underline{f}\times\underline{u}_h = +g'\nabla_h\eta_1$ (1)'

① (CONT'D)

FINALLY, RECAST RESULT IN TERMS OF UPPER-LAYER THICKNESS: $h = H_1 + \eta_0 - \eta_1 \approx H_1 - \eta_1$



SO ①' BECOMES $$\frac{D\underline{u}_h}{Dt} + \underline{f} \times \underline{u}_h = -g' \nabla_h h$$

OR, EXPLICITLY FOR SMALL-AMPLITUDE DISTURBANCES $(\|\eta_1\| \ll H_1)$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} - fv &= -g' \frac{\partial h}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -g' \frac{\partial h}{\partial y} \end{aligned}} \qquad \begin{aligned} &\text{(1a)} \\ &\text{(1b)} \end{aligned}$$

[COMPARE WITH SURFACE WAVE SHALLOW WATER EQ$^{\underline{N}}$S ON p.62 OF NOTES:

$$\frac{\partial u}{\partial t} - fv = -g \frac{\partial \eta}{\partial x} \; ; \; \frac{\partial v}{\partial t} + fu = -g \frac{\partial \eta}{\partial y}$$

SAME EQUATIONS EXCEPT $g \Rightarrow g' = g\left(\frac{\rho_2 - \rho_1}{\rho_1}\right)$ AND $\eta \Rightarrow h$]

LIKEWISE, MASS CONSERVATION IN THE THIN LAYER OF A $1\frac{1}{2}$ LAYER FLUID GIVES $\frac{Dh}{Dt} + h \nabla_h \cdot \underline{u}_h = 0$.

FOR SMALL-AMPLITUDE DISTURBANCES $\Rightarrow$

$$\boxed{\frac{\partial h}{\partial t} + H_1 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0} \qquad (2)$$

WHICH IS ANALOGOUS TO MASS CONSERVATION FOR SURFACE SHALLOW WATER EQUATIONS WITH $\eta \Rightarrow h$ AND $H \Rightarrow H_1$.

THE ANALOGY WITH SURFACE WAVES MEANS ALL THE RESULTS FOR INERTIAL (POINCARÉ), COASTAL KELVIN, ROSSBY AND EQUATORIAL WAVES ARE STRAIGHTFORWARDLY ADAPTED TO DESCRIBE INTERFACIAL WAVES IN A TWO-LAYER FLUID IN WHICH ONE LAYER IS MUCH THINNER THAN THE OTHER: SIMPLY REPLACE g WITH g' AND H WITH H_1

e.g. POINCARÉ WAVES:

DISP$^{\underline{N}}$ RELATION: $\omega^2 = c^2 |\underline{k}|^2 + f_0^2$ WITH $c = \sqrt{g' H_1}$

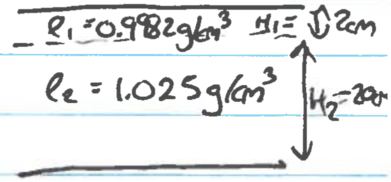
$\Rightarrow$ DEFORMATION RADIUS: $L_D = \frac{c}{|f_0|} = \sqrt{g' H_1} / |f_0|$

ENERGY $\langle E \rangle = \frac{1}{2} g' \eta_1^2 + \frac{1}{2} \frac{f_0^2}{k^2 H_1} \eta_1^2 = (\omega^2 / 2k^2 H_1) \eta_1^2 = \langle \underline{F}_E \rangle / c_g$

1 (CONT'D)

SPECIFIC EXAMPLES

(1) IN A TANK EXPERIMENT A 2cm DEEP LAYER OF FRESH WATER OVERLIES A 20cm DEEP LAYER OF SALT WATER WITH DENSITY $\rho_2 = 1.025\ \mathrm{g/cm^3}$



TREAT AS 1½ LAYER FLUID BECAUSE $H_2 \gg H_1$

$\Rightarrow$ SHALLOW WATER SPEED IS $c = \sqrt{g' H_1} = \left[(980\ \mathrm{cm/s^2})\left(\frac{1.025-0.998}{0.998}\right)(2\mathrm{cm})\right]^{1/2}$

$\Rightarrow c \simeq 7\ \mathrm{cm/s}$ (MORE ACCURATELY $c \simeq 7.3\ \mathrm{cm/s}$)

IF TANK ROTATES AT FREQUENCY $\Omega = 0.5\ \mathrm{s^{-1}} \Rightarrow L_D = \frac{c}{|f_0|} \approx \frac{7\ \mathrm{cm/s}}{2(0.5\mathrm{s^{-1}})} = 7\mathrm{cm}.$

So CRITICAL WAVENUMBER IS $k_c = L_D^{-1}$.

HENCE WAVES ARE STRONGLY INFLUENCED BY ROTATION IF THEIR WAVELENGTH $\lambda = \frac{2\pi}{k} > \frac{2\pi}{k_c} \approx 1\ \mathrm{cm}.$

(2) FOR THERMOCLINE IN OCEAN, SUPPOSE $H_1 = 100\mathrm{m}$ AND $\rho_1 \simeq 1024\ \mathrm{kg/m^3}$, $\rho_2 = 1028\ \mathrm{kg/m^3}$

SINCE OCEAN DEPTH OF $\approx 4\mathrm{km} \gg 100\mathrm{m} \Rightarrow$ 1½ LAYER FLUID

So $c = \sqrt{g' H_1} = \left[(9.80\ \mathrm{m/s^2})\left(\frac{1028-1024}{1024}\right)(100\mathrm{m})\right]^{1/2}$
$\simeq 2\ \mathrm{m/s}$

AT MID-LATITUDES $f_0 \approx 10^{-4}\ \mathrm{s^{-1}}$. So $L_D \simeq \frac{c}{|f_0|} \approx 2\times10^4\mathrm{m} = 20\mathrm{km}$

(THIS IS TYPICAL VALUE FOR "INTERNAL DEFORMATION RADIUS")

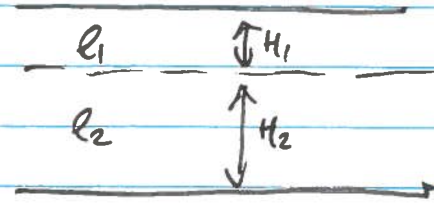
(3) AROUND THE EQUATOR FOUND THE EQUATORIAL DEFORMATION RADIUS TO BE $L_\beta = \sqrt{c/\beta} = (gH/\beta^2)^{1/4}$.

IN A 1½ LAYER FLUID $L_\beta = (c/\beta)^{1/2} = (g' H_1/\beta^2)^{1/4} \simeq \left((2\mathrm{m/s})/(2\times10^{-11}\tfrac{1}{\mathrm{ms}})\right)^{1/2}$

$\Rightarrow L_\beta \simeq 300\mathrm{km}$

INDEED, PHENOMENA SUCH AS WARM & COLD TONGUES ASSOCIATED WITH EL NIÑO & LA NIÑA ARE TYPICALLY CONFINED TO ~ 3° LATITUDE FROM EQUATOR.

② 2-LAYER FLUID



GENERALLY, IF THE DEPTH OF ONE LAYER IS NOT TOO DIFFERENT FROM THE OTHER, THEN THE SHALLOW WATER SPEED IS $\boxed{c = \sqrt{g'\bar{H}}}$

IN WHICH $g' = g\frac{\rho_2 - \rho_1}{\rho_1}$, AS BEFORE AND $\bar{H} = \frac{H_1 H_2}{H_1 + H_2}$

OF COURSE, IF $H_2 \gg H_1$ THEN $\bar{H} \simeq \frac{H_1 H_2}{H_2} = H_1$

IF THE LOWER LAYER IS SHALLOW $(H_1 \gg H_2)$: $\bar{H} \simeq \frac{H_1 H_2}{H_1} = H_2$

EXAMPLES

① GOING BACK TO EXAMPLE ① ON p. 75 WITH $H_1 = 2\,cm$, $H_2 = 20\,cm$

FIND $\bar{H} = (2\,cm)(20\,cm)/(22\,cm) \simeq 1.82\,cm$

So $c = \sqrt{g'\bar{H}} = \left[(980\,cm/s^2)\left(\frac{1.025 - 0.998}{0.998}\right)(1.82\,cm)\right]^{1/2}$
$\simeq 6.9\,cm/s$

THIS AGREES TO WITHIN 5% OF RESULT FOR 1½ LAYER FLUID.

② CONSIDER THERMOCLINE IN LESSER SLAVE LAKE IN WHICH TOP 10m IS 10°C WARMER THAN WATER BELOW THERMOCLINE WHERE THE TOTAL DEPTH IS 40m.

⇒ FROM THERMAL EXPANSION COEFFICIENT $\alpha_T \simeq 2.1 \times 10^{-4}\,K^{-1}$, ESTIMATE

$$g' = g(\Delta\rho/\rho_0) \simeq g\frac{1}{\rho_0}\left[(\rho_0) - (\rho_0(1 - \alpha_T \Delta T))\right] = g(\alpha_T \Delta T)$$
$$\simeq (9.8\,m/s^2)(2.1 \times 10^{-4}\,K^{-1}\,10\,K) \simeq 2.1 \times 10^{-2}\,m/s^2$$

AND $\bar{H} = (10m)(40m - 10m)/(40m) \simeq 7.5m$

So $c \simeq \sqrt{g'\bar{H}} \simeq 0.25\,m/s$.

NOTE $L_D \simeq c/|f_0| \simeq 0.25/10^{-4} \simeq 2.5\,km$. SO WAVES ALONG LENGTH FEEL ROTATION.

3) UNIFORMLY STRATIFIED FLUID

NOW SUPPOSE THE DENSITY/POTENTIAL TEMPERATURE VARIES CONTINUOUSLY WITH HEIGHT RATHER THAN JUMPING FROM ρ_1 TO ρ_2 AT SOME DEPTH H_1.

FOR SIMPLICITY, SUPPOSE THE FLUID IS "UNIFORMLY STRATIFIED"
$\Rightarrow$ $N^2 = N_0^2$ IS CONSTANT, IN WHICH $N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}$ FOR OCEAN AND $N^2 = +\frac{g}{\theta}\frac{d\bar{\theta}}{dz}$ FOR ATMOSPHERE

ALSO MAKE THE "BOUSSINESQ APPROXIMATION", SO THAT DENSITY VARIATIONS ARE ASSUMED SMALL EXCEPT IN BUOYANCY TERM.

SO, EQUATIONS OF MOTION ARE (SEE P.23 OF NOTES)

$$\nabla \cdot \underline{u} = 0$$
$$\frac{D\underline{u}_h}{Dt} + \underline{f} \times \underline{u}_h = -\frac{1}{\rho_0}\nabla_h p$$
$$\frac{Dw}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b$$
$$\frac{Db}{Dt} = -N_0^2 w$$

IN WHICH b $\left(= -g\frac{\rho}{\rho_0} \text{ FOR OCEAN}; = +g\frac{\theta}{\theta_0} \text{ FOR ATMOSPHERE}\right)$ IS BUOYANCY.

FOR SMALL AMPLITUDE DISTURBANCES OF f-PLANE

$$\Rightarrow \left\{ \begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ &\frac{\partial u}{\partial t} - f_0 v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ &\frac{\partial v}{\partial t} + f_0 u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\ &\frac{\partial w}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b \\ &\frac{\partial b}{\partial t} = -N_0^2 w \end{aligned} \right. \Rightarrow \left\{ \begin{pmatrix} \partial_t & -f_0 & 0 & 0 & +\frac{1}{\rho_0}\partial_x \\ f_0 & \partial_t & 0 & 0 & +\frac{1}{\rho_0}\partial_y \\ 0 & 0 & \partial_t & -1 & +\frac{1}{\rho_0}\partial_z \\ 0 & 0 & N_0^2 & \partial_t & 0 \\ \partial_x & \partial_y & \partial_z & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ b \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right.$$

ASSUMING $u = A_u e^{i(kx+ly+mz-\omega t)}$ etc

$$\Rightarrow \begin{pmatrix} -i\omega & -f_0 & 0 & 0 & \frac{1}{\rho_0}ik \\ f_0 & -i\omega & 0 & 0 & \frac{1}{\rho_0}il \\ 0 & 0 & & -1 & \frac{1}{\rho_0}im \\ 0 & 0 & N_0^2 & -i\omega & 0 \\ ik & il & im & 0 & 0 \end{pmatrix} \begin{pmatrix} A_u \\ A_v \\ A_w \\ A_b \\ A_p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

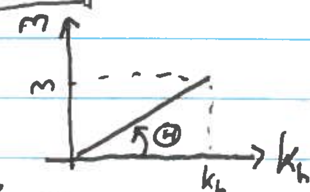
3 (cont'd)

TAKING DETERMINANT GIVES THE DISPERSION RELATION FOR "INTERNAL GRAVITY WAVES":

$$\boxed{\omega^2 = \frac{N_0^2(k^2+l^2)+f_0^2m^2}{k^2+l^2+m^2} = N_0^2\cos^2\Theta + f_0^2\sin^2\Theta} \qquad (*)$$

IN WHICH $\Theta = \tan^{-1}(m/k_h)$ WITH $k_h = \sqrt{k^2+l^2}$

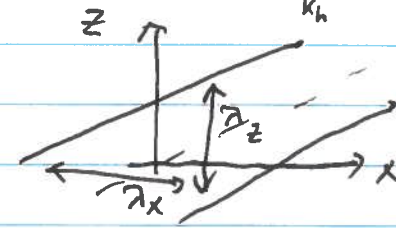
SO THAT $\cos\Theta = k_h/|\underline{k}|$ AND $\sin\Theta = m/|\underline{k}|$



IN TERMS OF PHASE LINES IN REAL SPACE

Θ IS THE ANGLE TO THE VERTICAL

e.g. $\tan\Theta = \frac{m}{k} = \frac{2\pi/\lambda_z}{2\pi/\lambda_x} = \frac{\lambda_x}{\lambda_z}$



SPECIAL CASES (FOR WAVES IN X-Z PLANE ⇒ $l=0$)

A] NEGLECTING ROTATION

IF $|N_0 k| \gg |f_0 m|$ (e.g. $\left|\frac{k}{m}\right| \gg \frac{f_0}{N_0} \approx 10^{-2}$)

$\Rightarrow \omega \approx N_0 k/\sqrt{k^2+m^2} = N_0\cos\Theta$

PHASE VELOCITY: $\underline{c}_p = \frac{\omega}{|\underline{k}|}\hat{k} = \frac{\omega}{|\underline{k}|^2}\underline{k} = \frac{N_0 k}{|\underline{k}|}\frac{1}{|\underline{k}|^2}(k,m) = \frac{N_0 k}{|\underline{k}|^3}(k,m)$

GROUP VELOCITY: $\underline{c}_g = \nabla_k\omega = \left(\frac{\partial\omega}{\partial k},\frac{\partial\omega}{\partial m}\right) = \left(N_0/|\underline{k}| - \frac{N_0 k^2}{|\underline{k}|^3}, \frac{N_0 k}{|\underline{k}|^3}(-m)\right)$

$= \frac{N_0 k m}{|\underline{k}|^3}(m,-k)$

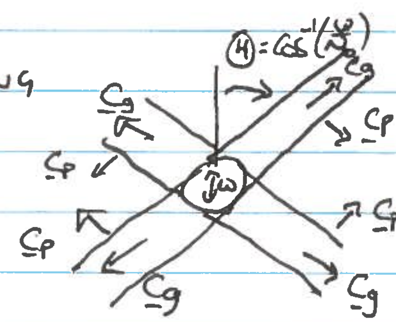
SO $\underline{c}_p \cdot \underline{c}_g = 0$ ⇒ PHASE & GROUP VELOCITY ARE PERPENDICULAR!

ALSO $c_{pz} = \frac{N_0 k m}{|\underline{k}|^3}$ AND $c_{gz} = -\frac{N_0 k m}{|\underline{k}|^3}$

SO IF PHASE GOES UP, GROUP GOES DOWN.

THESE DYNAMICS CAN BE SEEN BY OSCILLATING A HORIZONTAL CYLINDER IN STRATIFIED FLUID AT FREQUENCY $\omega < N_0$.

THE WAVES EMANATE IN THE PATTERN OF A CROSS.



③ (cont'd)

B] Rotation important

Now suppose $\left|\frac{k}{m}\right| \lesssim f_0/N_0 \approx 10^{-2}$.

So $\omega^2 = \dfrac{N_0^2 k^2 + f_0^2 m^2}{k^2+m^2} \approx \dfrac{N_0^2 k^2 + f_0^2 m^2}{m^2} = \left(\dfrac{N_0}{m}\right)^2 k^2 + f_0^2$

This is dispersion relation for "inertia gravity waves"

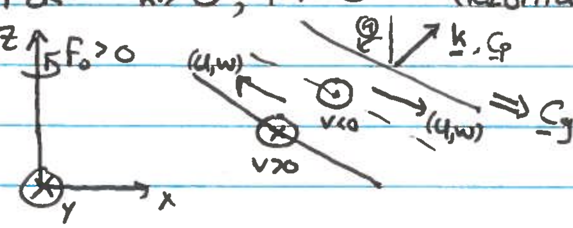
Compare with Poincaré (inertial) waves on shallow water $\omega^2 = c^2 k + f_0^2$. (with $c = \sqrt{gH}$ or $c = \sqrt{g'H}$ 2 layer)

Thus we can heuristically convert our shallow water results to hydrostatic waves in uniform stratification by replacing c with (N_0/m)

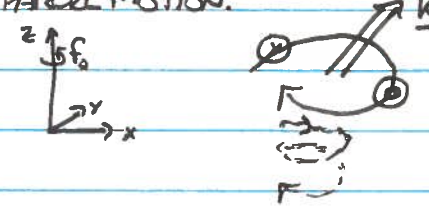
For polarization relations, suppose $\eta = \eta_0 \cos(kx+mz-\omega t)$ is vertical displacement field. So the velocity fields are

$$u = -\omega \frac{m}{k} \eta_0 \sin(kx+mz-\omega t)$$
$$v = f_0 \frac{m}{k} \eta_0 \cos(kx+mz-\omega t)$$
$$w = \omega \eta_0 \sin(kx+mz-\omega t)$$

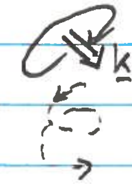
E.g. for $k>0$, $m>0$ (rightward & upward phase / downward group)



Parcel motion.



As with Poincaré waves, fluid parcels rotate anticyclonically at fixed depth. Motion is clockwise with increasing depth if $m>0$.



(If $m<0$, parcels still anticyclonic in horizontal, but rotation is anticlockwise with depth

(3) B] (cont'd)

MOMENTUM TRANSPORT BY INERTIA-GRAVITY WAVES

FROM X-MOMENTUM EQUATION: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - f_0 v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$

AND INCOMPRESSIBILITY $\nabla \cdot \underline{u} = 0$, CAN WRITE IN "FLUX FORM":

$$\frac{\partial u}{\partial t} + \partial_x(uu) + \partial_y(vu) + \partial_z(wu) - f_0 v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

NOW USE POLARIZATION RELATIONS, AVERAGING OVER A WAVELENGTH IN X: $\langle \cdot \rangle = \frac{1}{\lambda_x}\int_0^{\lambda_x} \cdot \, dx$

$$\Rightarrow \frac{\partial \langle u \rangle}{\partial t} + \underbrace{0}_{\langle \cdot \rangle_x = 0} + \underbrace{0}_{u \,\&\, v \text{ out of phase}} + \frac{\partial}{\partial z}\langle uw \rangle - \underbrace{0}_{\langle v \rangle = 0} = \underbrace{0}_{\langle \frac{\partial p}{\partial x} \rangle = 0}$$

i.e. THE FLOW ACCELERATES DUE TO A DIVERGENCE IN THE VERTICAL FLUX OF HORIZONTAL MOMENTUM PER MASS

$$\frac{\partial \langle u \rangle}{\partial t} = -\frac{\partial}{\partial z}\langle uw \rangle$$

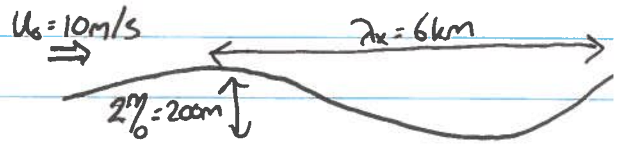
EXPLICITLY: $\langle uw \rangle = \frac{1}{2}\left(-\omega \frac{m}{k}\eta_0\right)(\omega \eta_0) = -\frac{1}{2}\omega^2 \frac{m}{k}\eta_0^2$

WHICH IS POSITIVE FOR RIGHTWARD WAVES IF $m < 0$

IN TERMS OF $\Theta = \tan^{-1}\left(\frac{m}{k}\right)$, $\langle uw \rangle = -\frac{1}{2}\left(N_0^2 \cos^2\Theta + f_0^2 \sin^2\Theta\right)\tan\Theta \; \eta_0^2$

EXAMPLE:

CONSIDER STEADY $U_0 = 10\,m/s$ WIND BLOWING OVER HILLS WITH CREST-CREST DISTANCE 6 km, AND VALLEY-TO-CREST HEIGHT 200 m, AND AMBIENT STRATIFICATION $N_0 = 2\times10^{-2}\,s^{-1}$, $f_0 = 10^{-4}\,s^{-1}$



SOL^N

$k = \frac{2\pi}{\lambda_x} \sim 10^{-3}\,m^{-1}$

FORCING FREQUENCY: $\omega = U_0 k \sim 10^{-2}\,s^{-1}$

WAVE PHASE LINES AT ANGLE $|\Theta| = \tan^{-1}\left(\sqrt{\frac{N_0^2 - \omega^2}{\omega^2 - f_0^2}}\right) \simeq \tan^{-1}(\sqrt{3}) = 60^0$

SINCE $\omega \gg f_0$, ROTATION IS NEGLIGIBLE

$\langle uw \rangle = \frac{1}{2}N_0^2 \cos\Theta \sin\Theta \, \eta_0^2 = \frac{1}{2}(2\times10^{-2}\,s^{-1})^2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)(100\,m)^2 \simeq 0.8\,m^2/s^2$

IF THE WAVES BREAK OVER 1 km HEIGHT, ACCELERATION IS $\frac{\partial \langle u \rangle}{\partial t} \sim \frac{0.8\,m^2/s^2}{1000\,m} \simeq 8\times10^{-4}\,m/s^2$

$\Rightarrow$ AFTER 1 HOUR THE WIND SPEED INCREASES BY $8\times10^{-4}\,m/s^2 \times 3600\,s \approx 3\,m/s$ //

4 ANELASTIC INTERNAL WAVES

NOW CONSIDER INTERNAL WAVES THAT PROPAGATE VERTICALLY IN THE ATMOSPHERE OVER DISTANCES LARGE ENOUGH THAT THE BACKGROUND DENSITY DECREASES SUBSTANTIALLY WITH HEIGHT. (i.e. DISTANCES $> H_\rho \approx 7\,km$)

FOR SIMPLICITY, NEGLECT ROTATION AND ASSUME 2D IN X-Z PLANE.

EQUATIONS OF MOTION (ANELASTIC APPROXIMATION)

$$\frac{Du}{Dt} = -\frac{\partial}{\partial x}(P/\bar\rho)$$
$$\frac{Dw}{Dt} = -\frac{\partial}{\partial z}(P/\bar\rho) + \frac{g}{\bar\theta}\theta \quad \overset{\text{SMALL AMPL.}}{\Rightarrow}$$
$$\frac{D\theta}{Dt} = -w\frac{d\bar\theta}{dz}$$
$$\nabla\cdot(\bar\rho\,\underline{u}) = 0$$

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(P/\bar\rho) \qquad (1)$$
$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}(P/\bar\rho) + \frac{g}{\bar\theta}\theta \qquad (2)$$
$$\frac{\partial \theta}{\partial t} = -w\frac{d\bar\theta}{dz} \qquad (3)$$
$$\nabla\cdot(\bar\rho\,\underline{u}) = 0 \qquad (4)$$

WHERE $\bar\rho(z)$ AND $\bar\theta(z)$ ARE BACKGROUND DENSITY AND POTENTIAL TEMPERATURE, RESPECTIVELY.

FROM (4) DEFINE THE "MASS STREAMFUNCTION", Ψ, SO THAT

$$u = -\frac{1}{\bar\rho}\frac{\partial\Psi}{\partial z}, \quad w = +\frac{1}{\bar\rho}\frac{\partial\Psi}{\partial x} \qquad (5)$$

ALSO, FIND VORTICITY EQUATION FROM (1) + (2) BY TAKING CURL AND DEFINING SPANWISE VORTICITY $\zeta \equiv \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$

$$\Rightarrow \frac{\partial\zeta}{\partial t} = -\frac{g}{\bar\theta}\frac{\partial\theta}{\partial x}$$

ELIMINATE θ USING (3)

$$\Rightarrow \frac{\partial^2}{\partial t^2}\zeta = -\frac{g}{\bar\theta}\frac{\partial^2\theta}{\partial x\partial t} = -\frac{g}{\bar\theta}\left(-\frac{d\bar\theta}{dz}\frac{\partial w}{\partial x}\right) = \frac{g}{\bar\theta}\frac{d\bar\theta}{dz}\left(\frac{1}{\bar\rho}\frac{\partial^2\Psi}{\partial x^2}\right) \qquad (*)$$

FINALLY, WRITE ζ IN TERMS OF Ψ USING (5):

$$\zeta = \frac{\partial}{\partial z}\left(-\frac{1}{\bar\rho}\frac{\partial\Psi}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{1}{\bar\rho}\frac{\partial\Psi}{\partial x}\right) = -\frac{1}{\bar\rho}\left(\nabla^2\Psi - \frac{\bar\rho'}{\bar\rho}\frac{\partial\Psi}{\partial z}\right)$$

$$\text{So } (*) \Rightarrow \boxed{\frac{\partial^2}{\partial t^2}(\nabla^2\Psi) + \frac{1}{H_\rho}\frac{\partial^3\Psi}{\partial t^2\partial z} + N^2\frac{\partial^2\Psi}{\partial x^2} = 0} \qquad (**)$$

WHERE $N^2 = \frac{g}{\bar\theta}\frac{d\bar\theta}{dz}$ IS SQUARED BOUYANCY FREQUENCY AND $H_\rho \equiv -\left(\frac{\bar\rho'}{\bar\rho}\right)^{-1}$ IS DENSITY SCALE HEIGHT.

NOTE: (**) REDUCES TO BOUSSINESQ EQUATION IF $H_\rho \to \infty$

4 (cont'd)

Now assume air is uniformly stratified. (E.g. for isothermal atmosphere $\frac{d\bar{p}}{dz} = -\bar{\rho} g$ and $\bar{p} = \bar{\rho} R_d T_0 \Rightarrow \bar{\rho} = \rho_0 e^{-z/H_\rho}$, $\bar{p} = p_0 e^{-z/H_\rho}$ with $H_\rho = RT_0/g$ and $\bar{\theta} = T_0 (\bar{p}/p_0)^{-\chi} = T_0 e^{z(\chi/H_\rho)} \Rightarrow N^2 = \frac{g\chi}{H_\rho}$)

So coefficients of (**) are all constants. So can look for solutions of form $\Psi(x,z,t) = A_\Psi \exp[i(kx + Mz - \omega t)]$. ①

$$\Rightarrow (-i\omega)^2((ik)^2 + (iM)^2) + \frac{1}{H_\rho}(-i\omega)^2(iM) + N^2(ik)^2 = 0$$

$$\Rightarrow \omega^2(k^2 + M^2 - iM/H_\rho) - N^2k^2 = 0 \qquad ②$$

We want a solution in which ω is real-valued. Suppose k is also real-valued. Then M must be complex-valued. I.e. $M = m + i\gamma$ such that the imaginary part of $M^2 - iM/H_\rho$ is 0

$$0 = \text{Im}\{M^2 - iM/H_\rho\} = \text{Im}\{(m+i\gamma)^2 - i(m+i\gamma)/H_\rho\}$$
$$= 2m\gamma - m/H_\rho$$

So $\gamma = \frac{1}{2H_\rho}$

And so dispersion relation is $\omega^2(k^2 + (m^2 - \gamma^2) + \gamma/H_\rho) - N^2k^2 = 0$

$$\Rightarrow \omega^2 = N^2k^2 / [k^2 + m^2 + \frac{1}{4H_\rho^2}]$$

And ① $\Rightarrow$ $\Psi = A_\Psi e^{i(kx+mz-\omega t)} e^{i(i\gamma z)}$

$$= A_\Psi e^{i(kx+mz-\omega t)} e^{-z/2H_\rho}$$

So

$$u = -\frac{1}{\bar{\rho}}\frac{\partial \Psi}{\partial z} = -\frac{1}{\rho_0 e^{-z/H_\rho}}\left[A_\Psi \left(im - \frac{1}{2H_\rho}\right) e^{i(kx+mz-\omega t)} e^{-z/H_\rho}\right.$$
$$= -\frac{1}{\rho_0}\left(im - \frac{1}{2H_\rho}\right) A_\Psi e^{i(kx+mz-\omega t)} e^{+z/2H_\rho}$$

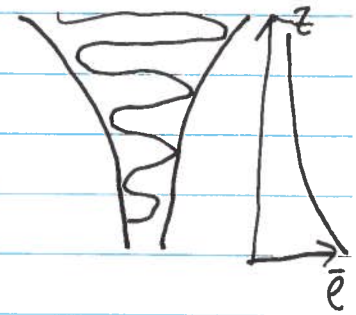
$$w = \frac{1}{\bar{\rho}}\frac{\partial \Psi}{\partial x} = \frac{1}{\rho_0}(ik) A_\Psi e^{i(kx+mz-\omega t)} e^{+z/2H_\rho}$$

$$\eta = -\frac{1}{\rho_0}\left(\frac{k}{\omega}\right) A_\Psi e^{i(kx+mz-\omega t)} e^{+z/2H_\rho} \overset{\text{real part}}{=} \eta_0 e^{z/2H_\rho} \cos(kx+mz-\omega t)$$

$$(A_\Psi = -\rho_0 \frac{\omega}{k} \eta_0)$$

(4) (CONT'D)

SO THE WAVES GROW IN AMPLITUDE AS THEY PROPAGATE UPWARD THROUGH PROGRESSIVELY LOWER DENSITY AIR: THIS IS CALLED "ANELASTIC GROWTH".



THE REASONS FOR ANELASTIC GROWTH CAN BE UNDERSTOOD FROM MOMENTUM CONSERVATION

$$\frac{Du}{Dt} = -\frac{\partial P}{\partial x}/\bar{\rho} \Rightarrow \frac{\partial}{\partial t}(\bar{\rho}u) + \bar{\rho}u\frac{\partial u}{\partial x} + \bar{\rho}w\frac{\partial u}{\partial z} = -\frac{\partial P}{\partial x}$$

$$\nabla\cdot(\bar{\rho}u) = 0 \Rightarrow \frac{\partial}{\partial t}(\bar{\rho}u) + \frac{\partial}{\partial x}(\bar{\rho}uu) + \frac{\partial}{\partial z}(\bar{\rho}uw) = -\frac{\partial P}{\partial x}$$

HORIZONAL AVERAGE: $\langle\cdot\rangle = \frac{1}{\lambda_x}\int_0^{\lambda_x}\cdot\, dx$

$$\Rightarrow \frac{\partial}{\partial t}(\bar{\rho}\langle u\rangle) = -\frac{\partial}{\partial z}(\bar{\rho}\langle uw\rangle)$$

SO THE WINDS ACCELERATE DUE TO THE DIVERGENCE OF THE MOMENTUM FLUX $\bar{\rho}\langle uw\rangle$.

CALCULATE EXPLICITLY IN TERMS OF η_0

$$u = Re\left\{-\frac{1}{\rho_0}\left(im - \frac{1}{2H_\rho}\right)A_\Psi e^{z/2H_\rho} e^{i(kx+mz-\omega t)}\right\}, \quad A_\Psi = -\rho_0\frac{\omega}{k}\eta_0$$

$$= \frac{1}{2}\left(im - \frac{1}{2H_\rho}\right)\frac{\omega}{k}\eta_0 e^{z/2H_\rho} e^{i(kx+mz-\omega t)} + c.c \text{ (COMPLEX CONJUGATE)}$$

$$w = Re\left\{\frac{1}{\rho_0}(ik)A_\Psi e^{z/2H_\rho} e^{i(kx+mz-\omega t)}\right\}$$

$$= \frac{1}{2}\left(-i\omega\eta_0 e^{z/2H_\rho} e^{i(kx+mz-\omega t)}\right) + c.c.$$

$$\text{So } \bar{\rho}\langle uw\rangle = \bar{\rho}\frac{1}{\lambda_x}\int_0^{\lambda_x}\left[\frac{1}{2}\left(im - \frac{1}{2H_\rho}\right)\frac{\omega}{k}\eta_0 e^{z/2H_\rho} e^{i()} + c.c.\right]\left[\frac{1}{2}(-i\omega\eta_0 e^{z/2H_\rho} e^{i()} + c.c\right]dx$$

$$= \bar{\rho}\frac{1}{\lambda_x}\int_0^{\lambda_x}\left[\left(\frac{1}{2}\left(im - \frac{1}{2H_\rho}\right)\frac{\omega}{k}\eta_0 e^{z/2H_\rho}\right)\left(\frac{1}{2}i\omega\eta_0 e^{z/2H_\rho}\right) + c.c\right]dx$$

$$\left(\text{SINCE } \int_0^{\lambda_x} e^{2i(kx)}dx = \int_0^{\lambda_x} e^{-2ikx}dx = 0\right)$$

$$\Rightarrow \bar{\rho}\langle uw\rangle = \bar{\rho}\left[\left(\frac{1}{4}(-m\omega^2/k)\eta_0^2 e^{z/H_\rho} - \frac{1}{8H_\rho}(i\omega^2/k)\eta_0 e^{z/H_\rho}\right) + c.c.\right]$$

$$= \bar{\rho}\left[-\frac{1}{2}\frac{m\omega^2}{k}\eta_0^2 e^{z/H_\rho}\right]$$

BUT $\bar{\rho} = \rho_0 e^{-z/H_\rho}$

$$\Rightarrow \bar{\rho}\langle uw\rangle = \rho_0\left[-\frac{1}{2}\frac{m\omega^2}{k}\eta_0^2\right] \quad \text{IS CONSTANT WITH HEIGHT.}$$