3] SYNOPTIC-SCALE FLOWS: BALANCE AND IMBALANCE

THE FULL EQUATIONS OF MOTION, JUST DERIVED, DESCRIBE ALMOST EVERYTHING FROM GYRES TO CURRENTS AND EDDIES TO WAVES AND TURBULENCE. DEPENDING ON THE SCALE (SIZE AND DURATION) OF THE PHENOMON IN QUESTION, SOME TERMS IN THE EQUATIONS MAY BE MORE IMPORTANT THAN OTHERS.

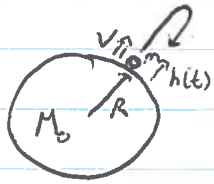
ONE CAN ASSESS A PRIORI WHICH TERMS DOMINATE THROUGH THE TECHNIQUES OF <u>DIMENSIONAL ANALYSIS</u> AND <u>NON-DIMENSIONALIZATION</u>.

ONCE THE EQUATIONS ARE IN NON-DIMENSIONAL FORM, ONE CAN APPLY <u>PERTURBATION THEORY</u> IN WHICH ONE SUCCESSIVELY SOLVES THE "LEADING ORDER" EQUATIONS COMING FROM THE DOMINANT TERMS ALONE AND THEN SOLVING THE EQUATIONS THAT INCLUDE THE NEXT-ORDER CORRECTIONS.

FOLLOWING THIS PROCEDURE FOR SYNOPTIC SCALE (LARGE-SCALE AND OF LONG DURATION) FLOWS, WE WILL FIND THE LEADING-ORDER TERMS DO NOT INVOLVE TIME-DERIVATIVES. THESE ARE "BALANCE EQUATIONS". EQUATIONS INVOLVING TIME-DEPENDENT "IMBALANCE" COME FROM PERTURBATION THEORY AT NEXT ORDER.

BEFORE PROCEDING WE FIRST CONSIDER A RELATIVELY SIMPLE PROBLEM WHOSE APPROXIMATE SOLUTION INVOLVES APPLICATION OF DIMENSIONAL ANALYSIS, NON-DIMENSIONALIZATION AND PERTURBATION THEORY.

1. AN INTRODUCTION TO DIMENSIONAL ANALYSIS AND PERTURBATION THEORY



CONSIDER A MASS m LAUNCHED WITH INITIAL SPEED V VERTICALLY FROM A PLANET OF RADIUS R AND MASS M_0. WHAT IS ITS HEIGHT $h(t)$?

- FIRST CONSIDER CIRCUMSTANCE IN WHICH V IS "SMALL" SO THAT GRAVITATIONAL ACCELERATION FELT THROUGHOUT MOTION IS APPROXIMATELY CONSTANT, $-g$. (e.g. $g = 9.8\ m/s^2$ ON EARTH NEAR SURFACE)

THEN NEWTON'S LAW PREDICTS $\underbrace{m\frac{d^2h}{dt^2}}_{\text{ACCELERATION}} = \underbrace{-mg}_{\text{FORCE}}$ (1)

SOLVING GIVES $h(t) = -\frac{1}{2}gt^2 + Vt$ (USING $h(0)=0$, $h'(0)=V$) (2)

- NOW SUPPOSE V IS SO LARGE THAT DECREASE IN GRAVITY WITH DISTANCE FROM THE PLANET IS FELT BY THE PROJECTILE.

GENERALLY, THE FORCE OF GRAVITY IS $F = -\frac{GM_0m}{(R+h)^2}$ ← UNIVERSAL GRAVITATIONAL CONSTANT

SO NEWTON'S LAW PREDICTS $m\frac{d^2h}{dt^2} = -\frac{GM_0m}{(R+h)^2}$

DEFINE $g \equiv \frac{GM_0}{R^2}$ (≈ 9.8 FOR EARTH)

$\Rightarrow \frac{d^2h}{dt^2} = -\frac{R^2g}{(R+h)^2}$ WITH $h(0)=0$, $h'(0)=V$ (3)

THIS EQUATION CANNOT BE SOLVED EXPLICITLY.

BUT NOTE, WE RECOVER (1), IF $\|h\| \ll R$.

WE WANT TO MAKE (3) NON-DIMENSIONAL BY CHOOSE CHARACTERISTIC HEIGHT AND TIME SCALES, h_c AND t_c, TO MAKE SMALLNESS OF h EXPLICIT IN THE EQUATIONS.

h_c AND t_c ARE FOUND BY CONSTRUCTING QUANTITIES WITH UNITS OF LENGTH AND TIME FROM THE PARAMETERS GIVEN IN THE PROBLEM: R [LENGTH], V [LENGTH/TIME], g [LENGTH/TIME^2]

1 (cont'd)

HERE ARE SOME POSSIBILITIES:

Ⓐ $h_c = R \;,\quad t_c = R/V$

Ⓑ $h_c = R \;,\quad t_c = \sqrt{R/g}$

Ⓒ $h_c = V^2/g \;,\quad t_c = V/g$

LET'S TRY EACH, DEFINING $\tilde{h} \equiv \frac{h}{h_c}$, $\tilde{t} \equiv \frac{t}{t_c}$ AND $\varepsilon = \frac{V^2}{gR}$

AND USE $\frac{d^2h}{dt^2} = \frac{h_c}{t_c^2}\frac{d^2\tilde{h}}{d\tilde{t}^2}$

Ⓐ $\Rightarrow \quad \varepsilon \frac{d^2\tilde{h}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{h})^2} \;;\quad \tilde{h}(0)=0,\ \tilde{h}'(0)=1$

Ⓑ $\Rightarrow \quad \frac{d^2\tilde{h}}{d\tilde{t}^2} = -\frac{1}{(1+\tilde{h})^2} \;;\quad \tilde{h}(0)=0,\ \tilde{h}'(0)=\varepsilon$

Ⓒ $\Rightarrow \quad \frac{d^2\tilde{h}}{d\tilde{t}^2} = -\frac{1}{(1+\varepsilon\tilde{h})^2} \;;\quad \tilde{h}(0)=0,\ \tilde{h}'(0)=1$

ALL THREE GIVE THE CORRECT ANSWER, BUT ONLY ONE IS REASONABLE IF WE CARE ABOUT CIRCUMSTANCE IN WHICH V IS "SMALL" ... MORE PRECISELY $\varepsilon \ll 1$.

CONSIDER SOLUTIONS OF ①, ② & ③ IN LIMIT $\varepsilon \to 0$

Ⓐ $\Rightarrow \quad 0 \approx -\frac{1}{(1+\tilde{h})^2} \;;\quad \tilde{h}(0)=0,\ \tilde{h}'(0)=1$

THIS HAS NO MATHEMATICAL SOLUTION (MUST HAVE $\tilde{h}(t) \to \infty$)

Ⓑ $\Rightarrow \quad \frac{d^2\tilde{h}}{dt^2} = -\frac{1}{(1+\tilde{h})^2} \;;\quad \tilde{h}(0)=0,\ \tilde{h}'(0)=0$

THIS HAS NO PHYSICAL SOLUTION ($\tilde{h}(t)^{<0}$ MUST MOVE INTO GROUND)

Ⓒ $\Rightarrow \quad \frac{d^2\tilde{h}}{d\tilde{t}^2} = -1 \;;\quad \tilde{h}(0)=0,\ \tilde{h}'(0)=1$

THIS CAN BE SOLVED: $\tilde{h}(\tilde{t}) = -\frac{1}{2}\tilde{t}^2 + \tilde{t}$

IN DIMENSIONAL FORM $\Rightarrow \quad \frac{1}{h_c}h(t) = -\frac{1}{2}\left(\frac{t}{t_c}\right)^2 + \left(\frac{t}{t_c}\right)$

$$\Rightarrow h(t) = \frac{V^2}{g}\left(-\frac{1}{2}\left(\frac{g}{V}\right)^2 t^2 + \left(\frac{g}{V}\right)t\right) = -\frac{1}{2}gt^2 + Vt \quad ✓$$

Ⓒ WORKS BECAUSE IT IS THE ONLY CHOICE OF h_c AND t_c THAT DOESN'T DEPEND ON R.

$h_c = \frac{V^2}{g}$ MEASURES MAXIMUM RISE HEIGHT, $t_c = \frac{V}{g}$ IS TIME TO REACH THIS HEIGHT IN LIMIT OF SMALL V.

(1) (CONT'd)

WE ARE NOW IN A POSITION TO FIND APPROXIMATE SOLUTIONS FOR (3) FOR V SMALL, BUT NOT TOO SMALL. I.E. $0<\varepsilon \ll 1$
(i.e. $V^2/gR \ll 1 \Rightarrow V \ll \sqrt{gR} \simeq 8000\,m/s$)

$$\frac{d^2\tilde h}{d\tilde t^2} = -\frac{1}{(1+\varepsilon\tilde h)^2} \; ; \quad \tilde h(0)=0,\ \tilde h'(0)=1$$

SUPPOSE $\tilde h = \tilde h^{(0)} + \varepsilon \tilde h^{(1)} + \varepsilon^2 \tilde h^{(2)} + \dots$ A "REGULAR" PERTURBATION EXPANSION

$$\Rightarrow \frac{d^2}{d\tilde t^2}\left(\tilde h^{(0)} + \varepsilon\tilde h^{(1)} + \varepsilon^2\tilde h^{(2)} + \dots\right) = -\left(1+\varepsilon(\tilde h^{(0)} + \varepsilon\tilde h^{(1)} + \varepsilon^2\tilde h^{(2)} + \dots)\right)^{-2} \qquad (4)$$

WITH $\tilde h^{(0)}(0) + \varepsilon\tilde h^{(1)}(0) + \varepsilon^2\tilde h^{(2)}(0) + \dots = 0$; $\tilde h^{(0)\prime}(0) + \varepsilon\tilde h^{(1)\prime}(0) + \varepsilon^2\tilde h^{(2)\prime}(0) + \dots = 1$

WANT TO COMPARE TERMS OF ORDER $\varepsilon^0, \varepsilon^1, \varepsilon^2, \dots$

TO DO THIS, NEED A TRICK FROM THE BINOMIAL THEOREM:

$$(1+x)^p = 1 + \binom{p}{1}x + \binom{p}{2}x^2 + \dots$$

IN WHICH $\binom{p}{k} = \frac{p(p-1)\cdots(p-k+1)}{k(k-1)\cdots(2)(1)}$, e.g. $\binom{p}{0}=1$, $\binom{p}{1}=p$, $\binom{p}{2}=\frac{p(p-1)}{2}$

$$\text{So } (4) \Rightarrow \frac{d^2}{d\tilde t^2}\left(\tilde h^{(0)} + \varepsilon\tilde h^{(1)} + \varepsilon^2\tilde h^{(2)} + \dots\right) = -\Big[1 + (-2)\left(\varepsilon(\tilde h^{(0)} + \varepsilon\tilde h^{(1)} + \varepsilon^2\tilde h^{(2)} + \dots)\right) + \overbrace{\frac{(-2)(-3)}{2}}^{3}\left(\varepsilon(\tilde h^{(0)} + \varepsilon\tilde h^{(1)} + \varepsilon^2\tilde h^{(2)} + \dots)\right)^2 + \dots$$

$$= -1 + 2\varepsilon\tilde h^{(0)} + \varepsilon^2\left(2\tilde h^{(1)} - 3(\tilde h^{(0)})^2\right) + \varepsilon^3(\dots) + \dots$$

SO AT ORDER ε^0 (WRITTEN $O(\varepsilon^0)$ ... i.e. MATCH TERMS WITH NO ε OUT FRONT)

$$\Rightarrow \frac{d^2}{d\tilde t^2}\tilde h^{(0)} = -1\,, \quad \tilde h^{(0)}(0) = 0\,, \quad \tilde h^{(0)\prime}(0) = 1$$

SOL^{N}: $\tilde h^{(0)}(\tilde t) = -\frac12 \tilde t^2 + \tilde t$, WHICH IS WHAT WE FOUND EARLIER W. $\varepsilon = 0$

AT $O(\varepsilon^1)$ (MATCH TERMS WITH JUST ε (NOT ε^2, etc) OUT FRONT)

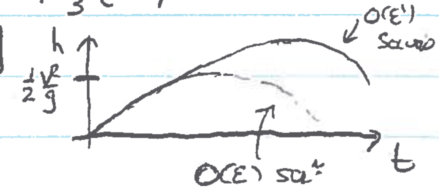
$$\Rightarrow \frac{d^2}{d\tilde t^2}\tilde h^{(1)} = 2\tilde h^{(0)} \quad (= -\tilde t^2 + 2\tilde t \text{ FROM } O(\varepsilon^0) \text{ SOLUTION}); \ \tilde h^{(1)}(0)=0,\ \tilde h^{(1)\prime}(0)=0$$

SOL^{N}: $\tilde h^{(1)}(\tilde t) = -\frac{1}{12}\tilde t^4 + \frac13 \tilde t^3$

SO, TO $O(\varepsilon)$ HAVE LEADING CORRECTION TO FEELING CHANGE IN GRAVITY:

$$\tilde h(\tilde t) \approx \tilde h^{(0)}(\tilde t) + \varepsilon\tilde h^{(1)}(\tilde t) = -\tfrac12\tilde t^2 + \tilde t + \varepsilon\left(-\tfrac{1}{12}\tilde t^4 + \tfrac13\tilde t^3\right)$$

$$\Rightarrow h(t) \approx -\tfrac12 g t^2 + Vt + \left[-\tfrac{1}{12}\frac{g^2}{R}t^4 + \tfrac13\frac{Vg}{R}t^3\right]$$



2 THE REYNOLDS NUMBER AND VISCOSITY

AS A SIMPLE FLUIDS EXAMPLE, WE ASSESS THE IMPORTANCE OF VISCOSITY UPON A UNIFORM-DENSITY, INCOMPRESSIBLE FLOW WITH NO ROTATION. THE MOMENTUM EQUATION ARE (SEE p.23)

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho_0}\nabla p + \underbrace{\nu}_{\text{KINEMATIC VISCOSITY} = \frac{\mu}{\rho_0}}\nabla^2\underline{u} \qquad (1)$$

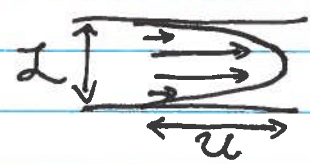
BECAUSE INCOMPRESSIBLE $(\nabla\cdot\underline{u}=0)$, TAKING THE DIVERGENCE OF (1) GIVES THE DIAGNOSTIC PRESSURE EQUATION:

$$\overbrace{\nabla^2}^{\nabla\cdot\nabla} p = -\rho_0\nabla\cdot(\underline{u}\cdot\nabla\underline{u}) \qquad (2)$$

NOW SUPPOSE THE FLOW HAS A CHARACTERISTIC VELOCITY AND LENGTH-SCALE

$\mathcal{U}$ (e.g. MAX FLOW SPEED)

$\mathcal{L}$ (e.g. CHANNEL WIDTH)



LET THE CHARACTERISTIC TIME SCALE BE $\mathcal{T} = \mathcal{L}/\mathcal{U}$.

DENOTE THE CHARACTERISTIC PRESSURE SCALE BY $\mathcal{P}$. WE CAN EXPRESS THIS IN TERMS OF $\mathcal{U}$ AND $\mathcal{L}$ USING (2):

DEFINE NON-DIMENSIONAL QUANTITIES $\tilde{x} = \frac{x}{\mathcal{L}}$, $\tilde{\underline{u}} = \frac{\underline{u}}{\mathcal{U}}$, $\tilde{t} = \frac{t}{\mathcal{T}}$, $\tilde{p} = \frac{p}{\mathcal{P}}$

(NOTE $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \frac{1}{\mathcal{L}}(\frac{\partial}{\partial \tilde{x}}, \frac{\partial}{\partial \tilde{y}}, \frac{\partial}{\partial \tilde{z}}) = \frac{1}{\mathcal{L}}\tilde{\nabla}$)

SO (2) $\Rightarrow$ $\frac{1}{\mathcal{L}^2}\mathcal{P}\,\tilde{\nabla}^2\tilde{p} = -\rho_0\frac{1}{\mathcal{L}^2}\mathcal{U}^2\,\tilde{\nabla}\cdot(\tilde{\underline{u}}\cdot\tilde{\nabla}\tilde{\underline{u}})$

ASSUMING $\mathcal{U}$ AND $\mathcal{L}$ WERE CHOSEN APPROPRIATELY SO THAT THE TILDE'D TERMS ON RIGHT AND LEFT ARE ORDER UNITY, FOR BALANCE OF BOTH SIDES, MUST HAVE $\frac{1}{\mathcal{L}^2}\mathcal{P} \approx \rho_0\frac{1}{\mathcal{L}^2}\mathcal{U}^2$

$$\Rightarrow \boxed{\mathcal{P} = \rho_0\mathcal{U}^2}$$

THIS SCALING FOR "DYNAMIC PRESSURE" ALSO FOLLOWS FROM BERNOULLI'S PRINCIPLE

2 (cont'd)

Now apply scalings to momentum equation ①:

$$\underbrace{\frac{\mathcal{U}}{\mathcal{T}}}_{\mathcal{U}/(\mathcal{L}/\mathcal{U})}\frac{\partial\tilde{\underline{u}}}{\partial\tilde{t}} + \frac{\mathcal{U}^2}{\mathcal{L}}\tilde{\underline{u}}\cdot\tilde{\nabla}\tilde{\underline{u}} = -\frac{1}{\rho_0}\underbrace{\frac{\mathcal{P}}{\mathcal{L}}}_{(\rho_0\mathcal{U}^2/\mathcal{L})}\tilde{\nabla}\tilde{p} + \nu\frac{\mathcal{U}}{\mathcal{L}^2}\tilde{\nabla}^2\tilde{\underline{u}}$$

$$\times\frac{\mathcal{L}}{\mathcal{U}^2} \Rightarrow \frac{\partial\tilde{\underline{u}}}{\partial\tilde{t}} + \tilde{\underline{u}}\cdot\tilde{\nabla}\tilde{\underline{u}} = -\tilde{\nabla}\tilde{p} + \frac{\nu}{\mathcal{U}\mathcal{L}}\tilde{\nabla}^2\tilde{\underline{u}} \qquad (3)$$

So define $\boxed{Re \equiv \frac{\mathcal{U}\mathcal{L}}{\nu}}$ to be the "Reynolds Number"

Thus we see that the viscous term $\nu\nabla^2\underline{u} \to \frac{1}{Re}\tilde{\nabla}^2\tilde{\underline{u}}$ is small compared to the other terms if $Re \gg 1$ ($\frac{1}{Re}$ plays the role of ε in perturbation theory)

To say whether or not a flow is viscous, more than just ν we need to specify the velocity & length scales

Examples:

Estimate Re for the following given $\nu \simeq 1.5\times10^{-5}\,m^2/s$ for air, $\nu \simeq 1.0\times10^{-6}\,m^2/s$ for water, $\nu \simeq 10^{-3}\,m^2/s$ for honey:

① Falling dust in air: $\mathcal{L}\sim10^{-6}\,m$, $\mathcal{U}\sim10^{-3}\,m/s$

$\Rightarrow Re \approx (10^{-6}\,m)(10^{-3}\,m/s)/(1.5\times10^{-5}\,m^2/s) \approx 7\times10^{-5} \Rightarrow$ very viscous

② Blowing air from a straw: $\mathcal{L}\sim10^{-3}\,m$, $\mathcal{U}\sim1\,m/s$ $\mathcal{L}\updownarrow$ ▭ $\to\mathcal{U}$

$\Rightarrow Re \approx (10^{-3})(1)/(1.5\times10^{-5}) \approx 70 \Rightarrow$ not very viscous

③ Drawing honey from a straw: $\mathcal{L}\sim10^{-3}\,m$, $\mathcal{U}\sim10^{-2}\,m/s$

$\Rightarrow Re \approx (10^{-3})(10^{-2})/(10^{-3}) \approx 10^{-2} \Rightarrow$ very viscous

④ Flow of jet stream in atmosphere: $\mathcal{L}\sim10^6\,m$, $\mathcal{U}\sim30\,m/s$ H $\mathcal{U}$ L $\updownarrow\mathcal{L}$

$\Rightarrow Re \approx (10^6)(30)/(1.5\times10^{-5}) \simeq 2\times10^{12} \Rightarrow$ viscosity negligible

⑤ Flow of Gulf Stream in ocean: $\mathcal{L}\sim10^5\,m$, $\mathcal{U}\sim1\,m/s$ $\mathcal{U}$ $\mathcal{L}$

$\Rightarrow Re \approx (10^5)(1)/(1.0\times10^{-6}) \simeq 10^{11} \Rightarrow$ viscosity negligible

3 THE FROUDE NUMBER AND TOTAL HYDROSTATIC BALANCE

RECALL DEFINITION OF BACKGROUND HYDROSTATIC BALANCE: $\frac{d\bar{p}}{dz} = -\bar{\rho} g$.
UNDER WHAT CIRCUMSTANCES ARE BUOYANCY FLUCTUATIONS BALANCED BY FLUCTUATION PRESSURE GRADIENTS? (I.e. $\frac{\partial p}{\partial z} \approx -\rho g$)

WE WILL SEE THIS DEPENDS ON HOW VERTICAL SCALES COMPARE TO HORIZONTAL SCALES (e.g. HOW z & w COMPARE WITH $\underline{x}_H = (x,y)$ AND $\underline{u}_H = (u,v)$) AS WELL AS IMPORTANCE OF BUOYANCY TERM $b = -g\,\rho/\rho_0$, $g\,\theta/\bar{\theta}$.

A] 1-LAYER OR 2-LAYER FLUID (AIR / WATER $\rho_0 \gg \rho_{AIR}$ 1-LAYER ; ρ_1 WARM/FRESH / $\rho_2 \gtrsim \rho_1$ COLD/SALTY 2-LAYER

DEFINE CHARACTERISTIC SCALES SO THAT

$$\|\underline{x}_H\| \sim \mathcal{L}, \quad \|z\| \sim \mathcal{H}, \quad \|\underline{u}_H\| \sim \mathcal{U}, \quad \|w\| \sim \mathcal{W}$$

RELATE THESE USING MASS CONSERVATION (INCOMPRESSIBLE OR ANELASTIC)

$$\left.\begin{aligned} \nabla\cdot\underline{u} = 0 &\Rightarrow \nabla_H\cdot\underline{u}_H = -\frac{\partial w}{\partial z} \\ \nabla\cdot(\bar{\rho}\underline{u}) = 0 &\Rightarrow \nabla_H\cdot(\bar{\rho}\underline{u}_H) = -\frac{\partial \bar{\rho} w}{\partial z} \end{aligned}\right\} \Rightarrow \frac{\mathcal{U}}{\mathcal{L}} \sim \frac{\mathcal{W}}{\mathcal{H}}$$

SO $\mathcal{W} = \left(\frac{\mathcal{H}}{\mathcal{L}}\right)\mathcal{U} = \alpha\,\mathcal{U}$ (1) WITH $\boxed{\alpha \equiv \frac{\mathcal{H}}{\mathcal{L}}}$ THE ASPECT RATIO

NOW CONSIDER VERTICAL MOMENTUM EQUATION (p.23) IGNORING VISCOSITY

$$\frac{\partial w}{\partial t} + \underline{u}_H\cdot\nabla_H w + w\frac{\partial}{\partial z}w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + b \quad (2)$$

WHERE $\|b\| = \|g\,\rho/\rho_0\| = g\frac{\Delta\rho}{\rho_0}$ ($= g$ FOR 1-LAYER, $= g' \equiv \frac{\rho_2-\rho_1}{\rho_0}g$ 2-LAYER, "REDUCED GRAVITY")

WANT TO COMPARE THE LEFT-HAND SIDE (LHS) OF (2) WITH b.

ASSUMING $\|t\| \sim \mathcal{L}/\mathcal{U}$, $\|\frac{\partial w}{\partial t}\| \sim \frac{1}{(\mathcal{L}/\mathcal{U})}\mathcal{W} \overset{(1)}{=} \alpha\,\mathcal{U}^2/\mathcal{L} = \alpha^2\,\mathcal{U}^2/\mathcal{H}$

ALSO $\|\underline{u}_H\cdot\nabla_H w\| \sim \frac{1}{\mathcal{L}}\mathcal{U}\mathcal{W} \overset{(1)}{=} \alpha\,\mathcal{U}^2/\mathcal{L} = \alpha^2\,\mathcal{U}^2/\mathcal{H}$

AND $\|w\frac{\partial w}{\partial z}\| \sim \frac{1}{\mathcal{H}}\mathcal{W}^2 \overset{(1)}{=} \frac{1}{\mathcal{H}}(\alpha\mathcal{U})^2 = \frac{\alpha^2}{\mathcal{H}}\mathcal{U}^2$

SO $\|L.H.S.\| \sim \alpha^2\,\mathcal{U}^2/\mathcal{H}$

COMPARED WITH b ON R.H.S: $\frac{\|L.H.S.\|}{\|b\|} = \underbrace{\alpha^2\frac{\mathcal{U}^2}{\mathcal{H}g}}_{\text{1-LAYER}}, \quad \underbrace{\alpha^2\frac{\mathcal{U}^2}{\mathcal{H}g'}}_{\text{2-LAYER}}$

③ A) (cont'd)

DEFINE $Fr = \frac{\mathcal{U}}{\sqrt{g\mathcal{H}}}$ AND $Fr = \frac{\mathcal{U}}{\sqrt{g'\mathcal{H}}}$ TO BE THE FROUDE NUMBER FOR A 1-LAYER & 2-LAYER FLUID, RESPECTIVELY

SO WE HAVE FOUND $\frac{Dw}{Dt}$ IS NEGLIGIBLY SMALL COMPARED WITH BUOYANCY, b, IF $\boxed{\alpha^2 Fr^2 \ll 1}$. IN THIS CASE b MUST BE COMPENSATED ENTIRELY BY PRESSURE GRADIENTS:

$$0 \approx -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + b \quad \Rightarrow \boxed{\frac{\partial P}{\partial z} \approx -g\rho}$$

TOGETHER WITH $\frac{d\bar{P}}{dz} = -\bar{\rho} g$ AND $P_T = \bar{P} + P$, $\rho_T = \bar{\rho} + \rho$

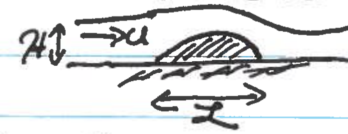
HAVE $\boxed{\frac{\partial P_T}{\partial z} = -\rho_T g}$ TOTAL HYDROSTATIC BALANCE

HENCE THE VERTICAL MOMENTUM EQUATION REDUCES TO $\frac{Dw}{Dt} = 0$ (VERTICAL ACCELERATION IS NEGLIGIBLE)

EXAMPLES

ASSESS THE FOLLOWING FOR TOTAL HYDROSTATIC BALANCE:

① A FAST-FLOWING STREAM OVER A BOULDER:



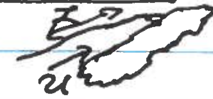
$\rho_0 \gg \rho_{AIR}$

$\mathcal{U} \sim 10\,m/s$, $\mathcal{L} \sim 1\,m$, $\mathcal{H} \sim 1\,m$

$\Rightarrow \alpha = \frac{\mathcal{H}}{\mathcal{L}} \sim 1$, $Fr = \frac{\mathcal{U}}{\sqrt{g\mathcal{H}}} \approx \frac{10\,m/s}{\sqrt{(9.81\,m/s^2)(1\,m)}} \sim 3$

$\Rightarrow \alpha^2 Fr^2 \sim 10 \gg 1$ SO "NONHYDROSTATIC"

② TIDES FLOWING INTO THE BAY OF FUNDY:



$\mathcal{U} \sim 10\,m/s$, $\mathcal{L} \sim 3\times10^5\,m$, $\mathcal{H} \sim 100\,m$

$\Rightarrow \alpha \sim \frac{100}{3\times10^5} \approx 3\times10^{-4}$, $Fr \sim \frac{10}{\sqrt{10\cdot100}} \approx 0.3$

$\Rightarrow \alpha^2 Fr^2 \sim 10^{-8}$ SO "HYDROSTATIC"

③ AN INTERFACIAL WAVE MADE BY A ROWBOAT IN 5m OF WARM WATER ($\rho_1 = 0.998\ g/cm^3$) OVER COLD WATER ($\rho_2 = 0.999\ g/cm^3$) IN A LAKE:

$\mathcal{U} \sim 1\,m/s$, $\mathcal{L} \sim 2\,m$, $\mathcal{H} \sim 5\,m$, $g' = g\frac{\rho_2 - \rho_1}{\rho_0} \approx (10\,m/s^2)\frac{0.001\,g/cm^3}{1\,g/cm^3} \approx 10^{-2}\,m/s^2$

$\Rightarrow \alpha \sim 2.5$, $Fr \sim \frac{1}{\sqrt{10^{-2}\cdot 5}} \sim 5 \Rightarrow \alpha^2 Fr^2 \sim 10^2$ SO "NONHYDROSTATIC"

3 (cont'd)

B] Continuously Stratified Fluids ($\bar{\rho}, \bar{\theta}$ vary continuously with z)

In this case vertical motion is inhibited by stratification.
Vertical scales are set by internal energy equation (p.23):

$$\frac{Db}{Dt} = -wN^2 \qquad \left(\text{with } N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz} \text{ or } \frac{g}{\theta}\frac{d\bar{\theta}}{dz}\right)$$

So $\mathcal{W} = \frac{1}{N^2}\frac{\mathcal{U}}{\mathcal{L}}\mathcal{B}$ with $\mathcal{B} \equiv \|b\| = \frac{g}{\rho_0}\|\rho\|$ or $\frac{g}{\theta_0}\|\theta\|$

Define $\varepsilon \equiv \frac{\mathcal{B}}{\mathcal{H}N^2} = \frac{\|\rho\|}{\mathcal{H}\|\frac{d\bar{\rho}}{dz}\|}$ or $\frac{\|\theta\|}{\mathcal{H}\|\frac{d\bar{\theta}}{dz}\|}$ which is usually $\ll 1$

Hence $\mathcal{W} = \alpha\varepsilon\mathcal{U}$ (smaller than 1- and 2-layer cases for which $\mathcal{W} = \alpha\mathcal{U}$)

So on L.H.S of vertical momentum equation (2) on p.30 have

$$\left\|\frac{\partial w}{\partial t}\right\| \sim \frac{\mathcal{W}}{\mathcal{L}/\mathcal{U}} = \alpha\varepsilon\mathcal{U}^2/\mathcal{L}, \quad \|u_H\cdot\nabla_H w\| \sim \alpha\varepsilon\mathcal{U}^2/\mathcal{L}, \quad \left\|w\frac{\partial w}{\partial z}\right\| \sim \alpha\varepsilon^2\mathcal{U}^2/\mathcal{L}$$

So dominant scale on LHS is of order $\alpha\varepsilon\mathcal{U}^2/\mathcal{L}$

Compare this with buoyancy scale on RHS, $\mathcal{B} = \varepsilon\mathcal{H}N^2$:

$$\|L.H.S\|/\|b\| \sim (\alpha\varepsilon\mathcal{U}^2/\mathcal{L})/(\varepsilon\mathcal{H}N^2) = \alpha^2\,\mathcal{U}^2/(N^2\mathcal{H}^2)$$

So here define <u>Froude number</u> as $\boxed{F_r \equiv \mathcal{U}/N\mathcal{H}}$

[My preference is to define <u>Long number</u> $L_o = N\mathcal{H}/\mathcal{U}$, but unconventional]

Hence condition for total hydrostatic balance is $\alpha^2 F_r^2 \ll 1$

Example

In atmosphere and ocean, typically $N \sim 0.01\,s^{-1}$

1. In planetary boundary layer $\mathcal{H} \sim 10^3 m$, $\mathcal{U} \sim 10 m/s \Rightarrow F_r \sim 1$
   So motion is hydrostatic if $\alpha^2 F_r^2 \sim \alpha^2 \ll 1$, e.g. $\left(\frac{\mathcal{H}}{\mathcal{L}}\right)^2 \lesssim 0.1$
   So hydrostatic for $\mathcal{L} \gtrsim \frac{1}{\sqrt{0.1}}\mathcal{H} \approx 3 km$
2. In ocean thermocline $\mathcal{H} \sim 10^2 m$, $\mathcal{U} \sim 0.1 m/s \Rightarrow F_r \sim 0.1$
   So motion hydrostatic for $\mathcal{L} \gtrsim \mathcal{H} \sim 100m$

4 THE ROSSBY NUMBER AND GEOSTROPHIC & THERMAL WIND BALANCE

A] THE ROSSBY NUMBER

NOW EXAMINE SCALES OF HORIZONTAL MOMENTUM EQUATIONS

$$\frac{\partial u}{\partial t} + \underline{u}\cdot\nabla u - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \underbrace{\underline{u}\cdot\nabla v}_{\mathcal{U}^2/\mathcal{L}} + \underbrace{fu}_{f_0\mathcal{U}} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

SCALES: $\mathcal{U}^2/\mathcal{L}$ $\quad f_0\mathcal{U}$

THE IMPORTANCE OF ADVECTIVE TO CORIOLIS TERMS IS GIVEN BY

$$(\mathcal{U}^2/\mathcal{L})/(f_0\mathcal{U}) = \mathcal{U}/(f_0\mathcal{L})$$

THIS WE DEFINE AS THE <u>ROSSBY NUMBER</u>: $\boxed{R_0 \equiv \frac{\mathcal{U}}{f_0\mathcal{L}}}$

R_0 CAN BE RECAST AS THE RATIO OF THE "INERTIAL TIME-SCALE", f_0^{-1} (ABOUT 3 HOURS AT MIDLATITUDES) AND THE "ADVECTIVE TIME-SCALE" $T = \mathcal{L}/\mathcal{U}$; $\quad R_0 = f_0^{-1}/T = 1/(f_0T)$

SO ROTATION IS IMPORTANT ($R_0 \ll 1$) IF $T \gg f_0^{-1}$ (≈ 3 HOURS)

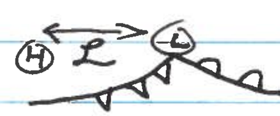
EXAMPLES

1. WATER DRAINING FROM A SINK

$\mathcal{L} \sim 1\,\text{cm}$, $\mathcal{U} \sim 10\,\text{cm/s}$, $f_0 \sim 10^{-4}\,\text{s}^{-1}$ AT MIDLATITUDES

$\Rightarrow R_0 \sim (10\,\text{cm/s})/(1\,\text{cm} \times 10^{-4}\,\text{s}^{-1}) = 10^5 \gg 1 \Rightarrow$ ROTATION UNIMPORTANT

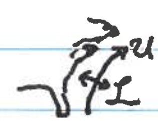
2. SYNOPTIC-SCALE WEATHER SYSTEMS



$\mathcal{L} \sim 10^6\,\text{m}$, $\mathcal{U} \sim 10\,\text{m/s}$, $f_0 \sim 10^{-4}\,\text{s}^{-1}$

$\Rightarrow R_0 \sim 10/(10^6 \times 10^{-4}) = 0.1 \Rightarrow$ ROTATION IMPORTANT

3. THE GULF STREAM



$\mathcal{L} \sim 10^5\,\text{m}$, $\mathcal{U} \sim 1\,\text{m/s}$, $f_0 \sim 10^{-4}\,\text{s}^{-1}$

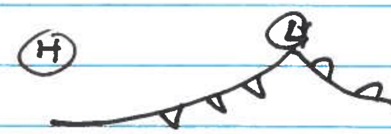
$\Rightarrow R_0 \sim 1/(10^5 \times 10^{-4}) = 0.1 \Rightarrow$ ROTATION IMPORTANT

④ (CONT'd)

B] GEOSTROPHIC BALANCE

FOR $Ro \ll 1$, THE DOMINANT TERM ON THE L.H.S OF THE HORIZONTAL MOMENTUM EQUATION IS THE CORIOLIS TERM. THIS MUST BE COMPENSATED BY THE PRESSURE GRADIENT ON THE R.H.S.

EXAMPLE



FOR SYNOPTIC-SCALE WEATHER

$$\|f u\| = f_0 U \sim (10^{-4}\,s^{-1})(10\,m/s) \sim 10^{-3}\,m/s^2$$

$$\left\|\frac{1}{\rho}\nabla_h p\right\| = \frac{1}{\rho_0}\frac{1}{L}(\delta P) \sim \frac{1}{1\,kg/m^3}\cdot\frac{1}{10^6\,m}\cdot(1\,kPa) \sim 10^{-3}\,m/s^2$$

SO, IF $Ro \ll 1$, WE HAVE "GEOSTROPHIC BALANCE"

$$\boxed{-f v_g = -\frac{1}{\rho}\frac{\partial P}{\partial x}\,, \quad f u_g = -\frac{1}{\rho}\frac{\partial P}{\partial y}} \qquad (I)$$

IN WHICH (u_g, v_g) IS THE "GEOSTROPHIC VELOCITY"

SPECIAL CASES:

i) ON THE f-PLANE, $f \approx f_0$ CONSTANT.

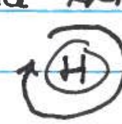
SO DEFINE $\boxed{\psi \equiv \left(\frac{1}{\rho f_0}\right) P}$ THE "GEOSTROPHIC STREAMFUNCTION"

HENCE $v_g = \frac{\partial \psi}{\partial x}$ AND $u_g = -\frac{\partial \psi}{\partial y}$.

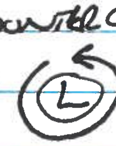
NOTE: $\nabla_H \cdot \underline{u}_g = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) = 0$ NON-DIVERGENT

VORTICITY IN VERTICAL IS $\zeta = \hat{z}\cdot(\nabla \times \underline{u}_g) = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi$

IN STEADY FLOW, FLUID MOVES ALONG LINES OF CONSTANT ψ, HENCE CONSTANT P (ISOBARS). IN THE NORTHERN HEMISPHERE $(f_0 > 0)$ FLOW IS CLOCKWISE ("ANTICYCLONIC") ABOUT HIGH PRESSURE AND COUNTERCLOCKWISE ("CYCLONIC") ABOUT LOW PRESSURE:



$f > 0$



④ B] (cont'd)

ii) On the β-plane at midlatitudes $f \approx f_0 + \beta y$

Take the curl of the momentum equations:

$$\frac{\partial}{\partial x}\left(f u_g = -\frac{1}{\rho}\frac{\partial P}{\partial y}\right) - \frac{\partial}{\partial y}\left(-f v_g = -\frac{1}{\rho}\frac{\partial P}{\partial x}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left((f_0 + \beta y)\, u_g \right) + \frac{\partial}{\partial y}\left((f_0 + \beta y)\, v_g \right) = 0$$

$$\Rightarrow (f_0 + \beta y)\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) + \beta v_g = 0 \qquad ①$$

Note, if $\beta = 0$ (f-plane) get $\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$ as before.

Here we can have vertical motion, $w \neq 0$

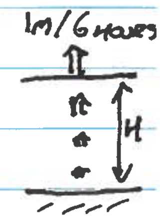
Assuming incompressible $\nabla \cdot \underline{u} = \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} + \frac{\partial w}{\partial z} = 0$

So can write ① as

$$\boxed{\beta v_g = f \frac{\partial w}{\partial z}} \quad ② \quad \text{"Sverdrup Balance"}$$

Streching a water column vertically in N.H. creates northward flow - an important process in the ocean

Example



Suppose the ocean surface rises by 1m in 6 hours resulting in a uniform stretch of waters below.

At surface $W_{TOP} \approx 1m/(6 \times 60 \times 60\,s) \approx 5 \times 10^{-5}\, m/s$.

Suppose $H \approx 5km \Rightarrow \frac{\partial w}{\partial z} \approx \frac{5 \times 10^{-5}\, m/s}{5 \times 10^{3}\, m} \sim 10^{-8}\, s^{-1}$.

At midlatitudes $f_0 \approx 10^{-4}$ and $\beta = 2\Omega_e \frac{\cos\theta_0}{R_e} \sim 10^{-11}\, \frac{1}{m\,s}$

$$\text{So } ② \Rightarrow V_g \approx \frac{1}{10^{-11}\,\frac{1}{m \cdot s}} \cdot (10^{-4}\, s^{-1})(10^{-8}\, s^{-1}) \sim 10^{-1}\, m/s.$$

Vertical stretch gives northward flow of 10cm/s //

④ (cont'd)

C] The Taylor - Proudman Effect

Assume the fluid has uniform density ρ_0, is geostrophic ($R_0 \ll 1$), on f-plane and hydrostatic. Then the equations of motion (p. 23) become:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad ① \qquad \text{(mass conservation, incompressible)}$$

$$-f_0 v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad ② \qquad \text{(x-mom}^{\underline{m}}\text{ conservation)}$$

$$f_0 u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \qquad ③ \qquad \text{(y-mom}^{\underline{m}}\text{ conservation)}$$

$$0 = -\frac{\partial p}{\partial z} - \rho_0 g \qquad ④ \qquad \text{(z-mom}^{\underline{m}}\text{ conservation)}$$

These are 4 equations in 4 unknowns u, v, w, p.

$$\frac{\partial}{\partial x}③ - \frac{\partial}{\partial y}② \Rightarrow f_0\frac{\partial u}{\partial x} + f_0\frac{\partial v}{\partial y} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho_0}\frac{\partial^2 p}{\partial y \partial x} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

as found before for geostrophic flow

So ① $\Rightarrow \frac{\partial w}{\partial z} = 0 \Rightarrow$ w is constant with height

If surface or bottom is rigid $\Rightarrow w = 0$ everywhere.

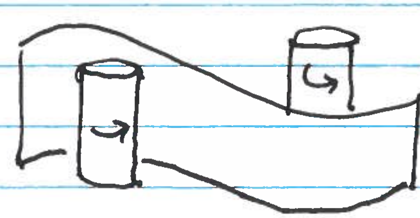
Also $\frac{\partial}{\partial z}③ \Rightarrow f_0\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial y} \overset{④}{=} -\frac{1}{\rho_0}\frac{\partial}{\partial y}(-\rho_0 g) = 0$

and $\frac{\partial}{\partial z}② \Rightarrow -f_0\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial x} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}(-\rho_0 g) = 0$

So $\frac{\partial u}{\partial z} = 0$ and $\frac{\partial v}{\partial z} = 0$.

Hence u and v are constant with height (though they may vary with x and y): $u = u(x,y)$, $v = v(x,y)$, $w = 0$

Hence fluid in a column has the same coherent flow from top to bottom. Such flow patterns are called "Taylor Columns"



4 (cont'd)

D] THERMAL WIND BALANCE

NOW RELAX ASSUMPTION THAT DENSITY IS UNIFORM AND ALLOW FOR POSSIBLY NON-CONSTANT f. So ① → ④ on p.36 REPLACED BY:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad ①$$

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \quad ②$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \quad ③$$

BOUSSINESQ APPROXIMATION, HENCE $\frac{1}{\rho_0}$ INSTEAD OF $\frac{1}{\rho}$

$$0 = -\frac{\partial p}{\partial z} - \rho g \quad ④$$

So

$$\frac{\partial}{\partial z}③ \Rightarrow f\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial y} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}(-\rho g) = \frac{g}{\rho_0}\frac{\partial \rho}{\partial y}$$

$$\frac{\partial}{\partial z}② \Rightarrow -f\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial^2 p}{\partial z \partial x} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}(-\rho g) = \frac{g}{\rho_0}\frac{\partial \rho}{\partial x}$$

HENCE VERTICAL SHEAR IS RELATED TO HORIZONTAL BUOYANCY VARIATIONS

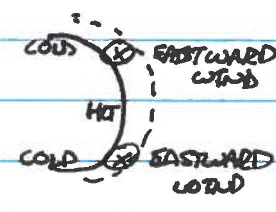
$$\boxed{\begin{aligned} \frac{\partial u}{\partial z} &= \frac{1}{f}\frac{g}{\rho_0}\frac{\partial \rho}{\partial y} = -\frac{1}{f}\frac{\partial b}{\partial y} \\ \frac{\partial v}{\partial z} &= -\frac{1}{f}\frac{g}{\rho_0}\frac{\partial \rho}{\partial x} = \frac{1}{f}\frac{\partial b}{\partial x} \end{aligned}}$$

"THERMAL WIND RELATIONS"

IN WHICH $b \equiv -\frac{g}{\rho_0}\rho \left(= +\frac{g}{\theta}\theta\right)$ IS BUOYANCY

EXAMPLE

i) JET STREAMS

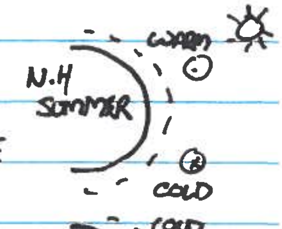


IN TROPOSPHERE AIR GETS COLDER (AND SO DENSITY INCREASES) FROM EQUATOR TO POLES.

- IN N.H. $f > 0$, $\frac{\partial \rho}{\partial y} > 0 \Rightarrow \frac{\partial u}{\partial z} > 0$
- IN S.H. $f < 0$, $\frac{\partial \rho}{\partial y} < 0 \Rightarrow \frac{\partial u}{\partial z} > 0$

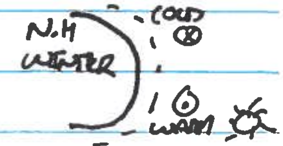
So WINDS INCREASE EASTWARD SPEED WITH HEIGHT: JET STREAMS

ii) POLAR NIGHT JET



IN STRATOSPHERE, AIR WARMER IN SUMMER HEMISPHERE

- IN N.H SUMMER $\frac{\partial \rho}{\partial y} < 0$ EVERYWHERE $\Rightarrow \frac{\partial u}{\partial z}$ < 0 N.H, > 0 S.H
- IN N.H WINTER $\frac{\partial \rho}{\partial y} > 0$ EVERYWHERE $\Rightarrow \frac{\partial u}{\partial z}$ > 0 N.H, < 0 S.H



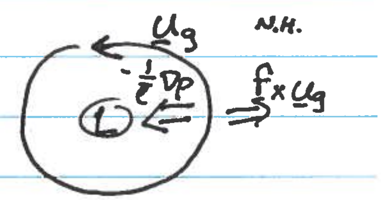
THE EASTWARD FLOW IN WINTER HEMISPHERE IS CALLED THE "POLAR NIGHT JET"

5 AGEOSTROPHY AND THE EKMAN LAYER & EKMAN TRANSPORT

AT LEADING ORDER SYNOPTIC-SCALE SYSTEMS (HAVING $Ro \ll 1$) ARE IN GEOSTROPHIC BALANCE. BUT THERMODYNAMIC AND MECHANICAL FORCES CAN ACT TO DRIVE THESE SYSTEMS AWAY FROM GEOSTROPHY.

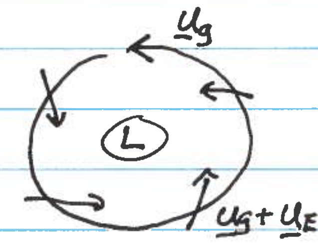
HERE WE FOCUS ON STRESSES EXERTED BY A HORIZONTAL BOUNDARY ACTING UPON A GEOSTROPHIC FLOW (e.g. WINDS BEING SLOWED BY THE SEA AND LAND; THE OCEAN SURFACE BEING DRIVEN BY WINDS)

QUALITATIVE EXAMPLE



CONSIDER GEOSTROPHIC WINDS SURROUNDING A LOW PRESSURE SYSTEM. WE HAVE FOUND

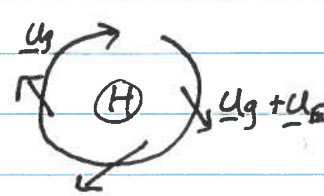
$$\underline{u} = \underline{u}_g \equiv (u_g, v_g) = \frac{1}{f\varrho}\left(-\frac{\partial p}{\partial y}, \frac{\partial p}{\partial x}\right)$$



NEAR THE EARTHS SURFACE THE WINDS ARE SLOWED AS THEY RUB OVER THE GROUND:
$\underline{u} = \underline{u}_g + \underline{u}_E$, IN WHICH $\underline{u}_E$ IS THE "EKMAN FLOW" OR "AGEOSTROPHIC VELOCITY".
(IN PERTURBATION THEORY $\underline{u}_g = \underline{u}^{(0)}$, THE LEADING-ORDER VELOCITY, AND $\underline{u}_E = \underline{u}^{(1)}$ IS THE NEXT ORDER (IN Ro) CORRECTION).

NOTE: $\underline{u}_E$ IS DIRECTED INWARDS TO A LOW PRESSURE SYSTEM. BECAUSE BOTTOM STRESS DOES NOT AFFECT INWARDS ∇p FORCE BUT SLOWER SPEEDS REDUCE THE OUTWARD CORIOLIS FORCE. THIS CONVERGENCE LEADS TO VERTICAL ASCENT ABOUT A LOW PRESSURE SYSTEM, WHICH LEADS TO CLOUDS AND STORM FORMATION

A HIGH PRESSURE SYSTEM ACTS IN REVERSE

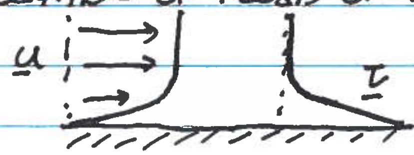


(5) (cont'd)

A] The Ekman number and Ekman layer depth

Denote stress $\left(\frac{\text{Force}}{\text{Area}}\right)$ by $\underline{\tau} = (\tau_{xz}, \tau_{yz}, 0)$. Like your hand sliding along a table it expresses how a horizontal force influences fluid vertically above and below.

The acceleration of fluid of density ρ_0 due to stress is $\frac{1}{\rho_0}\frac{\partial}{\partial z}\underline{\tau}$



In some cases (laminar flow in lab, approximation for turbulent flow) can assume $\boxed{\underline{\tau} \simeq \rho_0 A \frac{\partial \underline{u}}{\partial z}}$ in which the constant A is kinematic viscosity (ν) for laminar flow and "eddy viscosity" for turbulent.

So, for steady flow have

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + A\frac{\partial^2 u}{\partial z^2}$$

$$\underbrace{fu}_{f_0 U} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial y}}_{\frac{1}{\rho_0}\frac{1}{L}\delta P} + \underbrace{A\frac{\partial^2 v}{\partial z^2}}_{A\frac{1}{H^2}U}$$

Scales: $f_0 U \qquad \frac{1}{\rho_0}\frac{1}{L}\delta P \qquad A\frac{1}{H^2}U$

Define magnitude of stress to geostrophic terms to be the "Ekman number": $\boxed{Ek \equiv A/(f_0 H^2)}$ in which H is characteristic height for atmosphere/ocean. If $Ek \ll 1$, friction is negligible for flows at scale H.

The height, δ, over which stress is important is set by balance $f_0 U \sim A\frac{1}{\delta^2}U \Rightarrow \boxed{\delta = (A/f_0)^{1/2}}$

δ is the "Ekman layer depth".

Hence $Ek = (\delta/H)^2$ measures the relative depth of Ekman layer.

Example: For water ($\nu = 0.01\ cm^2/s$) in a rotating tank with $\Omega = 0.5\ s^{-1} \Rightarrow \delta \approx (\nu/f_0)^{1/2} = \left(\frac{0.01\ cm^2/s}{2\times 0.5\ s^{-1}}\right)^{1/2} = 0.1\ cm$ //

⑤ (CONT'D)

B] EKMAN TRANSPORT

CONSIDER A STEADY GEOSTROPHIC FLOW PERTURBED BY STRESS $\underline{\tau} = (\tau_{xz}, \tau_{yz}, 0)$. HORIZONTAL MOMENTUM EQUATIONS ARE

$$\underline{f} \times \underline{u} = -\frac{1}{\rho}\nabla P + \frac{1}{\rho}\frac{\partial \underline{\tau}}{\partial z}$$

WHERE $\underline{u} = \underline{u}_g + \underline{u}_E$ WITH

$$\underline{f} \times \underline{u}_g = -\frac{1}{\rho}\nabla P \quad (1) \qquad \text{(GEOSTROPHIC FLOW)}$$

$$\underline{f} \times \underline{u}_E = \frac{1}{\rho}\frac{\partial \underline{\tau}}{\partial z} \quad (2) \qquad \text{(EKMAN FLOW)}$$

EXPECTING THE EKMAN LAYER DEPTH TO BE SMALL, WE CAN APPROXIMATE $\rho \approx \rho_0$ IN (2). $\Rightarrow$ $\underline{f} \times (\rho_0 \underline{u}_E) = \frac{\partial \underline{\tau}}{\partial z}$ (3) IN WHICH $\underline{u}_E$ AND $\frac{\partial \underline{\tau}}{\partial z}$ ARE NEGLIGIBLY SMALL OUTSIDE EKMAN LAYER.

DEFINE "<u>EKMAN (MASS) TRANSPORT</u>" TO BE $\boxed{\underline{M}_E = \int \rho_0 \underline{u}_E \, dz}$ IN WHICH INTEGRAL IS FROM BOUNDARY WHERE STRESS IS APPLIED TO INTERIOR WHERE $|z| \gg \delta$ $\left(\approx (A/f_0)^{1/2}\right)$ AND $\underline{u}_E \approx \underline{0}$.

CASE i) STRESS OCCURS AT BOTTOM OF FLUID

VERTICALLY INTEGRATING BOTH SIDES OF (3) GIVES

$$\underline{f} \times \underline{M}_E = \int_0^\infty \frac{\partial \underline{\tau}}{\partial z}\, dz = -\underline{\tau}_B \quad (4) \quad \text{WITH } \underline{M}_E = \int_0^\infty \rho_0 \underline{u}_E \, dz$$

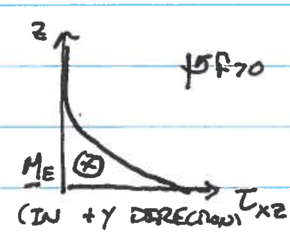
HERE $\underline{\tau}_B = \underline{\tau}|_{z=0}$ IS STRESS AT BOTTOM AND WE HAVE TAKEN UPPER BOUND OF INTEGRATION TO BE INFINITY SINCE THERE IS NEGLIGIBLE CONTRIBUTION OF INTEGRAL FROM $z \gg \delta$ TO ∞.

EXPLICITLY, (4) $\Rightarrow$ $(-f M_{Ey}, f M_{Ex}) = -(\tau_{Bxz}, \tau_{Byz})$

$$\Rightarrow (M_{Ex}, M_{Ey}) = \frac{1}{f}(-\tau_{Byz}, \tau_{Bxz})$$

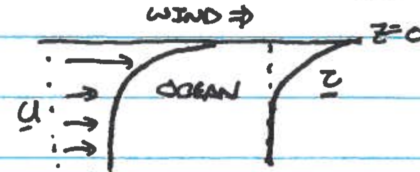
$$\Rightarrow \boxed{\underline{M}_E = \frac{1}{f}\hat{z} \times \underline{\tau}_B}$$

NOTE THAT MASS TRANSPORT IS PERPENDICULAR TO DIRECTION OF STRESS.



(5) B] (CONT'D)

CASE ii) STRESS OCCURS AT TOP OF FLUID

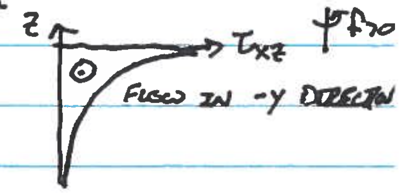


NOW VERTICALLY INTEGRATING (3) GIVES

$$\underline{f} \times \underline{M}_E = \int_{-\infty}^{0} \frac{\partial \underline{\tau}}{\partial z}\, dz = +\underline{\tau}_T \quad \text{WITH } \underline{M}_E = \int_{-\infty}^{0} \rho_0 \underline{u}_E\, dz$$

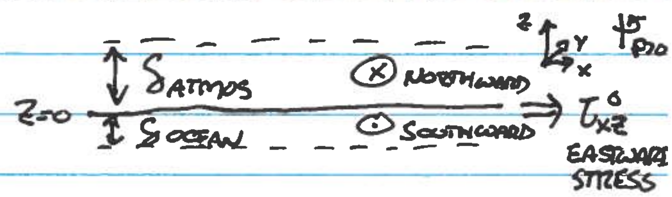
HERE $\underline{\tau}_T = \underline{\tau}|_{z=0}$ IS STRESS AT TOP OF FLUID.

MANIPULATING AS IN CASE i), FIND THE EKMAN TRANSPORT IS GIVEN EXPLICITLY BY $\boxed{\underline{M}_E = -\frac{1}{f}\,\hat{z} \times \underline{\tau}_T}$



AGAIN FLOW IS PERPENDICULAR TO STRESS BUT IS OPPOSITE TO FLOW ALOFT

IN PARTICULAR, CONSIDER THE STRESS IMPOSED BY THE WIND PASSING OVER THE OCEAN SURFACE.



IN N.H. ($f>0$) WIND STRESS ON OCEAN TRANSPORTS WATER RIGHTWARD
$\left(M_{Ey}^{OCEAN} = -\frac{1}{f}\tau_{xz}^{o}\right)$

OCEAN SURFACE PROVIDES STRESS ON WIND SO AIR TRANSPORTS LEFTWARD
$\left(M_{Ey}^{AIR} = +\frac{1}{f}\tau_{xz}^{o}\right)$

NOTE THAT THE TOTAL MASS TRANSPORT $M_{Ey}^{OCEAN} + M_{Ey}^{AIR} = 0$, WHICH IS A CONSEQUENCE OF MOMENTUM CONSERVATION.

EXAMPLE

A STANDARD ESTIMATE FOR WIND STRESS ON THE OCEAN IS $\tau = C_D \rho U_{10}^2$, WITH ρ THE DENSITY OF AIR, U_{10} THE WIND SPEED 10m ABOVE THE SURFACE AND "DRAG COEFFICIENT" $C_D \approx 10^{-3}$.

SUPPOSE $U_{10} = 10\,m/s$ AND TAKE $\rho \approx 1.2\,kg/m^3 \Rightarrow \tau \approx 0.12\,Pa$.

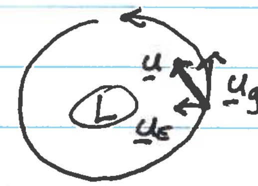
SO, AROUND 45°N EKMAN TRANSPORT IN OCEAN IS $|M_E| \approx \frac{1}{10^{-4}s}(0.12)$

$\Rightarrow |M_E| \approx 1.2 \times 10^3\, Pa \cdot s = 1.2 \times 10^3\, (kg\, m/s)/m^2$.

SO OVER EVERY $1m^2$ OF SURFACE MOMENTUM OF TRANSPORTED WATER BELOW IS $1.2 \times 10^3\, kg \frac{m}{s}$

(5) (CONT'D)

C] EKMAN PUMPING



CONSIDER A LOW PRESSURE SYSTEM IN N.H.

WE HAVE SEEN THAT EKMAN TRANSPORT AT BOTTOM CARRIES FLUID LEFTWARD ... TOWARD CENTER OF LOW. THE RESULTING HORIZONTAL CONVERGENCE OF AIR IN EKMAN LAYER MUST RESULT IN VERTICAL ASCENT OF AIR INTO GEOSTROPHIC FLOW ALOFT.

TOTAL FLOW IS $\underline{u} = \underline{u}_g + \underline{U}_E$ IN WHICH $\nabla \cdot \underline{u}_g = 0$ ON f-PLANE

SO, FROM CONSERVATION OF MASS FOR INCOMPRESSIBLE FLUID, $\nabla \cdot \underline{u} = 0$

$$\Rightarrow \frac{\partial u_E}{\partial x} + \frac{\partial v_E}{\partial y} = -\frac{\partial w}{\partial z}$$

MULTIPLY THROUGH BY ρ_0 AND INTEGRATE VERTICALLY

$$\Rightarrow \frac{\partial}{\partial x} M_{Ex} + \frac{\partial}{\partial y} M_{Ey} = -\rho_0 \int \frac{\partial w}{\partial z}\, dz = -\rho_0 [\underbrace{W_T}_{\text{TOP}} - \underbrace{W_B}_{\text{BOTTOM}}] \qquad (1)$$

USING $\underline{M}_E \equiv \int \rho_0 \underline{u}_E \, dz$

CASE i) STRESS AT RIGID BOTTOM

$$\Rightarrow \underline{M}_E = \frac{1}{f_0} \hat{z} \times \underline{\tau}_B = \frac{1}{f_0}(-\tau_{Byz}, \tau_{Bxz})$$ (SEE P.40)

AND $W_B = 0$ (NO DOWNWARD FLOW THROUGH BOTTOM)

$$\text{So } (1) \Rightarrow \frac{\partial}{\partial x}\left(-\frac{1}{f_0}\tau_{Byz}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f_0}\tau_{Bxz}\right) = -\rho_0 W_T$$

$$\Rightarrow W_T = \frac{1}{\rho_0}\frac{1}{f_0}\left(\frac{\partial \tau_{Byz}}{\partial x} - \frac{\partial \tau_{Bxz}}{\partial y}\right)$$

$$\Rightarrow \boxed{W_T = \frac{1}{\rho_0}\frac{1}{f_0}\, \hat{z} \cdot \nabla \times \underline{\tau}_B}$$

THIS IS THE VERTICAL VELOCITY OF FLOW LEAVING THE TOP OF THE EKMAN LAYER

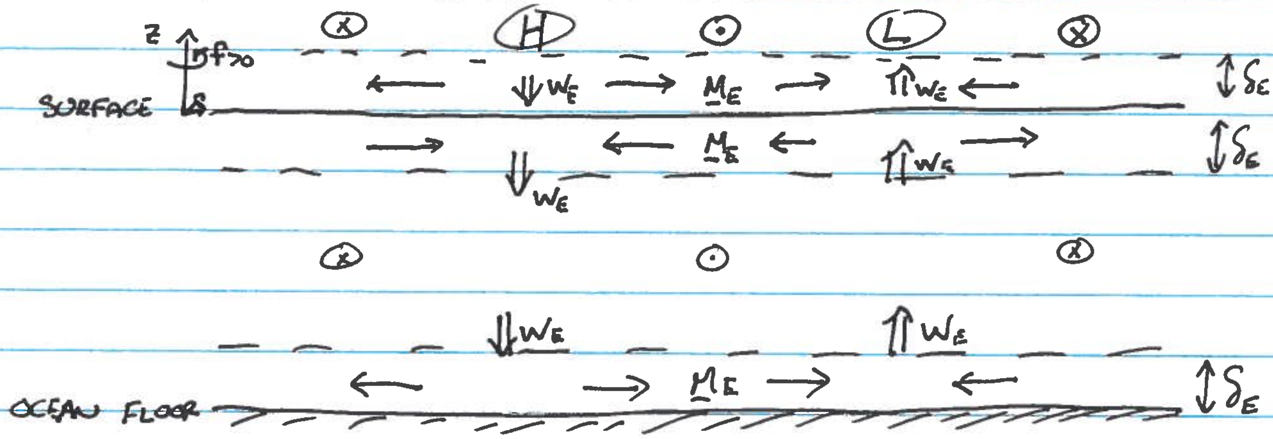
CASE ii) STRESS AT TOP

NOW $W_T = 0$ IN (1) AND $\underline{M}_E = -\frac{1}{f_0}(-\tau_{Tyz}, \tau_{Txz})$

PROCEEDING AS ABOVE, THE VERTICAL VELOCITY DESCENDING BELOW THE BOTTOM OF THE EKMAN LAYER IS $\boxed{W_B = \frac{1}{\rho_0 f_0}\, \hat{z} \cdot \nabla \times \underline{\tau}_T}$

5 c] (cont'd)

Combining these results we have a picture of ageostrophic transport due to stresses at horizontal boundaries of the atmosphere and ocean



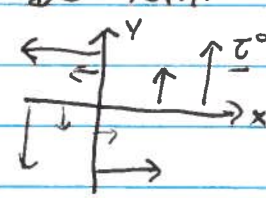
$$\boxed{w_E = \frac{1}{\rho_0}\frac{1}{f_0}\,\hat{z}\cdot\nabla\times\underline{\tau}\Big|_{\text{BOUNDARY}}}$$ is the "Ekman Pumping Velocity"

Example

Consider a flow about a low pressure system in N.H. at midlatitudes that gives a surface stress

$\underline{\tau}^{\circ} = \left(-\tau_0 \frac{y}{R},\ \tau_0 \frac{x}{R}\right)$ in which

$\tau_0 = 0.1$ Pa, $R = 100$ km.



The Ekman pumping velocity of air rising into geostrophic flow aloft is

$$w_E = \frac{1}{\rho_{AIR}}\frac{1}{f_0}\left(\frac{\partial}{\partial x}\tau^{\circ}_y - \frac{\partial}{\partial y}\tau^{\circ}_x\right) = \frac{1}{\rho_{AIR}}\frac{1}{f_0}\left(\frac{\tau_0}{R} - \left(-\frac{\tau_0}{R}\right)\right) = \frac{2}{\rho_{AIR}}\frac{1}{f_0}\frac{\tau_0}{R}$$

$$\simeq 2/(1.2\,\text{kg/m}^3)\ \ 1/(10^{-4}\,\text{s}^{-1})\ \ (0.1\,\text{Pa})/(10^5\,\text{m})$$

$$\simeq 1.7\times10^{-2}\ \text{m/s}$$

The corresponding velocity of water below rising to surface is

$$w_E = \frac{2}{\rho_{WATER}}\frac{1}{f_0}\tau_0/R$$

$$\simeq 2/(10^3\,\text{kg/m}^3)\ \ 1/(10^{-4}\,\text{s}^{-1})\ \ (0.1\,\text{Pa})/(10^5\,\text{m})$$

$$\simeq 2\times10^{-5}\ \text{m/s}\ \ //$$

5 (CONT'd)

D] AN EXPLICIT SOLUTION: THE SPIN-DOWN PROBLEM

FOR LAMINAR FLOW OR ASSUMING EDDY VISCOSITY $\underline{\tau} = A \frac{\partial \underline{u}_E}{\partial z}$

SO EKMAN BALANCE GIVES $\underline{f}_0 \times \underline{u}_E = \frac{\partial}{\partial z}\underline{\tau} = A \frac{\partial^2 \underline{u}_E}{\partial z^2}$ ON f-PLANE

$$\Rightarrow \begin{cases} -f_0 V_E = A \frac{\partial^2}{\partial z^2} u_E & \text{(1a)} \\ f_0 u_E = A \frac{\partial^2}{\partial z^2} V_E & \text{(1b)} \end{cases}$$

COMBINE (1a) & (1b) INTO 1 EQUATION WITH A SLICK TRICK:

DEFINE $\mathcal{U} \equiv u_E + i V_E$ WHERE $i \equiv \sqrt{-1}$

SO (1b) $- i$(1a) $\Rightarrow f_0 u_E + i f_0 V_E = A \frac{\partial^2}{\partial z^2} V_E - i A \frac{\partial^2}{\partial z^2} u_E$

$$= -i A \frac{\partial^2}{\partial z^2}(u_E + i V_E)$$

HENCE $f_0 \mathcal{U} = -i A \frac{\partial^2}{\partial z^2}\mathcal{U}$ (2)

ASSUME "NO-SLIP" BOTTOM BOUNDARY $\Rightarrow \underline{u}|_{z=0} = 0 \Rightarrow \underline{u}_E|_{z=0} = -\underline{u}_g|_{z=0}$

SO $\mathcal{U}|_{z=0} = -(u_g + i V_g)$ (3)

FAR ABOVE EKMAN LAYER $\underline{u}_E|_{z\to\infty} = 0 \Rightarrow \mathcal{U}|_{z\to\infty} = 0$ (4)

FOR GENERAL SOLUTION OF (2), SUPPOSE $\mathcal{U} \propto e^{pz}$ FOR SOME CONSTANT p.

(2) $\Rightarrow f_0 e^{pz} = -i A p^2 e^{pz} \Rightarrow f_0 = -i A p^2$

$\Rightarrow p^2 = i(f_0/A) \Rightarrow p = \pm\sqrt{|f_0|/A}\,\sqrt{\pm i}$ $\begin{cases} + & \text{N.H.} \\ - & \text{S.H.} \end{cases}$

WRITE $\pm i = e^{\pm i\pi/2} \Rightarrow \sqrt{\pm i} = e^{\pm i\pi/4} = \cos\frac{\pi}{4} \pm i \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$

SO $p = \pm\sqrt{|f_0|/A}\,\frac{1}{\sqrt{2}}(1 \pm i)$ $\begin{cases} + & \text{NH} \\ - & \text{SH} \end{cases}$

SO GENERAL SOLUTION IS $\mathcal{U} = \mathcal{C}_1 \exp\left[+\sqrt{\frac{|f_0|}{2A}}(1\pm i)z\right] + \mathcal{C}_2 \exp\left[-\sqrt{\frac{|f_0|}{2A}}(1\pm i)z\right]$

FROM (4) MUST HAVE $\mathcal{C}_1 = 0$ SO THAT $\mathcal{U}$ IS BOUNDED AS $z\to\infty$

FROM (3) MUST HAVE $\mathcal{C}_2 = -(u_g + i V_g)$

SO SOLUTION IS $\mathcal{U} = -(u_g + i V_g)\exp\left[-\sqrt{\frac{|f_0|}{2A}}(1\pm i)z\right]$

DEFINE $\delta_A \equiv \sqrt{2A/|f_0|} \Rightarrow \mathcal{U} = -(u_g + i V_g)\exp\left[-(1\pm i)\,z/\delta_A\right]$

5 D] (CONT'D)

WE FOUND $\mathcal{U} = u_E + i v_E = -(u_g + i v_g)\, e^{-z/\delta_A}\, e^{\mp i z/\delta_A}$ $\left\{ \begin{matrix} - & \text{N.H} \\ + & \text{S.H} \end{matrix} \right\}$

$$= -(u_g + i v_g)\, e^{-z/\delta_A} \left(\cos(z/\delta_A) \mp i \sin(z/\delta_A) \right) \quad (5)$$

EXTRACTING REAL AND IMAGINARY PARTS GIVES

$$\boxed{\begin{aligned} u_E &= -\left(u_g \cos(z/\delta_A) \pm v_g \sin(z/\delta_A) \right) e^{-z/\delta_A} \\ v_E &= \left(\pm u_g \sin(z/\delta_A) - v_g \cos(z/\delta_A) \right) e^{-z/\delta_A} \end{aligned}} \quad (6)$$

IN PARTICULAR, FOR AN EASTWARD WIND BLOWING OVER A BOTTOM BOUNDARY IN N.H. ($f_0 > 0$) $(u_g, v_g) = (u_0, 0)$

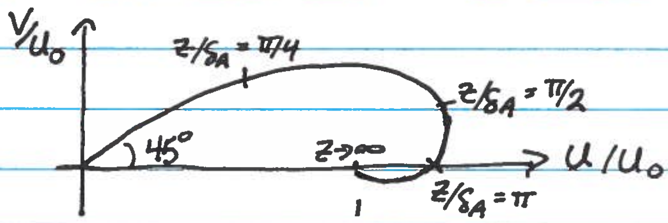
So (6) $\Rightarrow$ $u_E = -u_0 \cos(z/\delta_A)\, e^{-z/\delta_A}$

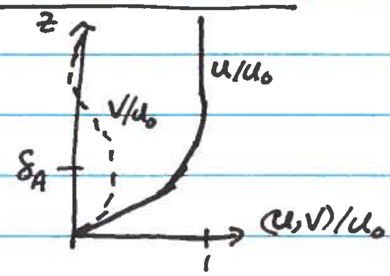
$v_E = u_0 \sin(z/\delta_A)\, e^{-z/\delta_A}$

THE TOTAL FLOW IS $u = u_g + u_E = u_0\left(1 - \cos(z/\delta_A)\, e^{-z/\delta_A}\right)$

$v = v_g + v_E = u_0 \sin(z/\delta_A)\, e^{-z/\delta_A}$

THESE ARE THE FLOWS OF THE "<u>EKMAN SPIRAL</u>"





NOTE $\delta_A = \sqrt{\dfrac{2A}{|f_0|}} = \sqrt{2}\sqrt{\dfrac{A}{|f_0|}} = \sqrt{2}\,\delta$ WITH δ THE EKMAN LAYER DEPTH

So, AS EXPECTED, BOUNDARY EFFECTS ARE FELT ONLY OVER A DISTANCE OF ORDER δ.

EXAMPLES

(1) FOR WATER IN A TANK ROTATING WITH ANGULAR FREQUENCY $\Omega = 0.5\,s^{-1}$

$A = \nu = 0.01\ cm^2/s$ (KINEMATIC VISCOSITY OF WATER)

$\Rightarrow \delta_A = \sqrt{2(0.01\ cm^2/s)/(2(0.5\ s^{-1}))} \approx 0.14\ cm$

(2) FOR ATMOSPHERE IN N.H. AT MIDLATITUDES SUPPOSE EDDY VISCOSITY IS $A \approx 100\ m^2/s$ (ABOUT 10^7 LARGER THAN KINEMATIC VISCOSITY OF AIR)

$\Rightarrow \delta_A \approx \sqrt{2(100\ m^2/s)/(10^{-4}\ s^{-1})} \approx 1.4\ km$

5 D] (CONT'D)

NOW COMPUTE THE EKMAN PUMPING VELOCITY, W_E.

i) CALCULATION USING STRESS FORMULA $\quad W_E = \frac{1}{\rho_0 f_0}\hat{z}\cdot\nabla\times\underline{\tau}$ (p.43)

$$\underline{\tau}|_{BNDY} = \rho_0 A \frac{\partial \underline{u}_E}{\partial z}\Big|_{z=0}$$

$$⑥ \Rightarrow \quad = \rho_0 A\Big\{\Big[\tfrac{1}{\delta_A}u_g \sin\left(\tfrac{z}{\delta_A}\right) \mp \tfrac{1}{\delta_A}v_g\cos\left(\tfrac{z}{\delta_A}\right)\Big]e^{-z/\delta_A} - \Big[u_g\cos\left(\tfrac{z}{\delta_A}\right) \pm v_g \sin\left(\tfrac{z}{\delta_A}\right)\Big]\Big(\tfrac{-1}{\delta_A}e^{-z/\delta_A}\Big),$$
$$\Big[\pm\tfrac{1}{\delta_A}u_g\cos\left(\tfrac{z}{\delta_A}\right) + \tfrac{1}{\delta_A}v_g\sin\left(\tfrac{z}{\delta_A}\right)\Big]e^{-z/\delta_A} + \Big[\pm u_g \sin\left(\tfrac{z}{\delta_A}\right) - v_g\cos\left(\tfrac{z}{\delta_A}\right)\Big]\Big(\tfrac{-1}{\delta_A}e^{-z/\delta_A}\Big)\Big\}\Big|_{z=0}$$

$$= \rho_0 A \frac{1}{\delta_A}\{(\mp v_g + u_g), (\pm u_g + v_g)\}$$

$$\text{SO } W_E = \frac{1}{\rho_0 f_0}\left(\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y}\right)$$
$$= \frac{A}{f_0}\frac{1}{\delta_A}\left(\frac{\partial}{\partial x}(\pm u_g + v_g) - \frac{\partial}{\partial y}(\mp v_g + u_g)\right)$$
$$= \frac{A}{f_0}\frac{1}{\delta_A}\left(\pm\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}\right)$$

BUT $\nabla\cdot\underline{u}_g = 0$ AND $\frac{A}{f_0} = \underset{S.H.}{\overset{N.H.}{\pm}}\frac{A}{|f_0|} = \pm\frac{1}{2}\delta_A^2$ (USING $\delta_A \equiv \sqrt{\frac{2A}{|f_0|}}$)

ALSO DEFINING GEOSTROPHIC VORTICITY $\zeta_g \equiv \hat{z}\cdot\nabla\times\underline{u}_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y}$

$\Rightarrow \quad \boxed{W_E = \pm\frac{1}{2}\delta_A\zeta_g}$ WITH $\pm = \text{SIGN}(f_0)$

ii) ALTERNATELY, FROM $\nabla\cdot\underline{u} = 0 \Rightarrow \frac{\partial w}{\partial z} = -\left(\frac{\partial u_E}{\partial x} + \frac{\partial v_E}{\partial y}\right) \Rightarrow W_E = -\int_0^\infty \nabla\cdot\underline{u}_E\,dz$

BUT ⑤ $\Rightarrow \int_0^\infty \mathcal{U}\,dz = -(u_g + iv_g)\int_0^\infty e^{-(1\pm i)z/\delta_A}dz$
$$= -(u_g + iv_g)[\delta_A/(1\pm i)] \qquad \text{WHERE } \tfrac{1}{1\pm i} = \tfrac{1}{2}(1\mp i)$$
$$= -\tfrac{1}{2}\delta_A\left[(u_g \pm v_g) + i(v_g \mp u_g)\right]$$

$$\Rightarrow \int_0^\infty u_E\,dz = -\tfrac{1}{2}\delta_A(u_g \pm v_g), \qquad \int_0^\infty v_E\,dz = -\tfrac{1}{2}\delta_A(v_g \mp u_g)$$

$$\text{SO } W_E = -\frac{\partial}{\partial x}\left[-\tfrac{1}{2}\delta_A(u_g \pm v_g)\right] - \frac{\partial}{\partial y}\left[-\tfrac{1}{2}\delta_A(v_g \mp u_g)\right]$$
$$= +\tfrac{1}{2}\delta_A\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) \mp \tfrac{1}{2}\delta_A\left(\frac{\partial u_g}{\partial y} - \frac{\partial v_g}{\partial x}\right)$$
$$= \pm\tfrac{1}{2}\delta_A\zeta_g \quad \text{AS ABOVE.}$$

EXAMPLE: IN TANK OF DEPTH $H = 10\,cm$ ROTATING AT $\Omega = 1\,s^{-1}$ HAVE

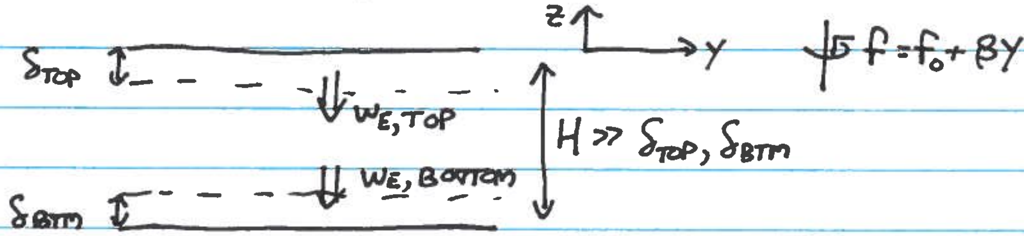
$\delta_A = \sqrt{2\nu/f_0} = \sqrt{\frac{2(0.01\,cm^2/s)}{2(1\,s^{-1})}} = 0.1\,cm$ AND $\zeta_g = 0$ IN SOLID BODY ROTATION

IF ANGULAR FREQUENCY OF TANK DECREASED TO $0.9\,s^{-1}$, RELATIVE VORTICITY IS $\zeta_g = 2(1.0 - 0.9) = 0.2\,s^{-1}$. SO $W_E = \frac{1}{2}(0.1\,cm)(0.2\,s^{-1}) = 0.01\,cm/s$

SO SPIN-DOWN TIME IS $\frac{H}{|W_E|} \sim \frac{10\,cm}{0.01\,cm/s} = 1000\,s$ //

(6) Sverdrup Transport

Now consider response of whole ocean depth to Ekman pumping at surface and bottom



Assume a uniform-density ocean $\bar{\rho} \approx \rho_0$ and motion on β-plane
Away from Ekman layers have $\underline{f} \times \underline{u}_g = -\frac{1}{\rho_0}\nabla p$
Taking curl of equation gives Sverdrup balance (see p. 35)

$$\boxed{\beta v_g = f_0 \frac{\partial w}{\partial z}}$$

Integrate vertically from $z = -H + \delta_{BTM}\ (\approx -H)$ to $z = -\delta_{TOP}\ (\approx 0)$

$$\Rightarrow \beta \int_{-H}^{0} v_g \, dz = f_0 \left(w_{E,TOP} - w_{E,BTM} \right)$$

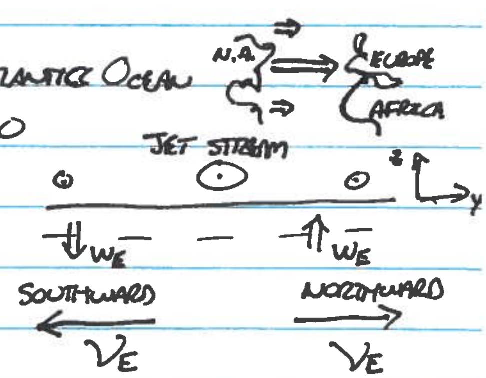
Define vertically integrated meridional flow by $\mathcal{V}_E \equiv \int_{-H}^{0} v_g \, dz$
Hence Ekman pumping results in a meridional flow

$$\mathcal{V}_E = \frac{f_0}{\beta}\left(w_{E,TOP} - w_{E,BTM} \right)$$

This is northward in N.H. if $w_{E,TOP} - w_{E,BTM} > 0$
But $w_{E,TOP} > 0$ in N.H. if $\hat{z} \cdot \nabla \times \underline{\tau} > 0$, e.g. if $\zeta_g > 0$ for winds above surface. $w_{E,TOP} < 0$ if $\zeta_g < 0$ for winds above surface.

Example:

(1) Consider jet stream passing over North Atlantic Ocean
North of jet stream $\zeta_g \approx -\frac{\partial u_g}{\partial y} > 0 \Rightarrow \mathcal{V}_E > 0$
South of jet stream $\zeta_g \approx -\frac{\partial u_g}{\partial y} < 0 \Rightarrow \mathcal{V}_E < 0$



The divergence of waters from mid-latitudes is filled by the Gulf Stream.

⑥ (CONT'D)

EXAMPLE

② SUPPOSE SURFACE WIND STRESS DECREASES LINEARLY FROM 1.0 Pa TO 0.1 Pa ($\approx$ 30m/s WINDS TO 10m/s WINDS ··· SEE p.41) GOING NORTHWARD FROM 45°N TO 55°N. ESTIMATE SVERDRUP TRANSPORT NORTHWARD ACROSS 50°N OVER SPAN OF ATLANTIC OCEAN

SOL$^{\underline{n}}$

ESTIMATE $f_0 = 2\Omega_e \sin 50° \approx 1.1 \times 10^{-4}\, s^{-1}$

$$\beta = 2\Omega_e \cos 50°/R_e \simeq \frac{9.3\times10^{-5}\, s^{-1}}{6.4\times10^{6}\, m} \approx 1.5\times10^{-11}\, \frac{1}{m \cdot s}$$

$$W_E = \frac{1}{\rho_0 f_0}\left(\frac{\partial \tau_{y}}{\partial x} - \frac{\partial \tau_{x}}{\partial y}\right) \approx \frac{1}{\rho_0 f_0}\frac{-\Delta\tau}{L} \approx \frac{1}{(1000\, kg/m^3)}\frac{1}{(1.1\times10^{-4} s^{-1})}\frac{[-(0.1 - 1.0 Pa)]}{10° \times 111 km/°}$$

$$\simeq 7.4\times10^{-6}\, m/s$$

IGNORING BOTTOM DRAG ($W_{E,BTM} \approx 0$) HAVE

$$V_E = \frac{f_0}{\beta} W_E \approx \frac{1.1\times10^{-4}\, s^{-1}}{1.5\times10^{-11}\, (ms)^{-1}}(7.4\times10^{-6}\, m/s) \approx 5.4\times10^{1}\, m^2/s$$

THE SPAN OF THE ATLANTIC AT 50°N IS ABOUT $L_x \simeq 3.4\times10^3$ km (e.g. SEE WWW.MOVABLE-TYPE.CO.UK/SCRIPTS/LATLONG.HTML)

SO THE VOLUME FLUX ACROSS 50°N IS $V_E L_x \approx 1.8\times10^{8}\, m^3/s$ //

VOLUME TRANSPORT IN THE OCEAN IS TYPICALLY MEASURED IN "SVERDRUPS" : $1\, Sv = 10^6\, m^3/s$.

SO THE ABOVE EXAMPLE GIVES A VOLUME TRANSPORT OF 1.8×10^{2} Sv