Dynamics of the Atmosphere and Ocean II

Chapter 6: Instability of Stratified Shear Flows (supplemental)

Chapter Overview

- Here we include the effects of background shear in a stratified fluid showing in some circumstances that waves are unstable, growing exponentially in time. Rotation is ignored.
- After deriving the basic equations and interface conditions, we examine:
  - **interfacial waves in a stationary unbounded fluid**
  - **waves in a uniform-density semi-infinite shear flow (Rayleigh waves)**
  - **instability of waves in a uniform-density and stratified shear layer (Kelvin-Helmholtz instability)**
- But first, let's look at examples of unstable flows in general …

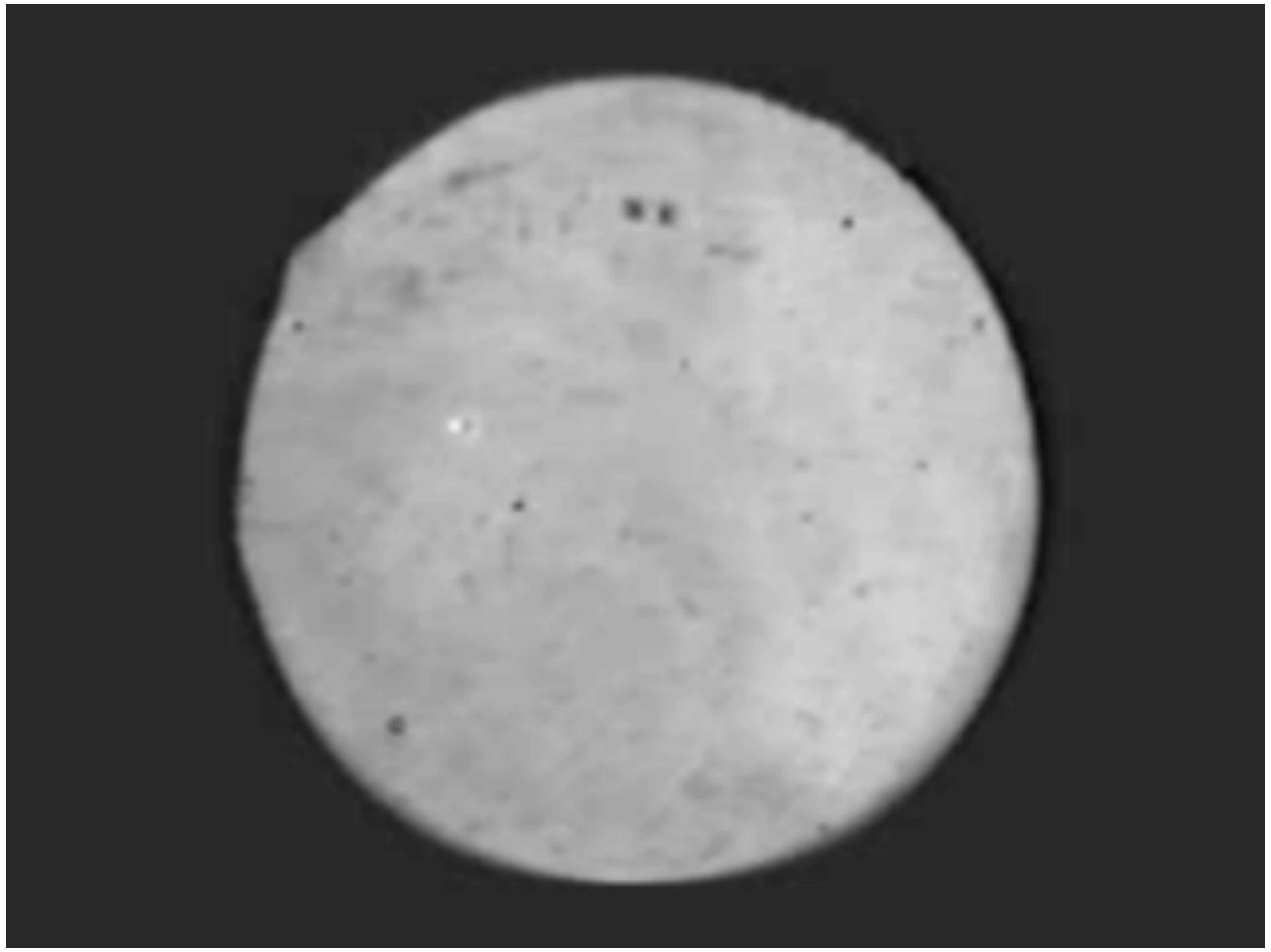
Break up of a stream of water into drops

Beading of water on a thread

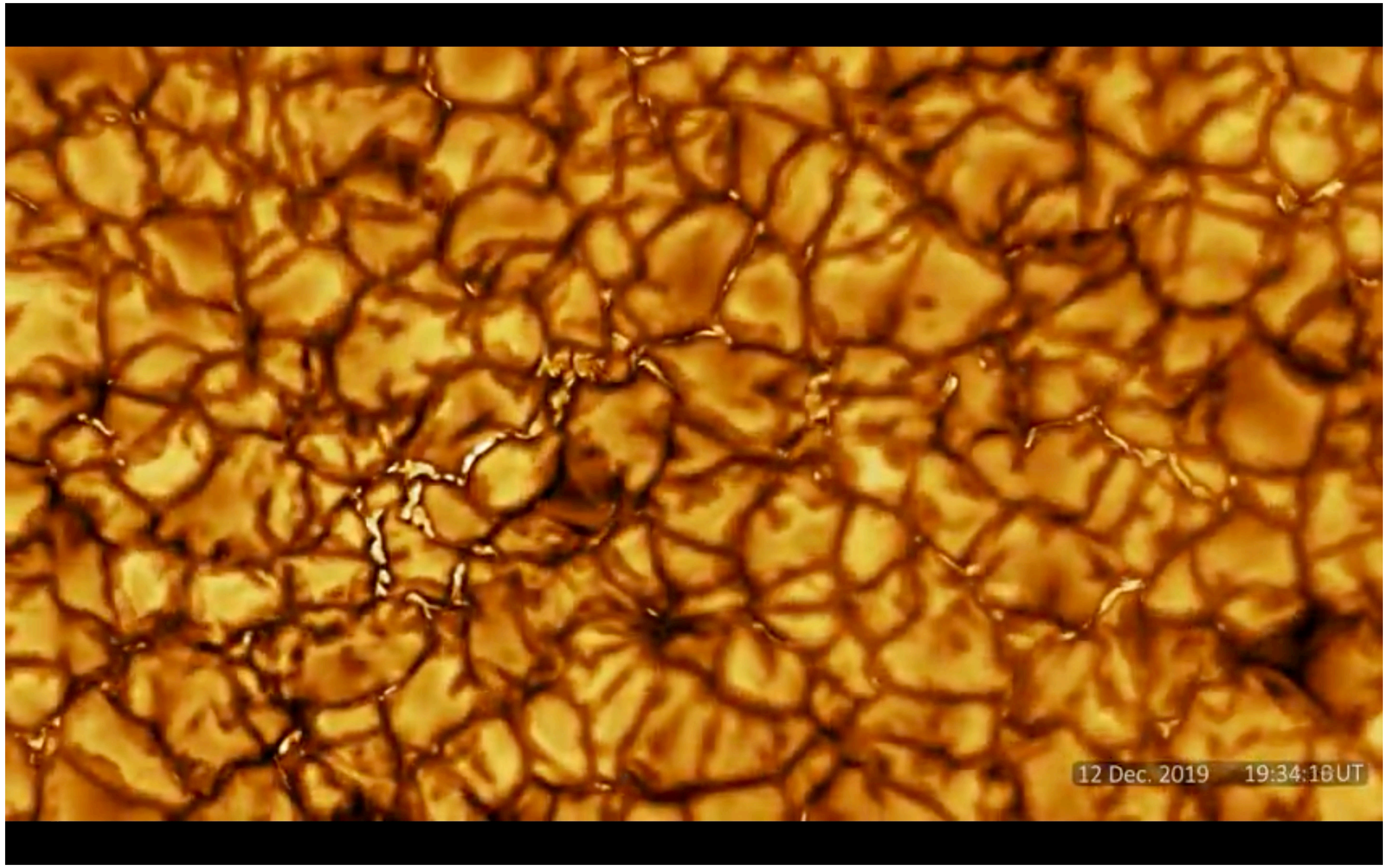
A good glass of wine develops “tears” (“legs”)

This movie shows their development

If a fluid is moderately heated from below, it forms convection cells



Convection occurs on very large scales



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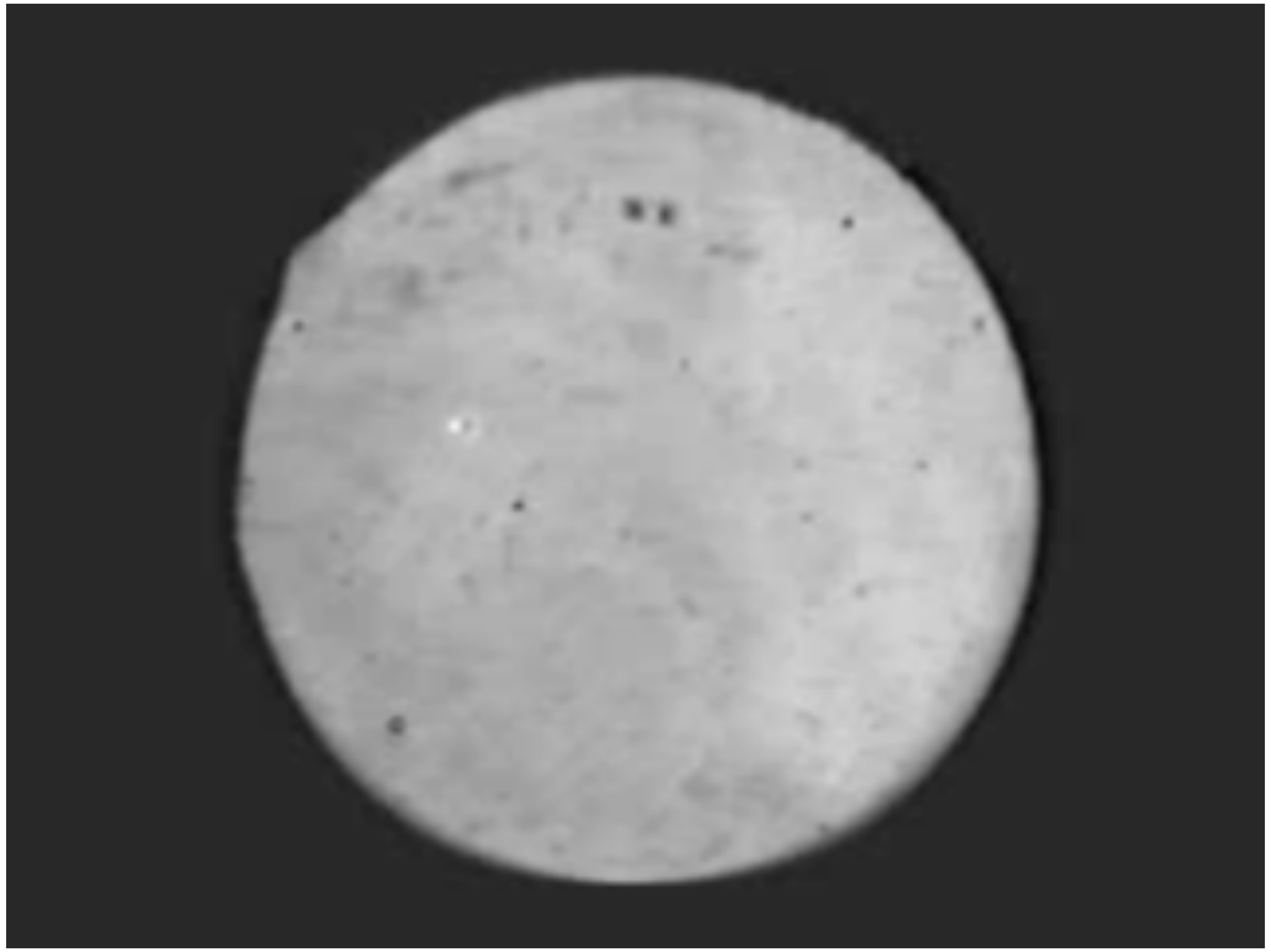
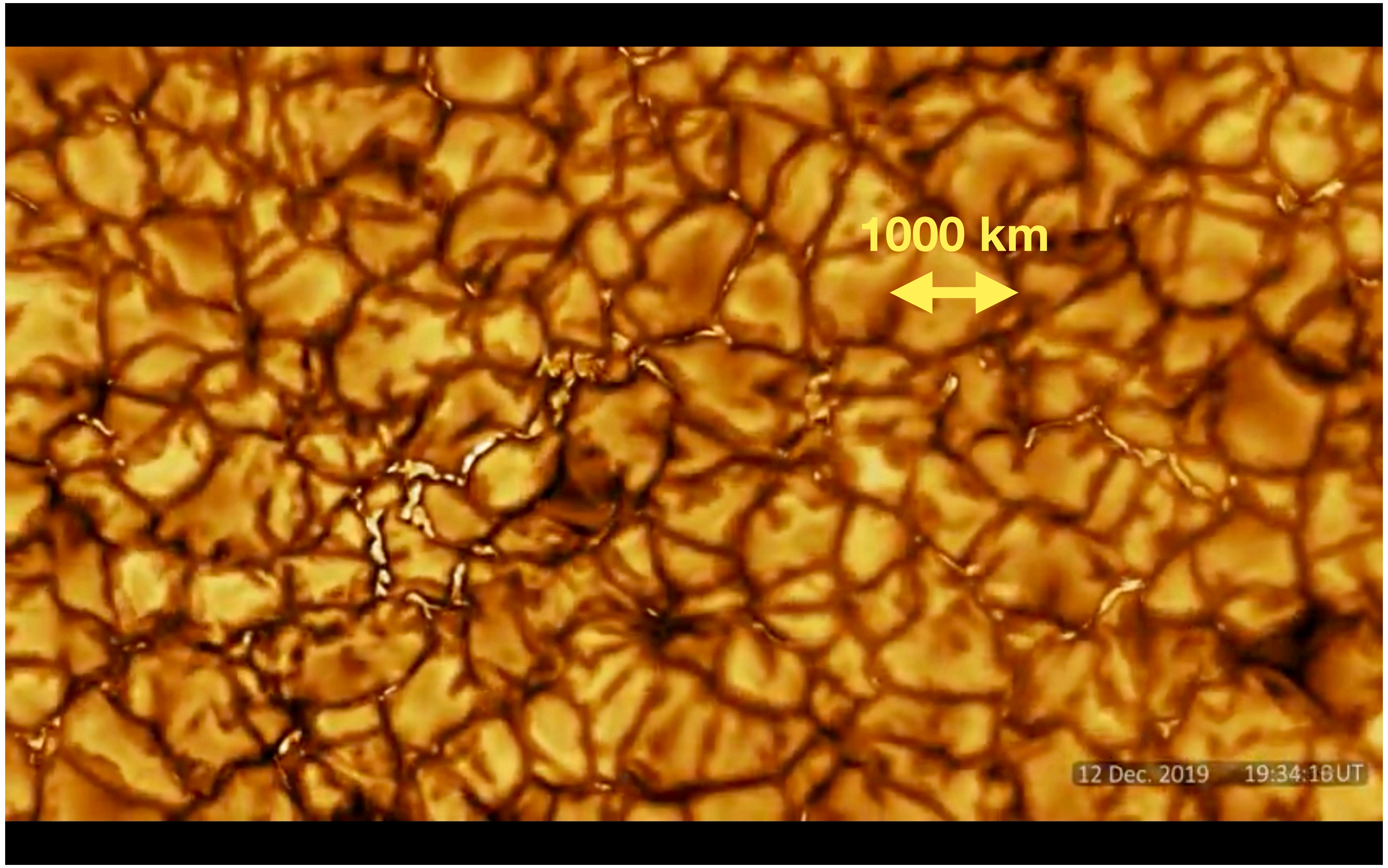
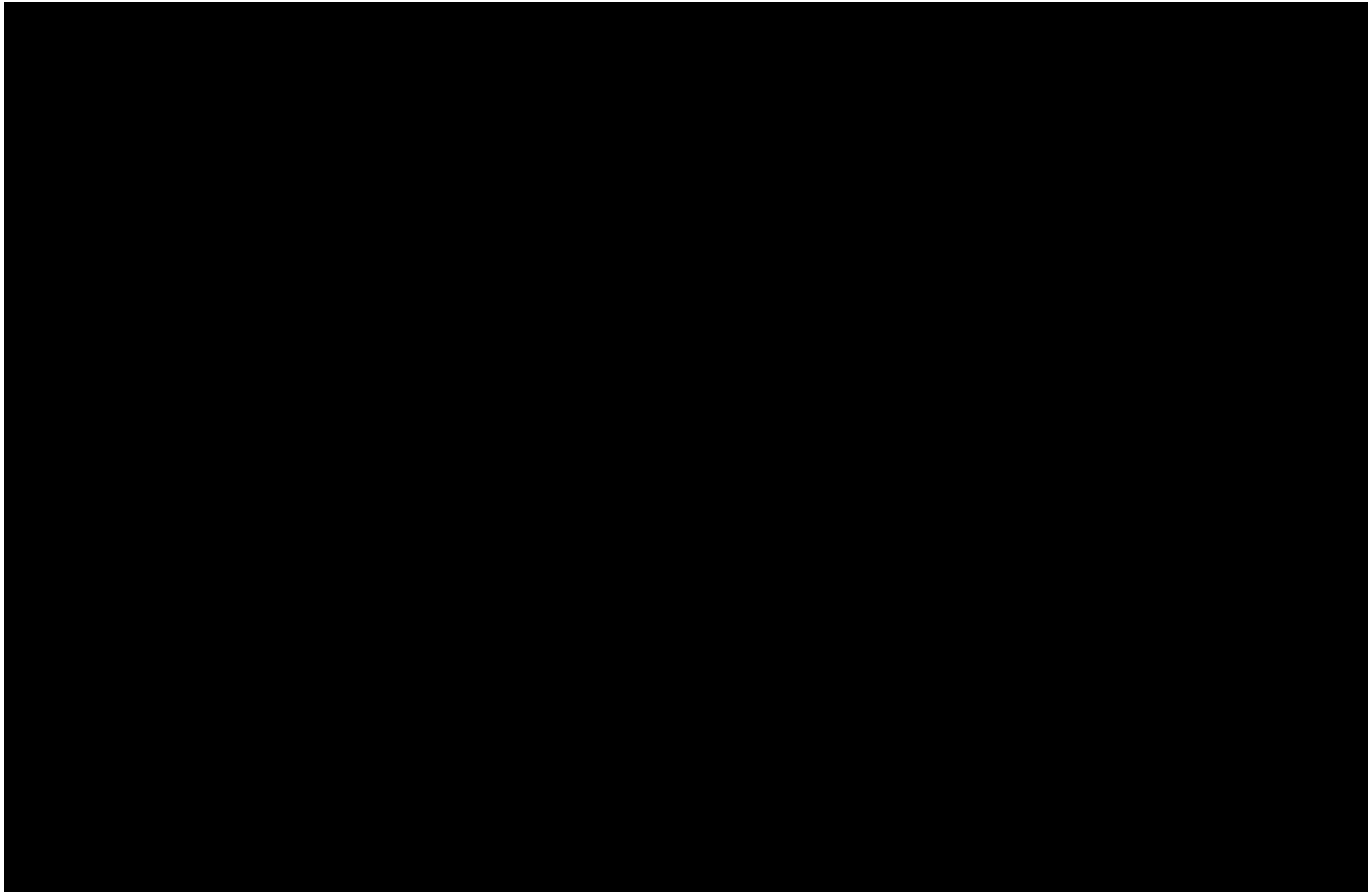


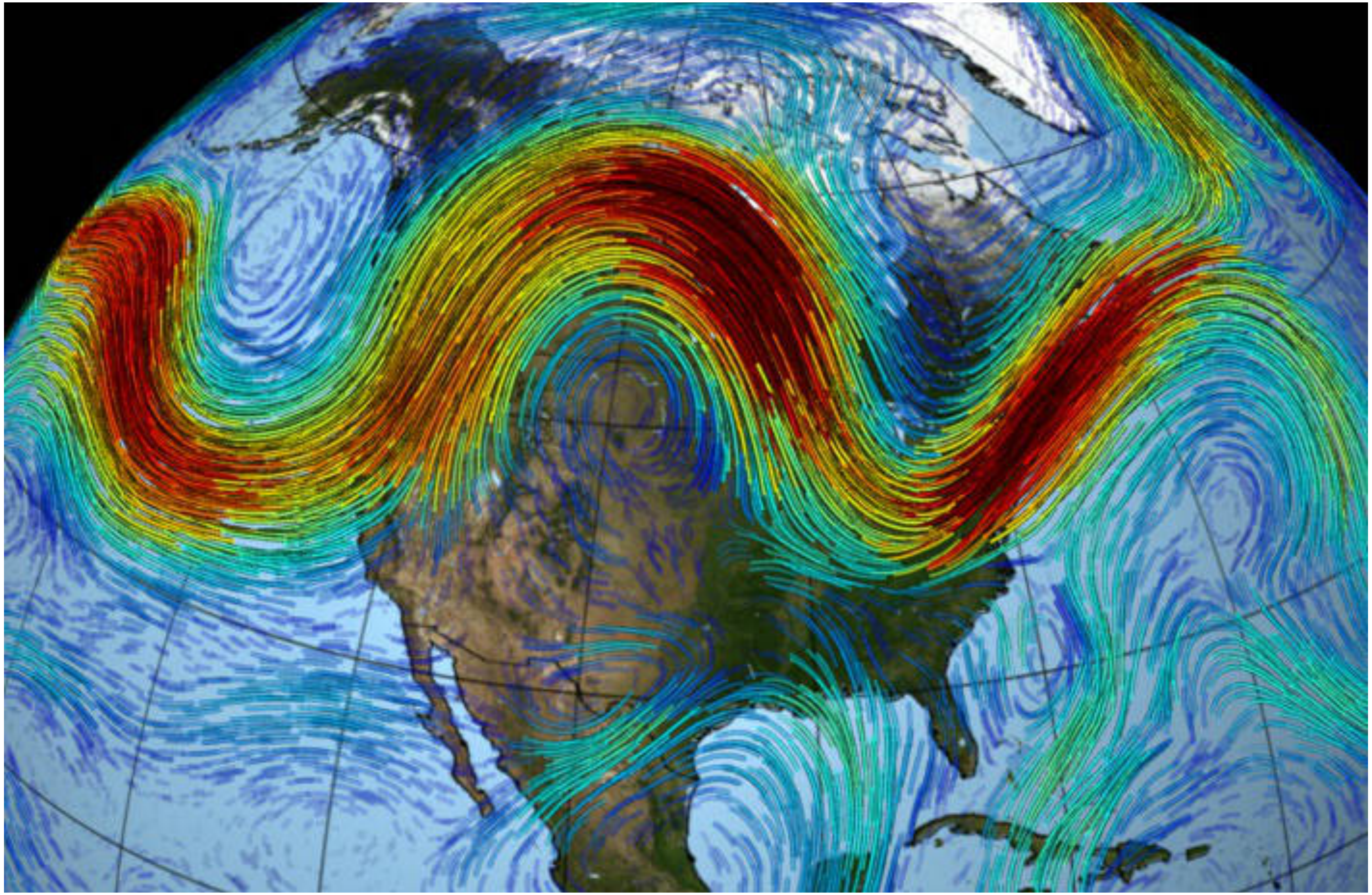
Convection occurs on very large scales … on the sun



“Salt fingers” form when as hot and salty water cools over fresh water



A combination of northward cooling, shear and the Earth's rotation leads to instability of the Jet Stream: a "Baroclinic Wave"

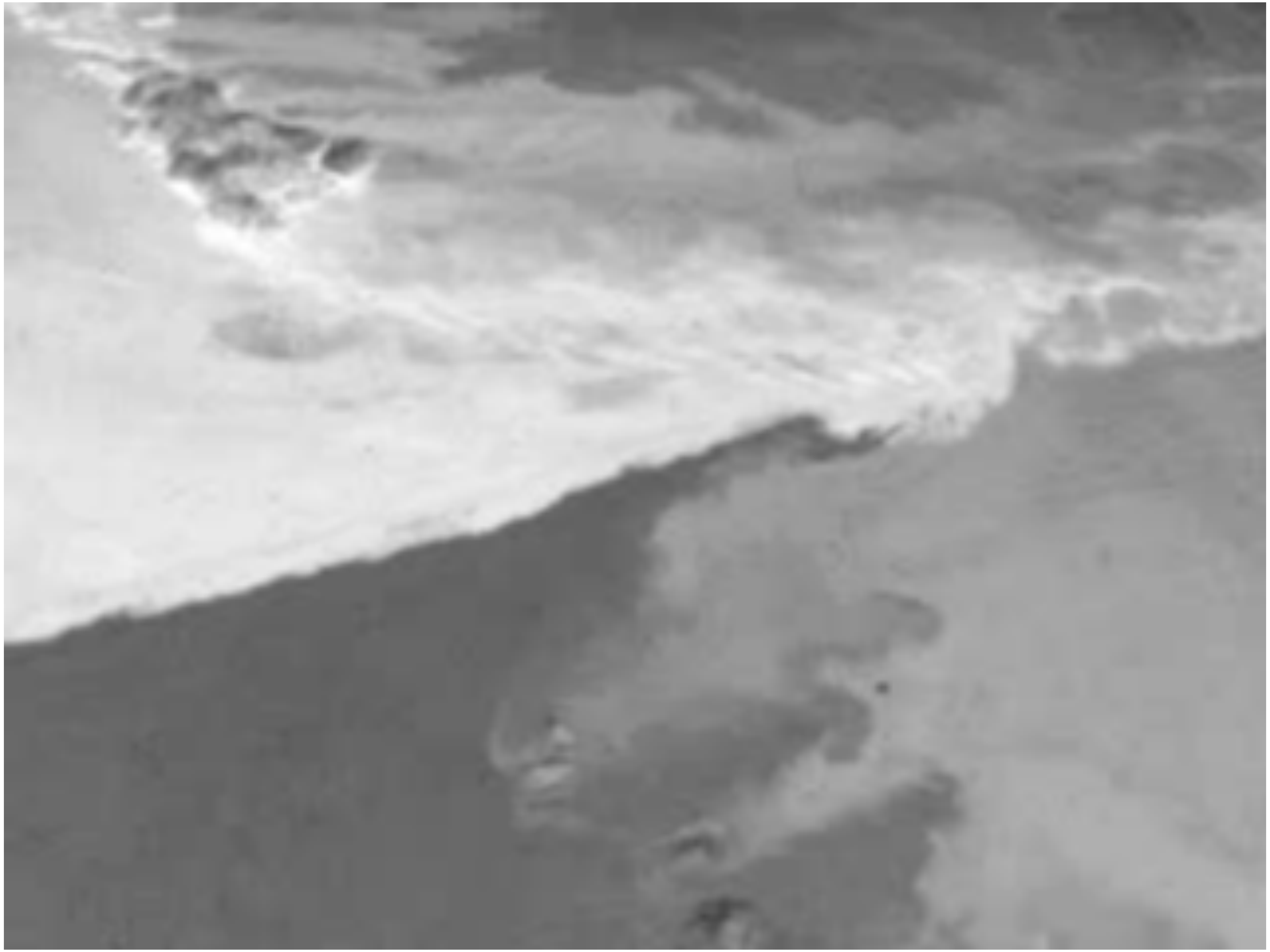


[Visualization of winds associated with the northern Jet Stream by NASA]

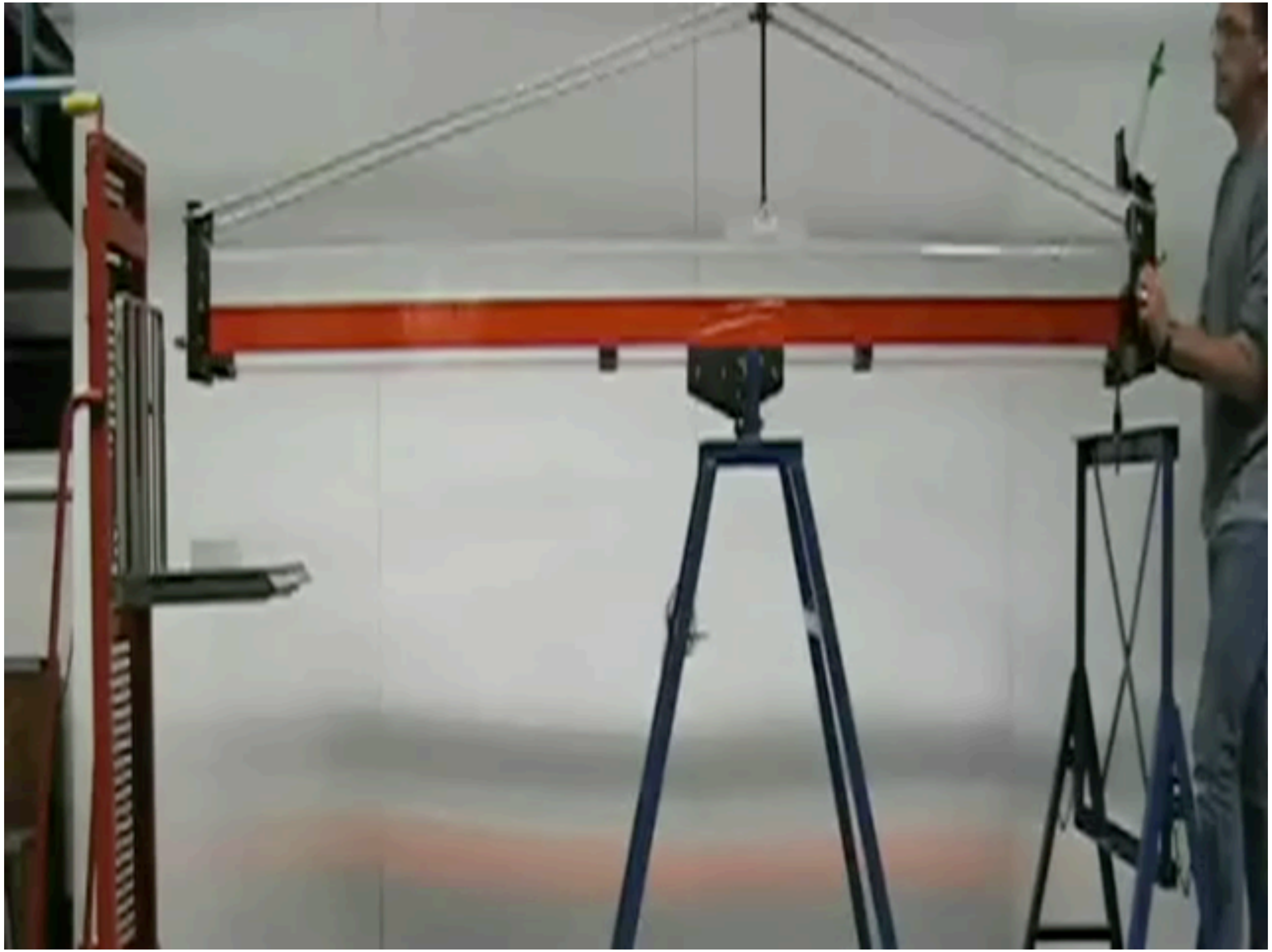
Shear instability causes a shear flow to wrap into vortices

[Credit: NASA]

Instability causes a shear flow to wrap into vortices



Shear instability causes a shear flow to wrap into vortices



[Laboratory Experiment by G. Worster, U. Cambridge]

Shear instability causes a shear flow to wrap into vortices

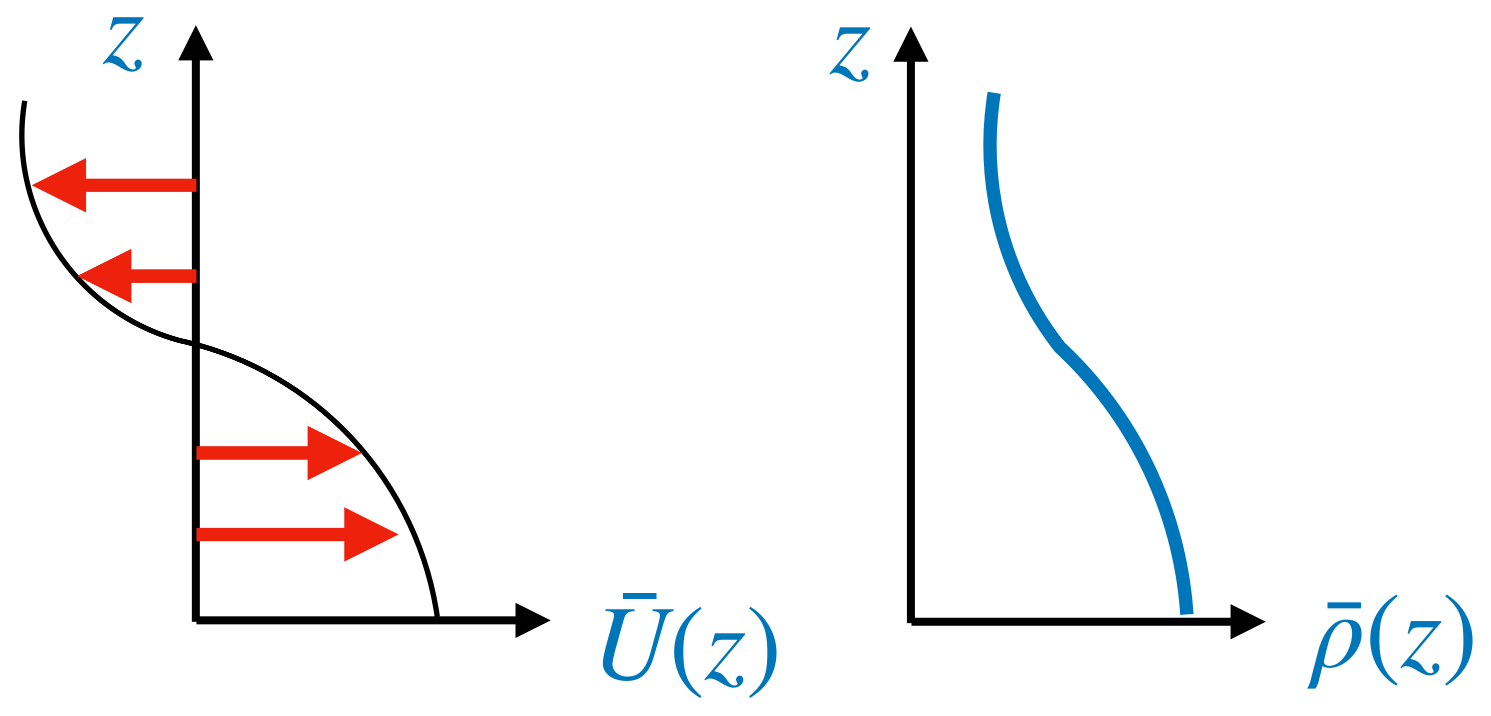
6.1] Equations with Non-uniform Background Flow

- So far we have considered waves in an otherwise stationary fluid.
- Now suppose there is a background flow whose velocity changes in space.
- Specifically, we will look at background flows oriented only in the x-direction and which varies in the z-direction.

  This is called a "parallel flow".

Equations with a Parallel Background Flow:

- In the absence of any perturbations:
  - the background flow is $\bar{\mathbf{u}} = \bar{U}(z)\,\hat{x}$
  - the background density is $\bar{\rho}(z)$



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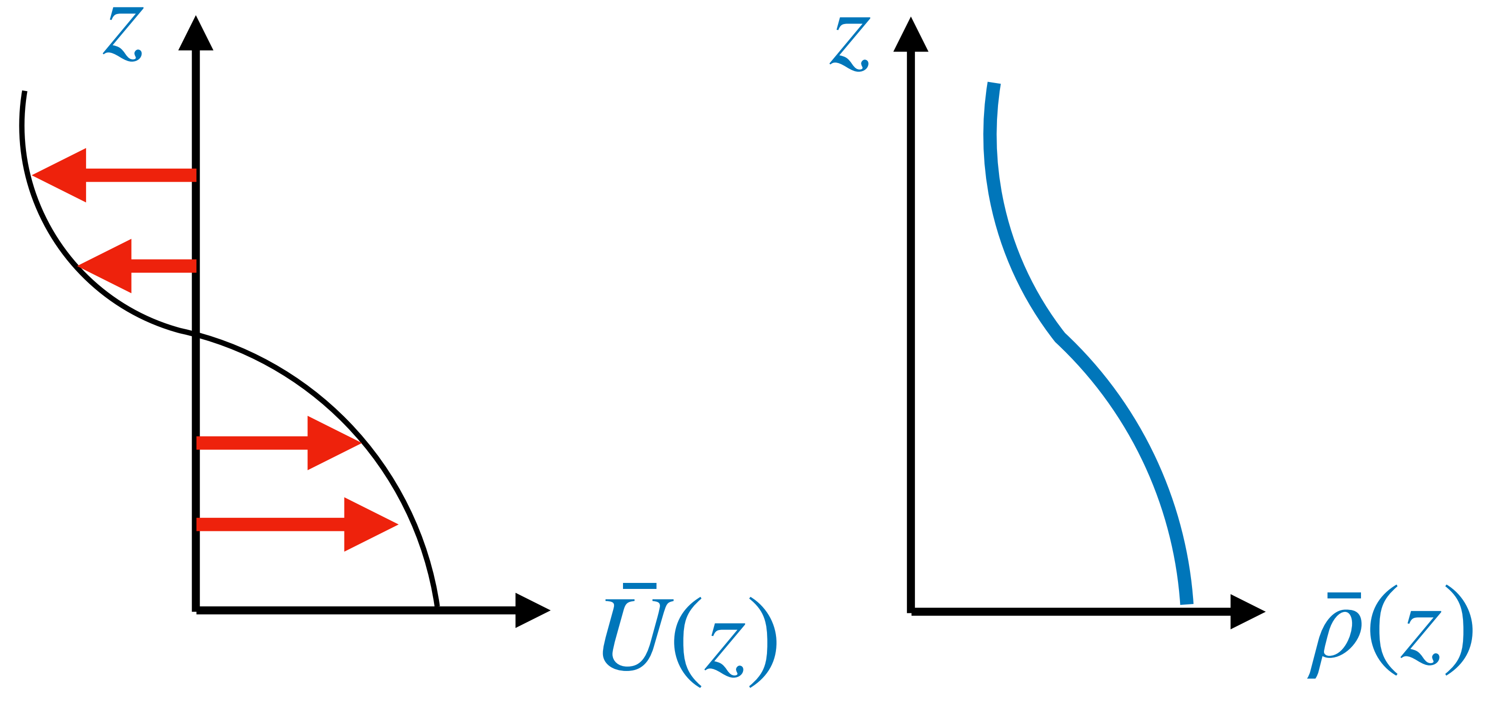

- Write equations with 2D perturbations superimposed on the background:
  $\mathbf{u} = \big(u(x,z,t),\ w(x,z,t)\big)$, $\rho(x,z,t)$ and $p(x,z,t)$ $\quad \Rightarrow \mathbf{u}_T = (\bar{U} + u,\ w)$

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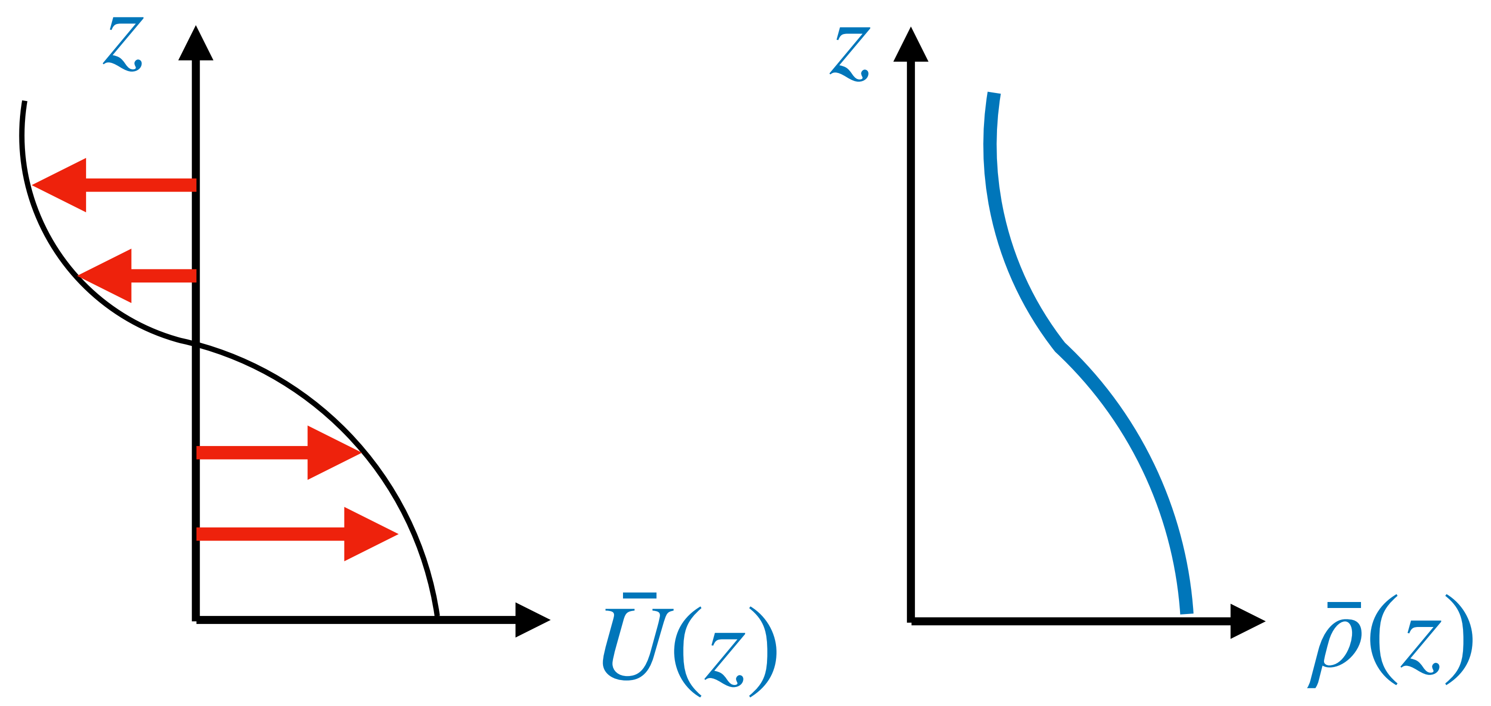

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- Assume small perturbations so that squared perturbation terms can be neglected
  - incompressible: $\nabla \cdot \mathbf{u}_T = 0$ $\quad\quad \partial_x u + \partial_z w = 0$

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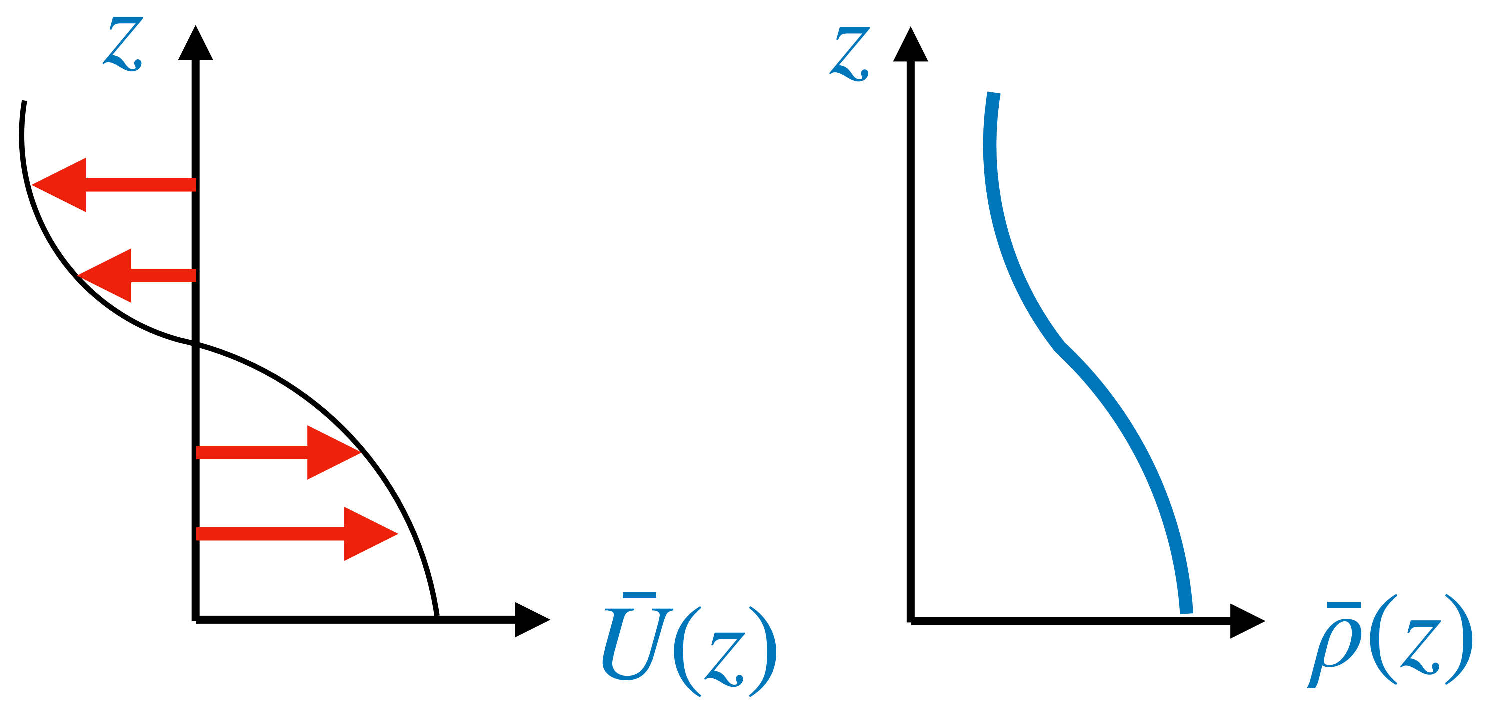

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  - x-momn: $D(\bar{U} + u)/Dt = -\,(1/\rho_0)\,\partial_x p$ $\qquad \partial_t u + \bar{U}\partial_x u + w\bar{U}' = -\,(1/\rho_0)\,\partial_x p$

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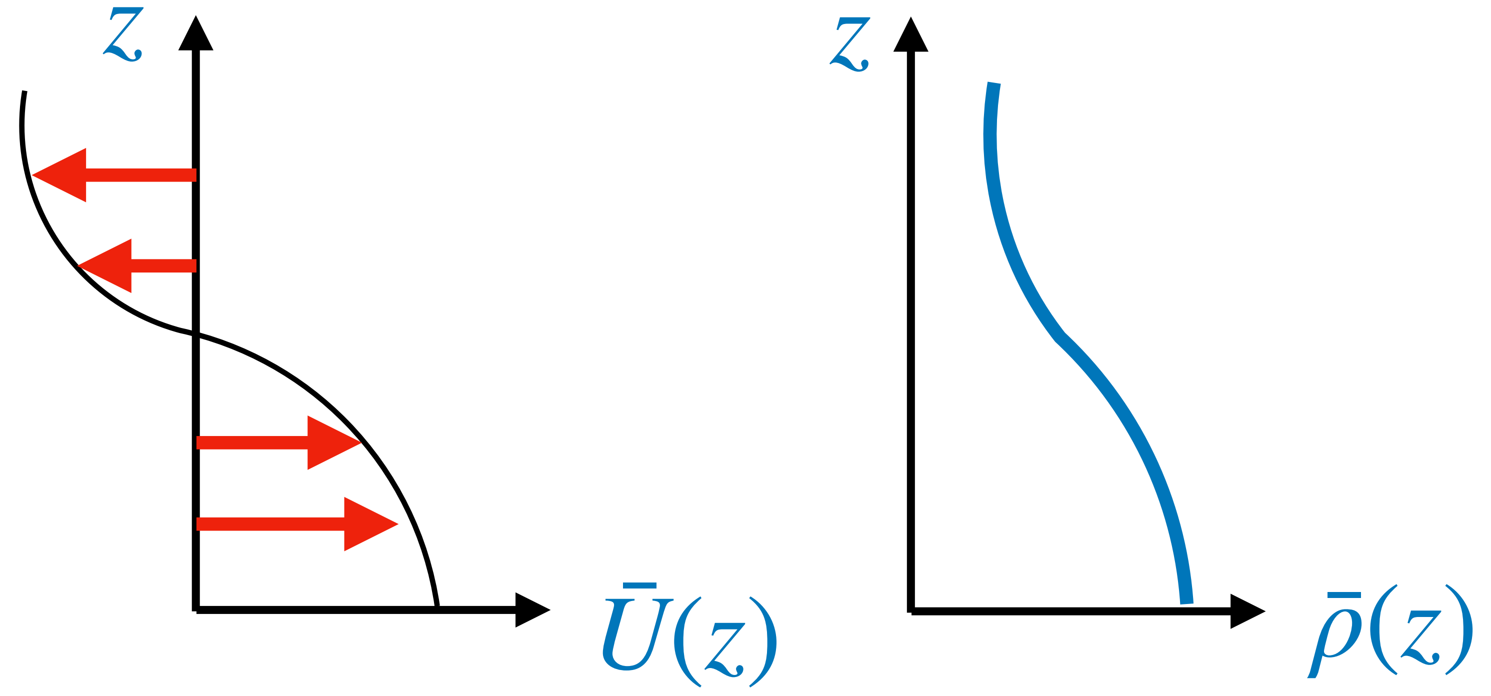

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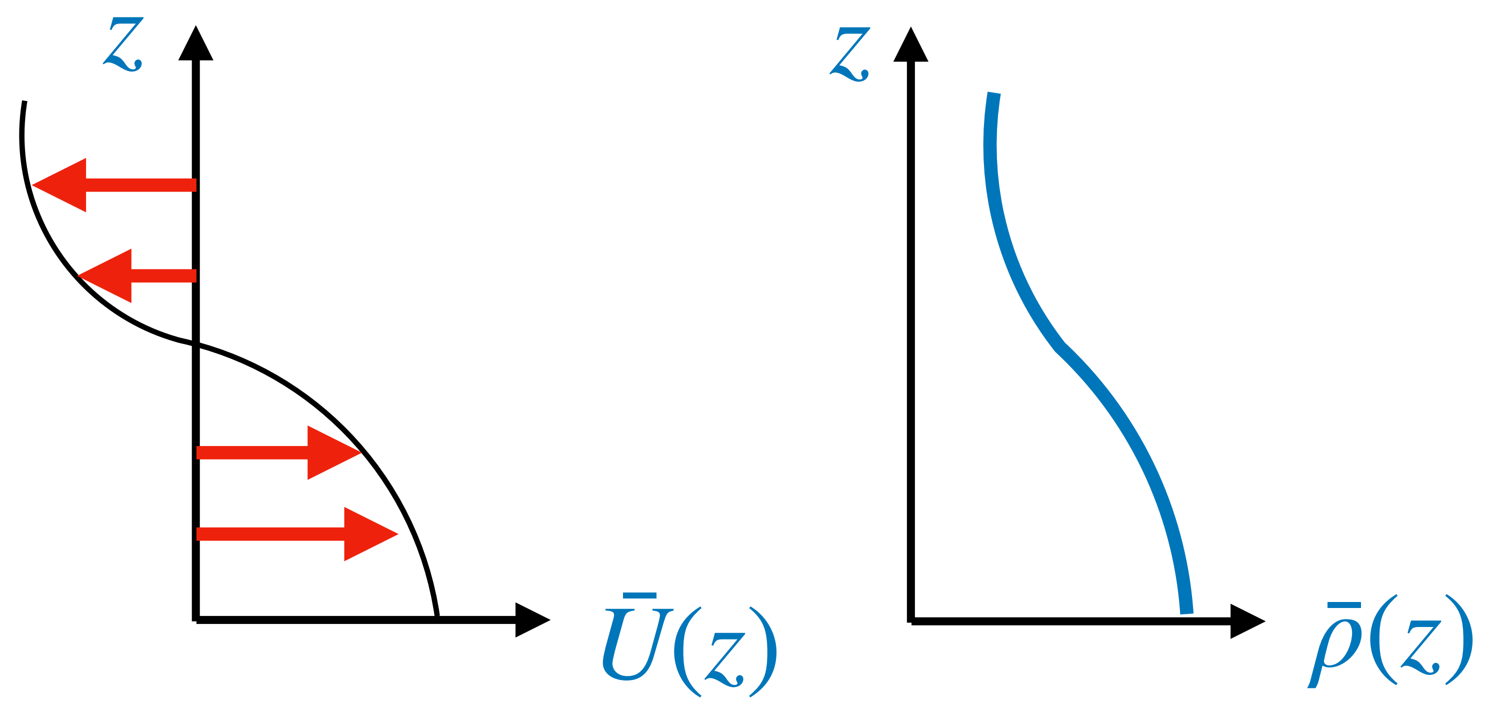

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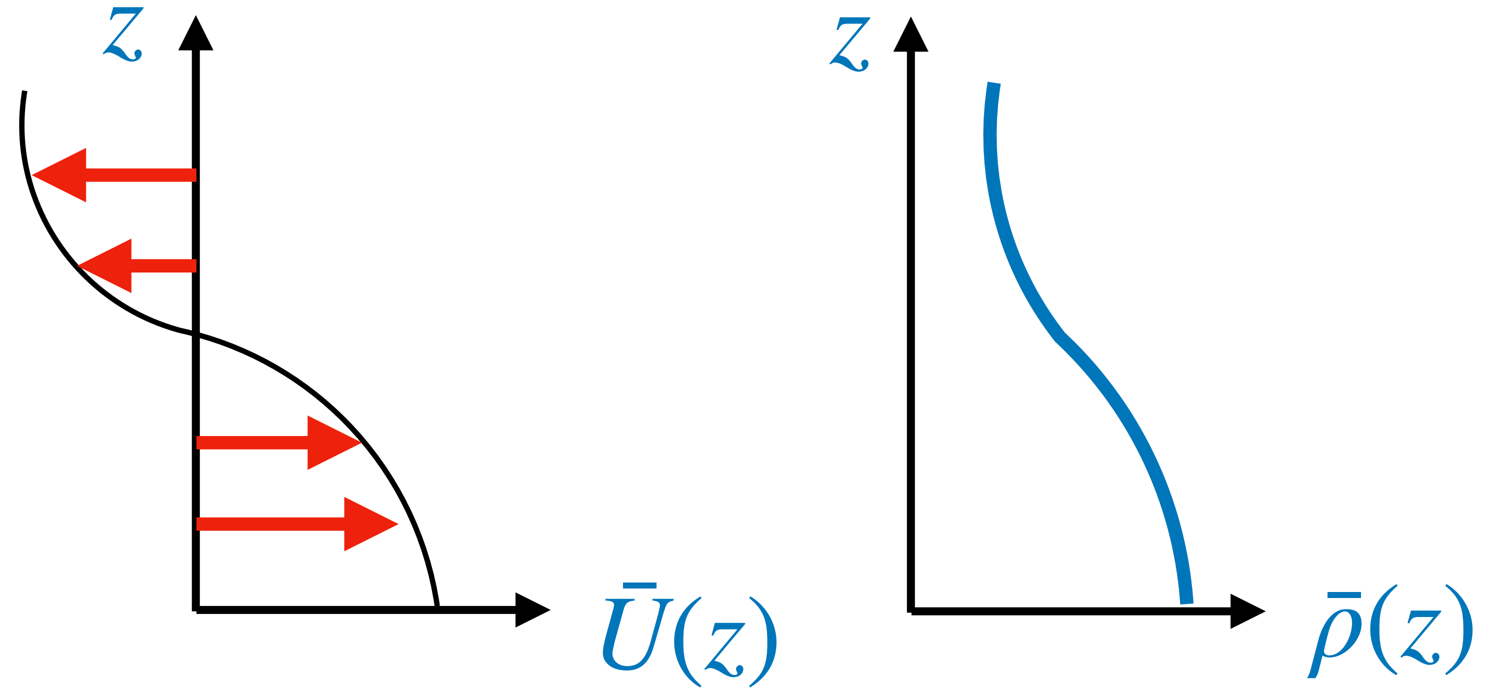

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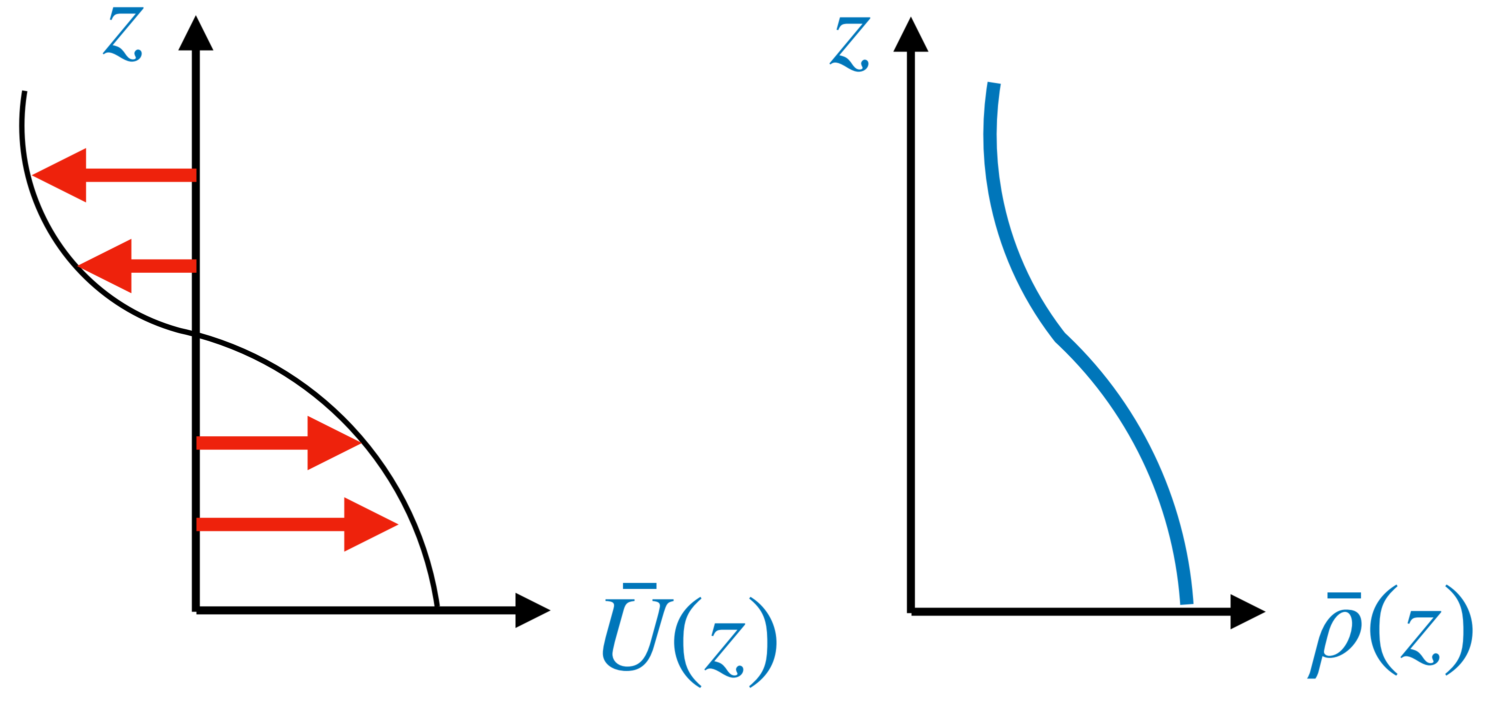
- Using incompressibility, can write $u = -\dfrac{\partial \psi}{\partial z},\ w = \dfrac{\partial \psi}{\partial x}$

Equations with a Parallel Background Flow: vertical structure equations



$$\partial_t u + \bar{U}\partial_x u + w\bar{U}' = -\,(1/\rho_0)\ \partial_x p$$

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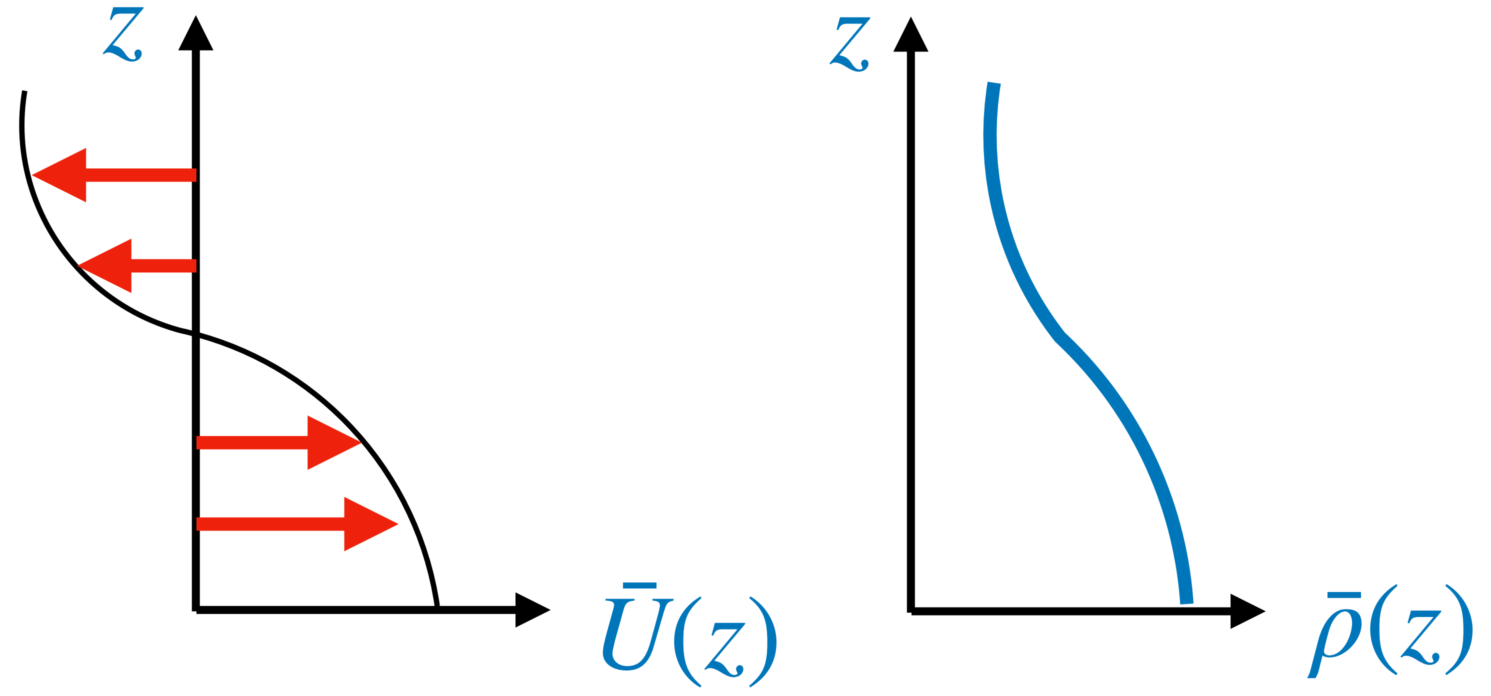
Equations with a Parallel Background Flow: vertical structure equations

- Assume perturbations are periodic in x and t:



$$\Rightarrow \psi = \hat{\psi}(z)\, e^{\imath(kx-\omega t)},\ p = \hat{p}(z)\, e^{\imath(kx-\omega t)},\ \rho = \hat{\rho}(z)\, e^{\imath(kx-\omega t)}$$

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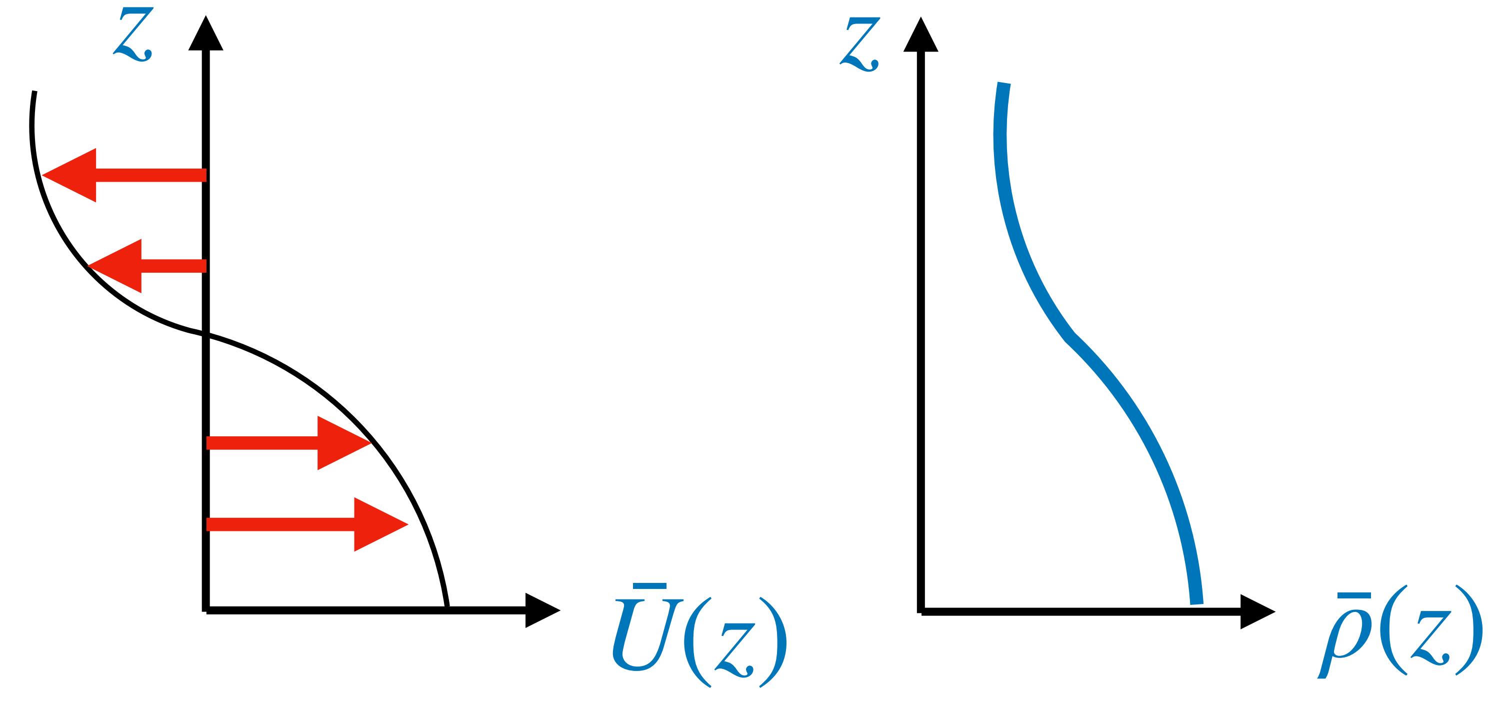

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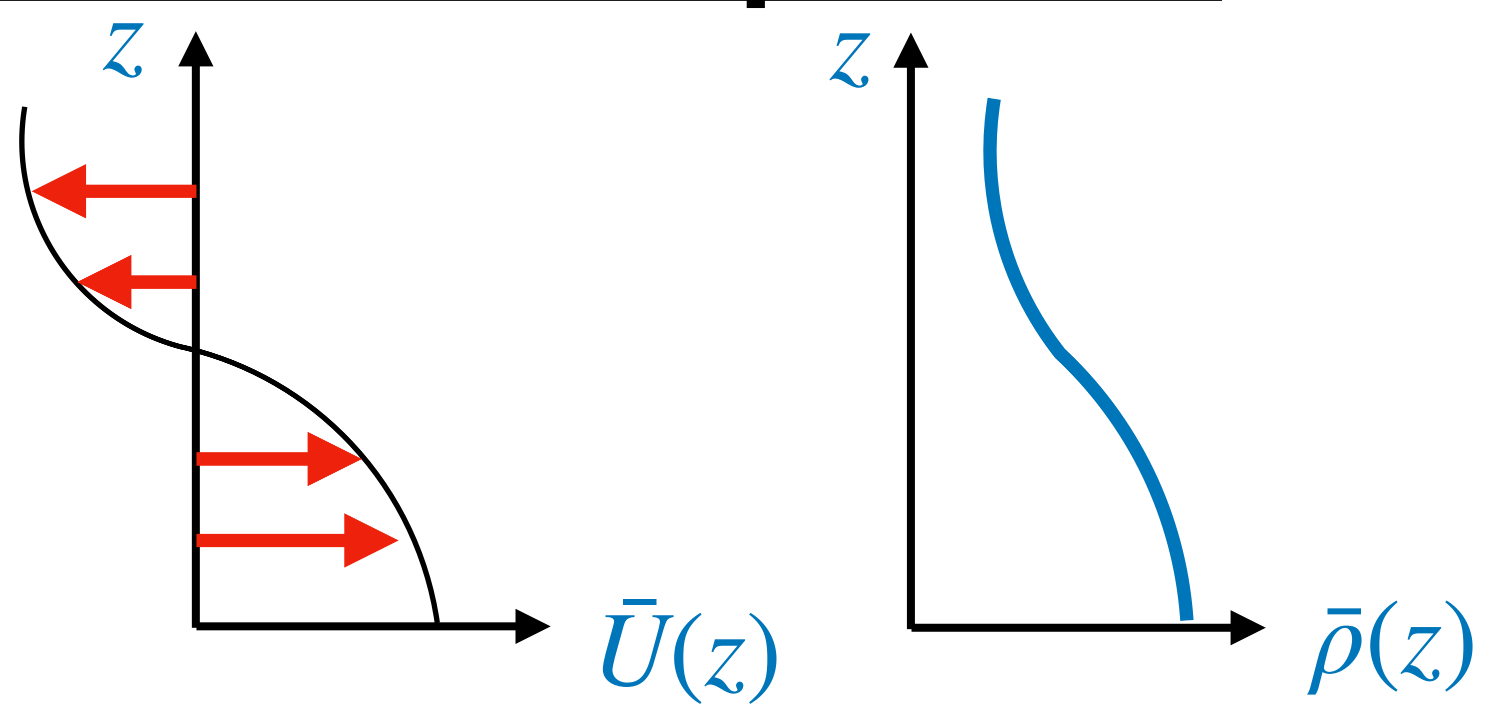

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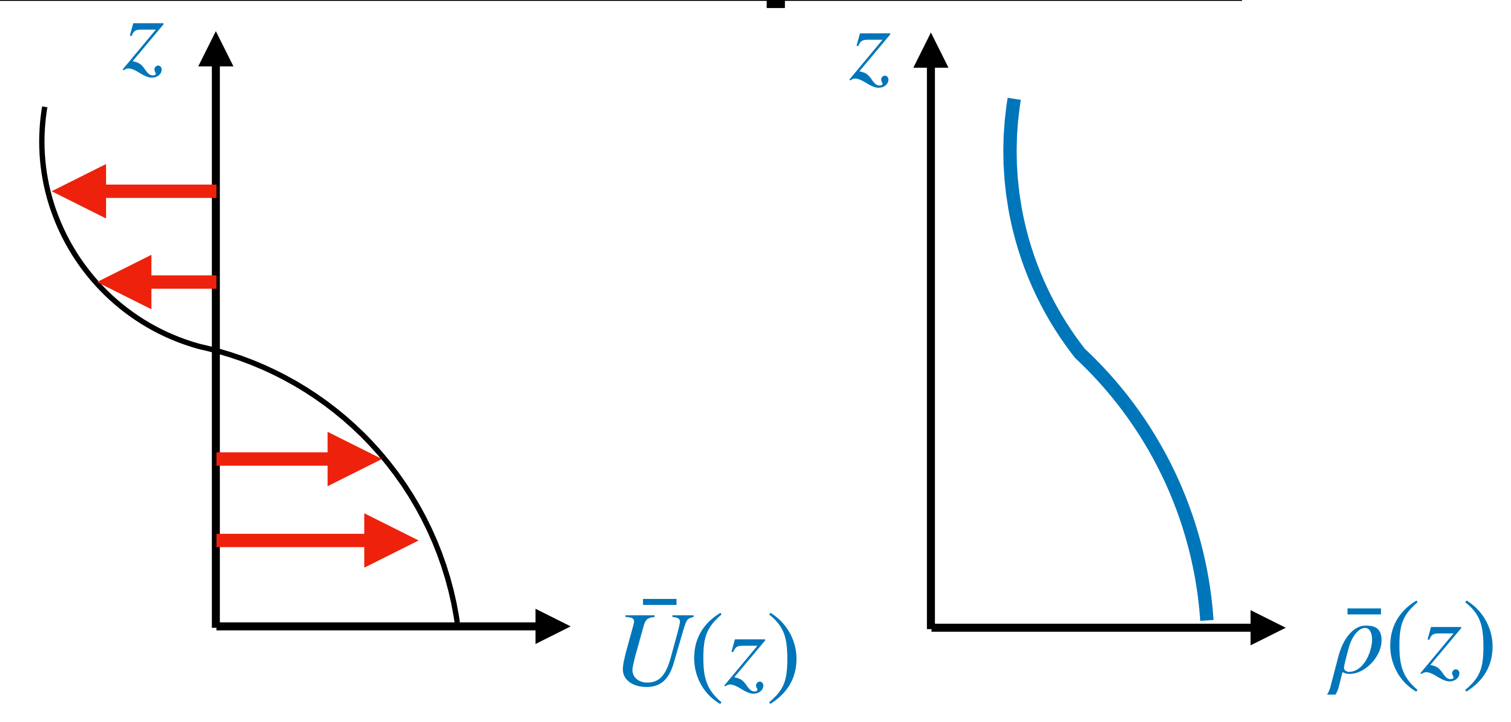

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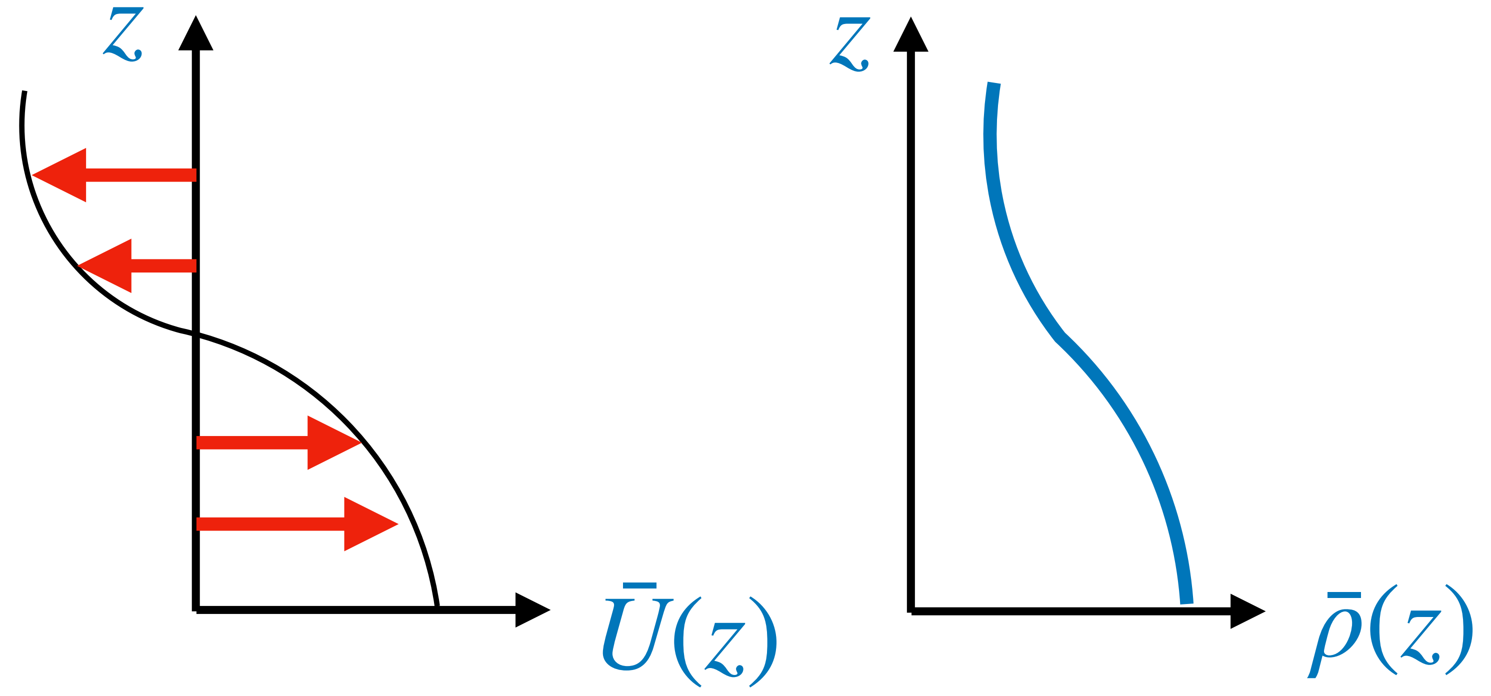

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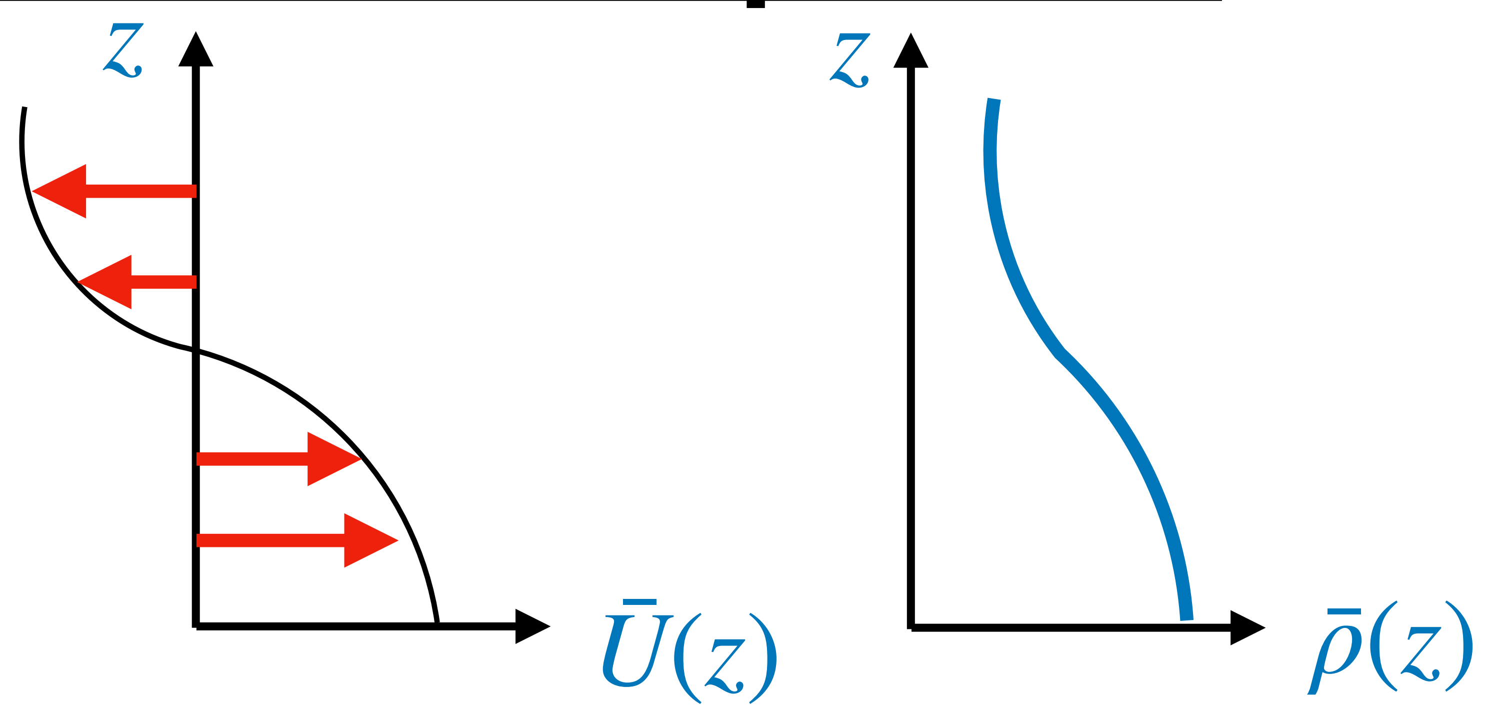

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Equations with a Parallel Background Flow: vertical structure equations

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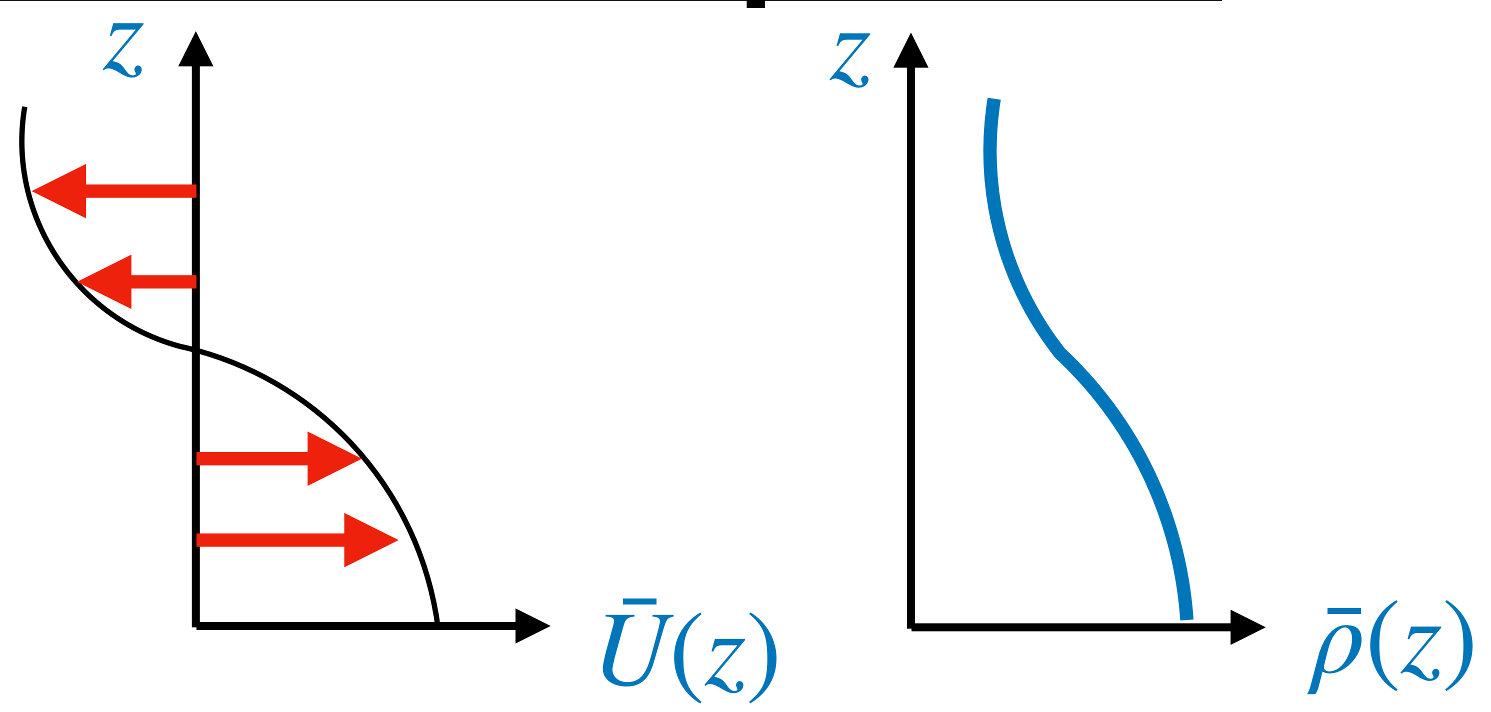

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$$\Rightarrow \boxed{(\bar{U}-c)\,\hat{\rho} = -\,\hat{\psi}\,\bar{\rho}'}$$

- Next eliminate $\hat{p}$ and $\hat{\rho}$ from these 3 equations, to get one equation for $\hat{\psi}$ alone.

Equations with a Parallel Background Flow: Taylor-Goldstein equation

- Eliminating $\hat{p}$ and $\hat{\rho}$ from

$$(\bar{U}-c)\,\hat{\psi}' - \bar{U}'\hat{\psi} = (1/\rho_0)\,\hat{p}$$

$$k^2(\bar{U}-c)\,\hat{\psi} = (1/\rho_0)\,\hat{p}' + (g/\rho_0)\,\hat{\rho}$$

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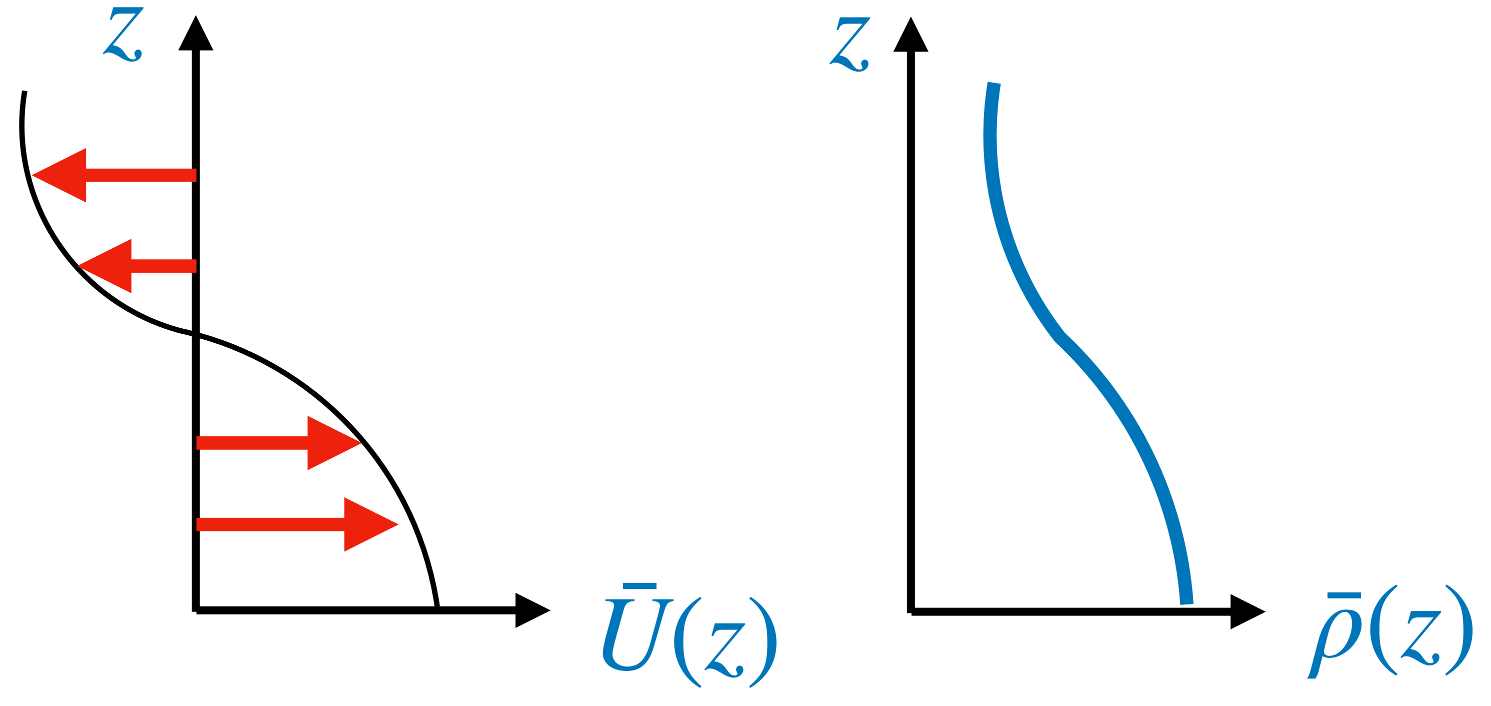

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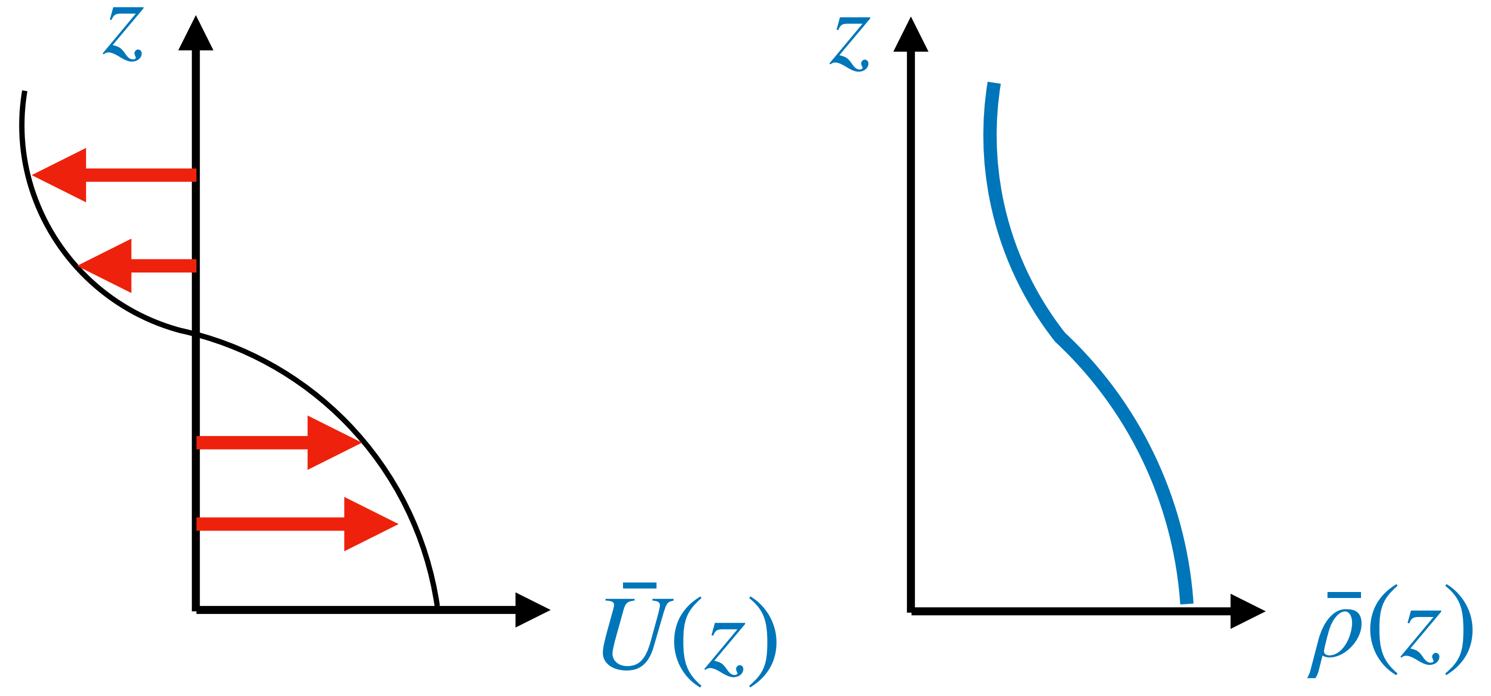

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$$(\bar{U}-c)\,\hat{\rho} = -\,\hat{\psi}\,\bar{\rho}'$$

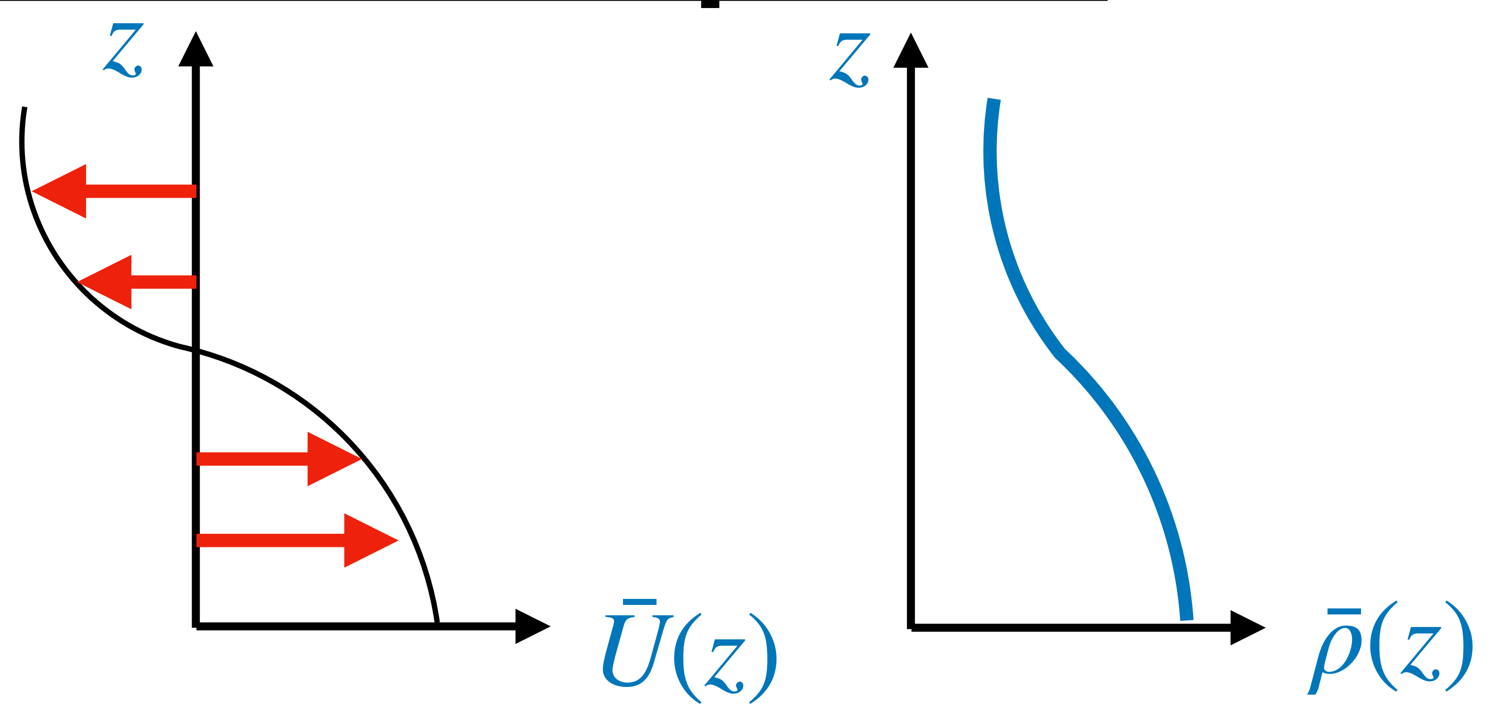
gives

$$\hat{\psi}'' - \left[-\frac{N^2}{(\bar{U}-c)^2} + \frac{\bar{U}''}{\bar{U}-c} + k^2\right]\hat{\psi} = 0$$

with $c \equiv \omega/k$ and $N^2(z) \equiv -\dfrac{g}{\rho_0}\dfrac{d\bar{\rho}}{dz}$ (the squared buoyancy frequency)

- This is the "Taylor-Goldstein" equation (first derived ~ 1931).

Equations with a Parallel Background Flow: Taylor-Goldstein equation



- Eliminating $\hat{p}$ and $\hat{\rho}$ from

$$(\bar{U}-c)\,\hat{\psi}' - \bar{U}'\hat{\psi} = (1/\rho_0)\,\hat{p}$$

$$k^2(\bar{U}-c)\,\hat{\psi} = (1/\rho_0)\,\hat{p}' + (g/\rho_0)\,\hat{\rho}$$

$$(\bar{U}-c)\,\hat{\rho} = -\,\hat{\psi}\,\bar{\rho}'$$

gives

$$\hat{\psi}'' - \left[-\frac{N^2}{(\bar{U}-c)^2} + \frac{\bar{U}''}{\bar{U}-c} + k^2\right]\hat{\psi} = 0$$

with $c \equiv \omega/k$ and $N^2(z) \equiv -\dfrac{g}{\rho_0}\dfrac{d\bar{\rho}}{dz}$ (the squared buoyancy frequency)

- This is the "Taylor-Goldstein" equation (first derived ~ 1931).

- For given $\bar{U}(z)$, $N^2(z)$, the equation is a differential eigenvalue problem for each k.

  Its solution gives the vertical structure of the disturbance and corresponding phase speed c (hence $\omega = ck$).

6.2] Interface Conditions

- We found that horizontally and time-periodic disturbances in stratified shear flows have vertical structure given by the Taylor-Goldstein equation:

$$\hat{\psi}'' - \left[-\frac{N^2}{(\bar{U}-c)^2} + \frac{\bar{U}''}{\bar{U}-c} + k^2 \right] \hat{\psi} = 0$$

with $c \equiv \omega / k$ and $N^2(z) \equiv -\dfrac{g}{\rho_0}\dfrac{d\bar{\rho}}{dz}$

- Generally this eigenvalue problem needs to be solved numerically.

  However, analytic solutions can be found if we assume
  - the background flow is "piecewise-linear"
  - the background density is "piecewise-constant".

Piecewise-Linear Flows and Piecewise-Constant Density

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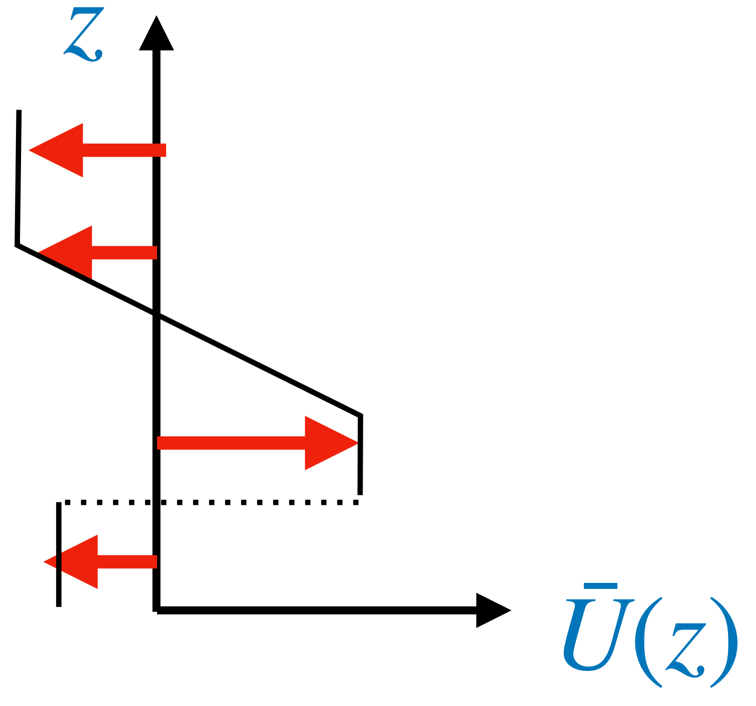

with $c \equiv \omega/k$ and $N^2(z) \equiv -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}$.

- Suppose $\bar{U}(z)$ is a "piecewise-linear" flow, being composed of segments that are constant or having constant change with height.

  Then $\bar{U}'' = 0$ *within* each segment.

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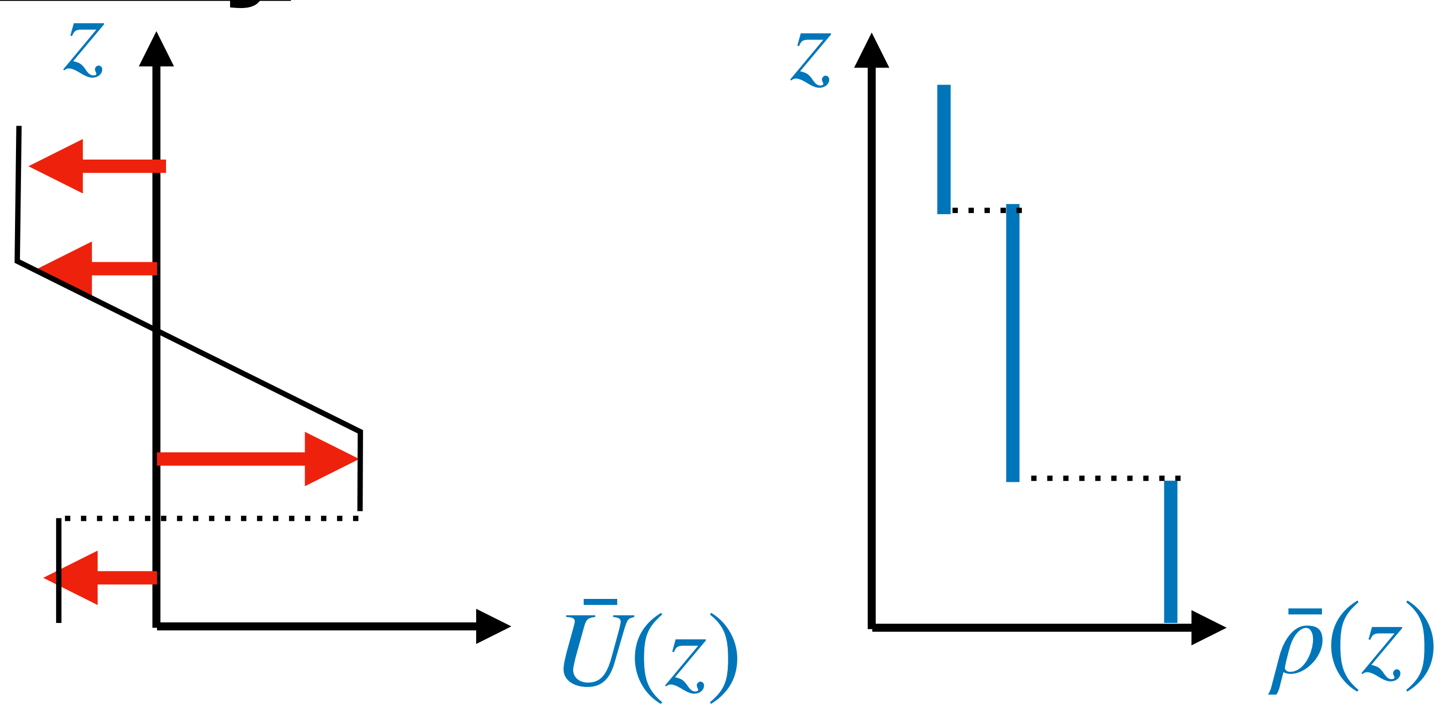

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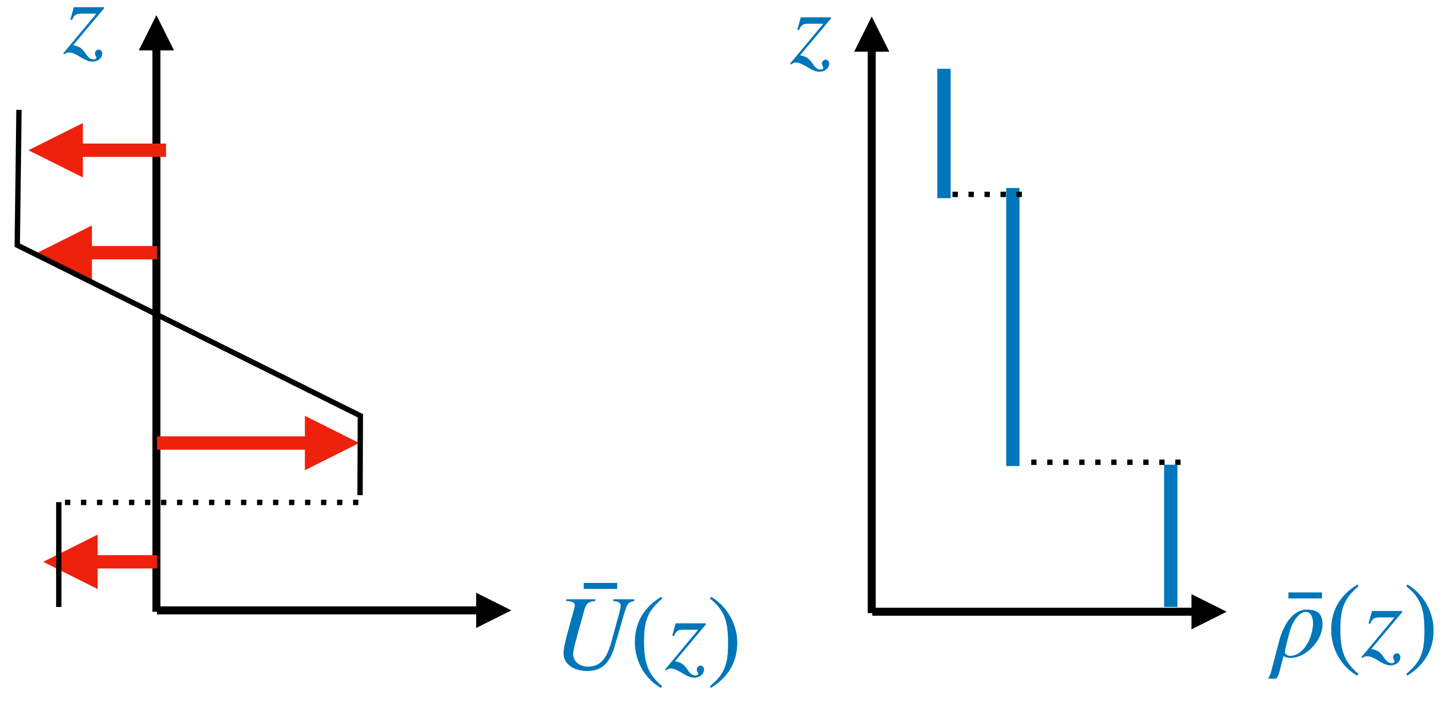

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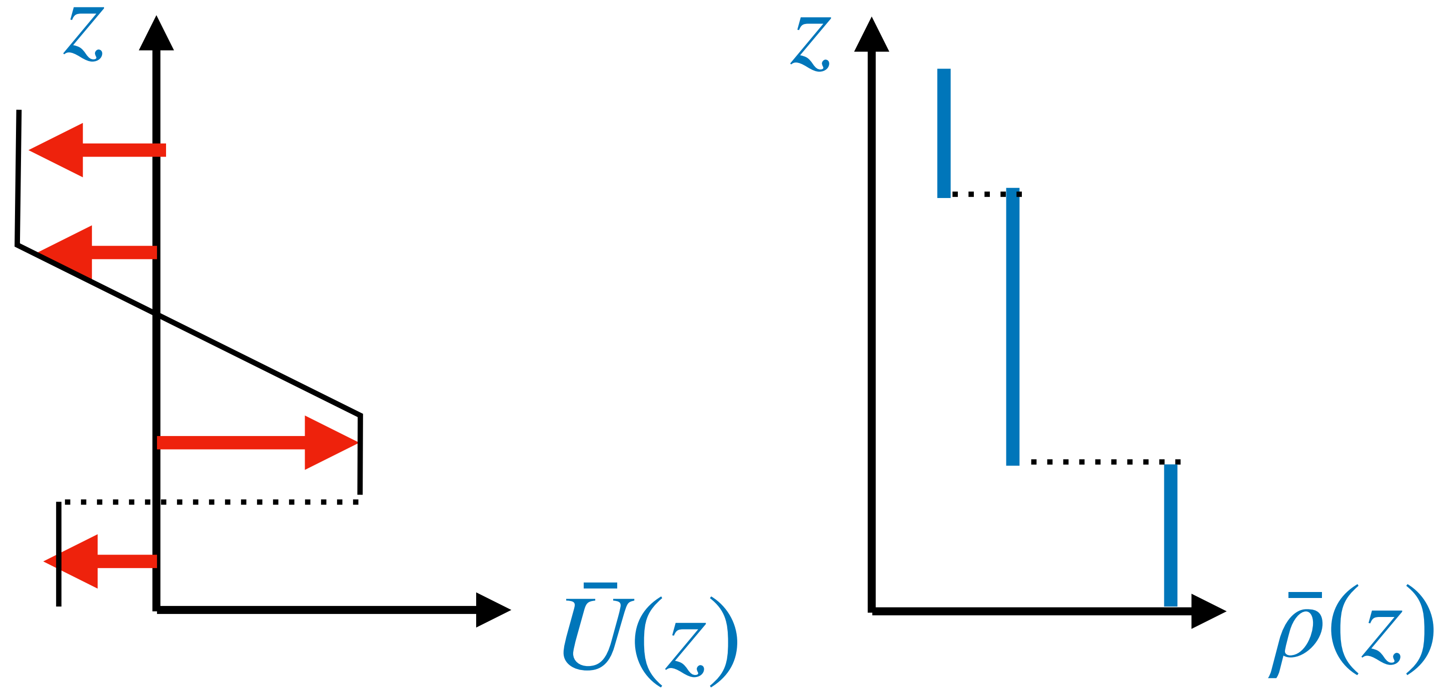

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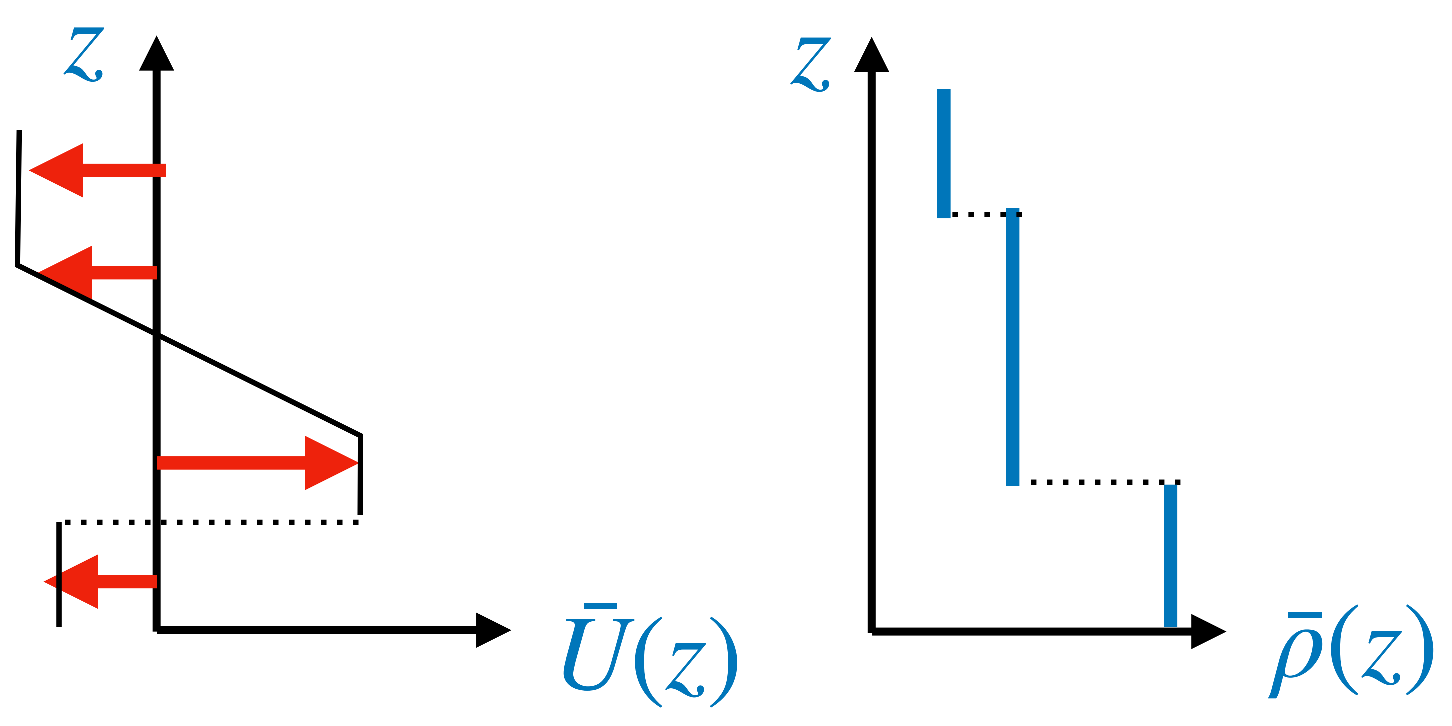
- If both hold, the Taylor-Goldstein equation reduces to $\hat{\psi}'' - k^2 \hat{\psi} = 0$

- This can be solved within each segment/layer. But need interface conditions to match solutions from one layer to the next.

Interface Conditions

- For every change in $\bar{U}(z)$ or in $\bar{\rho}(z)$ we need 2 interface conditions.
- For simplicity, suppose the interface is at $z = 0.$

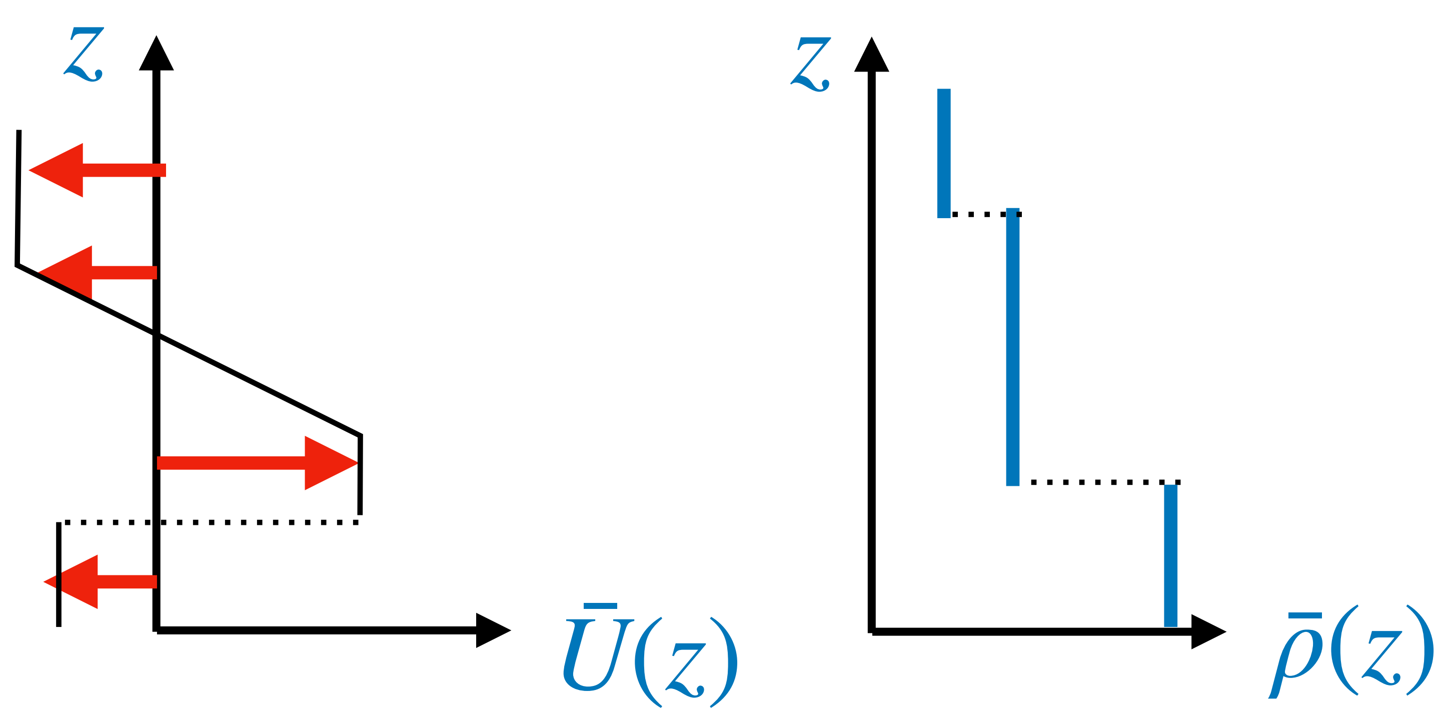


Interface conditions:

1) Fluid at the interface stays there:

2) Pressure is continuous crossing the interface:

Interface Conditions



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<u>Interface conditions</u>:

1) Fluid at the interface stays there: $w|_{z=0^+} = w|_{z=0^-} = \dfrac{D\eta}{Dt} \simeq \dfrac{\partial \eta}{\partial t} + \bar{U}\dfrac{\partial \eta}{\partial x}$

$$\eta = Ae^{\imath(kx-\omega t)}$$

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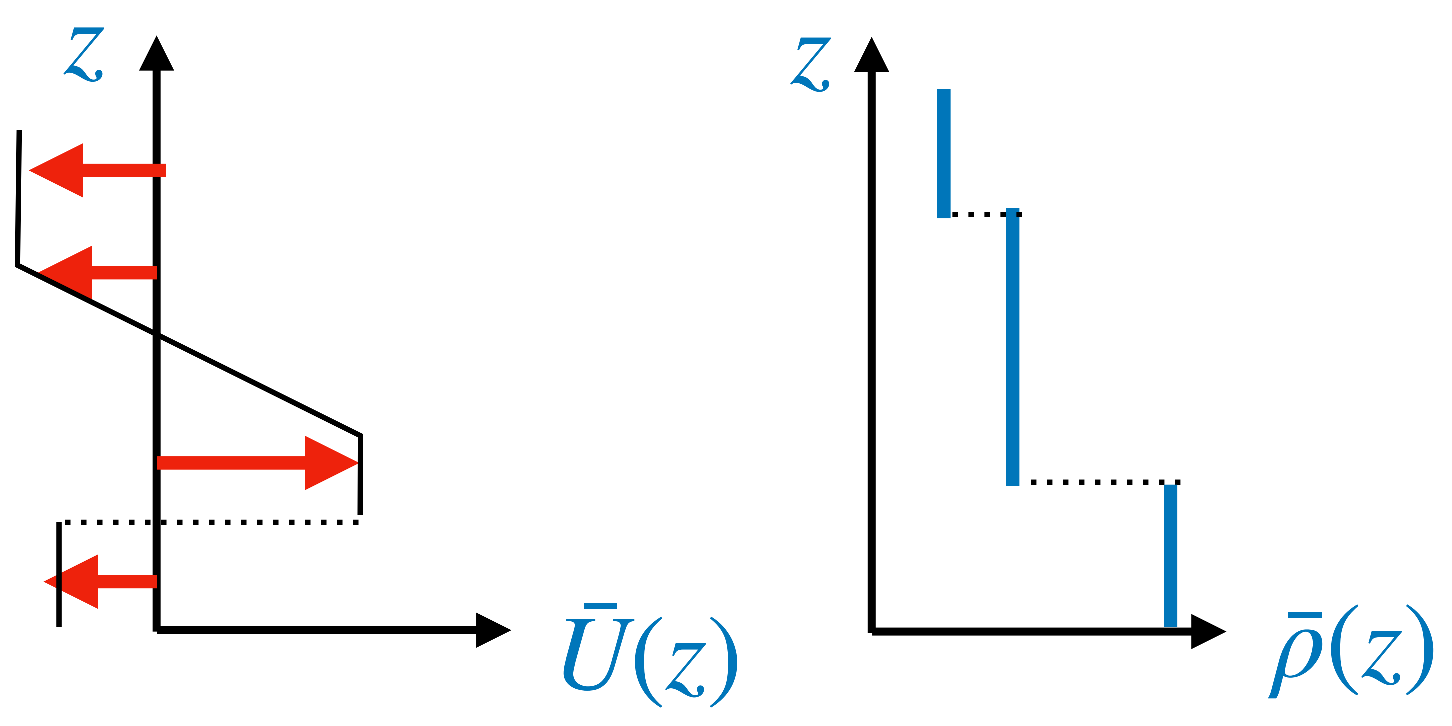

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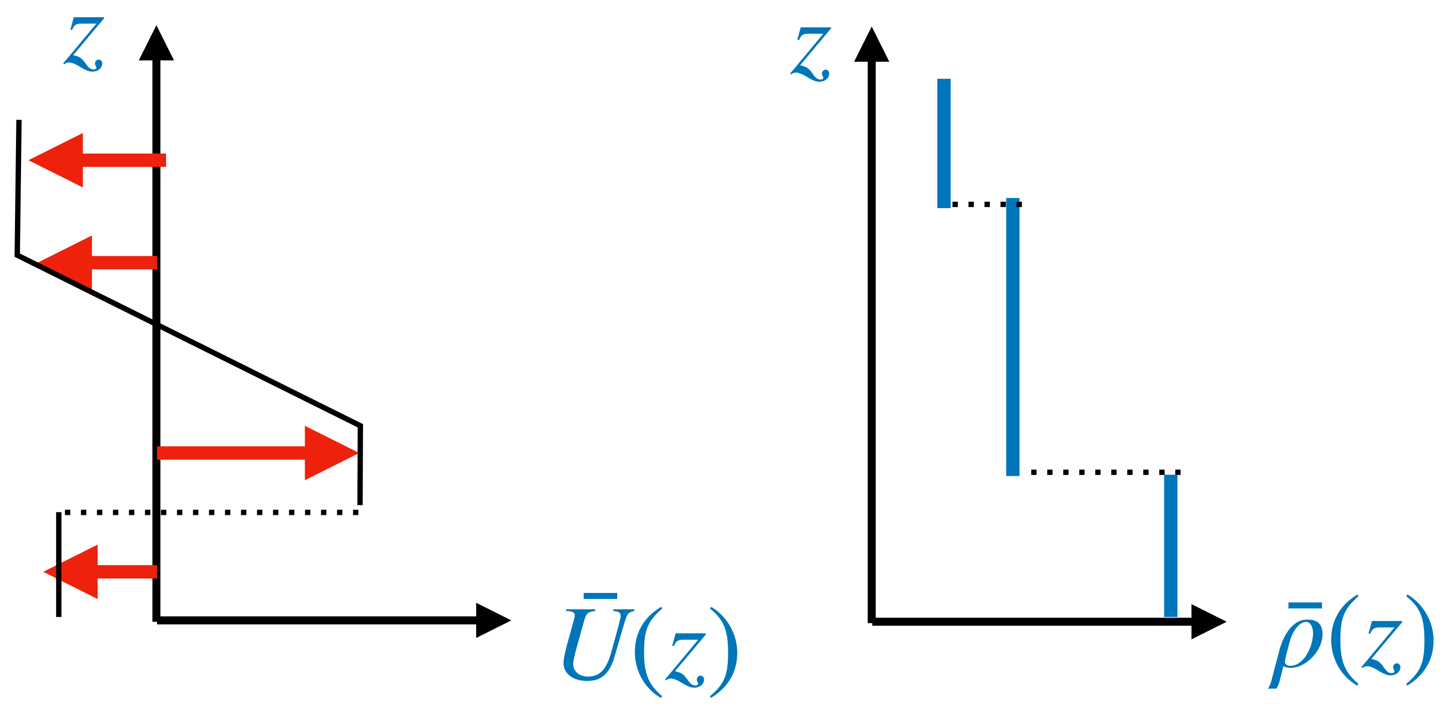

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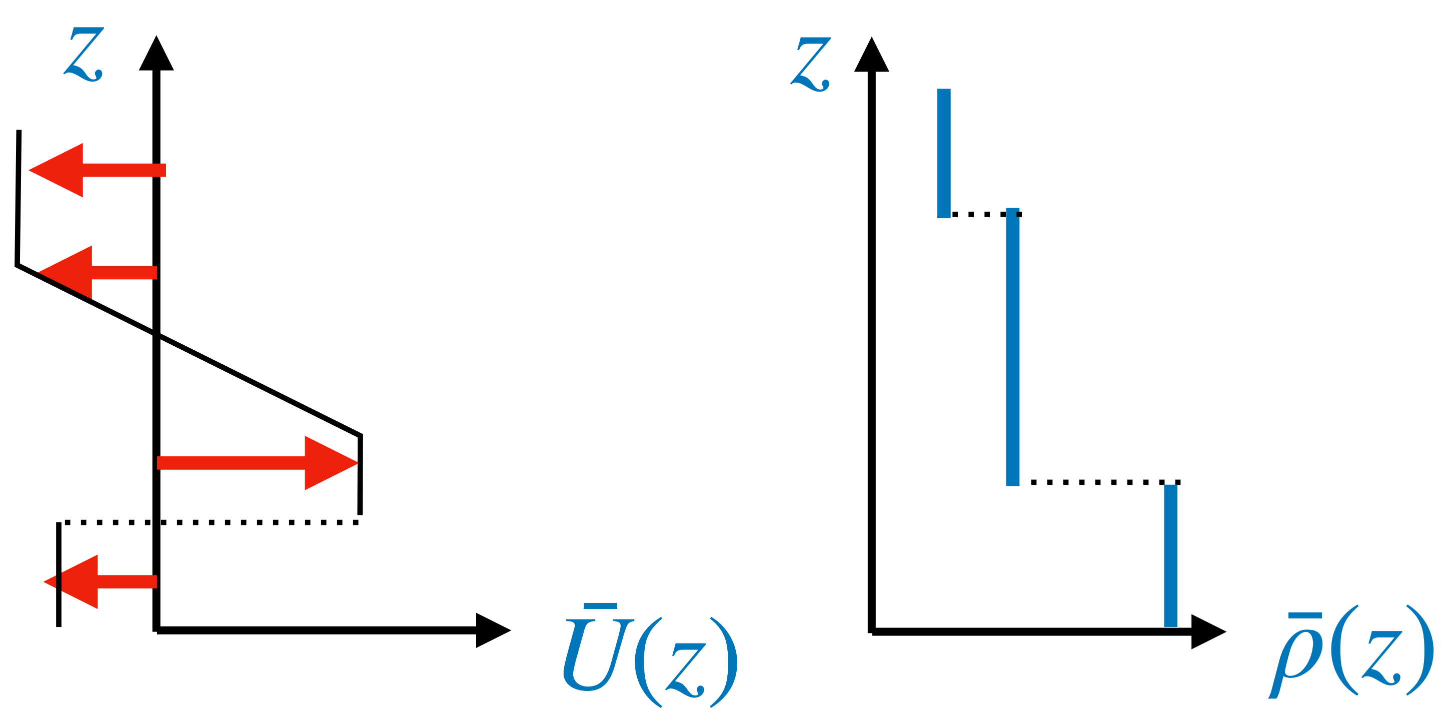

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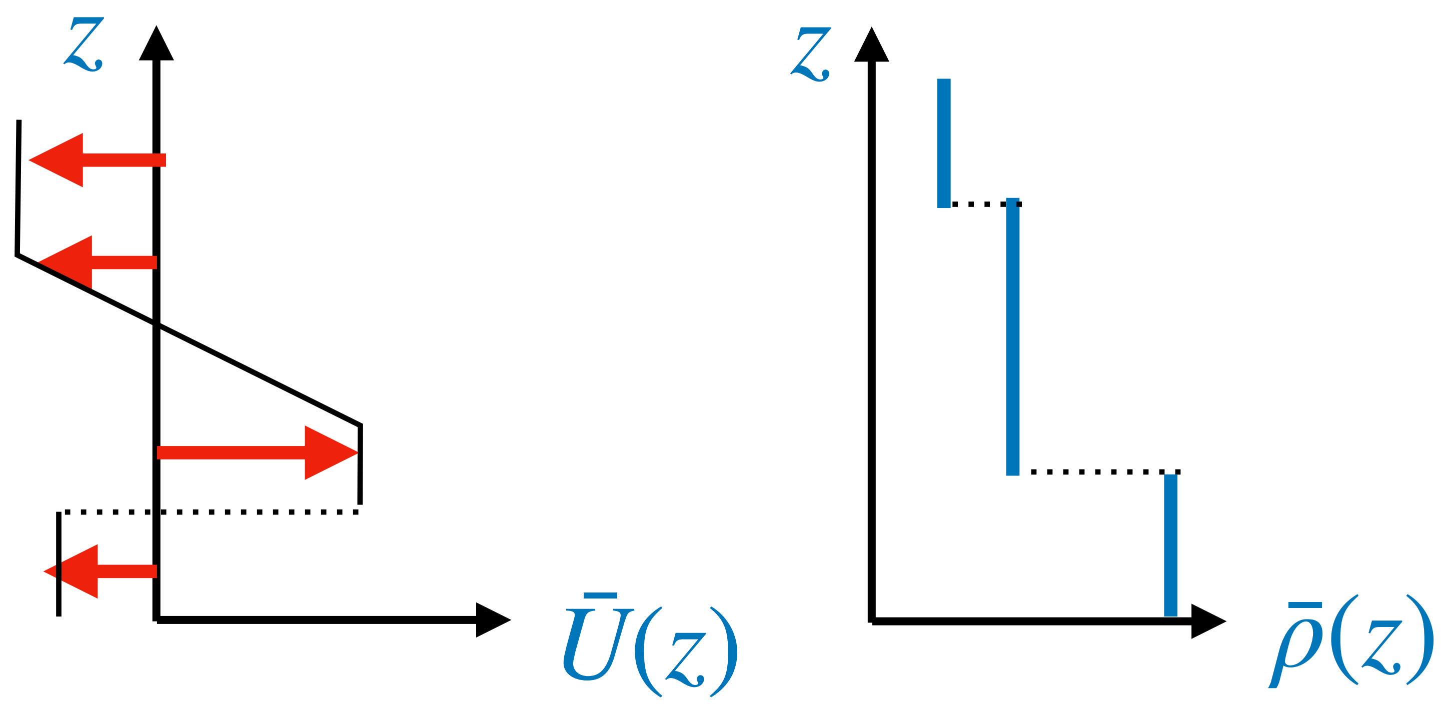

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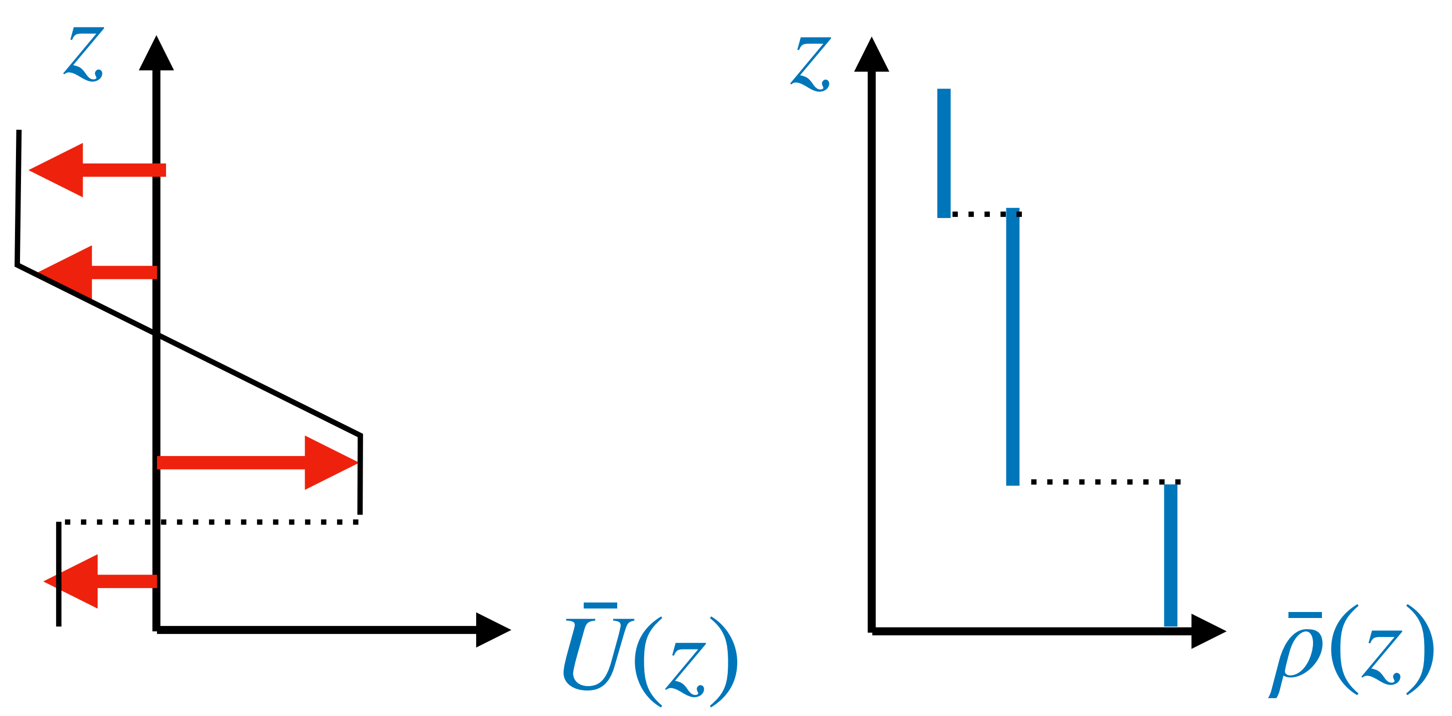

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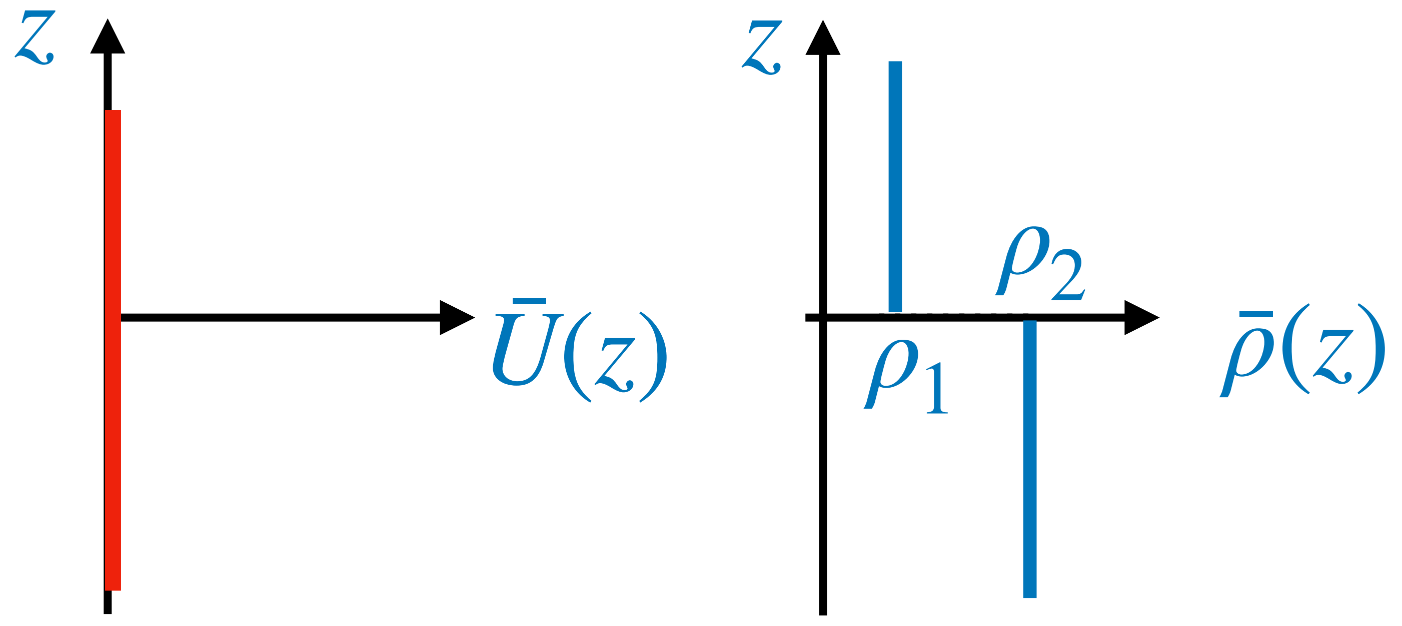
$$\Rightarrow \quad \hat{p}\,|_{z=\eta} = \bar{\rho}\left.\left[(\bar{U}-c)\,\hat{\psi}' - \bar{U}'\,\hat{\psi} - \frac{g}{\bar{U}-c}\,\hat{\psi}\right]\right|_{z=0^+} = \bar{\rho}\left.\left[(\bar{U}-c)\,\hat{\psi}' - \bar{U}'\,\hat{\psi} - \frac{g}{\bar{U}-c}\,\hat{\psi}\right]\right|_{z=0^-}$$

Example of a Stationary Two-layer Fluid



- Consider an unbounded, stationary, 2-layer fluid:
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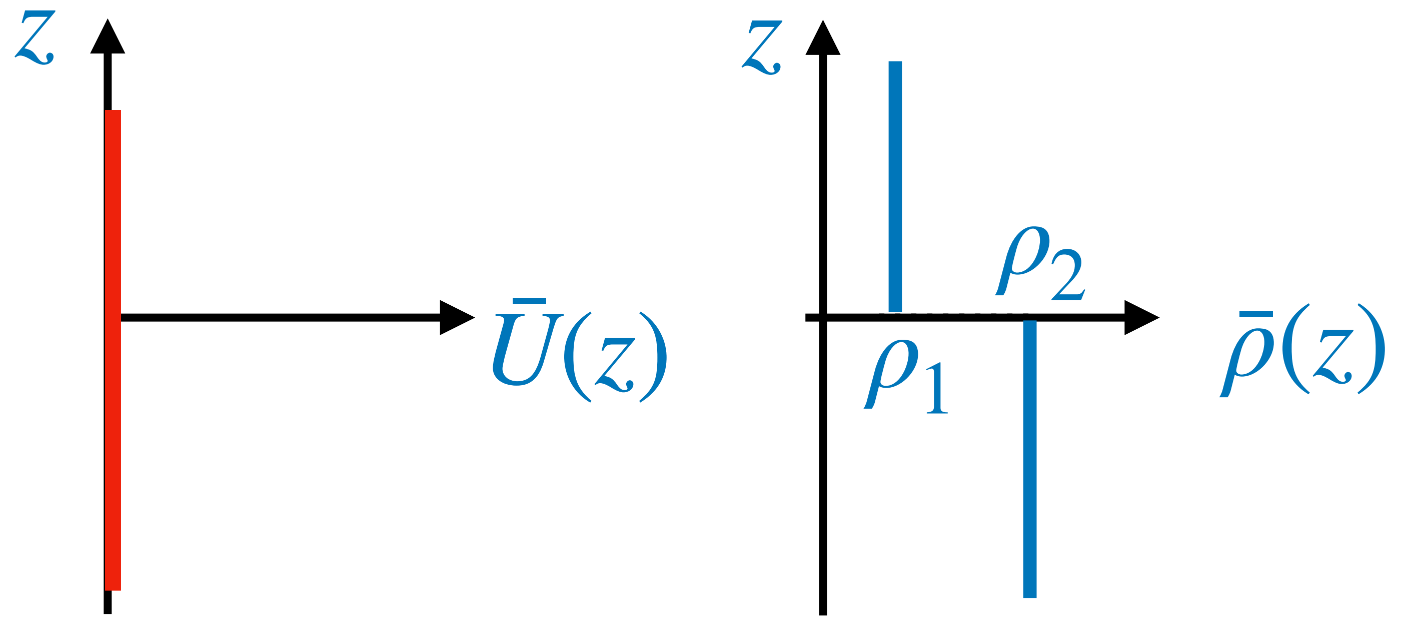

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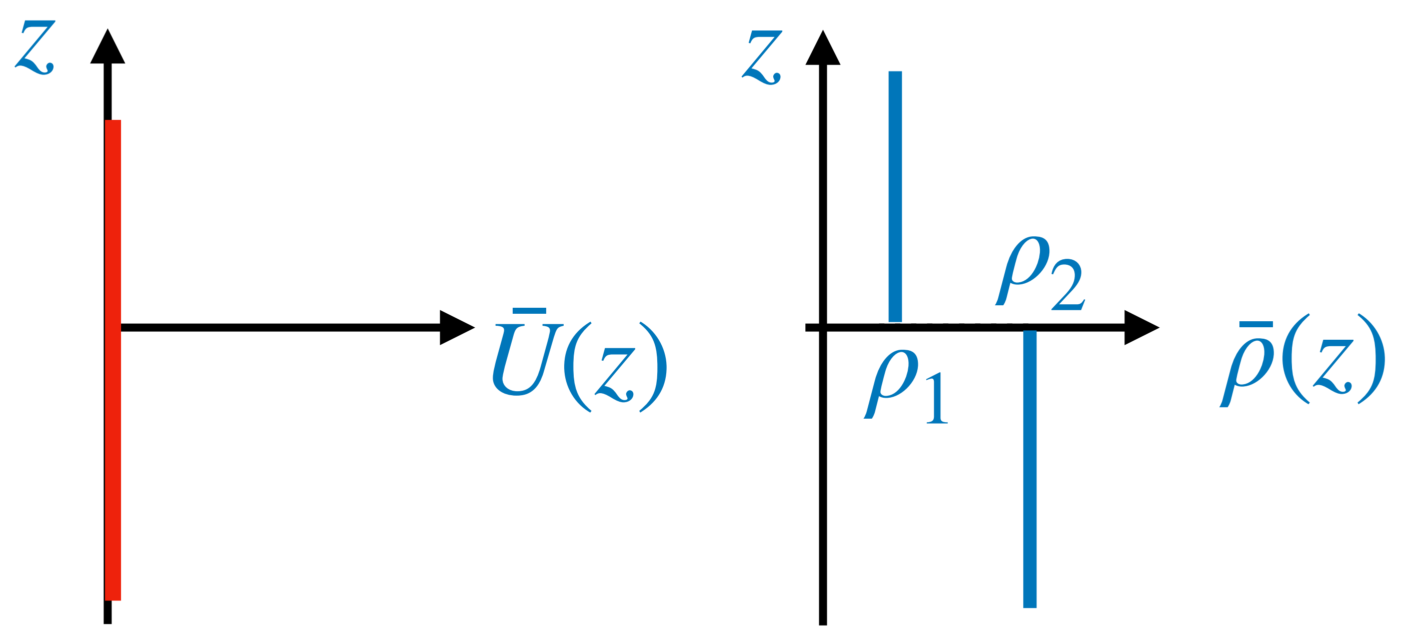

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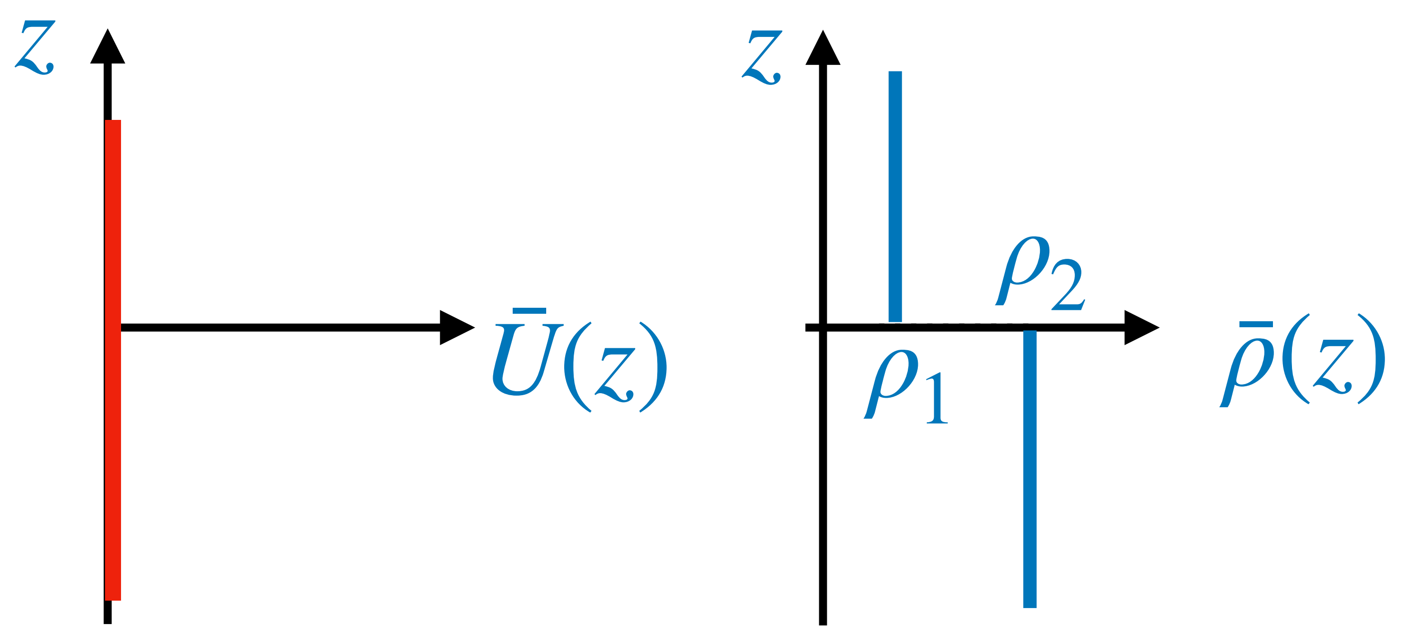

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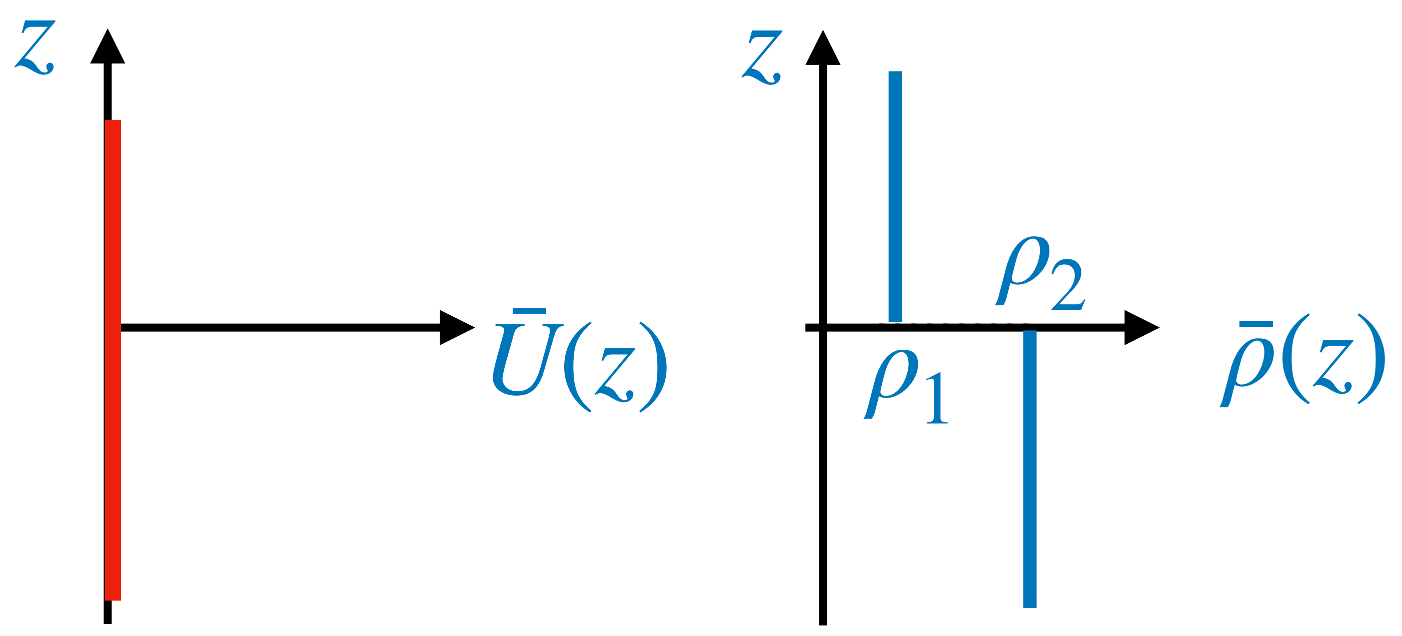

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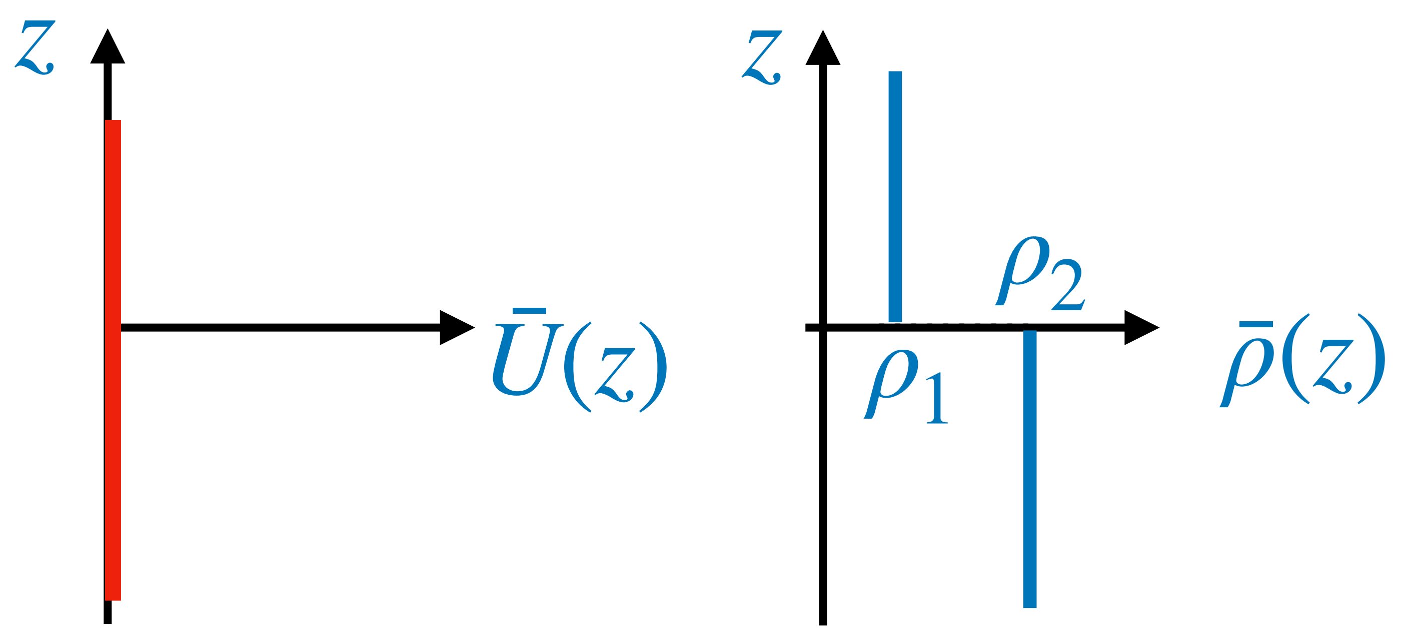

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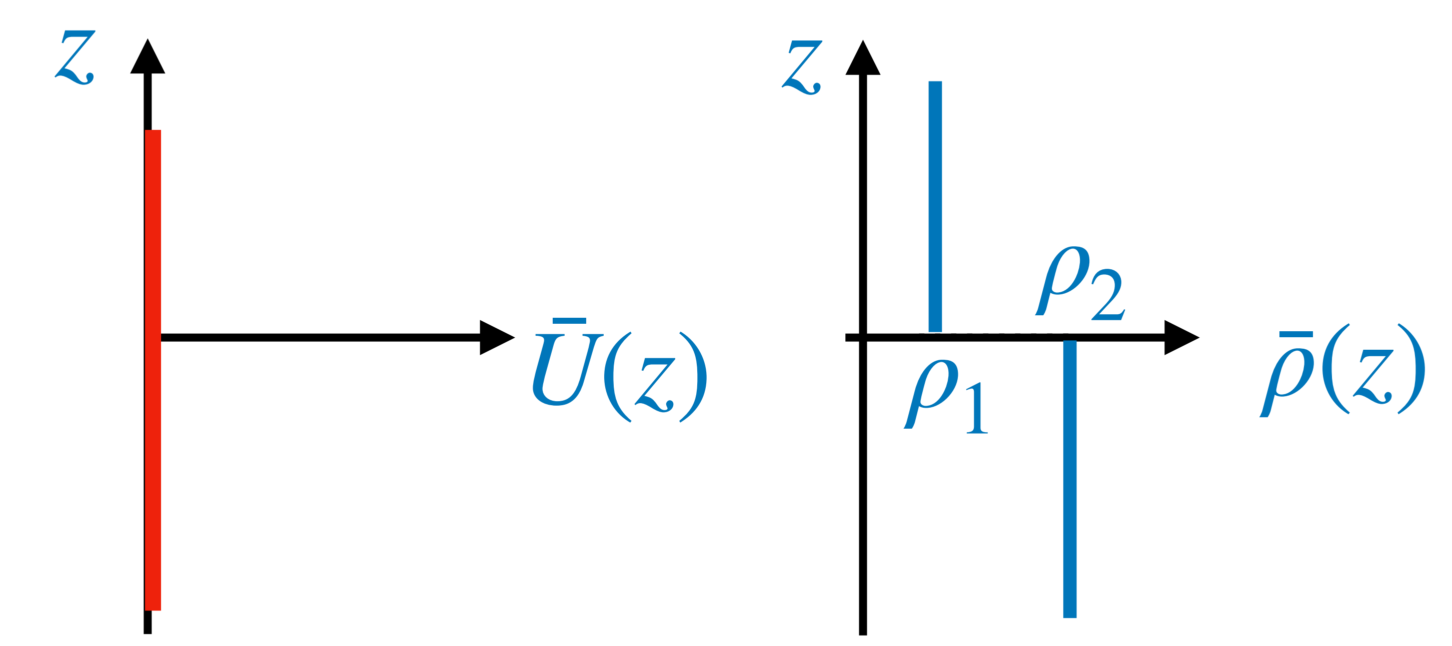

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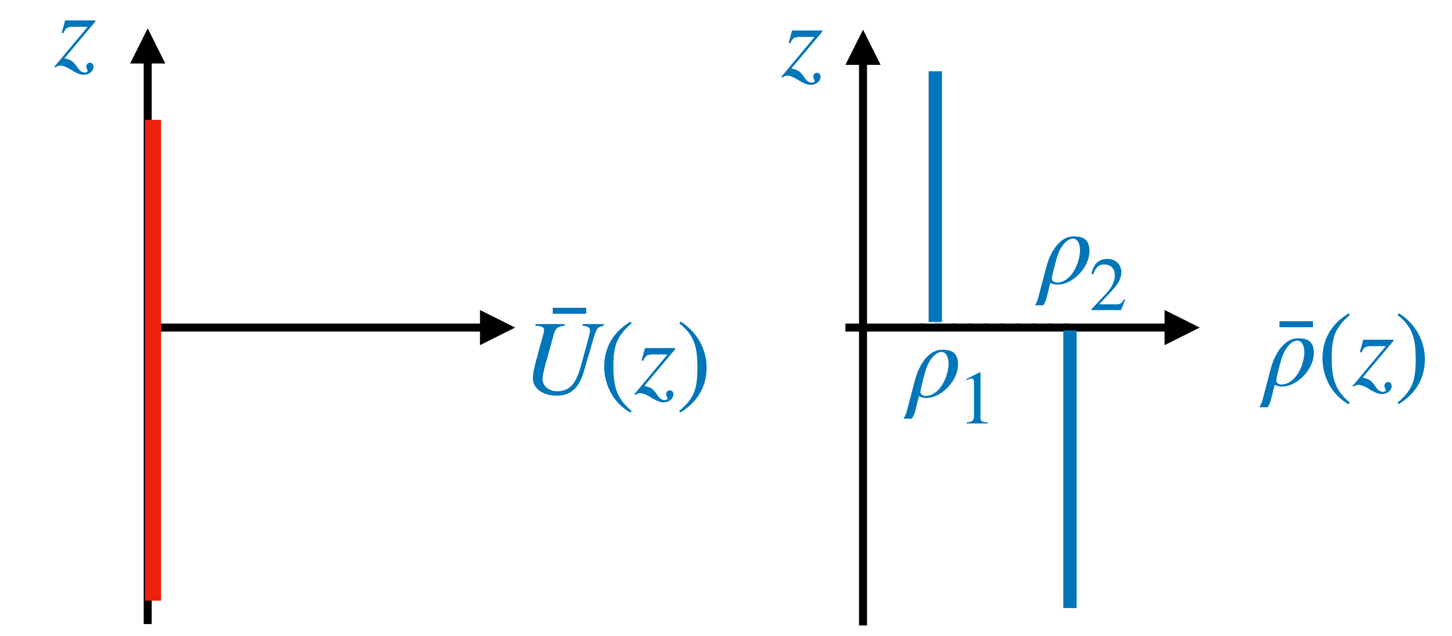

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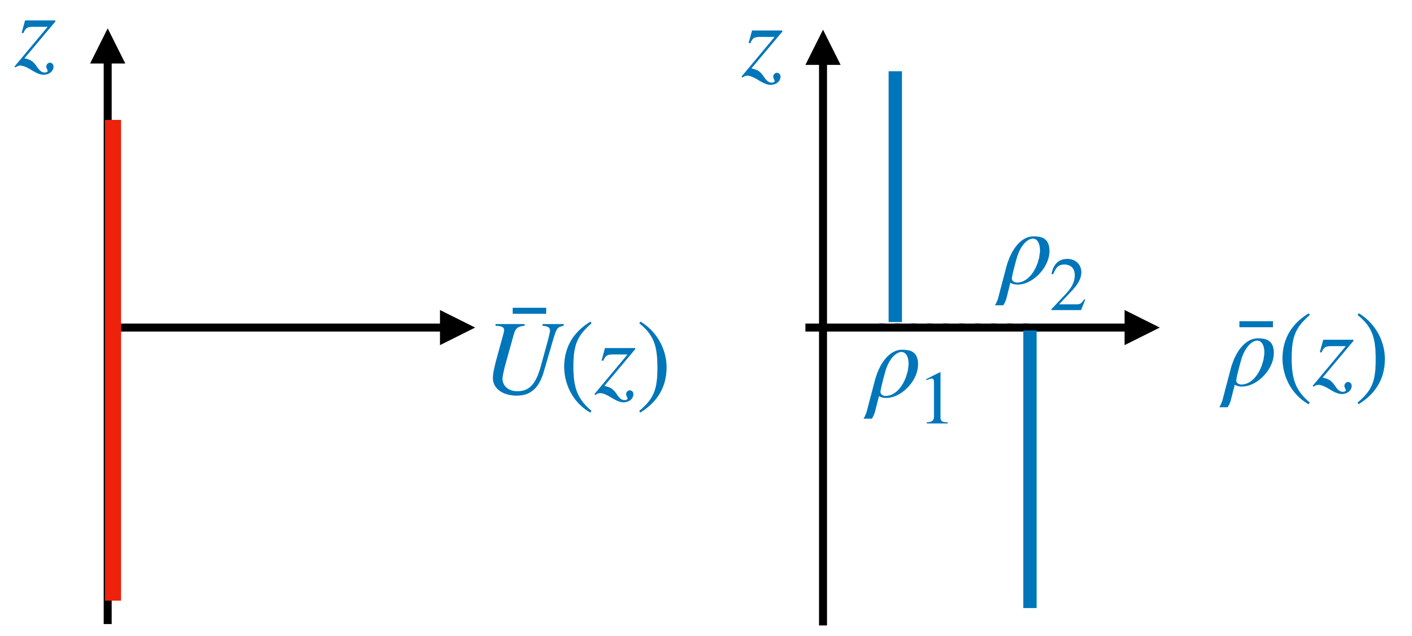

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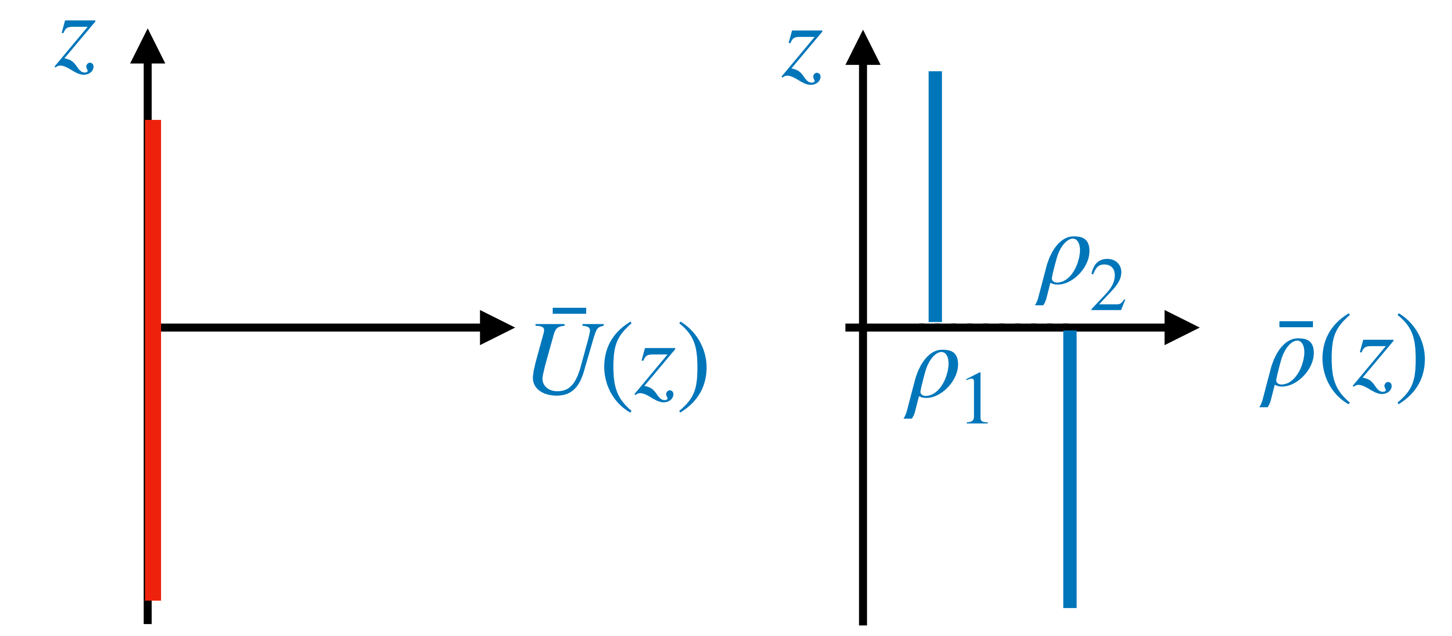
$$\Rightarrow \quad \rho_1\left[(-c)(-k\mathscr{A}_1) - 0 - \frac{g}{-c}\mathscr{A}_1\right] = \rho_2\left[(-c)(k\mathscr{A}_2) - 0 - \frac{g}{-c}\mathscr{A}_2\right]$$

Example of a Stationary Two-layer Fluid



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- Denote the streamfunction in the upper and lower layer respectively by $\psi_1(x,z,t) = \hat{\psi}_1(z)\, e^{\imath(kx-\omega t)}$ and $\psi_2(x,z,t) = \hat{\psi}_2(z)\, e^{\imath(kx-\omega t)}$
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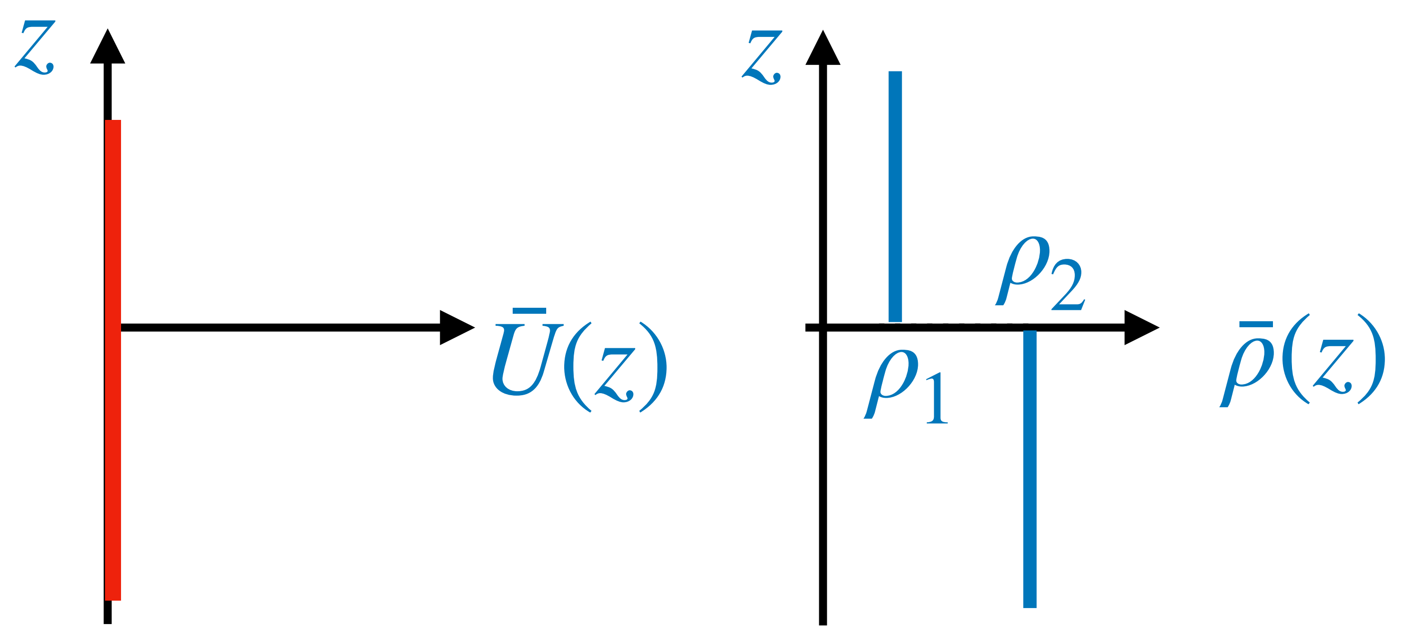
$$\Rightarrow \quad \rho_1\left[(-c)(-k\mathscr{A}_1) - 0 - \frac{g}{-c}\mathscr{A}_1\right] = \rho_2\left[(-c)(k\mathscr{A}_2) - 0 - \frac{g}{-c}\mathscr{A}_2\right] \Rightarrow \quad \rho_1[c^2k + g]\mathscr{A}_1 = \rho_2[-c^2k + g]\mathscr{A}_2$$

Example of a Stationary Two-layer Fluid



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using $\mathscr{A}_1 = \mathscr{A}_2$

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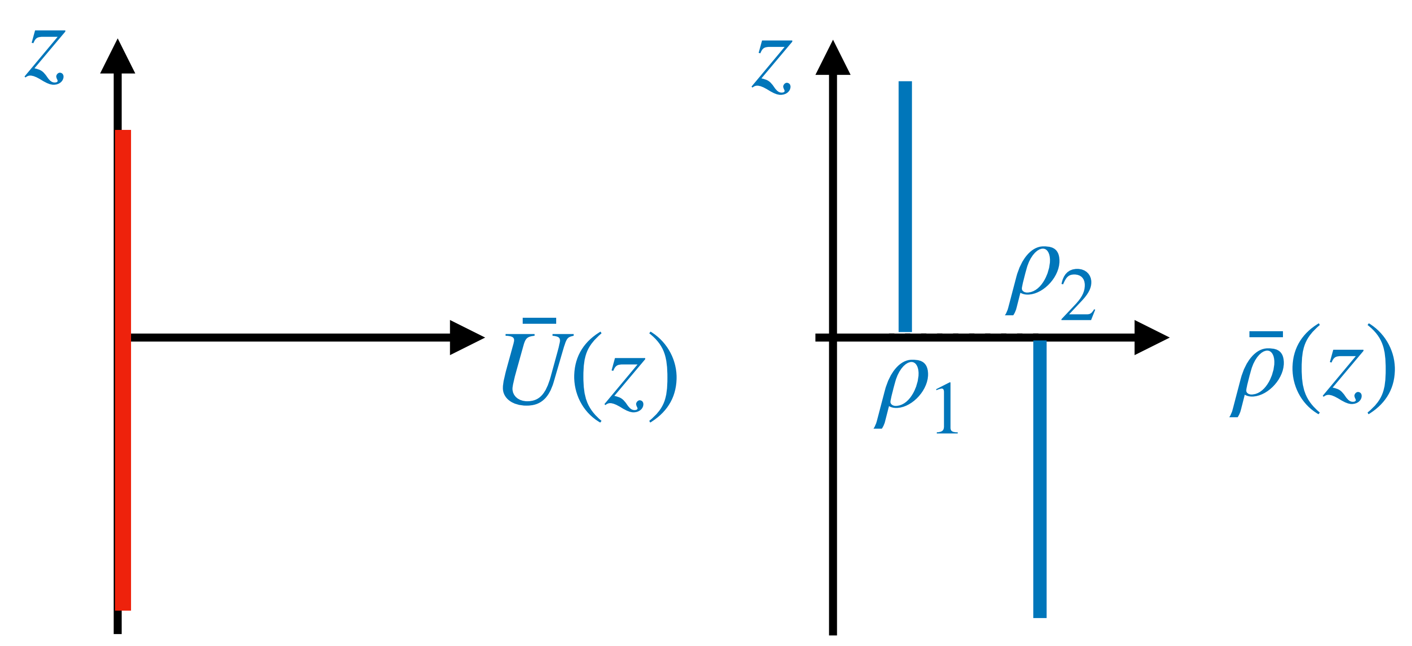

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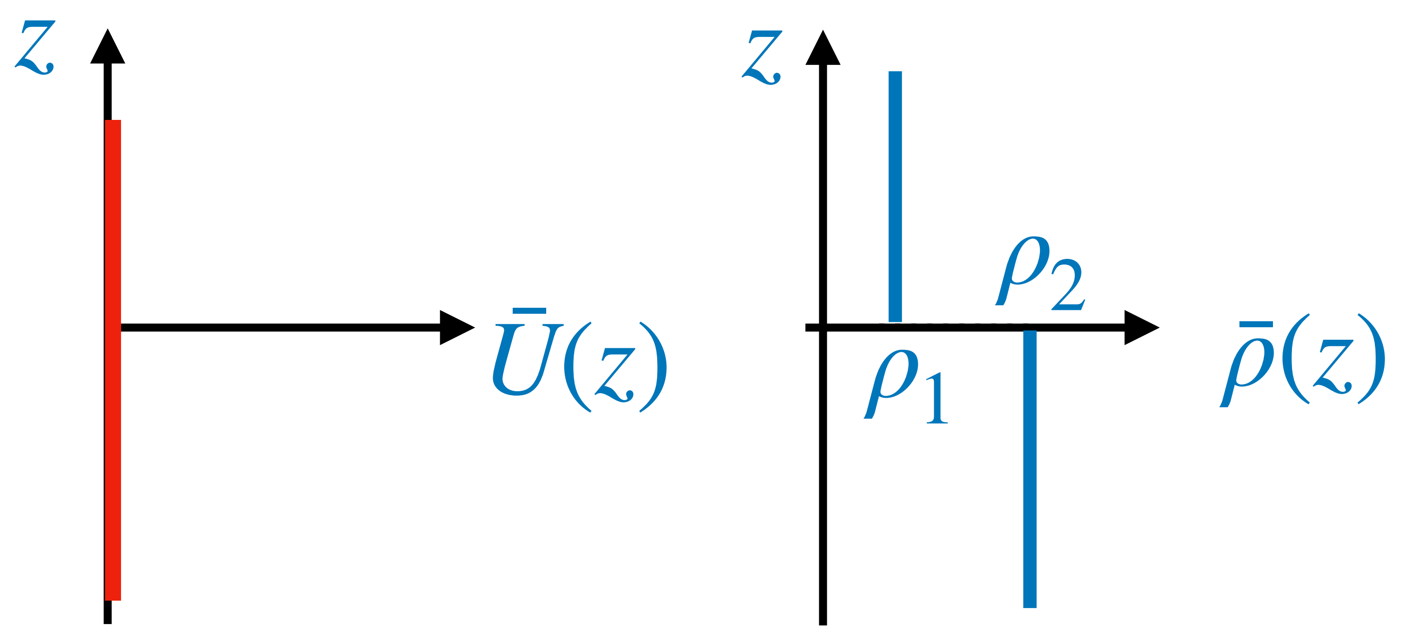
$$\Rightarrow \quad \rho_1[c^2k + g] = \rho_2[-c^2k + g] \quad \Rightarrow \quad c^2 = \frac{g}{k}\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$$

using $\mathscr{A}_1 = \mathscr{A}_2$

Example of a Stationary Two-layer Fluid



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$$\Rightarrow \rho_1[c^2k+g] = \rho_2[-c^2k+g] \Rightarrow c^2 = \frac{g}{k}\frac{\rho_2-\rho_1}{\rho_2+\rho_1} \Rightarrow \omega^2 = gk\frac{\rho_2-\rho_1}{\rho_2+\rho_1}$$

using $\mathscr{A}_1 = \mathscr{A}_2$

using $c \equiv \omega/k$

This is the dispersion relation found before.

6.3] Rayleigh Waves (in fluids)

- For piecewise-linear flows, $\bar{U}(z)$, and piecewise-constant density, $\bar{\rho}(z)$, we found that the vertical structure $\hat{\psi}(z)$ of the streamfunction $\psi(x,z,t) = \hat{\psi}\, e^{\imath(kx-\omega t)}$ satisfies

$$\boxed{\hat{\psi}'' - k^2\hat{\psi} = 0}$$

subject to interface conditions

$$\left[\frac{\hat{\psi}}{\bar{U}-c}\right]\Bigg|_{z=0^+} = \left[\frac{\hat{\psi}}{\bar{U}-c}\right]\Bigg|_{z=0^-}$$

$$\bar{\rho}\left[(\bar{U}-c)\,\hat{\psi}' - \bar{U}'\,\hat{\psi} - \frac{g}{\bar{U}-c}\,\hat{\psi}\right]\Bigg|_{z=0^+} = \bar{\rho}\left[(\bar{U}-c)\,\hat{\psi}' - \bar{U}'\,\hat{\psi} - \frac{g}{\bar{U}-c}\,\hat{\psi}\right]\Bigg|_{z=0^-}$$

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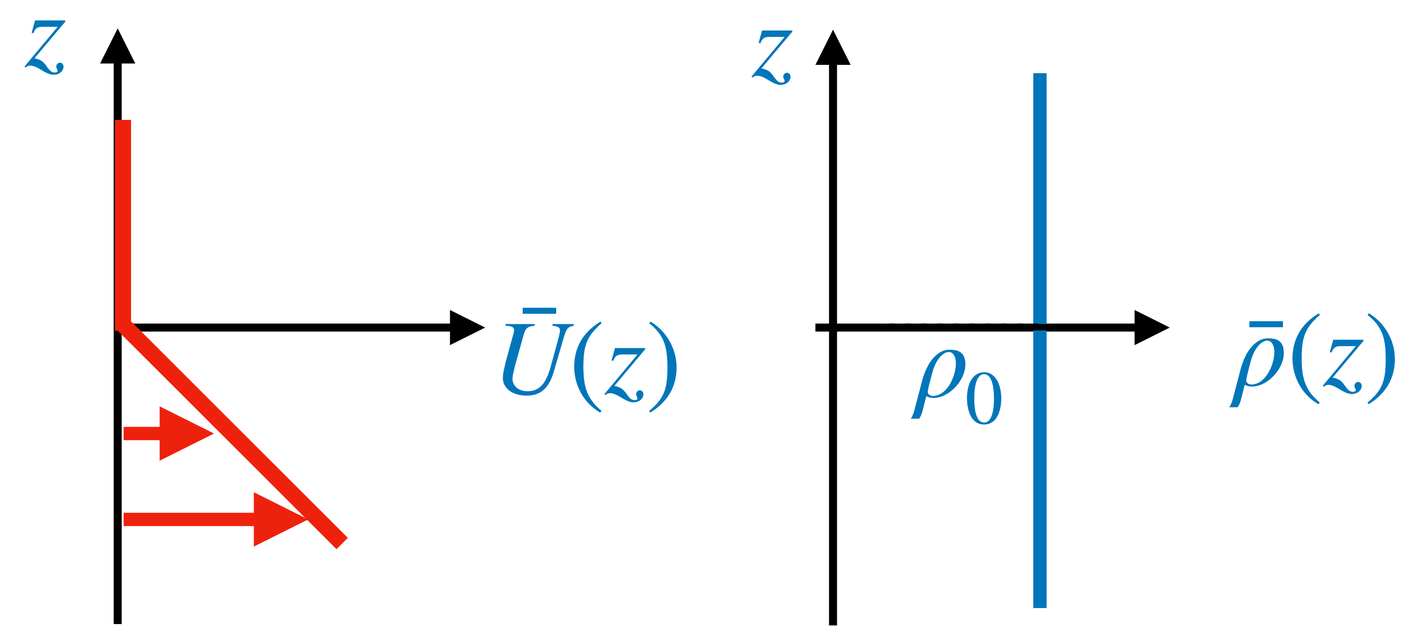
Rayleigh Waves (in fluids)



- Consider an unbounded, uniform-density fluid:
  - $\bar{\rho}(z) = \rho_0$

  in which $\bar{U}$ is a “kinked-shear” flow:
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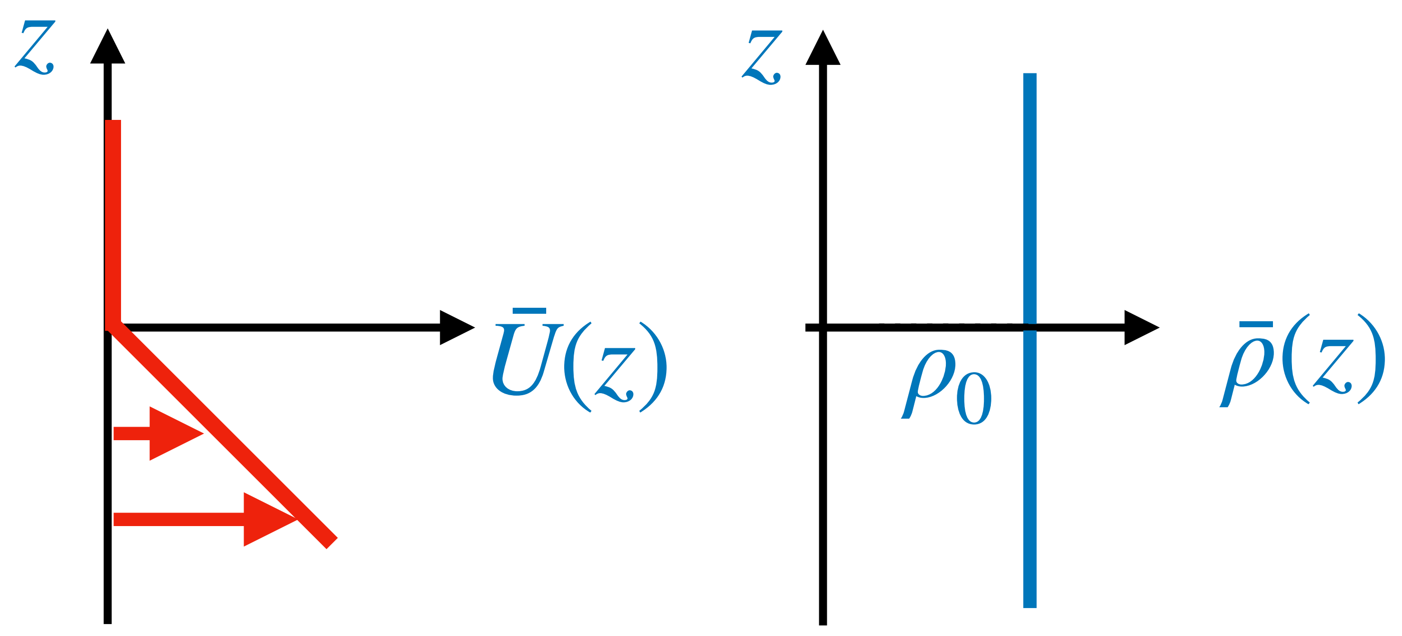
Rayleigh Waves (in fluids)



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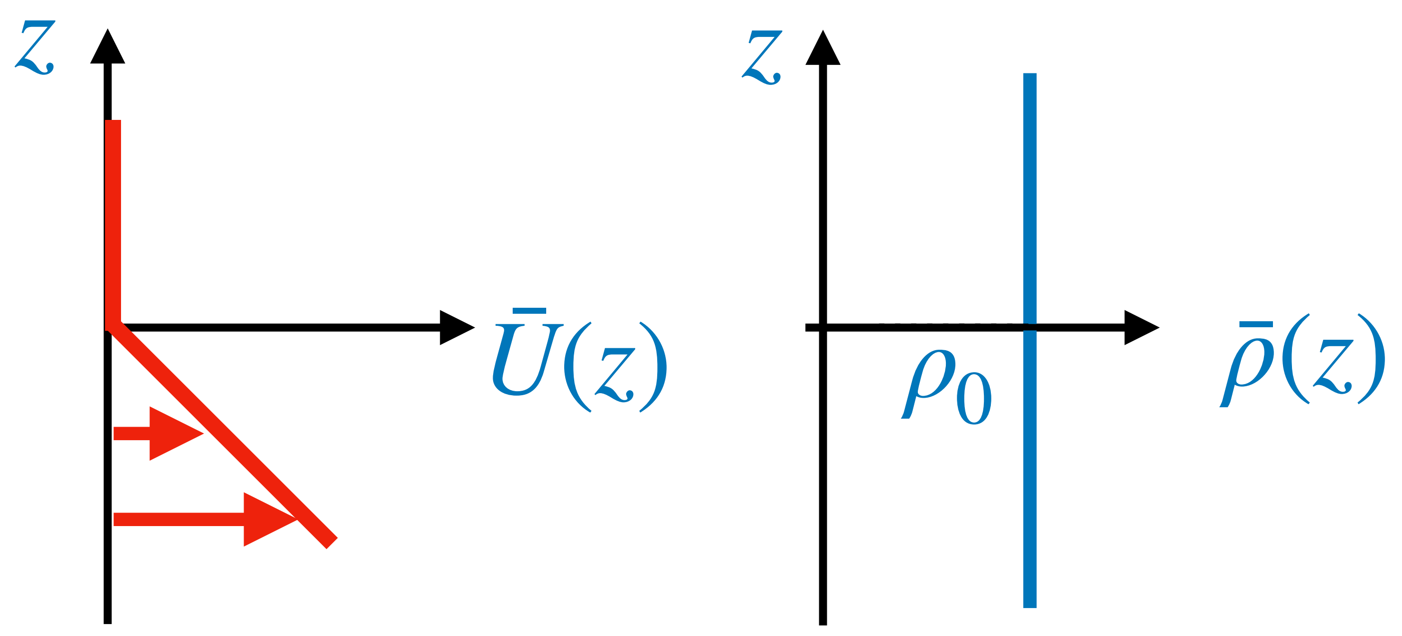
Rayleigh Waves (in fluids)



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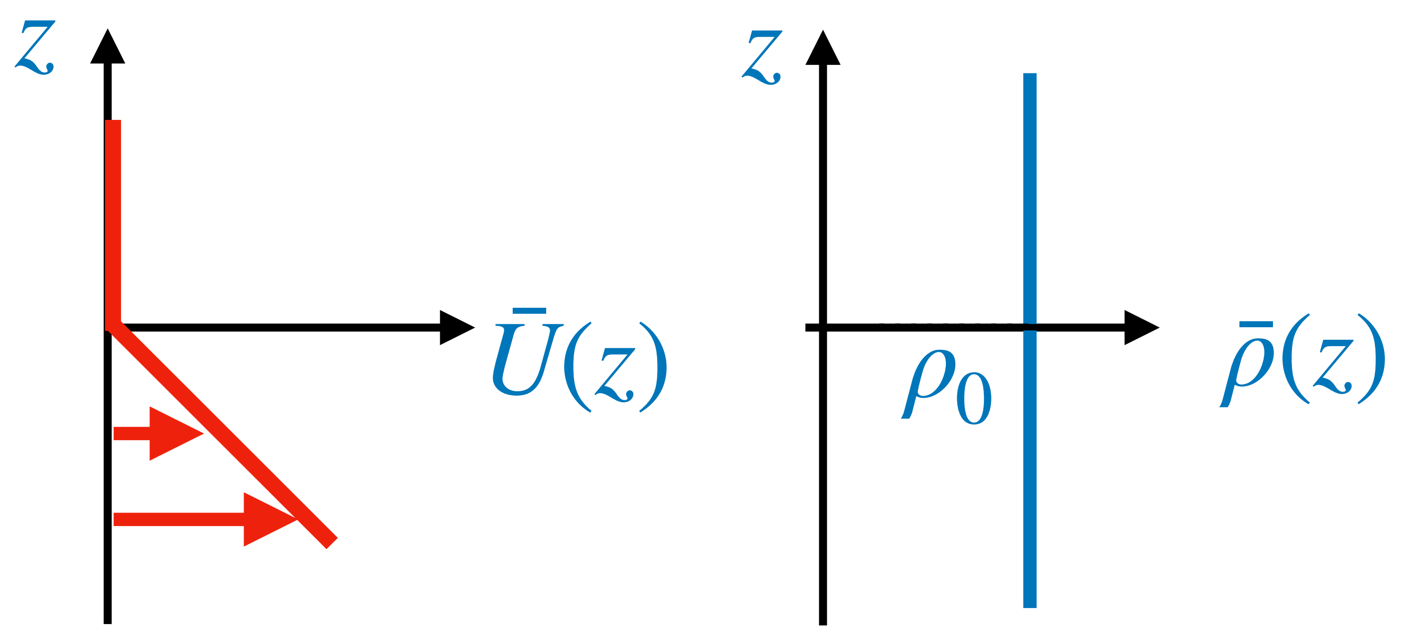
Rayleigh Waves (in fluids)



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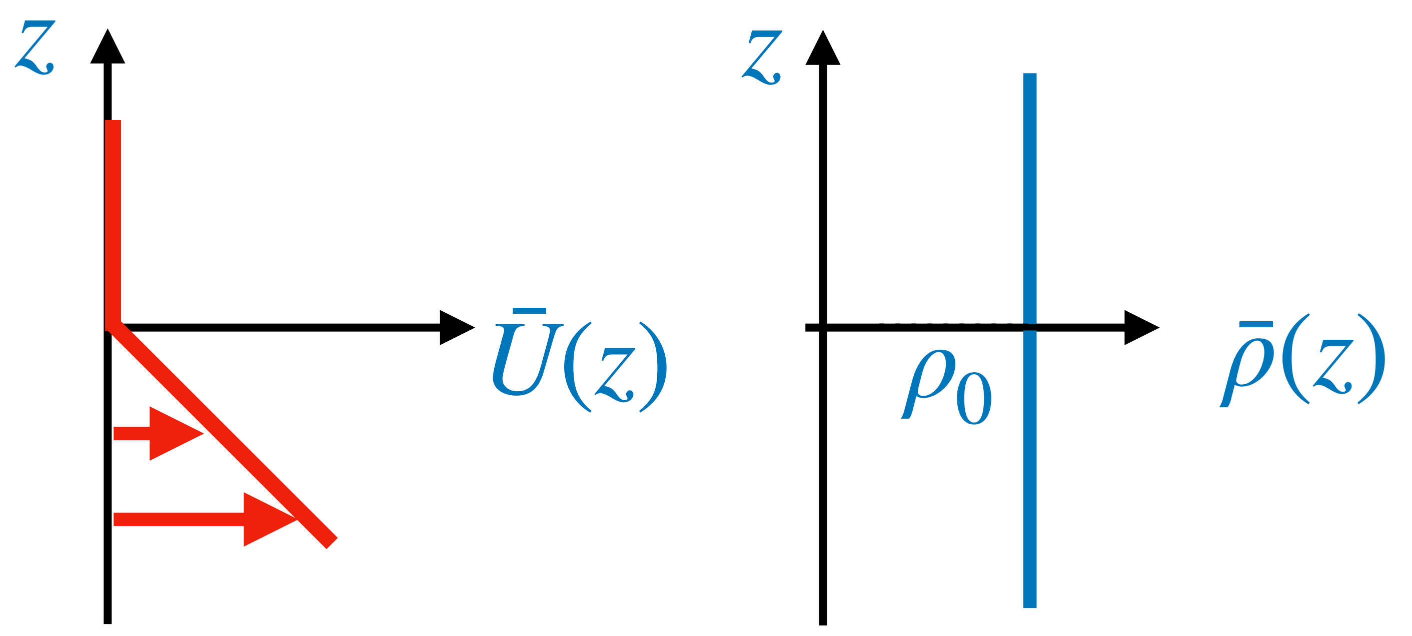
Rayleigh Waves (in fluids)



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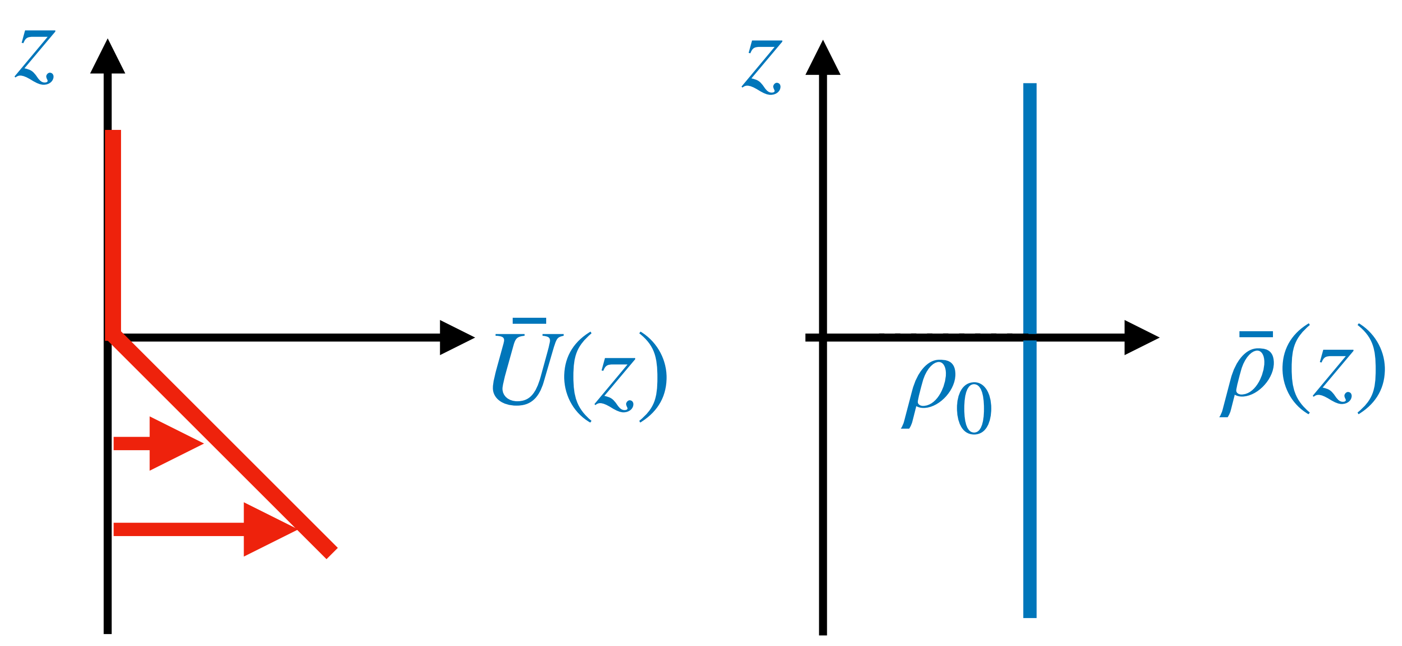

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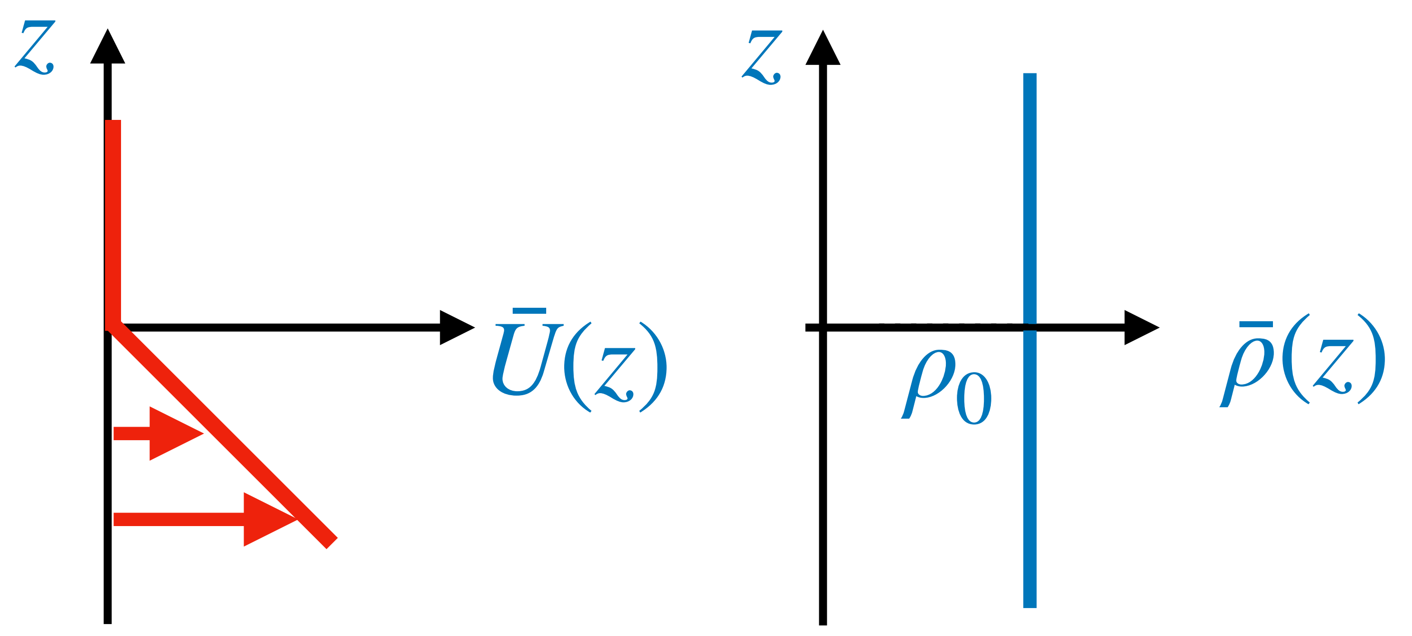
Rayleigh Waves (in fluids)



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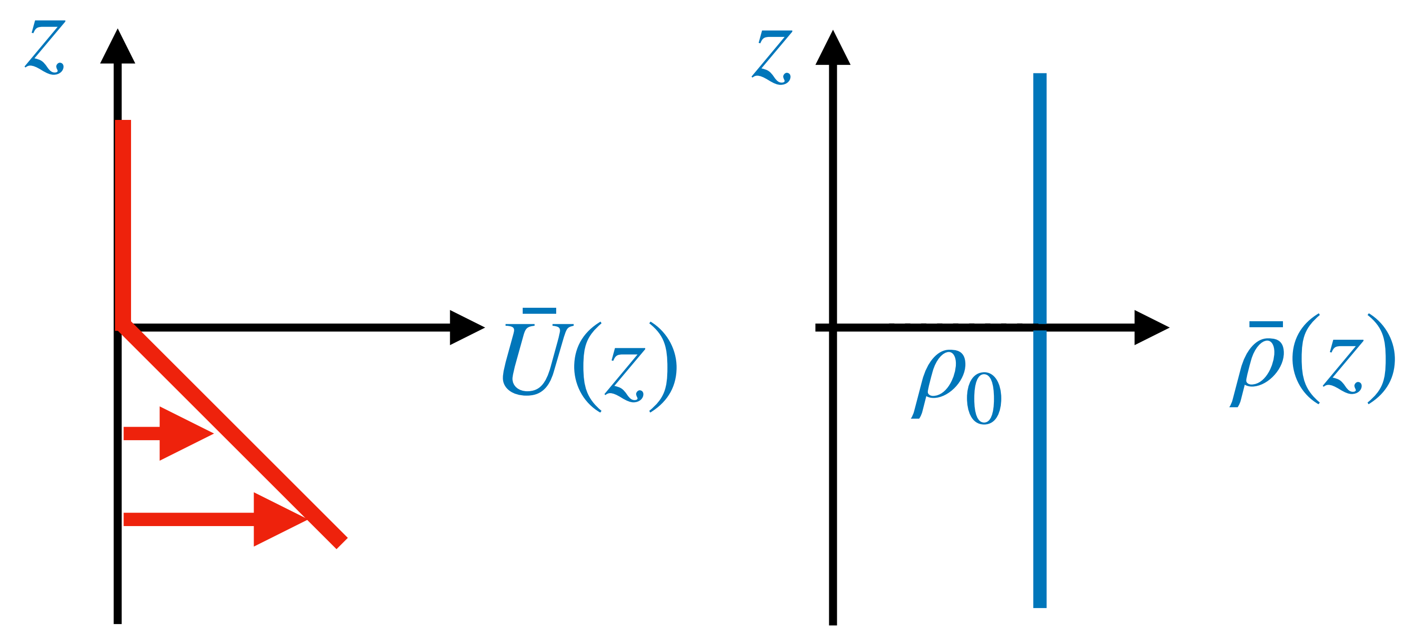
Rayleigh Waves (in fluids)



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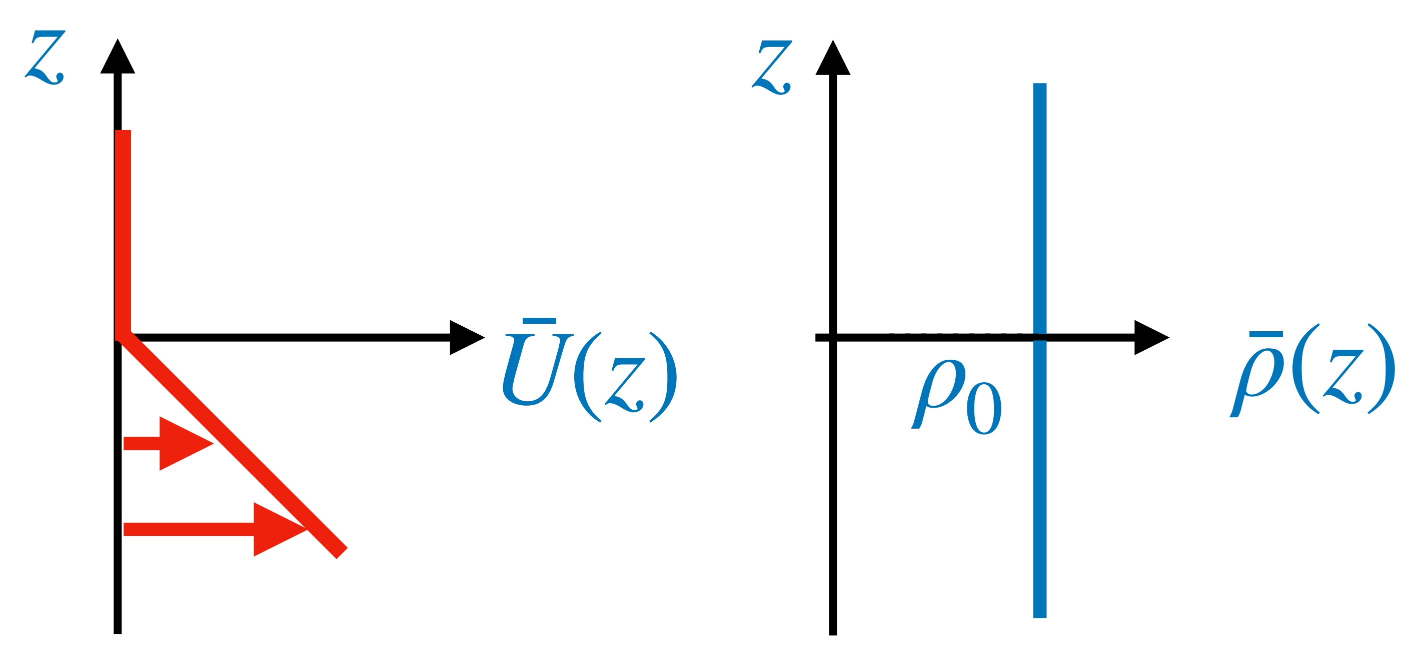
using $\mathscr{A}_1 = \mathscr{A}_2$

Rayleigh Waves (in fluids)



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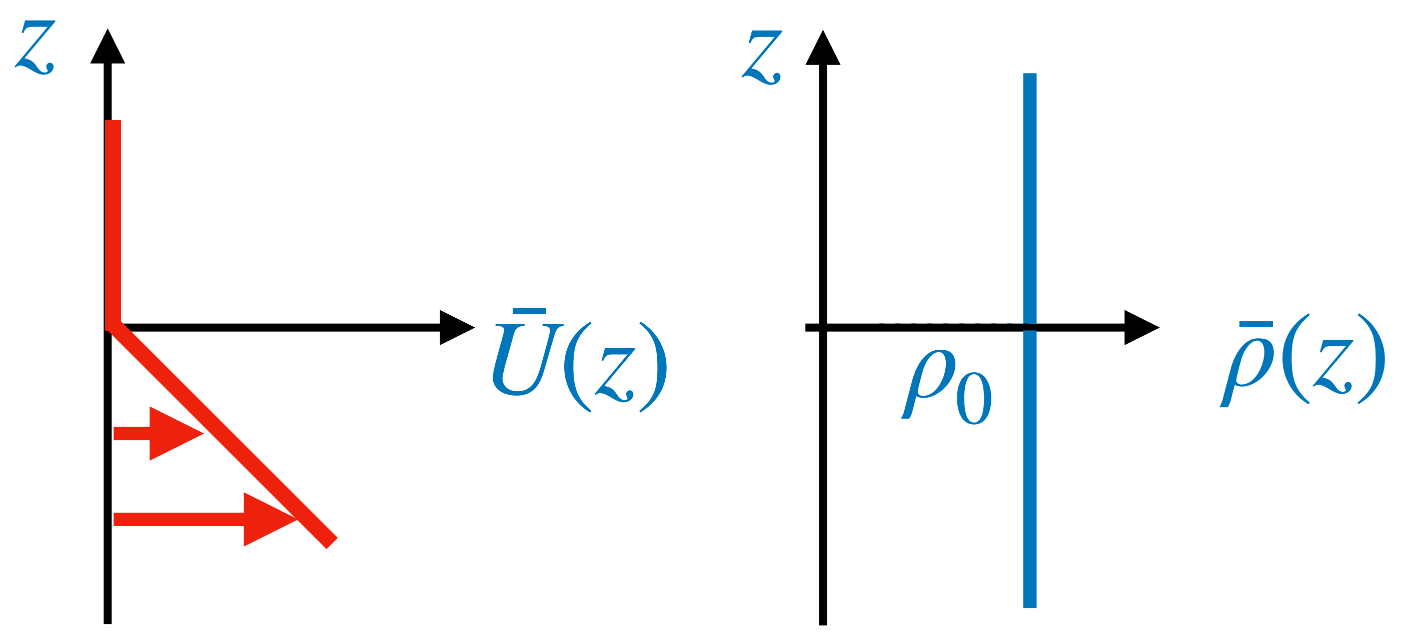
using $\mathscr{A}_1 = \mathscr{A}_2$

$$\Rightarrow \quad c = s_0/(2k)$$

Rayleigh Waves (in fluids)



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using $\mathscr{A}_1 = \mathscr{A}_2$

$$\Rightarrow \quad c = s_0/(2k) \quad \Rightarrow \quad \omega = s_0/2$$

using $c \equiv \omega/k$

Rayleigh Waves (in fluids)

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  - $\bar{\rho}(z) = \rho_0$

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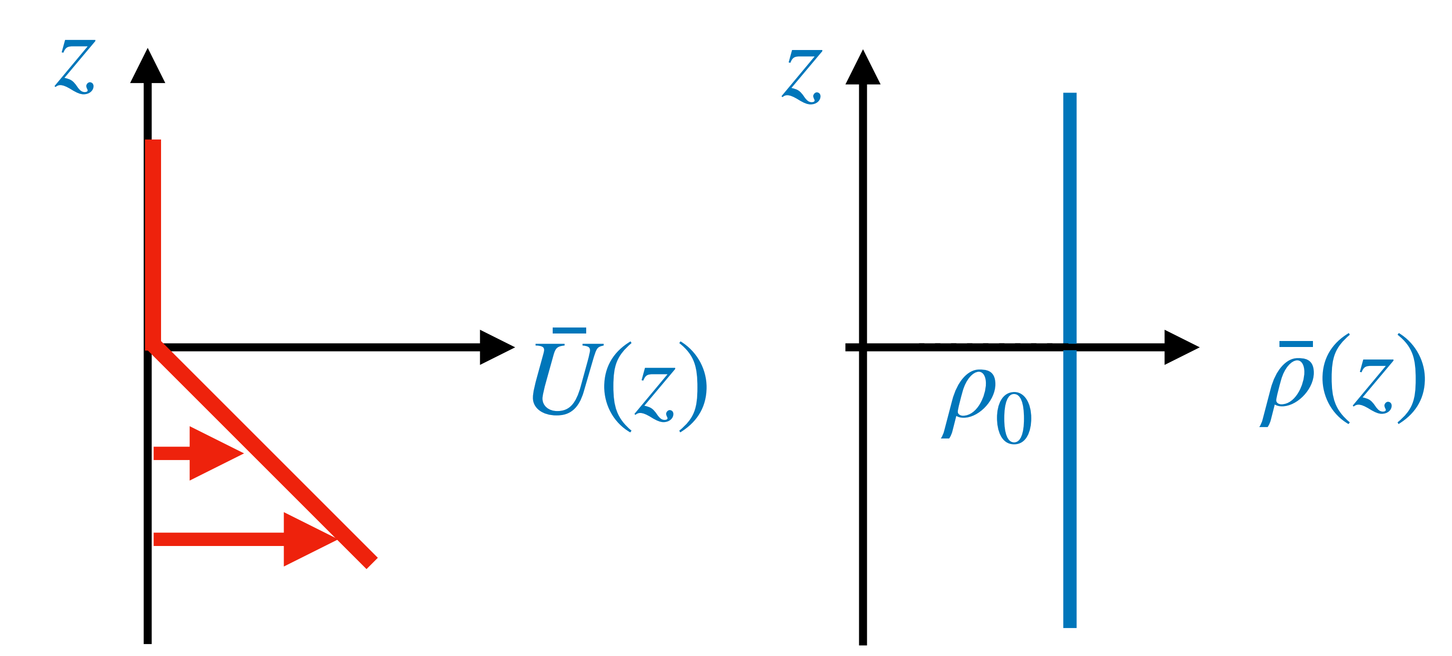
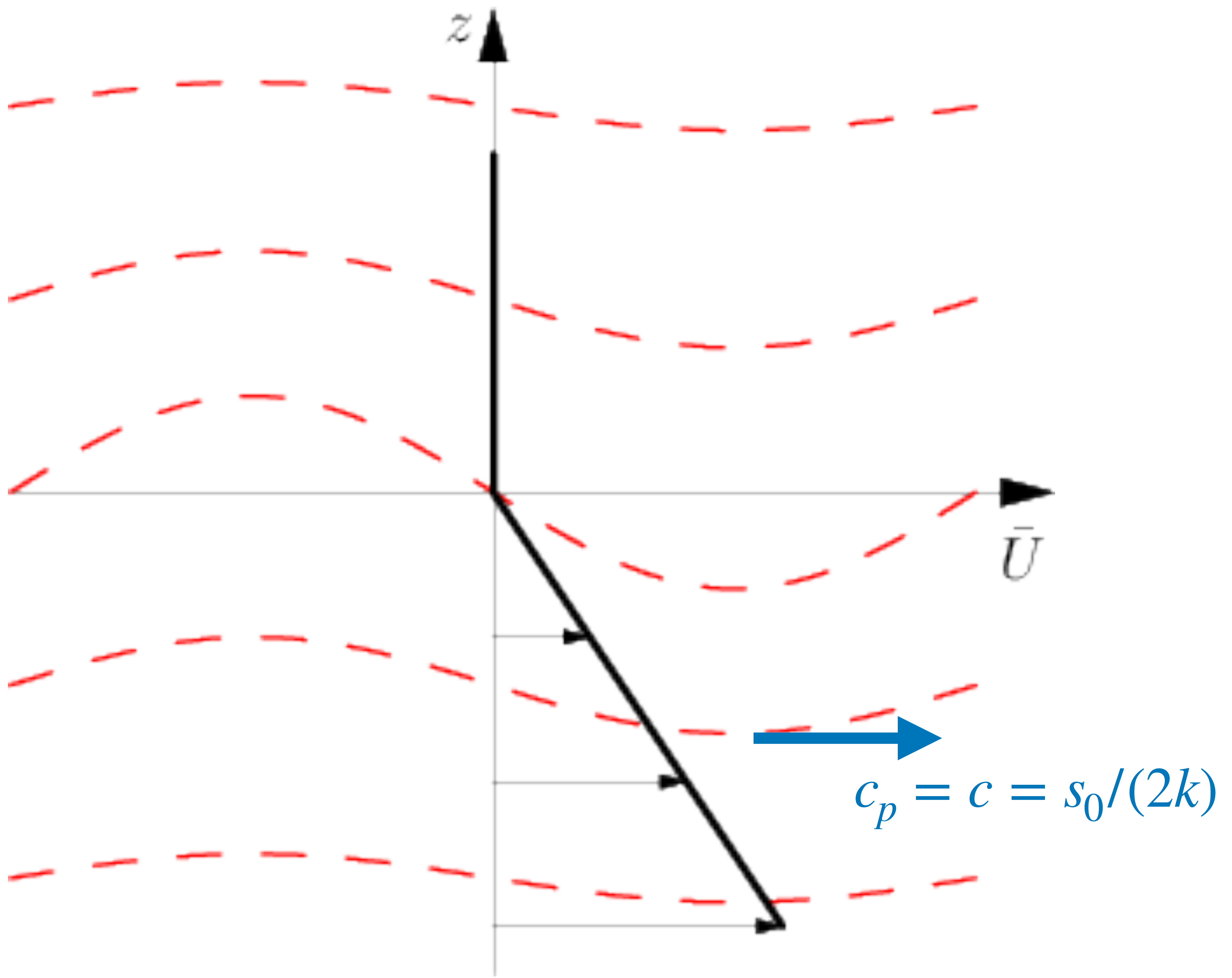


$$\psi(x, z, t) = \mathscr{A} e^{-k|z|} e^{\imath(kx-\omega t)}$$

$$\omega(k) = s_0/2$$



6.4] Kelvin-Helmholtz Instability

- We proceed as in the previous examples but not considering piecewise-linear flows and piecewise-constant density profiles that can admit two wave solutions.

- In some circumstances, the waves can resonate leading to their amplitude growth in time, thus rendering the flow unstable

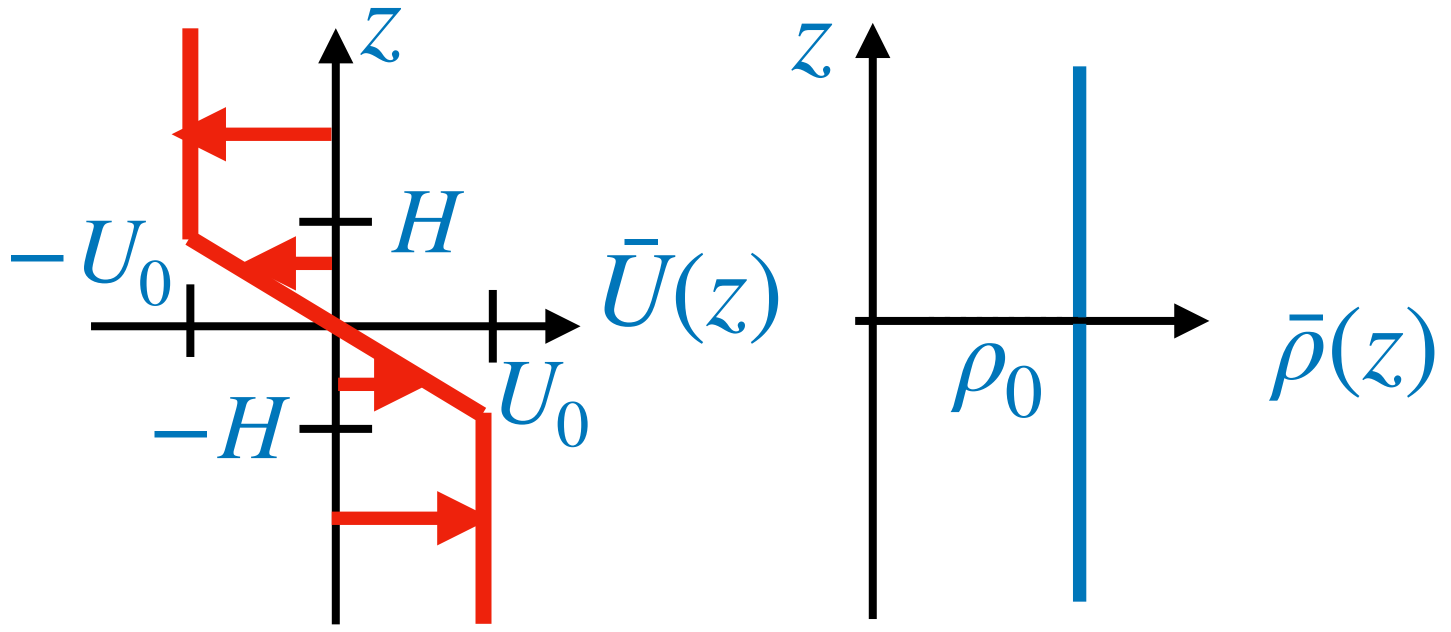
Kelvin-Helmholtz Instability: structure and interface conditions

- This results from resonantly coupled Rayleigh Waves in a shear layer:

  - $\bar{U}(z) = \quad \{-U_0,\ z > H;\ -s_0\, z,\ |z| < H;\ \ U_0,\ z < -H\}$

  where $s_0 \equiv U_0/H$

  For now assume that $\bar{\rho} = \rho_0$, constant.



Kelvin-Helmholtz Instability: structure and interface conditions

- This results from resonantly coupled Rayleigh Waves in a shear layer:

  - $\bar{U}(z) = \{-U_0,\ z > H;\ -s_0 z,\ |z| < H;\ U_0,\ z < -H\}$

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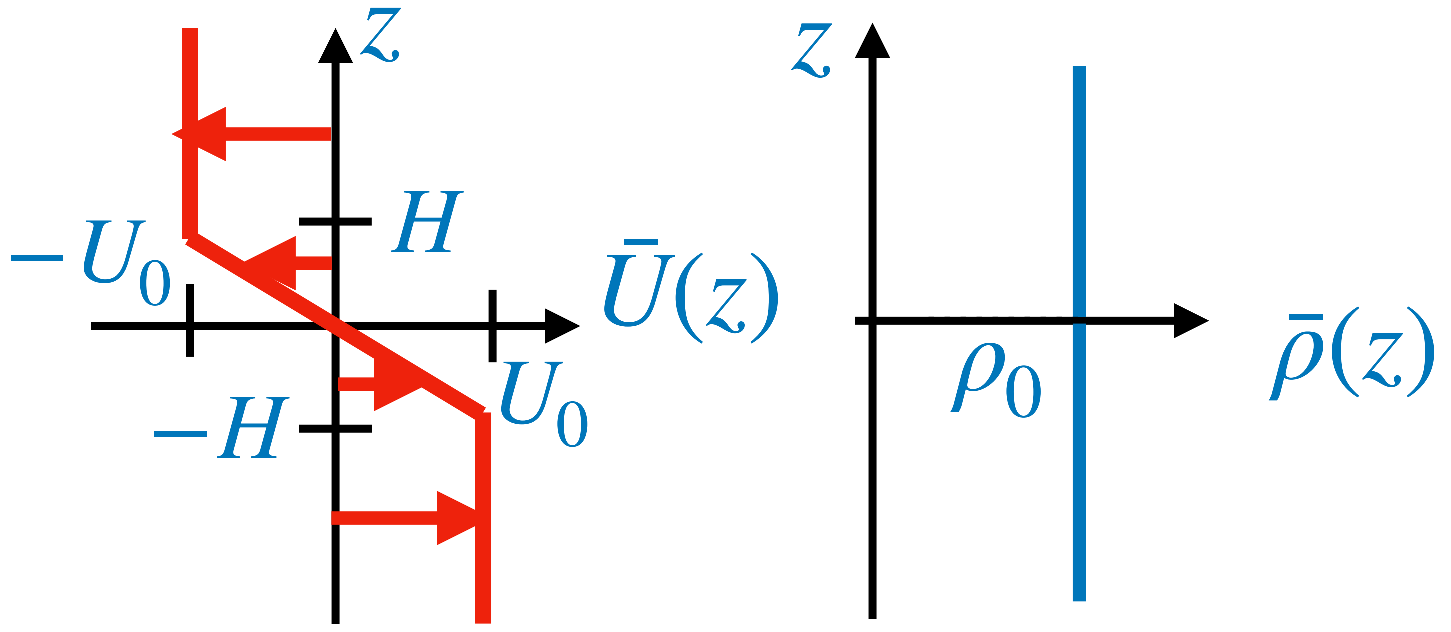
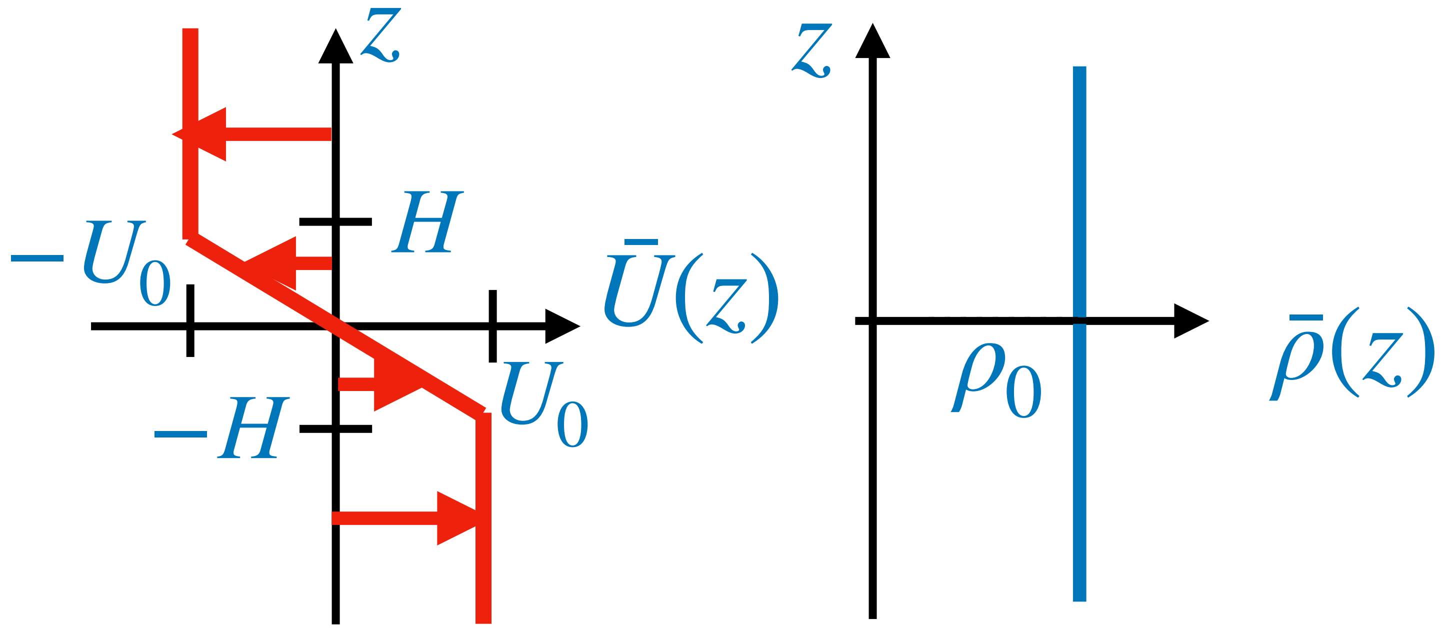


- Solve $\hat{\psi}'' - k^2 \hat{\psi} = 0$, require bounded solutions for $z \to \pm\infty$, and take $k > 0$:

Kelvin-Helmholtz Instability: structure and interface conditions



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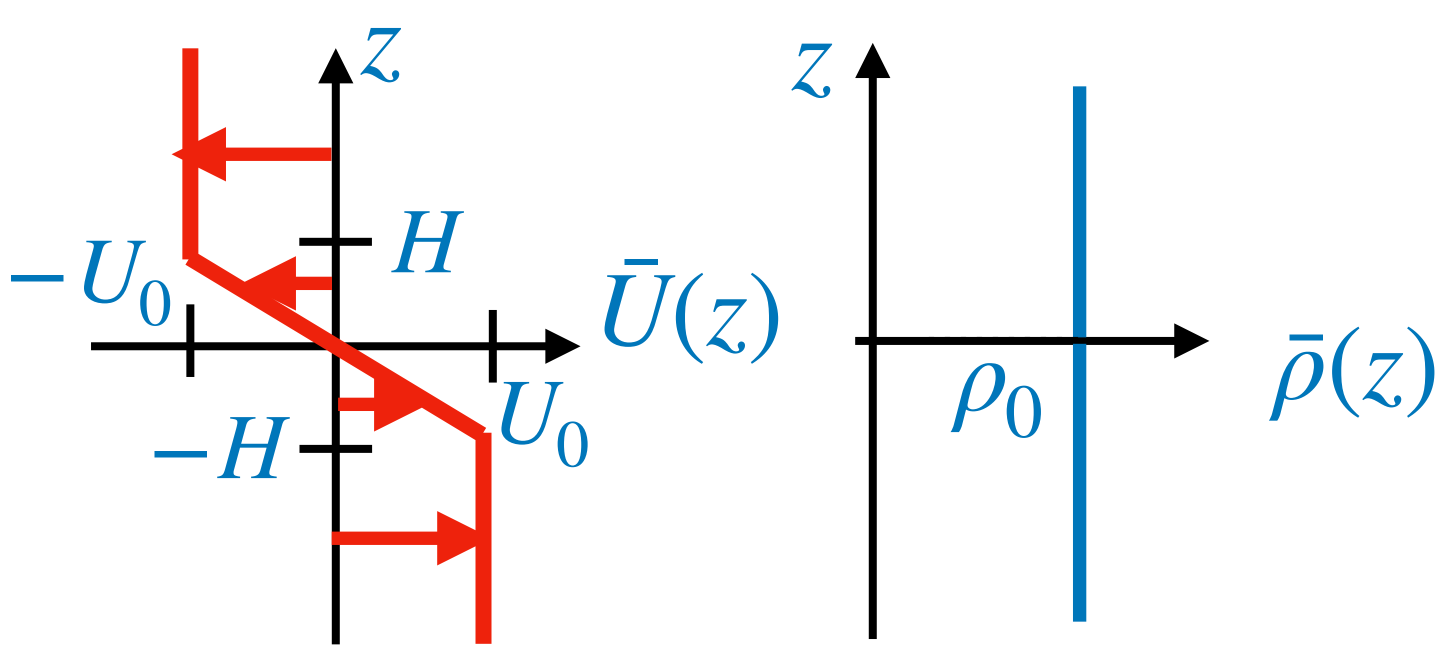

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- Because $\bar{U}(z)$ is continuous and $\bar{\rho}(z) = \rho_0$,

  1): $\dfrac{\hat{\psi}}{\bar{U} - c}$ continuous $\Rightarrow$ $\hat{\psi}$ continuous

Kelvin-Helmholtz Instability: structure and interface conditions

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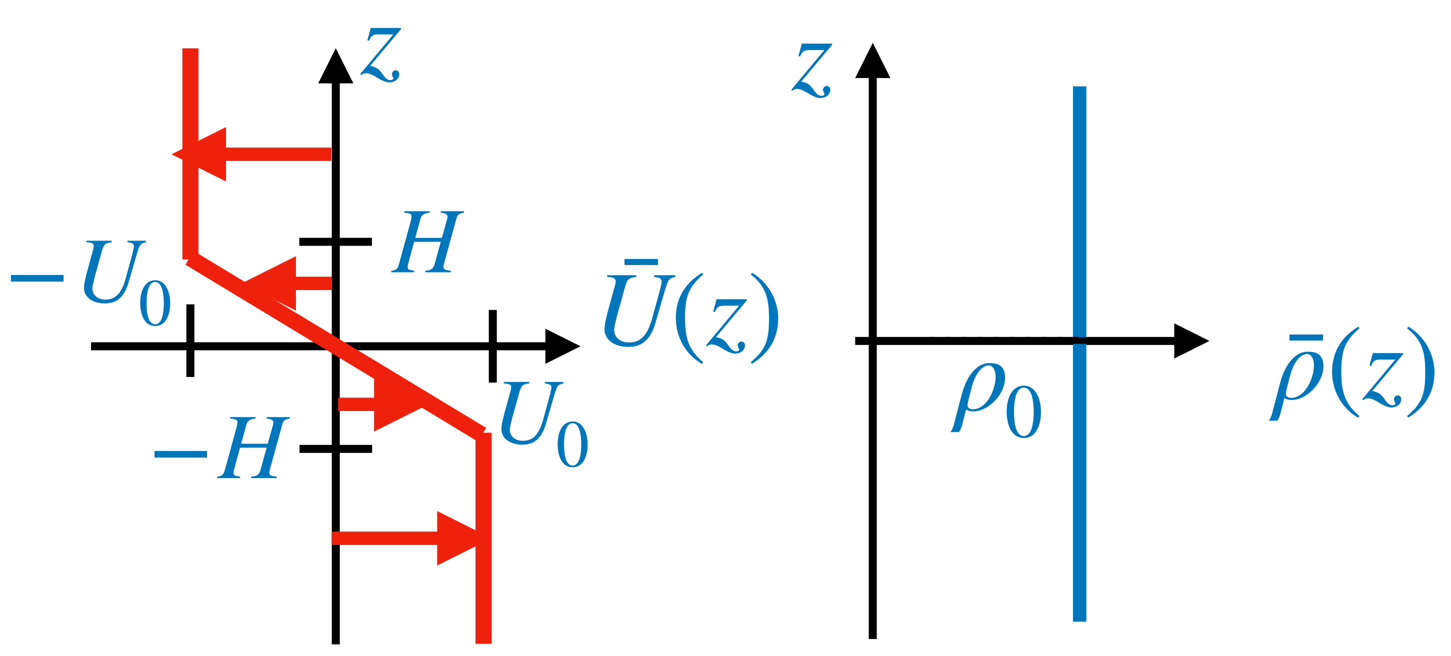

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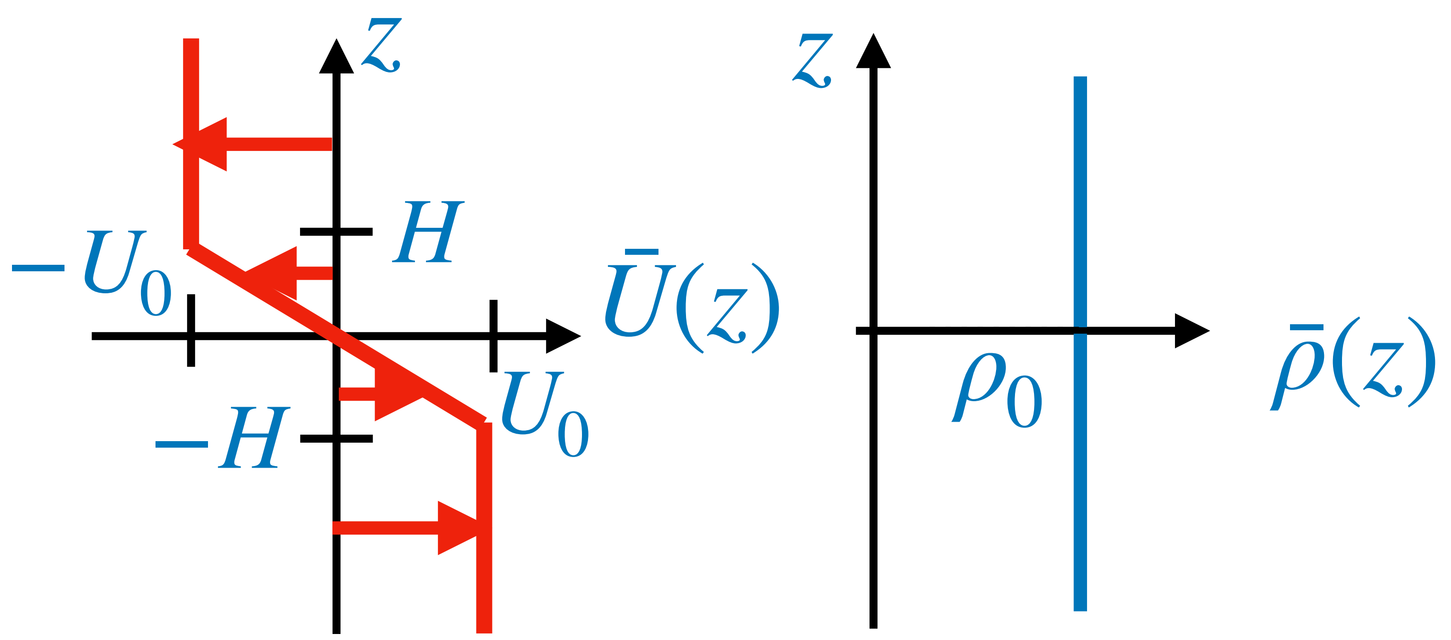
  1): $\dfrac{\hat{\psi}}{\bar{U} - c}$ continuous $\Rightarrow$ $\hat{\psi}$ continuous

  2): $\bar{\rho}\left[(\bar{U} - c)\hat{\psi}' - \bar{U}'\hat{\psi} - \dfrac{g}{\bar{U} - c}\hat{\psi}\right]$ continuous $\Rightarrow$ $(\bar{U} - c)\hat{\psi}' - \bar{U}'\hat{\psi}$ continuous

Kelvin-Helmholtz Instability: eigenvalue problem

$$\hat{\psi} = \begin{cases} \mathscr{A} e^{-kz} & z > H \\ \mathscr{B}_1 e^{kz} + \mathscr{B}_2 e^{-kz} & |z| < H \\ \mathscr{C} e^{kz} & z < -H \end{cases}$$



$\bar{U}(z) = \{-U_0,\ z > H;\ -s_0 z,\ |z| < H;\ U_0,\ z < -H\}$

1) $\hat{\psi}$ continuous

at $z = H$:

Kelvin-Helmholtz Instability: eigenvalue problem

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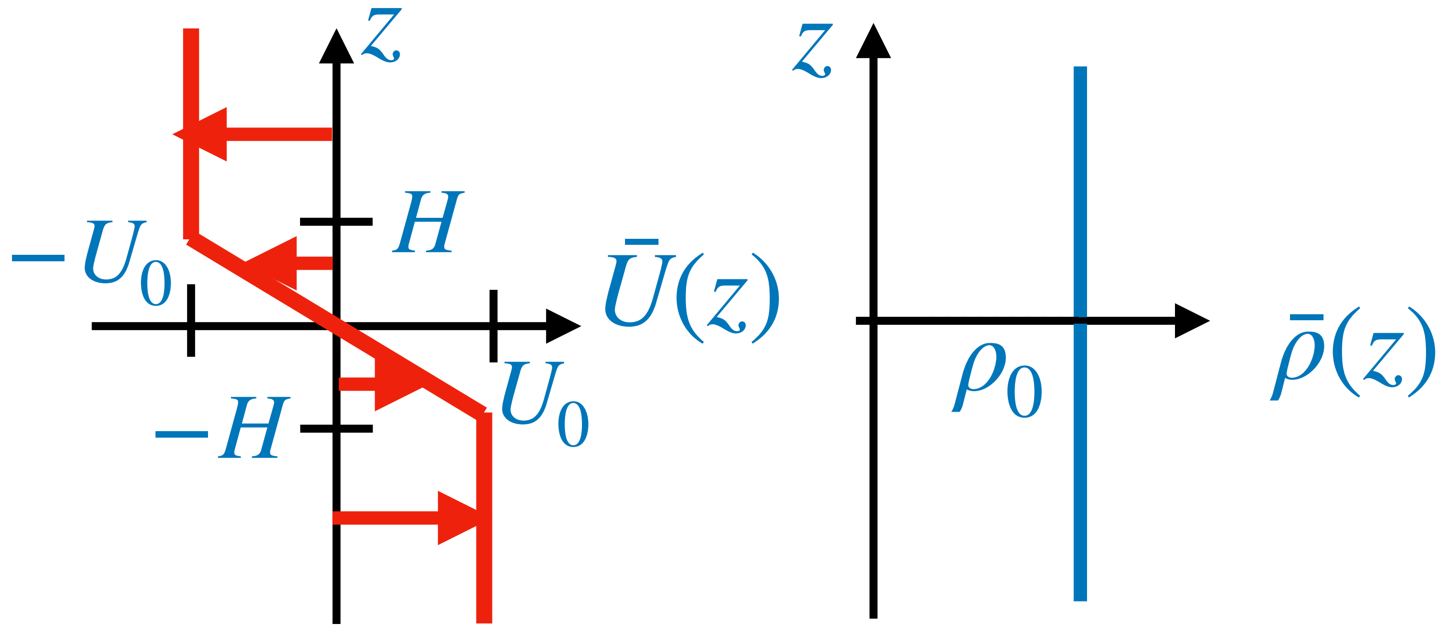

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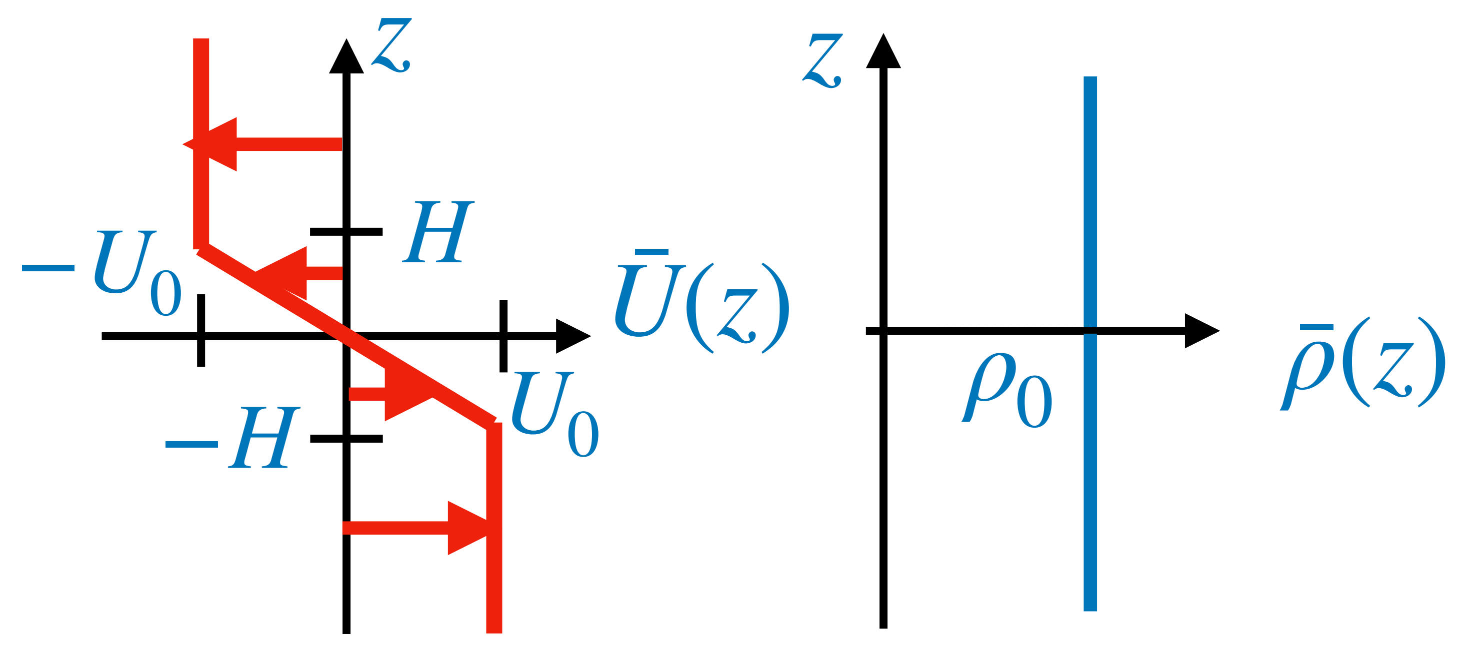

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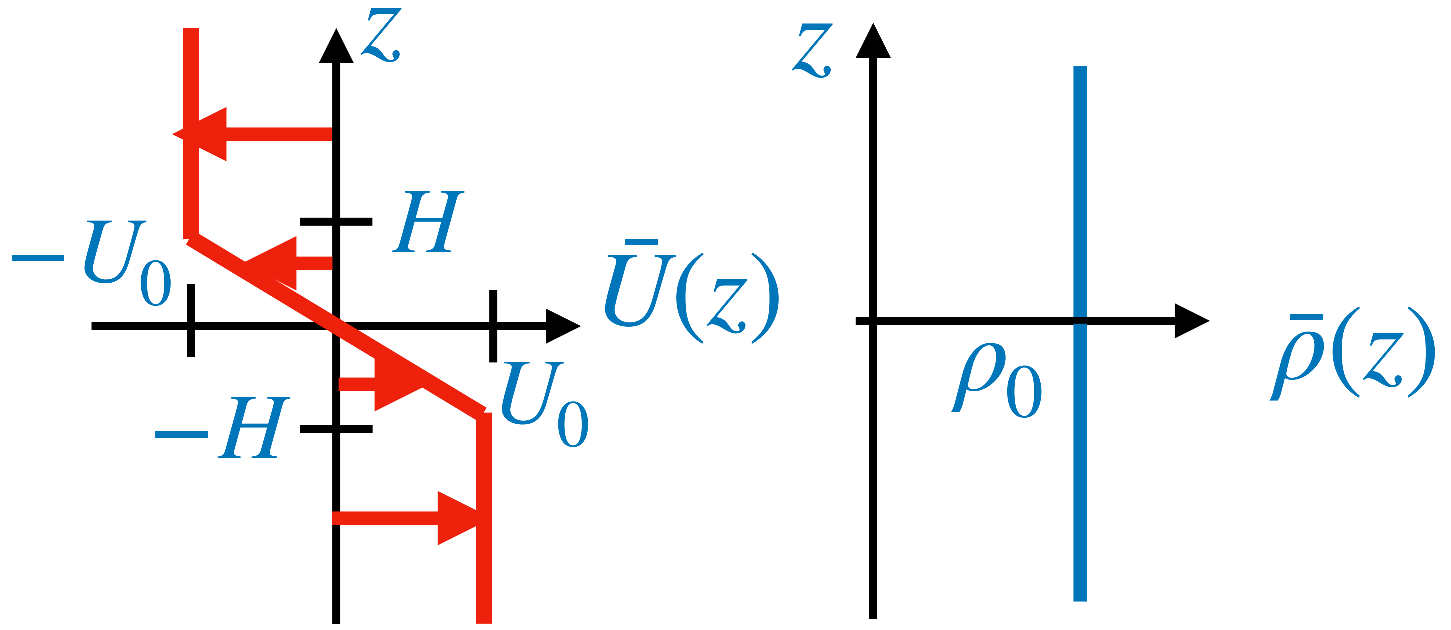

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Kelvin-Helmholtz Instability: eigenvalue problem

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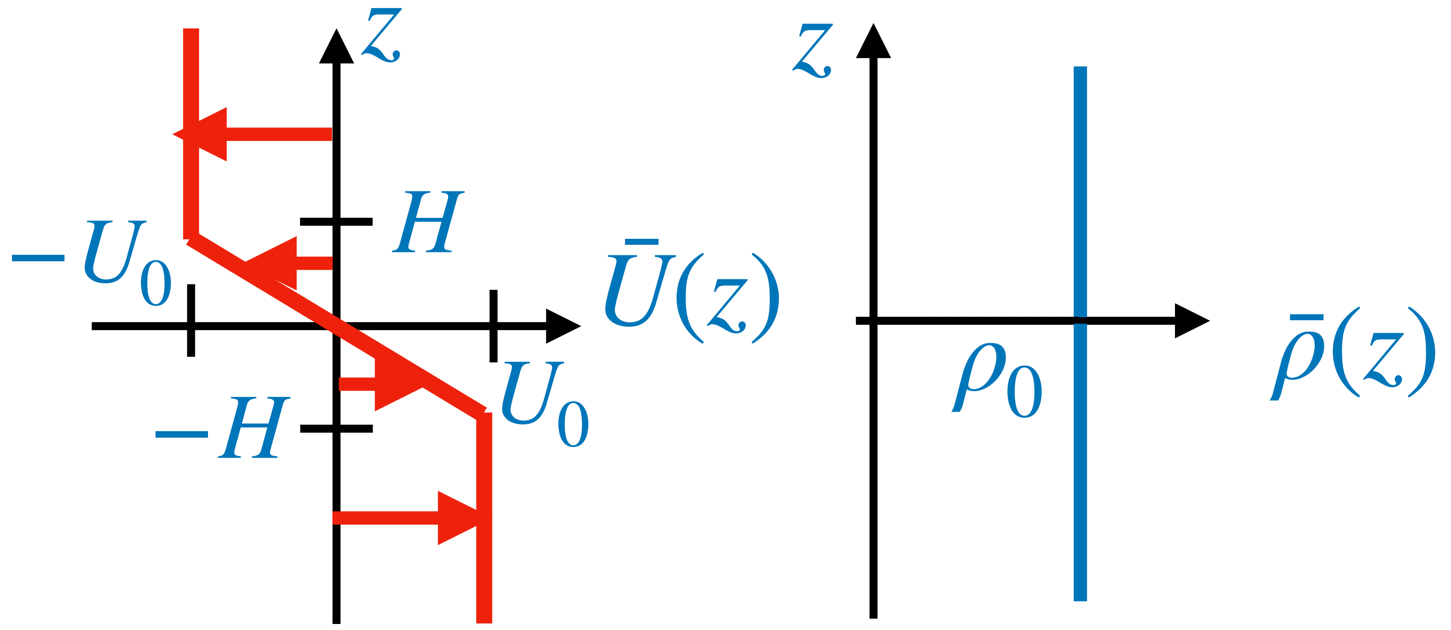

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at $z = H$: $(-U_0 - c)\left[-k\mathscr{A} e^{-kH}\right] = (-U_0 - c)\left[k\mathscr{B}_1 e^{kH} - k\mathscr{B}_2 e^{-kH}\right] - (-s_0)\left[\mathscr{B}_1 e^{kH} + \mathscr{B}_2 e^{-kH}\right]$

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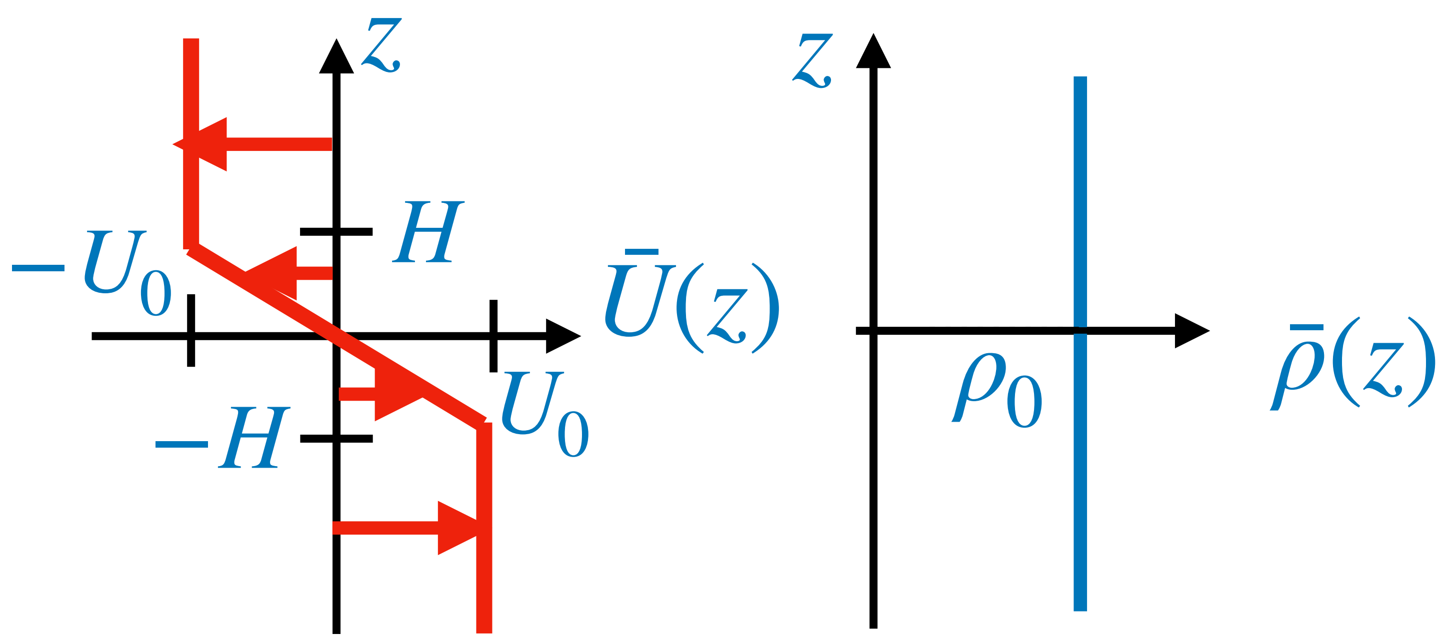
$$\hat{\psi} = \begin{cases} \mathscr{A}e^{-kz} & z > H \\ \mathscr{B}_1 e^{kz} + \mathscr{B}_2 e^{-kz} & |z| < H \\ \mathscr{C}e^{kz} & z < -H \end{cases}$$



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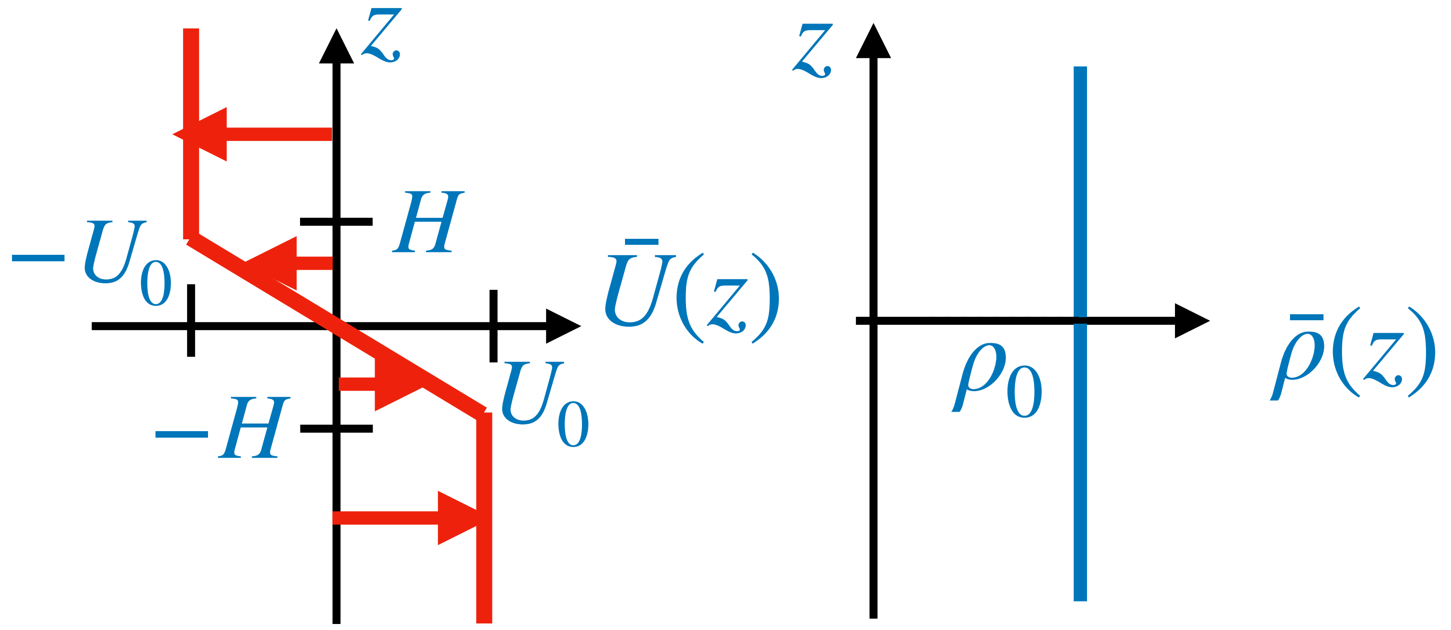

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$$\Rightarrow (U_0 k + \omega)\left[\mathscr{B}_1 e^{kH} + \mathscr{B}_2 e^{-kH}\right] = -(U_0 k + \omega)\left[\mathscr{B}_1 e^{kH} - \mathscr{B}_2 e^{-kH}\right] + s_0\left[\mathscr{B}_1 e^{kH} + \mathscr{B}_2 e^{-kH}\right]$$

$c \equiv \omega / k$

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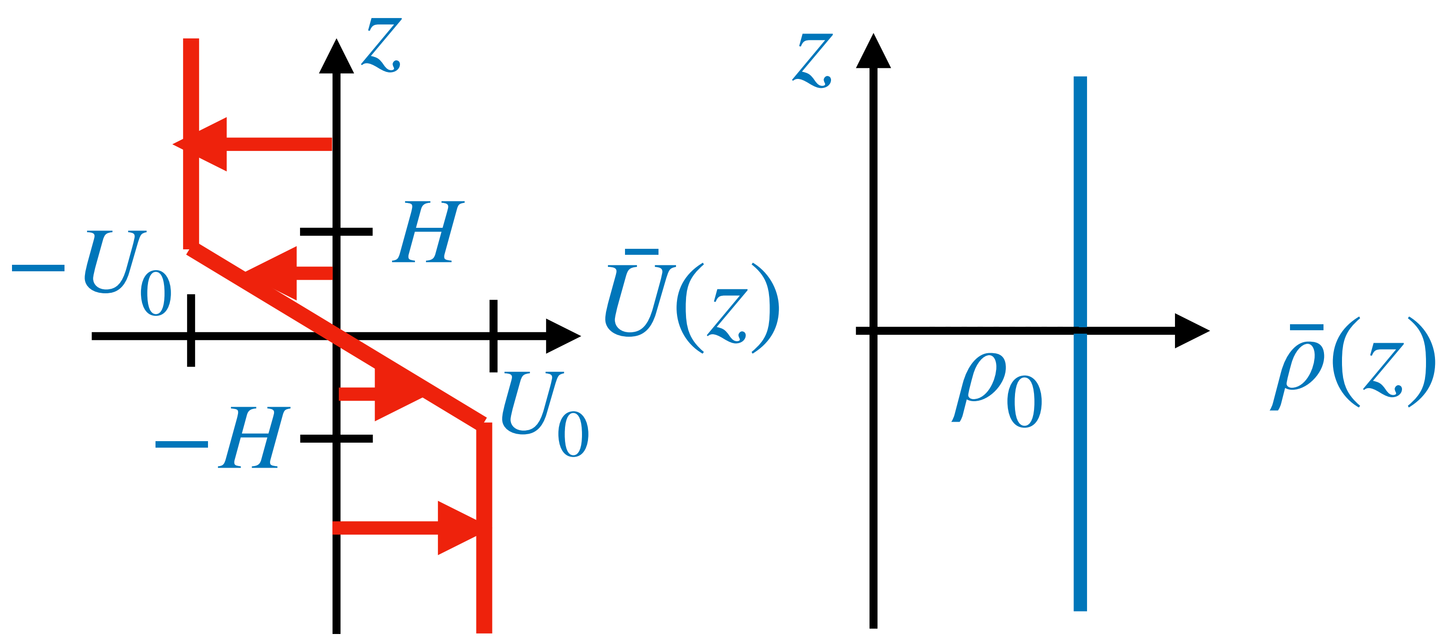
$$\hat{\psi} = \begin{cases} \mathscr{A} e^{-kz} & z > H \\ \mathscr{B}_1 e^{kz} + \mathscr{B}_2 e^{-kz} & |z| < H \\ \mathscr{C} e^{kz} & z < -H \end{cases}$$



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Kelvin-Helmholtz Instability: dispersion relation

$$\hat{\psi} = \begin{cases} \mathscr{A} e^{-kz} & z > H \\ \mathscr{B}_1 e^{kz} + \mathscr{B}_2 e^{-kz} & |z| < H \\ \mathscr{C} e^{kz} & z < -H \end{cases}$$

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$$\Rightarrow \quad (2U_0 k + 2\omega - s_0)\, \mathscr{B}_1 e^{kH} + (-s_0)\, \mathscr{B}_2 e^{-kH} = 0$$

$$\Rightarrow \quad (-s_0)\, \mathscr{B}_1 e^{-kH} + (2U_0 k - 2\omega - s_0)\, \mathscr{B}_2 e^{kH} = 0$$

Kelvin-Helmholtz Instability: dispersion relation

$$\hat{\psi} = \begin{cases} \mathscr{A} e^{-kz} & z > H \\ \mathscr{B}_1 e^{kz} + \mathscr{B}_2 e^{-kz} & |z| < H \\ \mathscr{C} e^{kz} & z < -H \end{cases}$$

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$$\boxed{U_0 = s_0 H} \ \Rightarrow \ \begin{pmatrix} [2\omega + s_0(2kH - 1)]\, e^{kH} & -s_0\, e^{-kH} \\ -s_0\, e^{-kH} & -[2\omega - s_0(2kH - 1)]\, e^{kH} \end{pmatrix} \begin{pmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Kelvin-Helmholtz Instability: dispersion relation

$$\hat{\psi} = \begin{cases} \mathscr{A} e^{-kz} & z > H \\ \mathscr{B}_1 e^{kz} + \mathscr{B}_2 e^{-kz} & |z| < H \\ \mathscr{C} e^{kz} & z < -H \end{cases}$$

$$(U_0 k + \omega)\left[\mathscr{B}_1 e^{kH} + \mathscr{B}_2 e^{-kH}\right] = -(U_0 k + \omega)\left[\mathscr{B}_1 e^{kH} - \mathscr{B}_2 e^{-kH}\right] + s_0\left[\mathscr{B}_1 e^{kH} + \mathscr{B}_2 e^{-kH}\right]$$

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$$\boxed{U_0 = s_0 H} \;\Rightarrow\; \begin{pmatrix} [2\omega + s_0(2kH - 1)]\,e^{kH} & -s_0\,e^{-kH} \\ -s_0\,e^{-kH} & -[2\omega - s_0(2kH - 1)]\,e^{kH} \end{pmatrix} \begin{pmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\omega^2 = \frac{1}{4} s_0^2 \left[(1 - 2kH)^2 - e^{-4kH}\right]}$$

6.4] Unstable Coupled Wave Solutions (cont'd)

- We proceed as in the previous examples but not considering piecewise-linear flows and piecewise-constant density profiles that can admit two wave solutions.

- In some circumstances, the waves can resonate leading to their amplitude growth in time, thus rendering the flow unstable

- We are considering instability occurring in a "shear layer" in uniform density fluid.

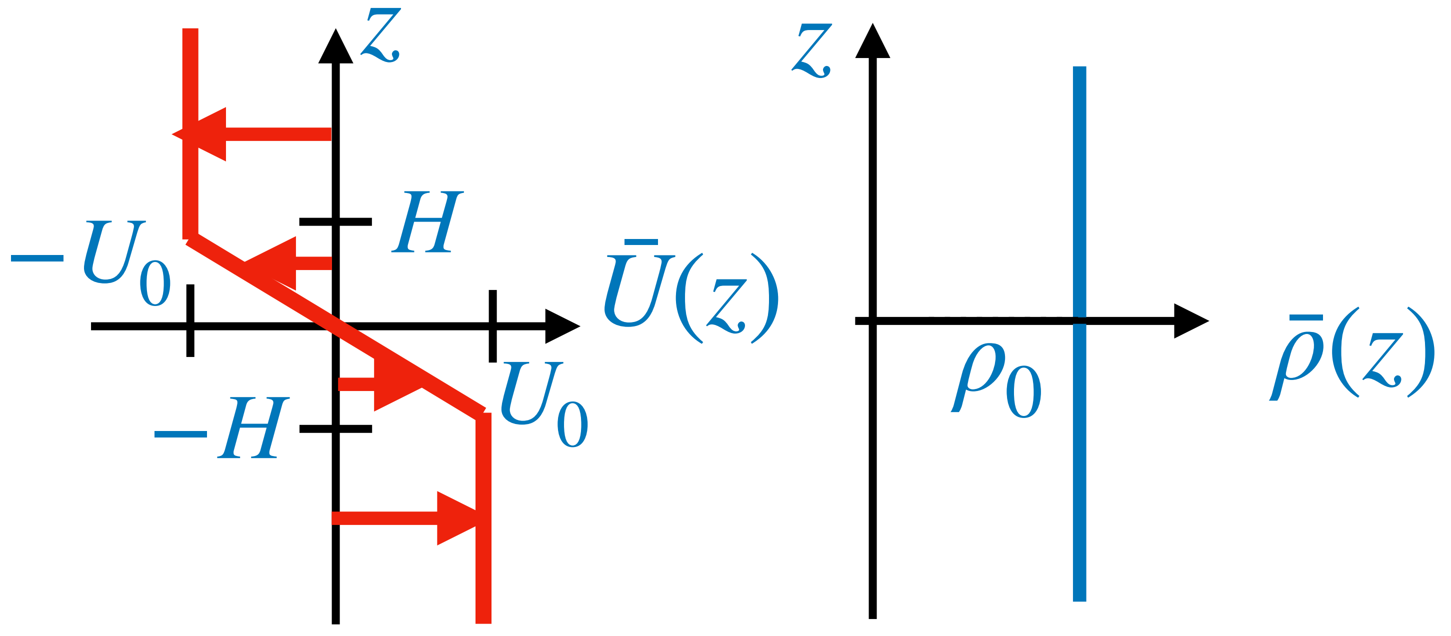
  This is "Kelvin-Helmholtz instability".

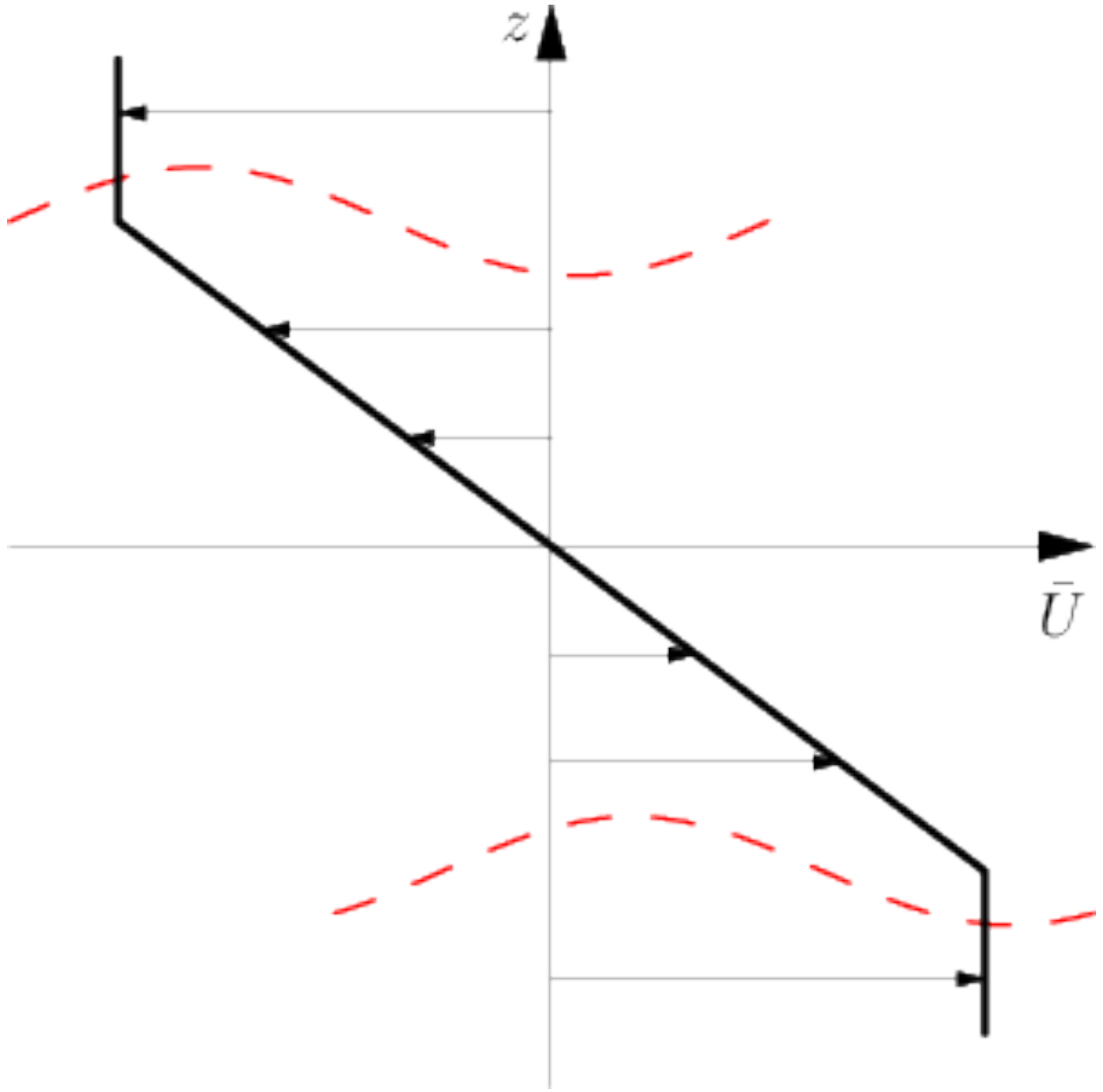
Kelvin-Helmholtz instability: coupled Rayleigh waves

- We found that waves in a shear layer satisfy the dispersion relation

$$\omega^2 = \frac{1}{4} s_0^2 \left[(1 - 2kH)^2 - e^{-4kH} \right] \quad \text{with} \quad s_0 \equiv U_0 / H$$

- To help interpret this, first consider the limit $kH \gg 1$





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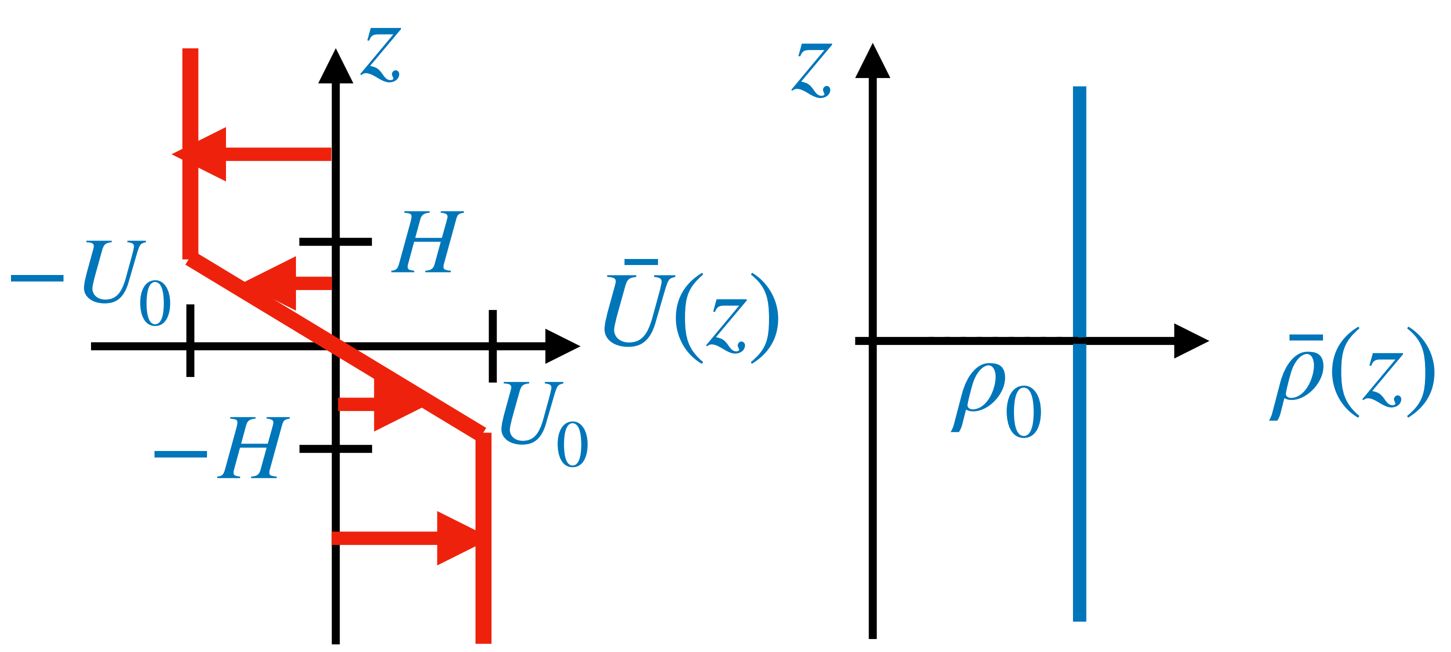
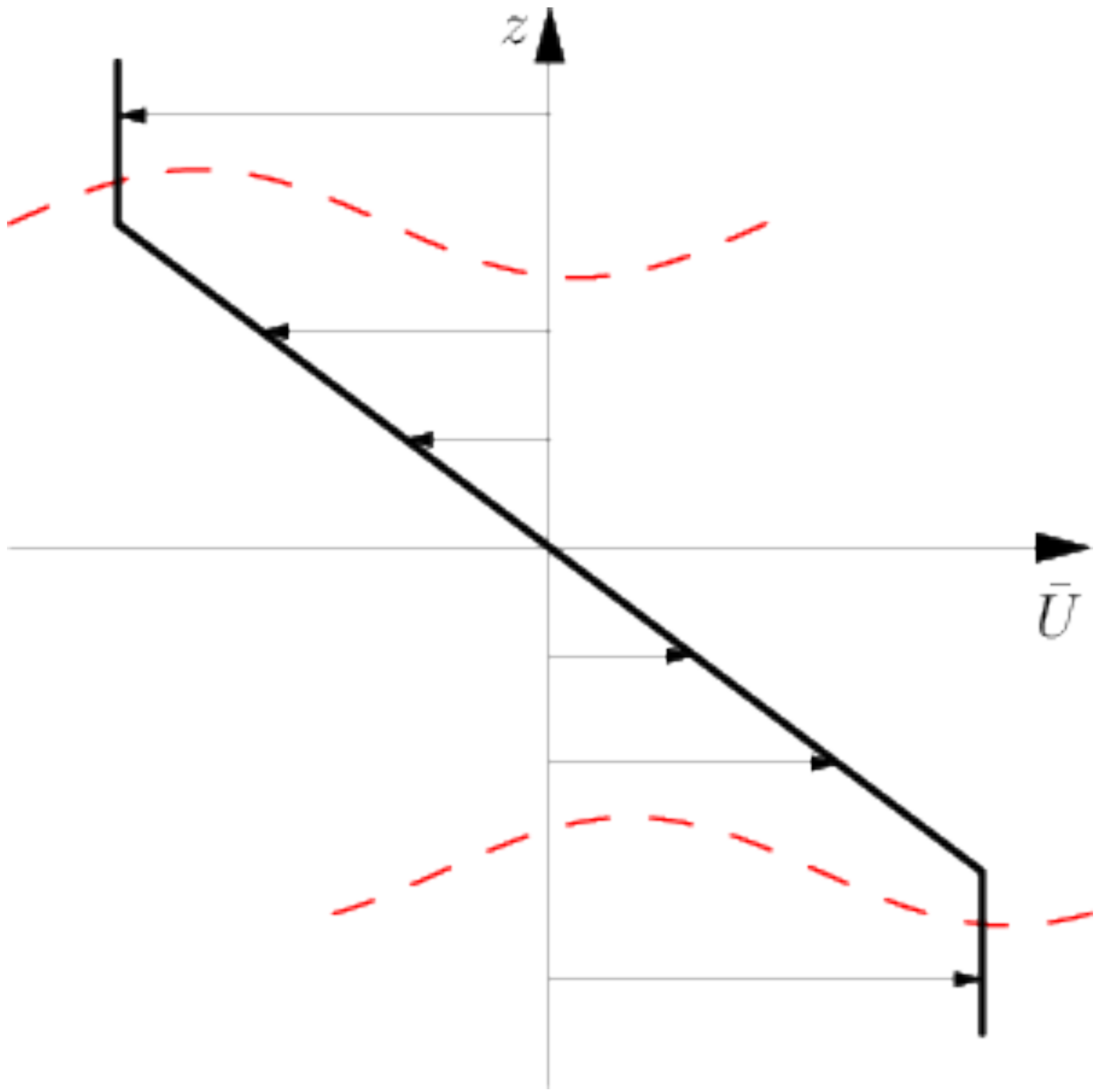
- Then $\omega \simeq \pm\frac{1}{2}s_0(1-2kH)$





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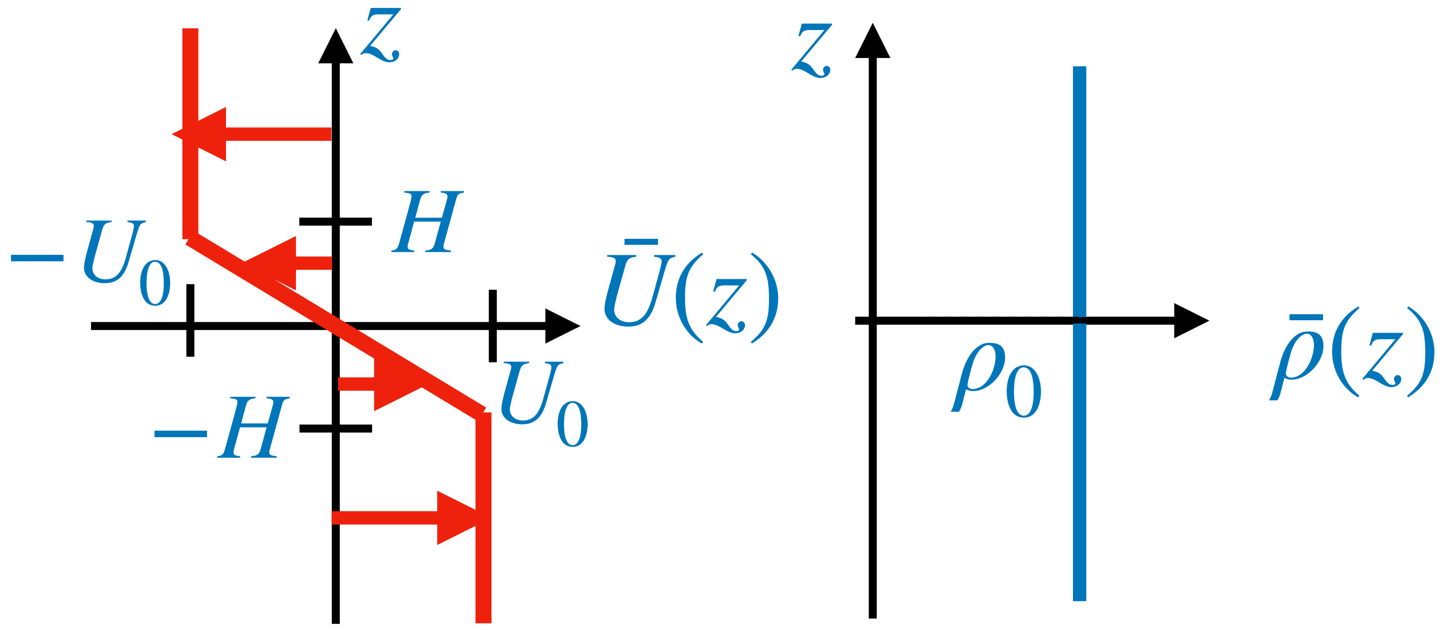
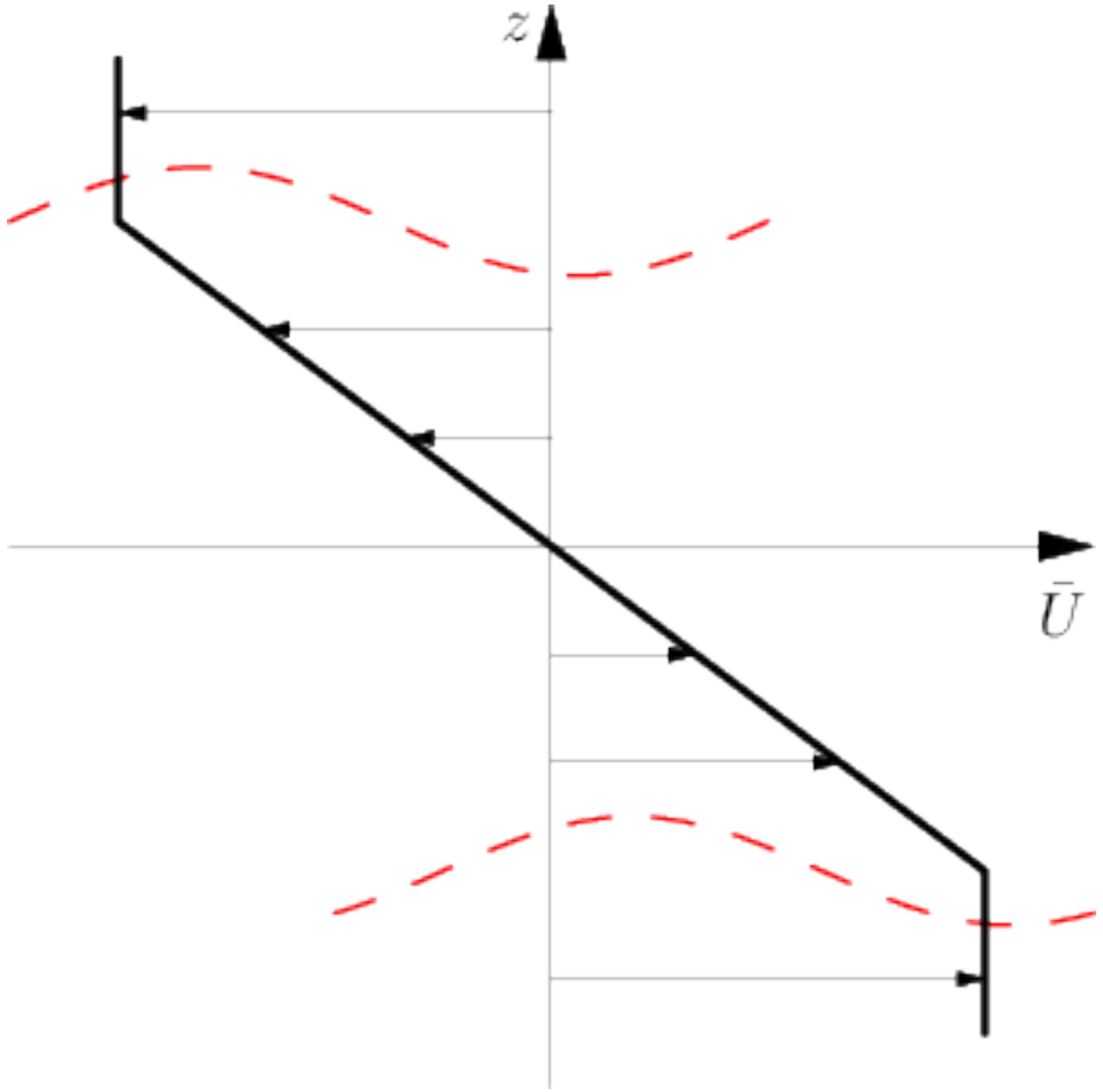


- To help interpret this, first consider the limit $kH \gg 1$
- Then $\omega \simeq \pm\frac{1}{2}s_0(1-2kH) \quad = \pm\left(\frac{s_0}{2} - U_0 k\right)$

  This describes 2 Rayleigh waves:



Kelvin-Helmholtz instability: coupled Rayleigh waves

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$$\omega^2 = \frac{1}{4}s_0^2\left[(1-2kH)^2 - e^{-4kH}\right] \quad \text{with} \quad s_0 \equiv U_0/H$$

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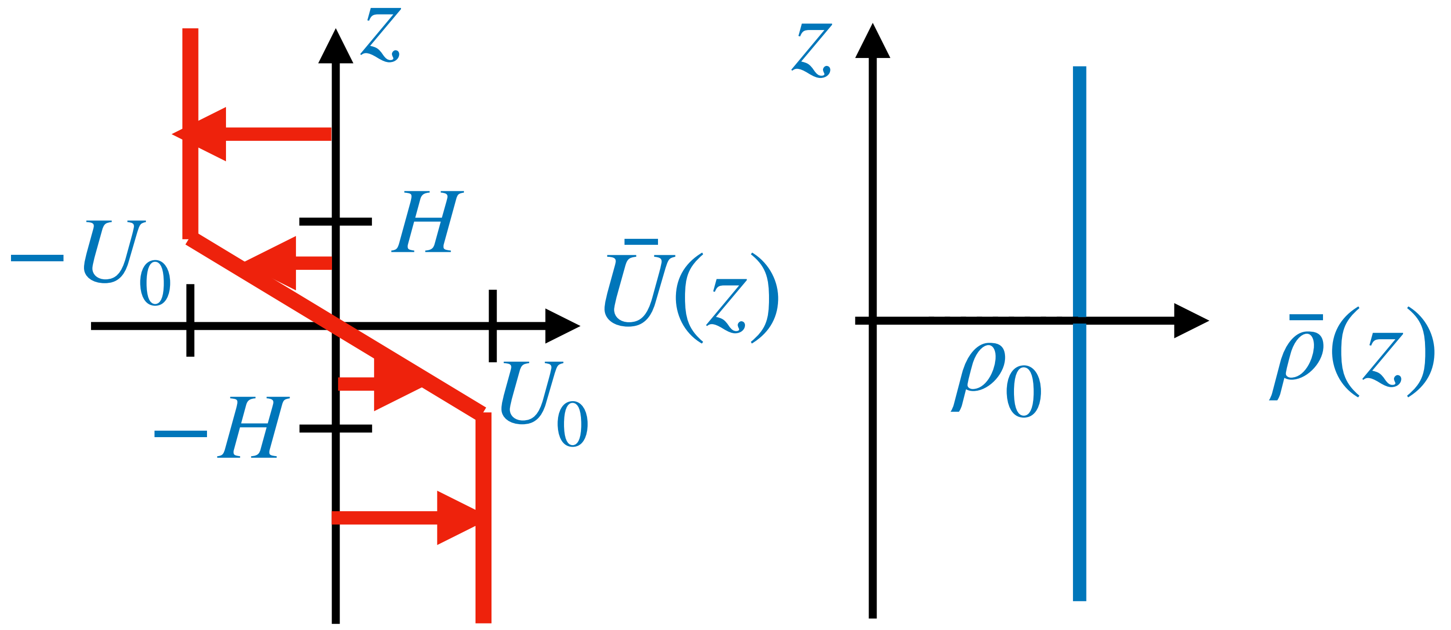

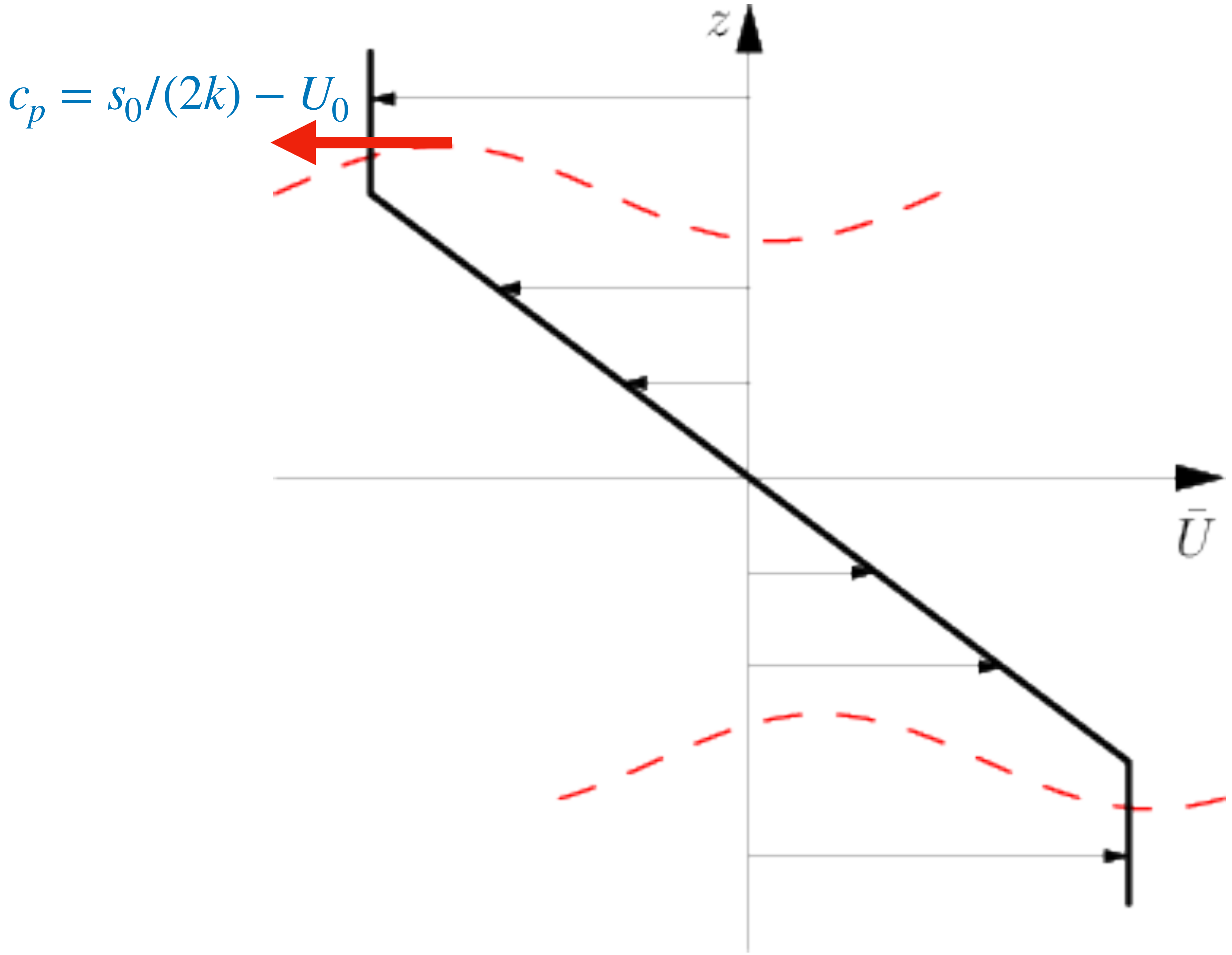

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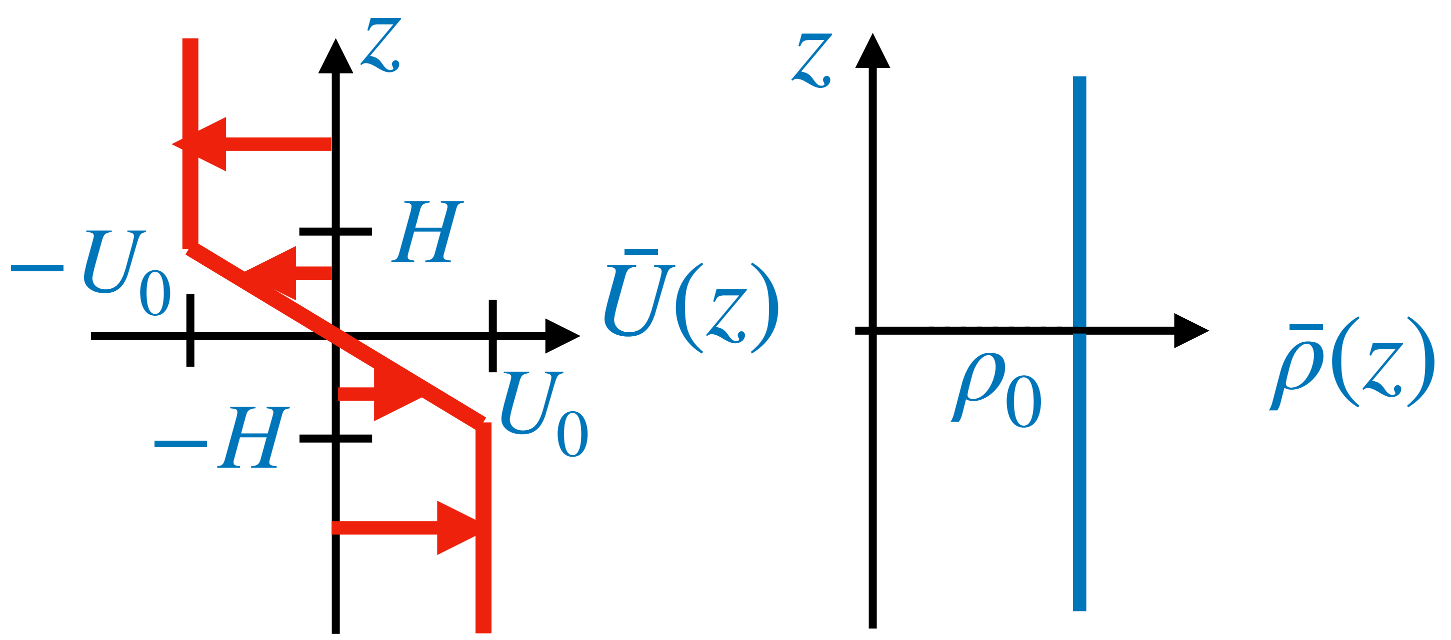

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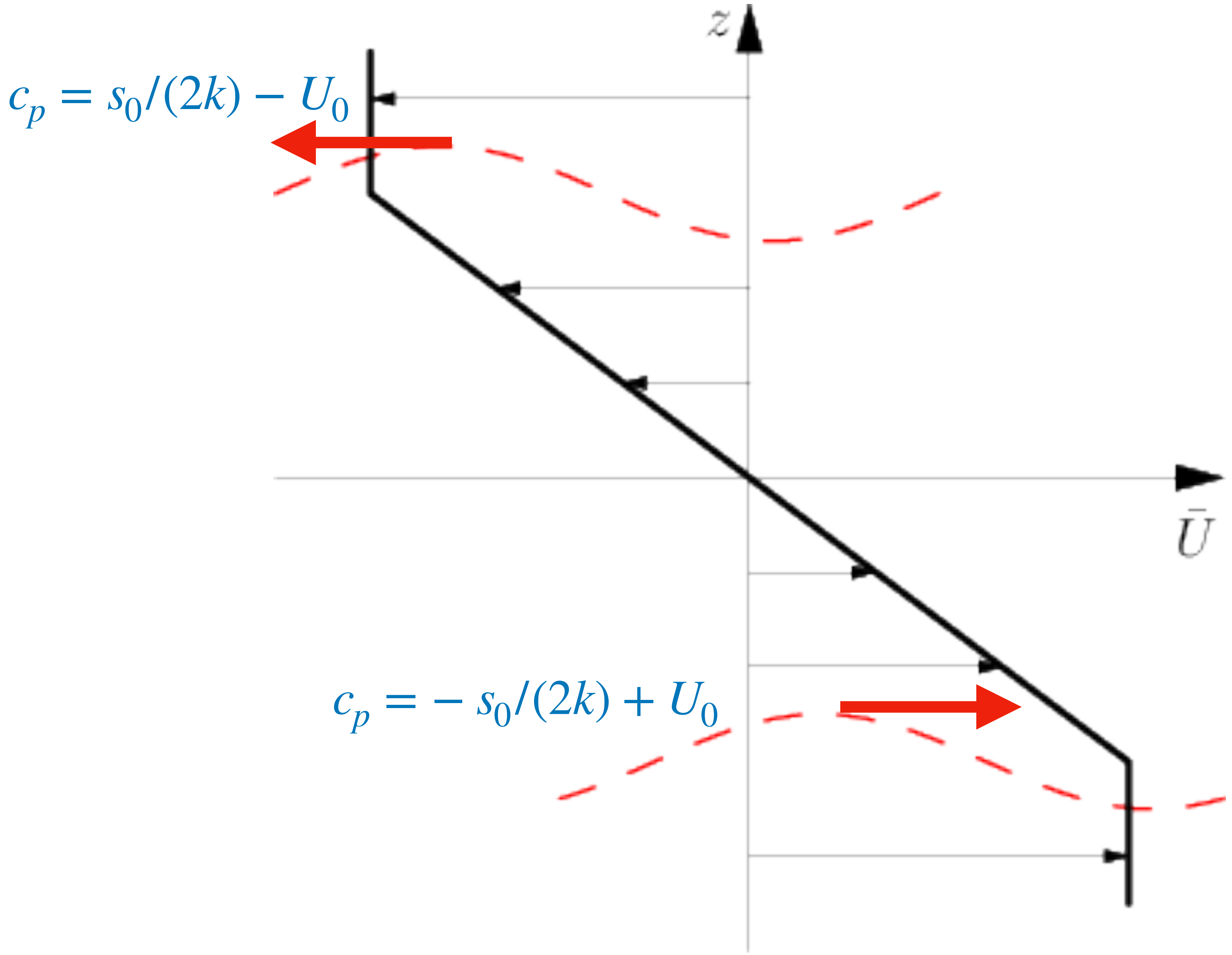
Doppler-shift by $+U_0$



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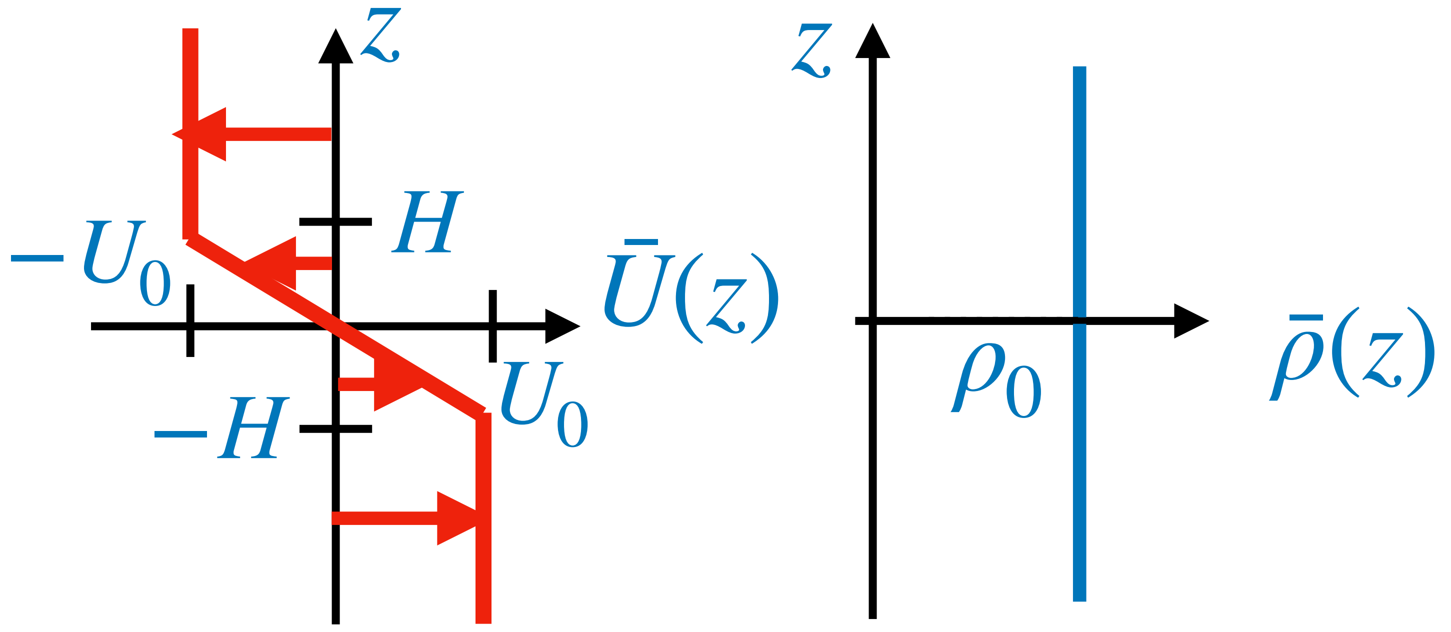

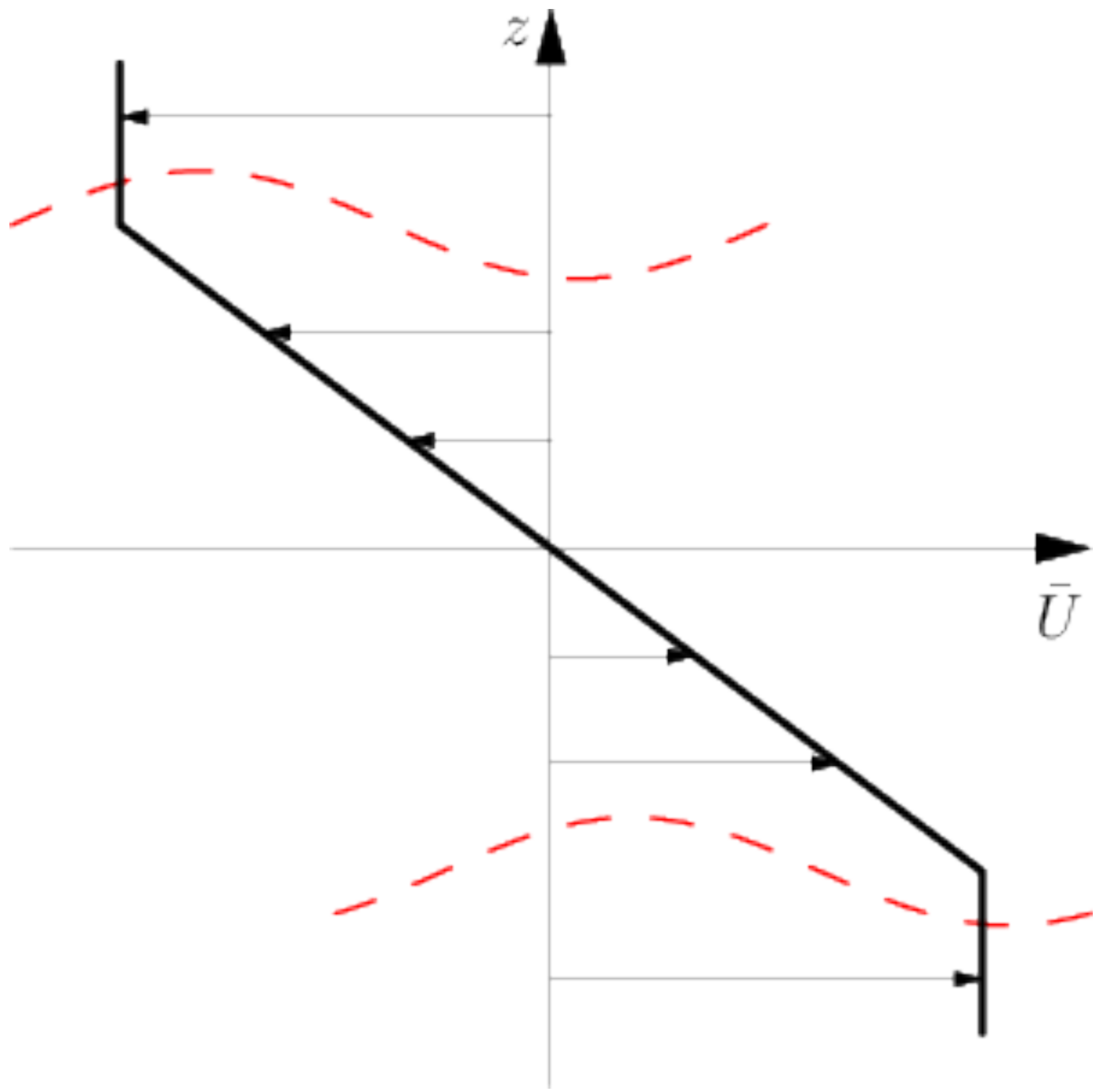

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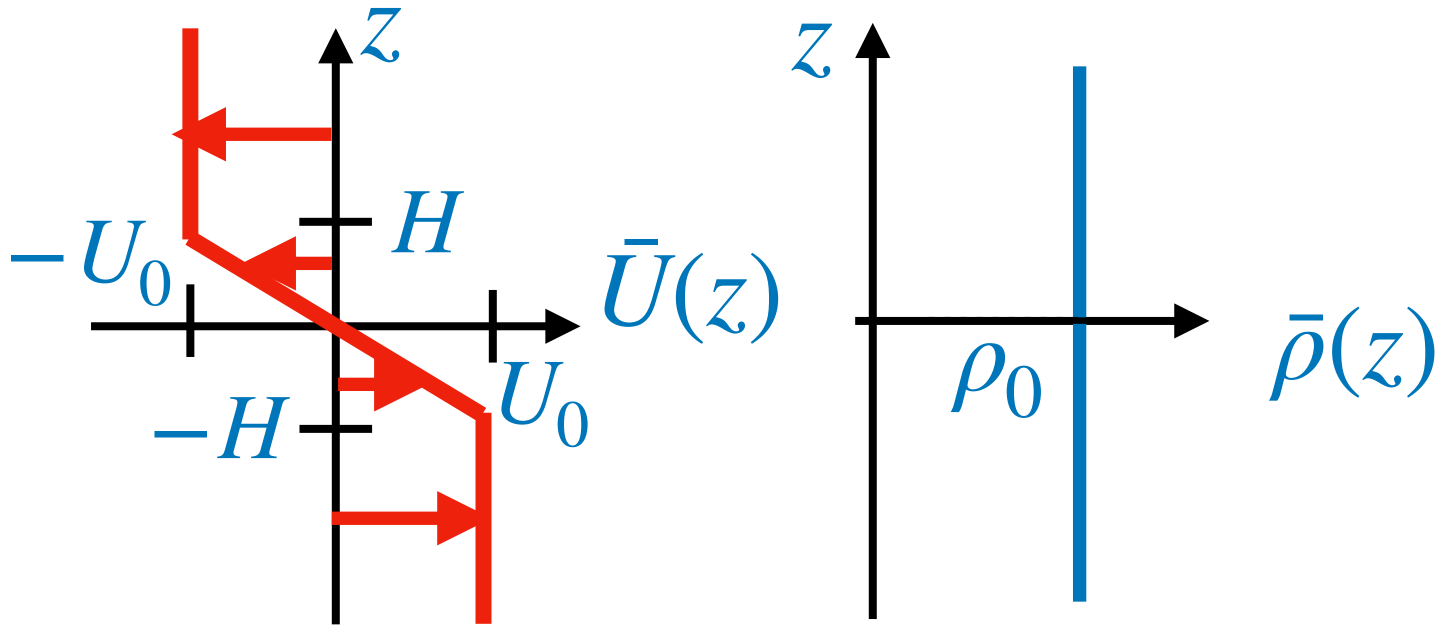

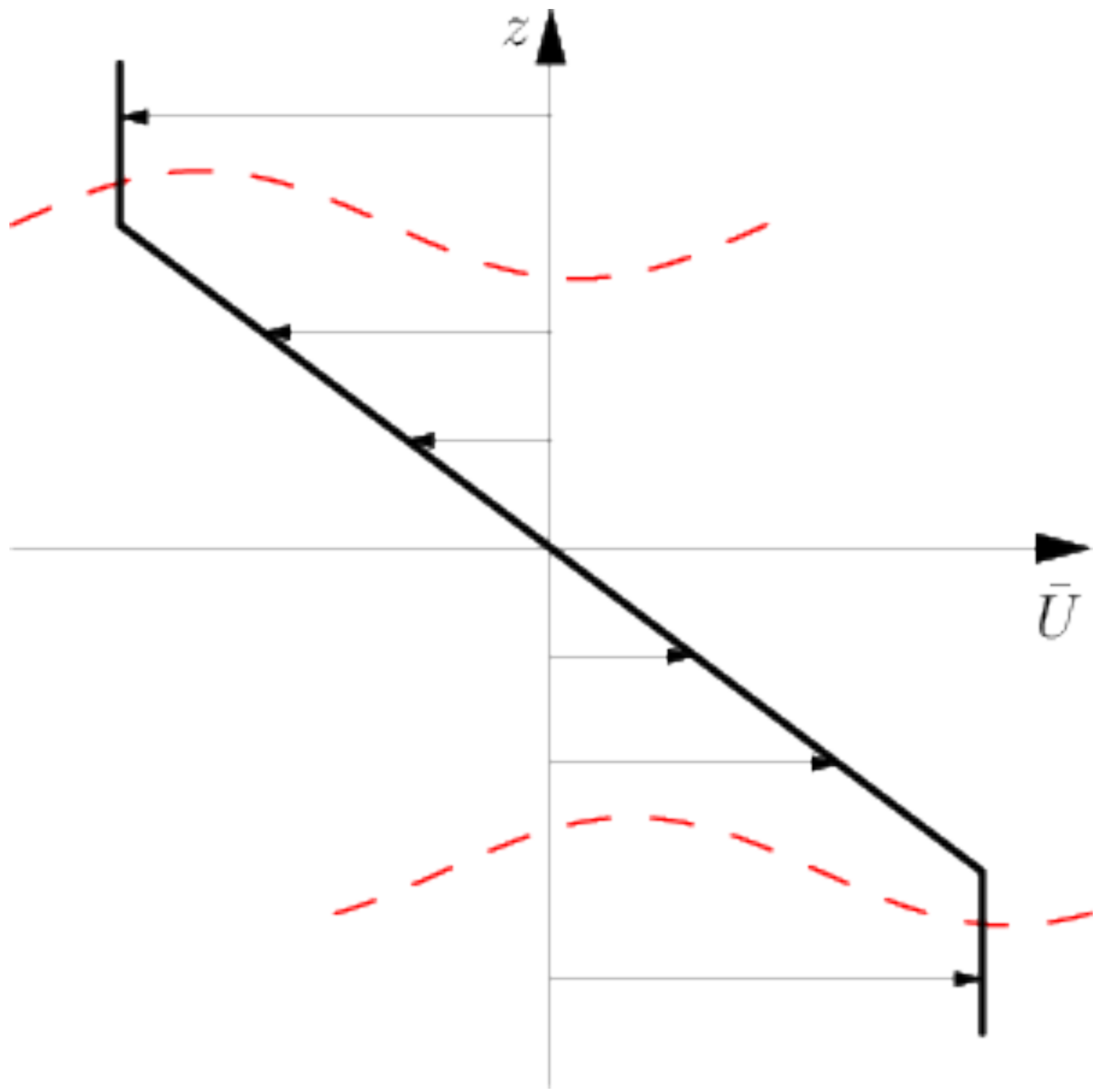

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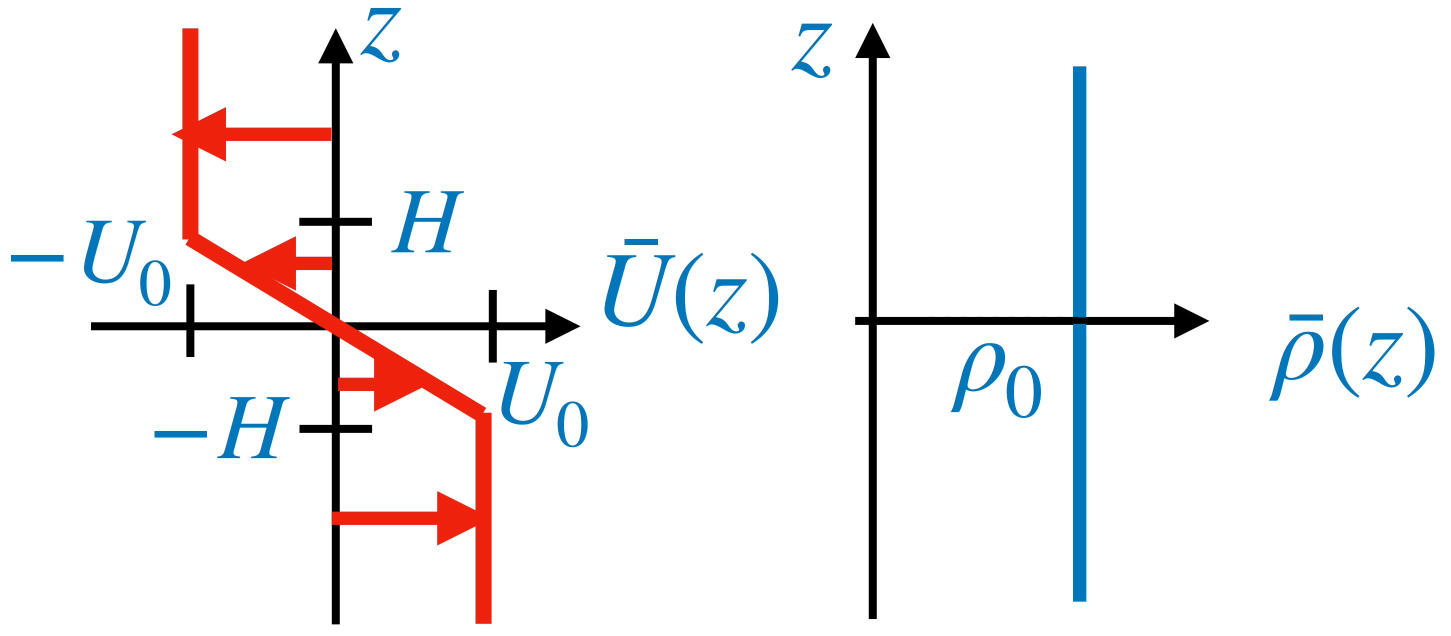

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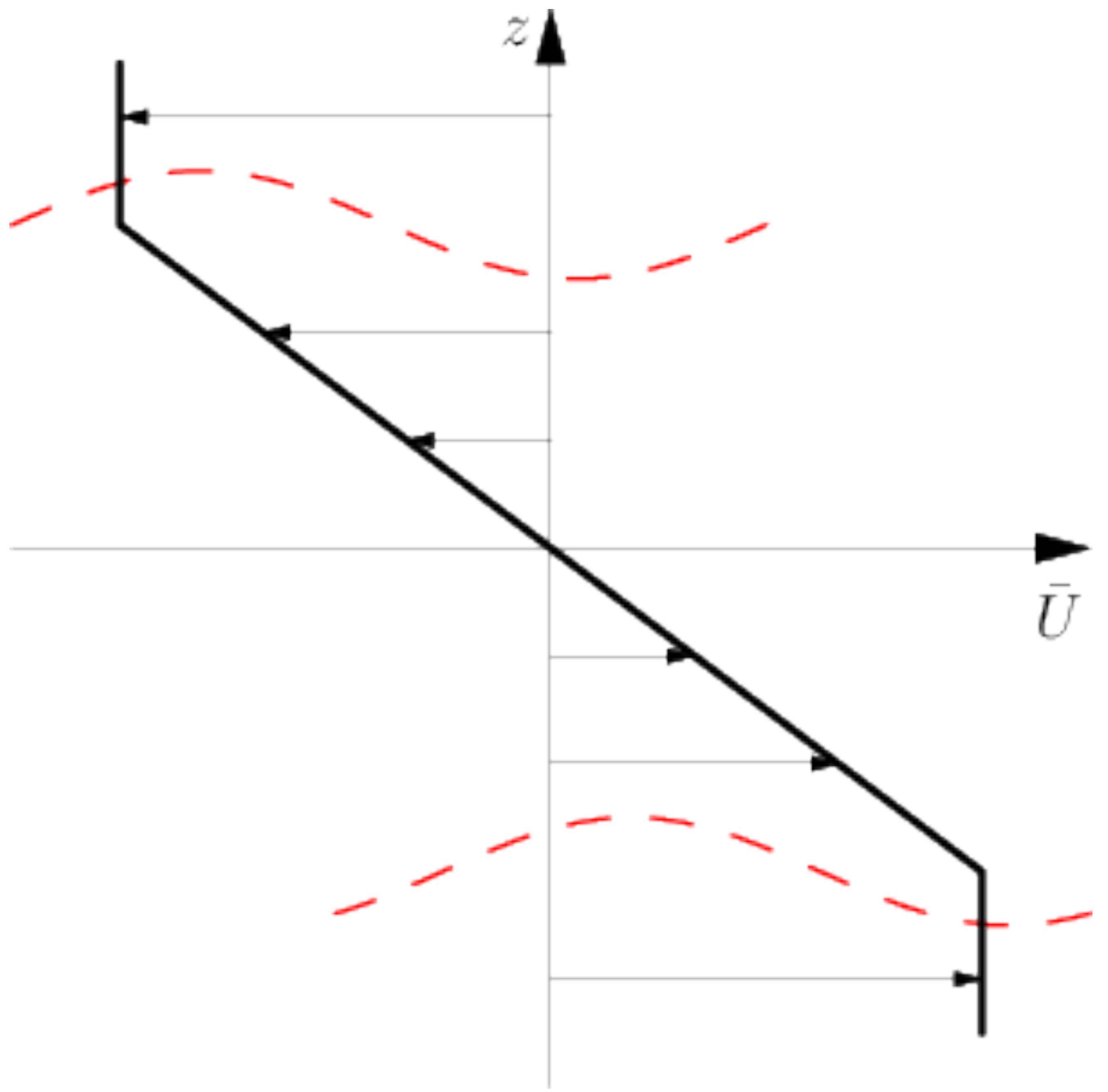

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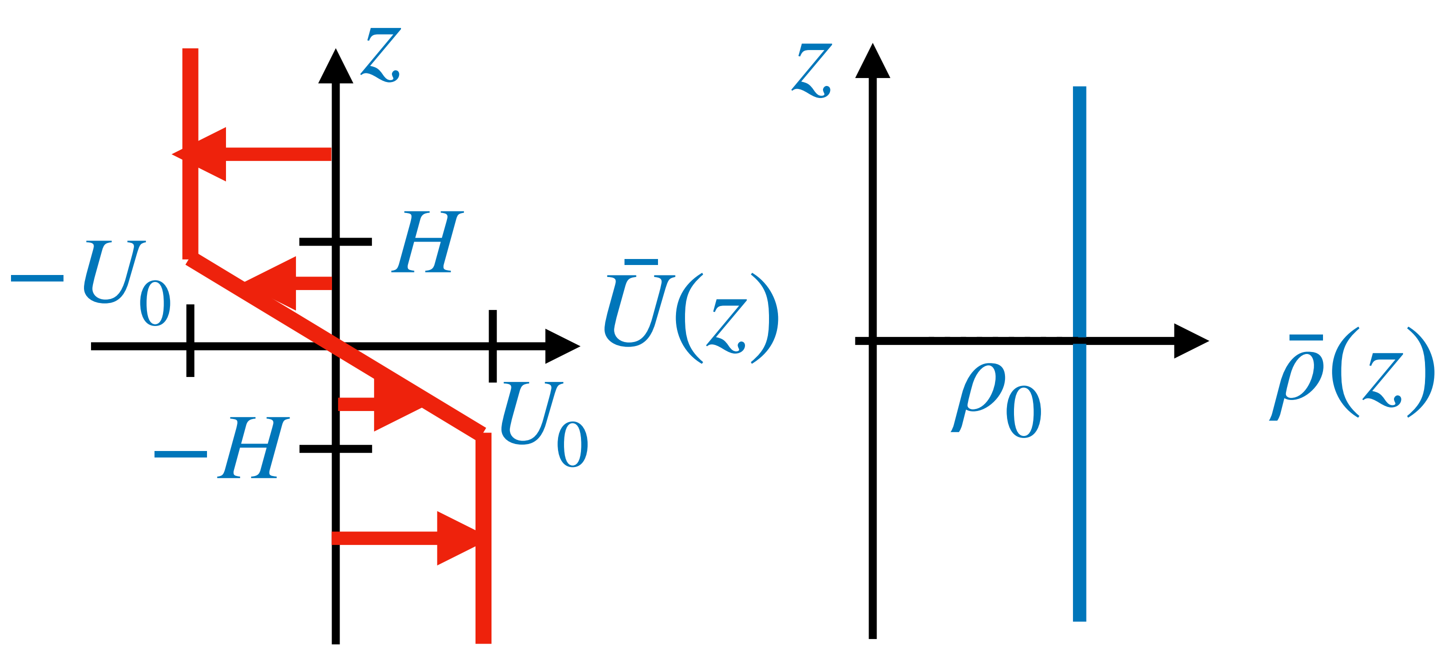

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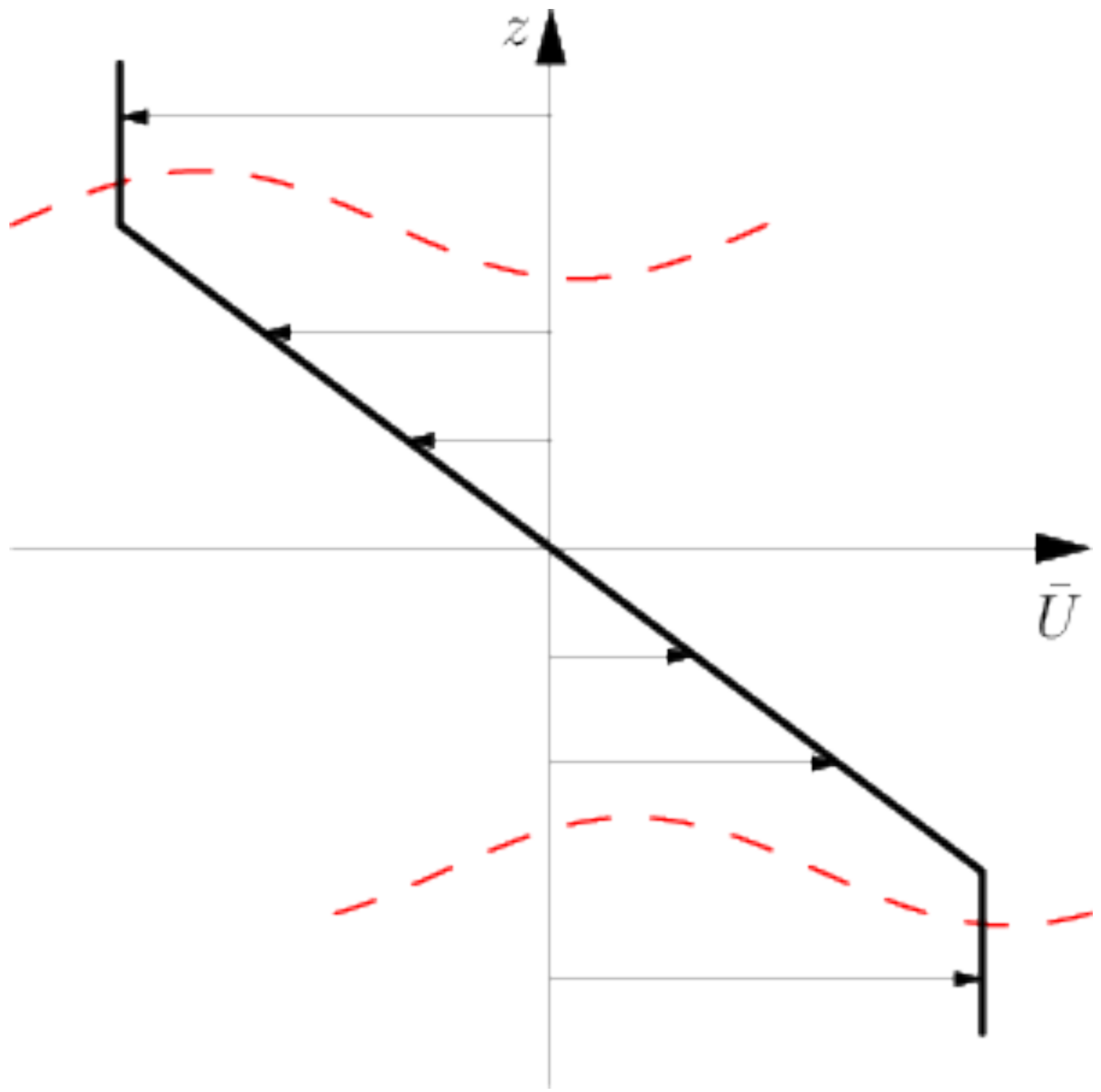

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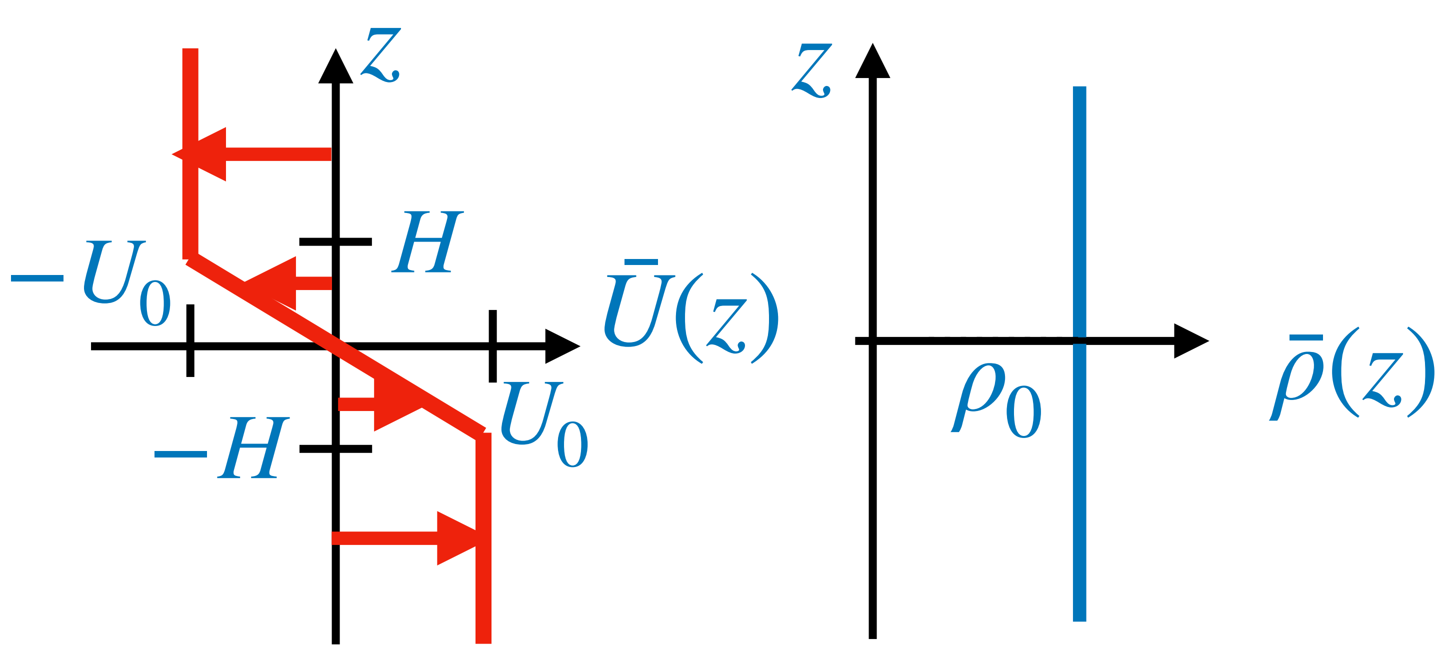

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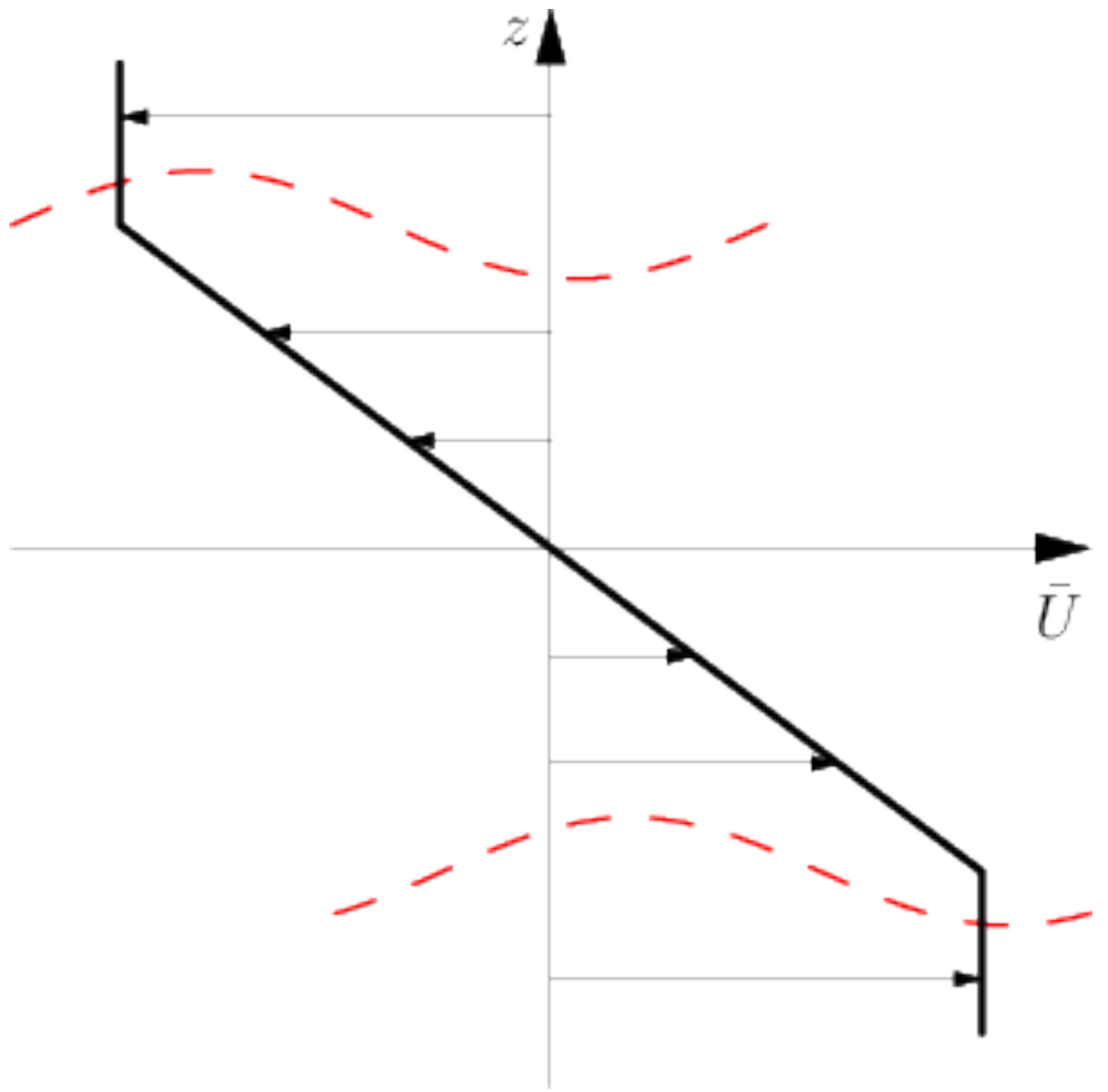

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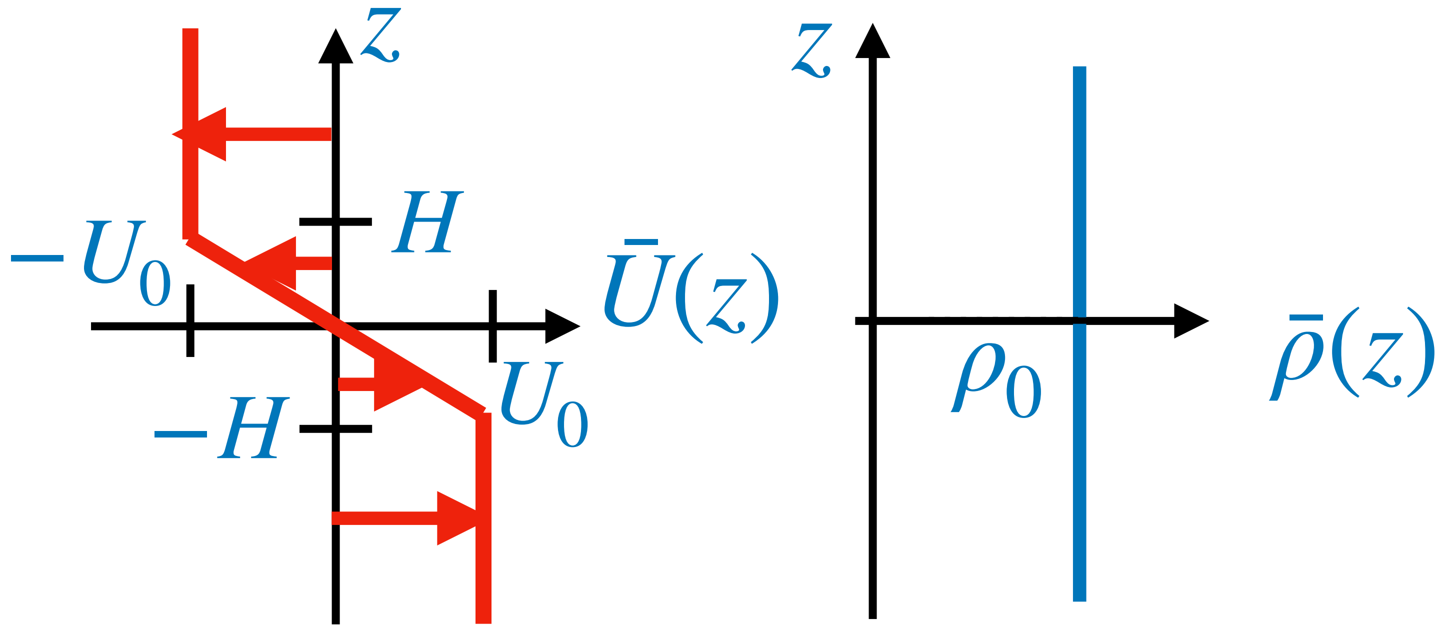

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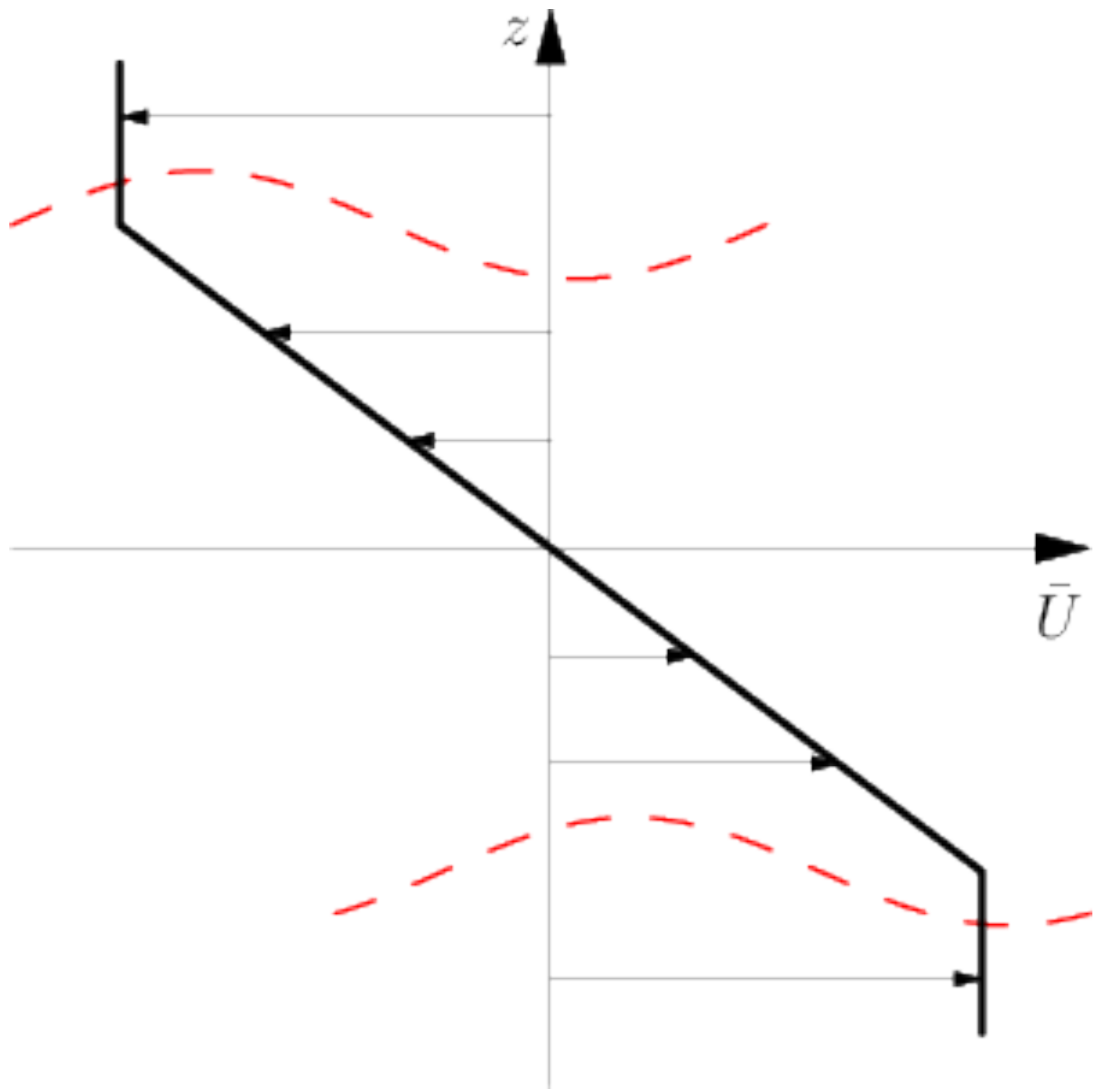

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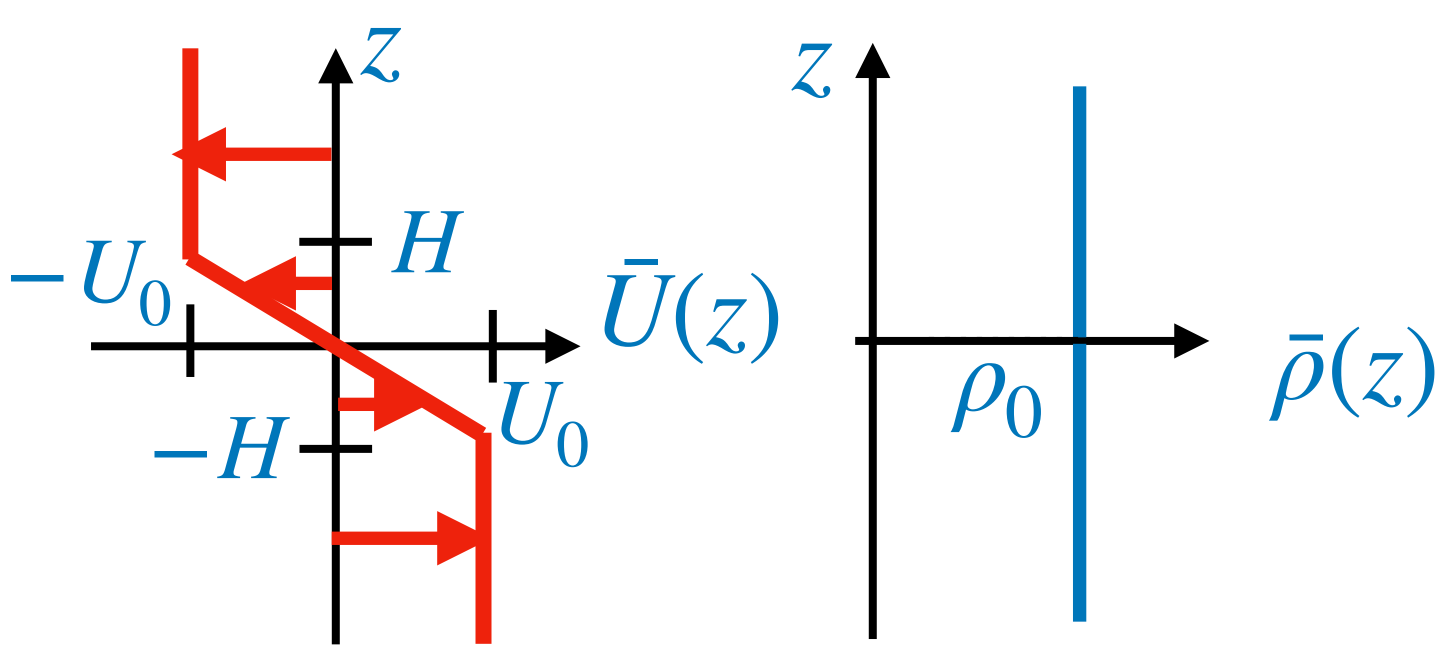

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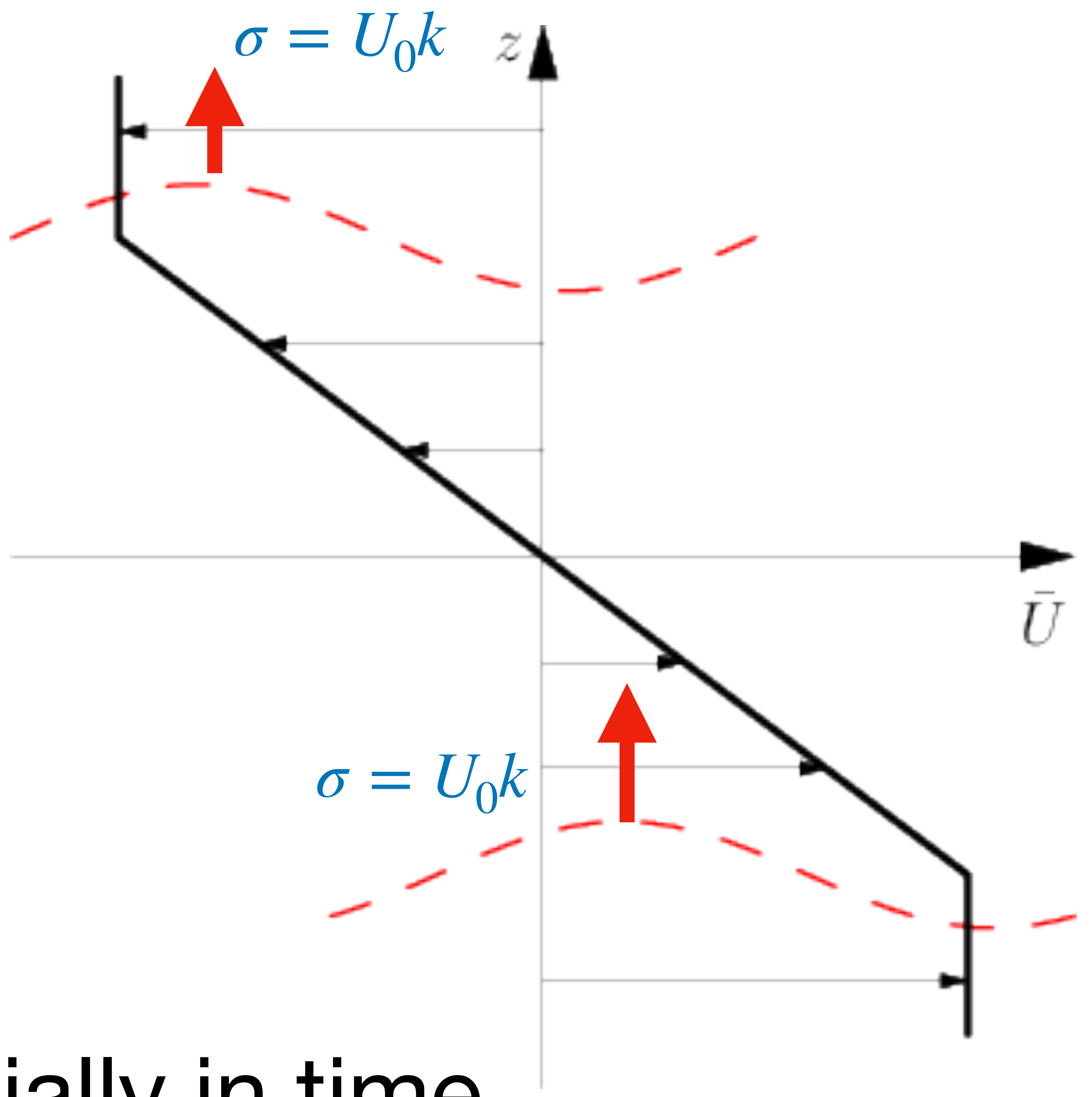

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So this corresponds to waves that decay or grow exponentially in time.

Kelvin-Helmholtz instability: most unstable mode

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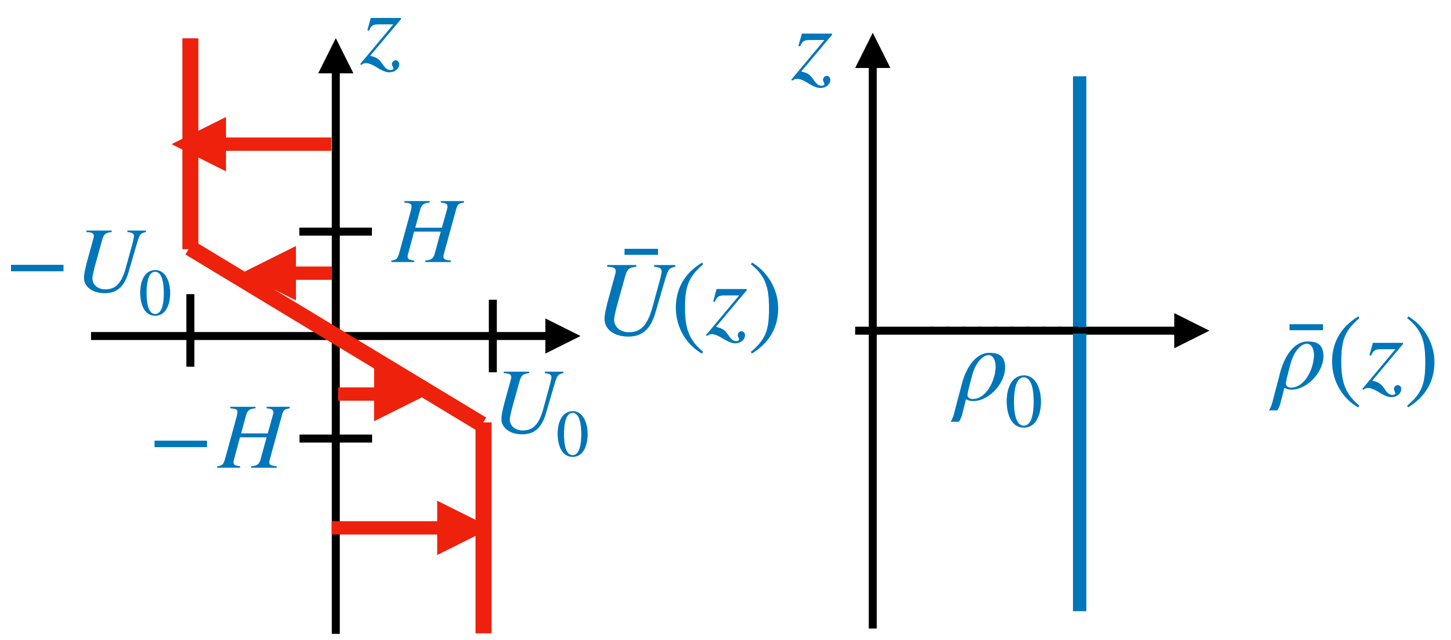

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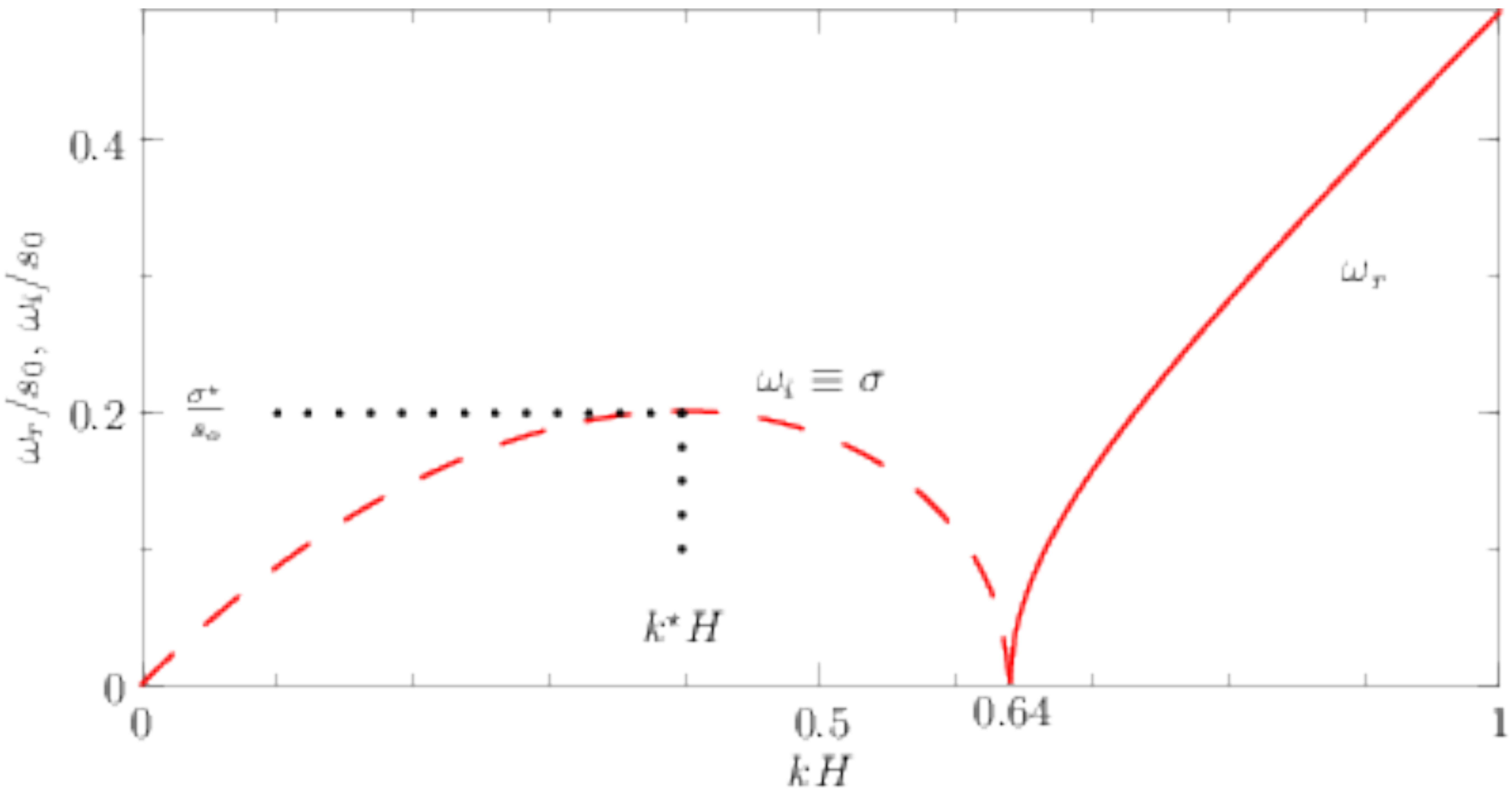

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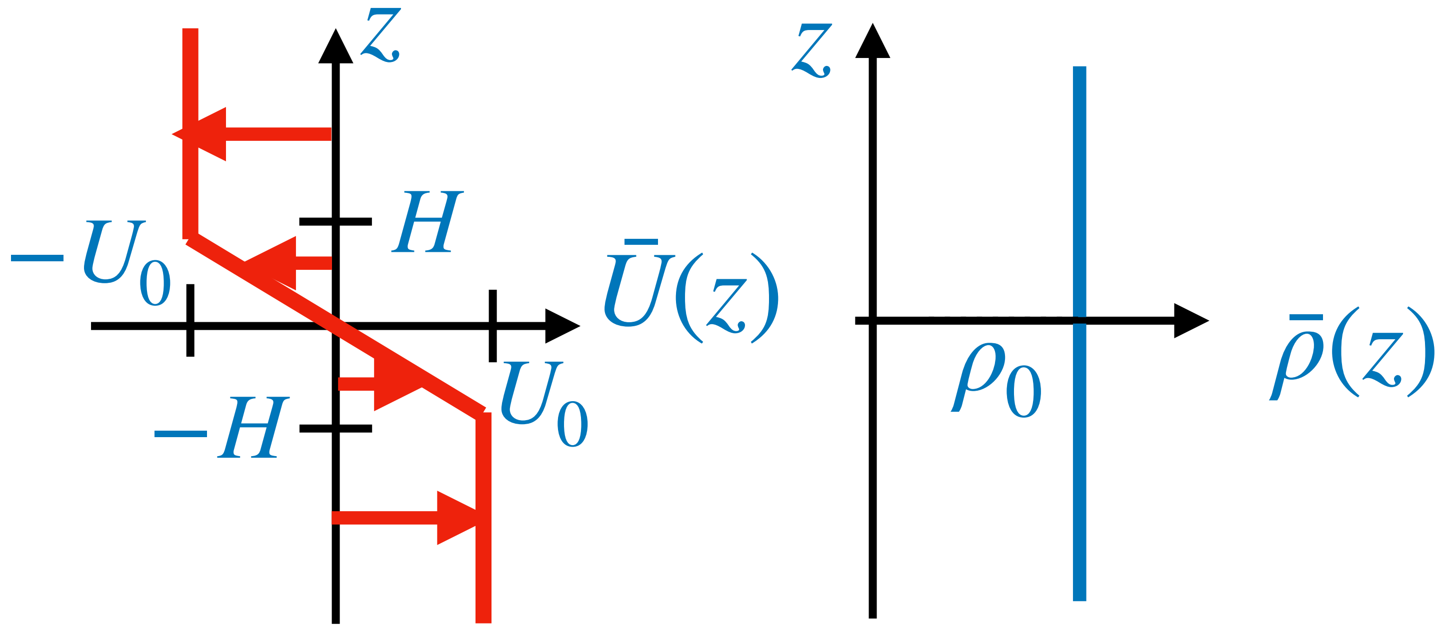

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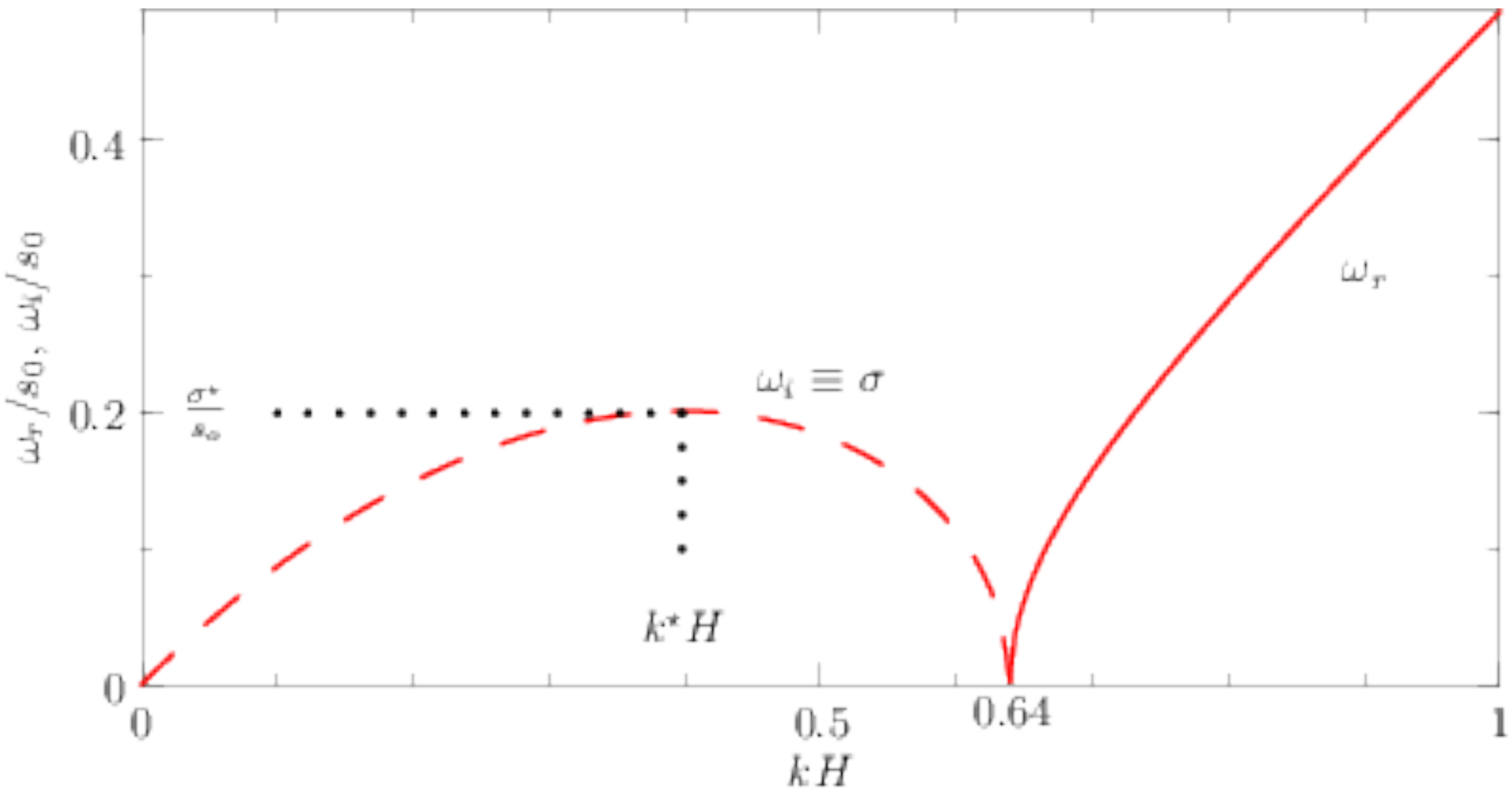

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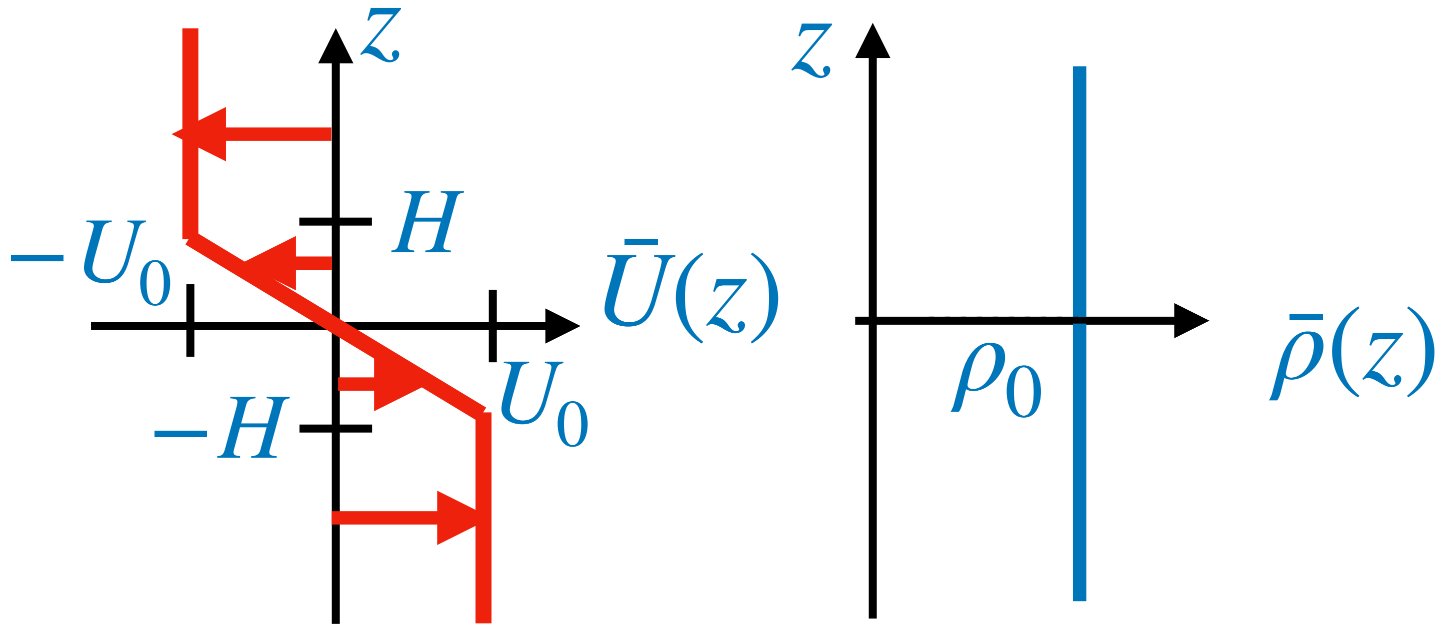

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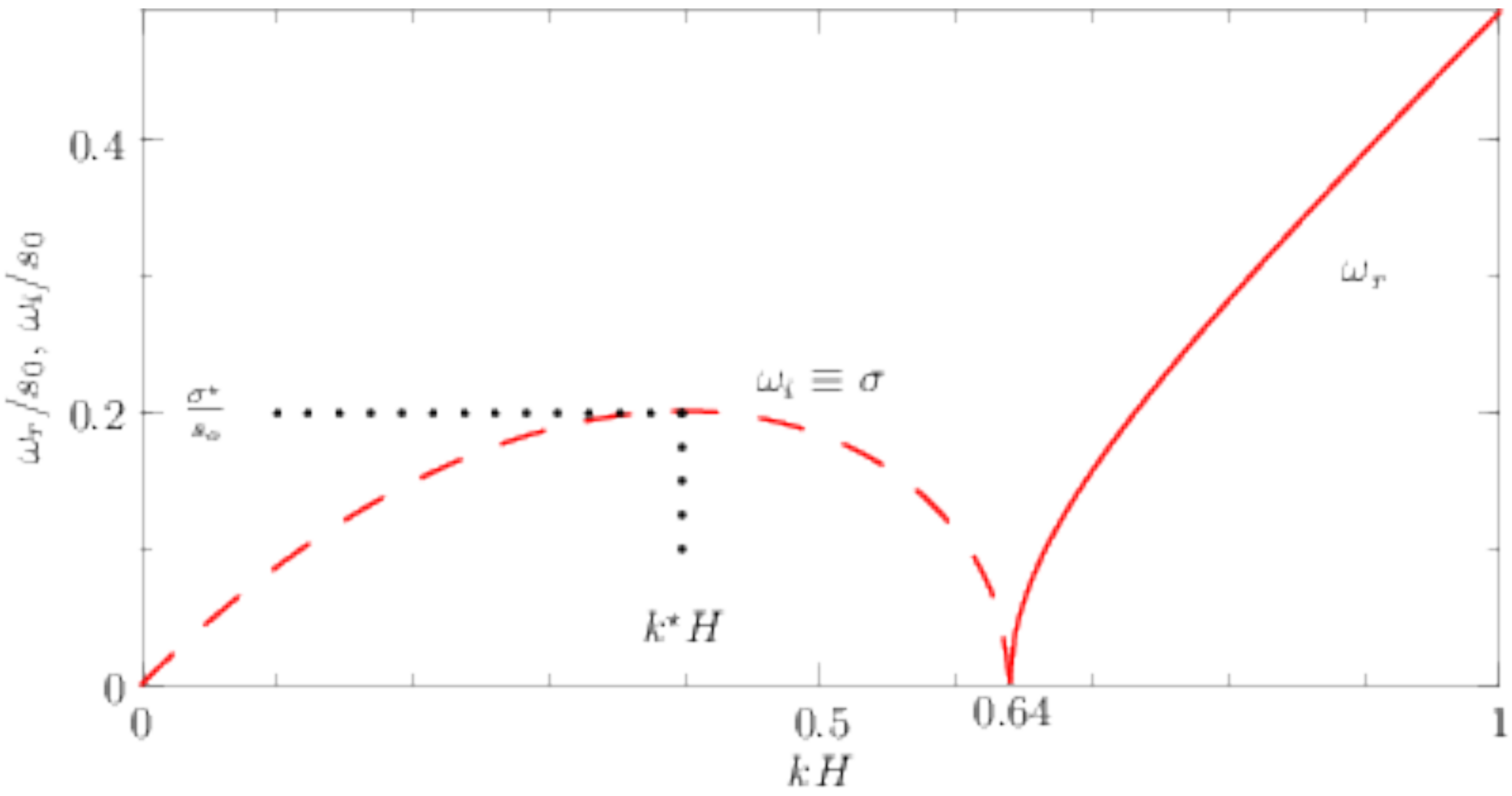

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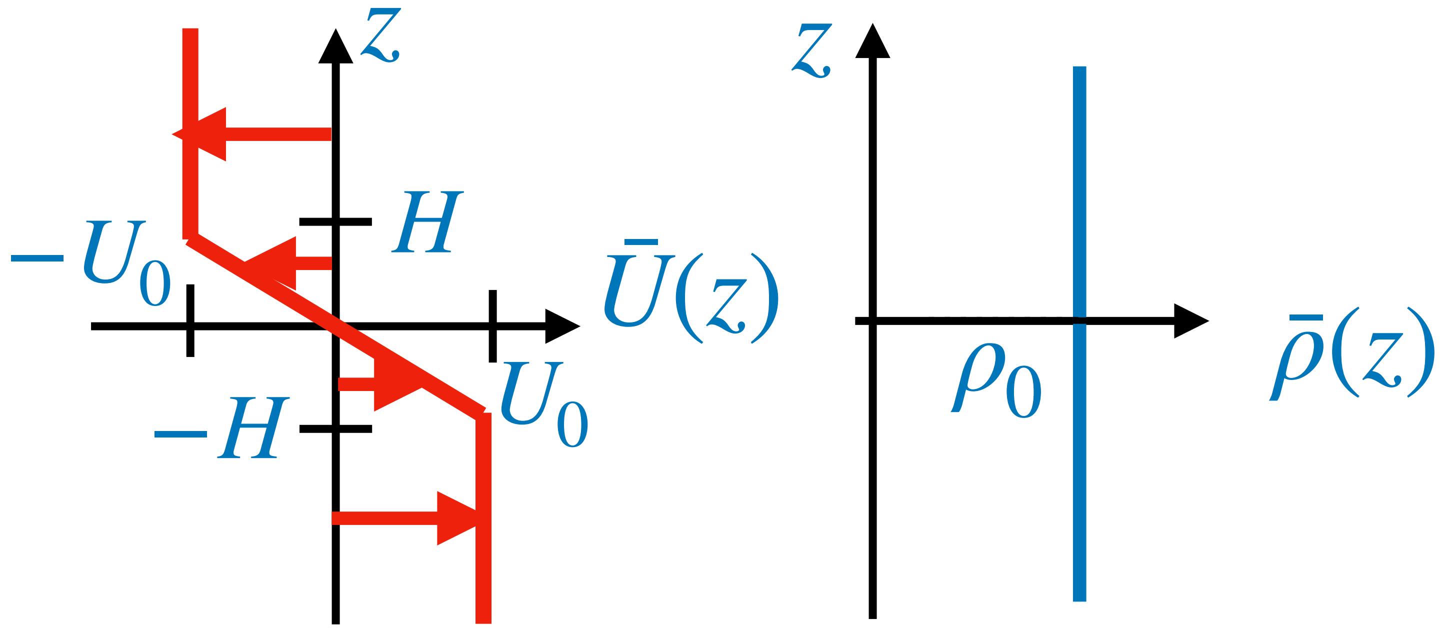

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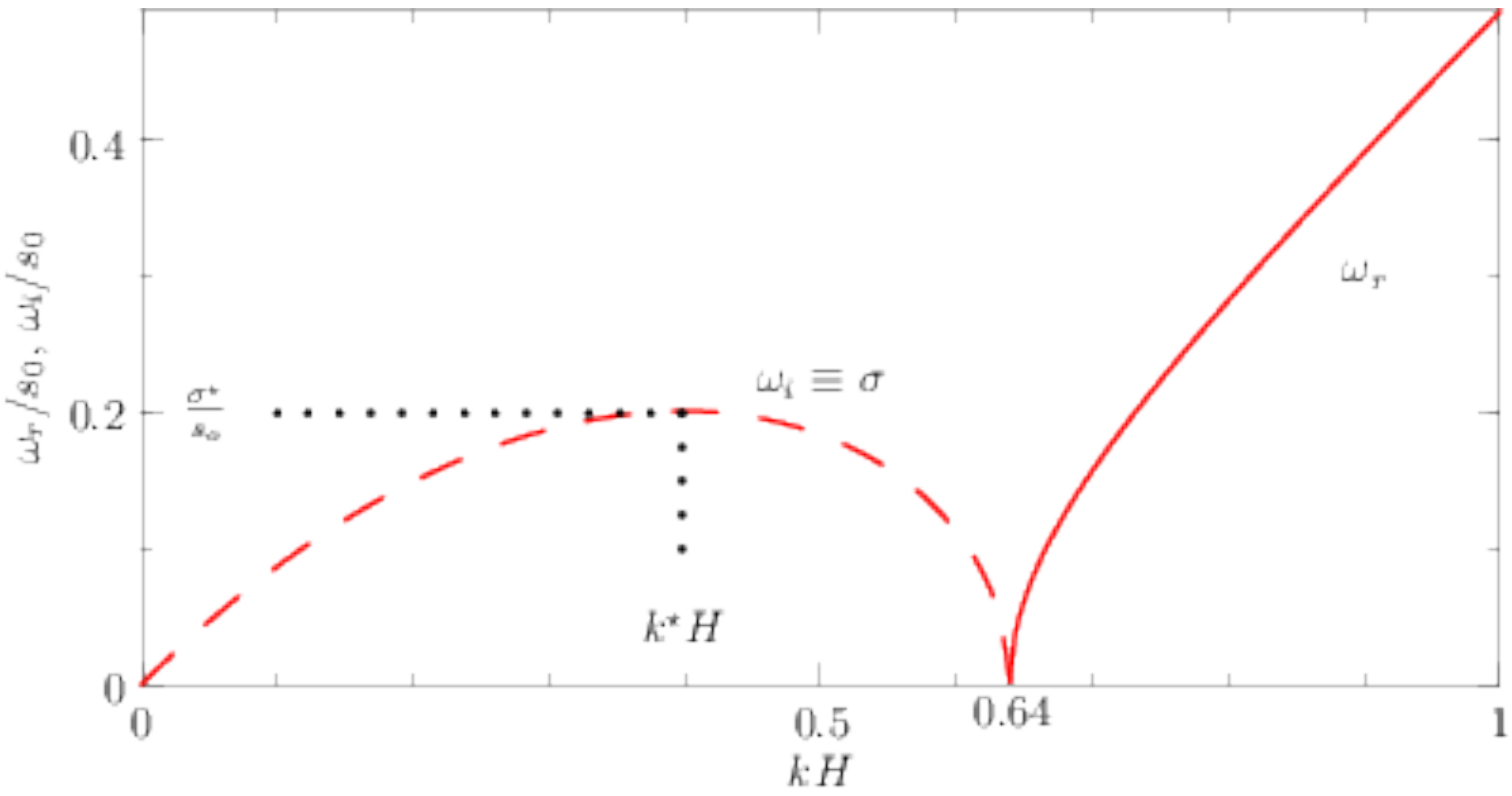

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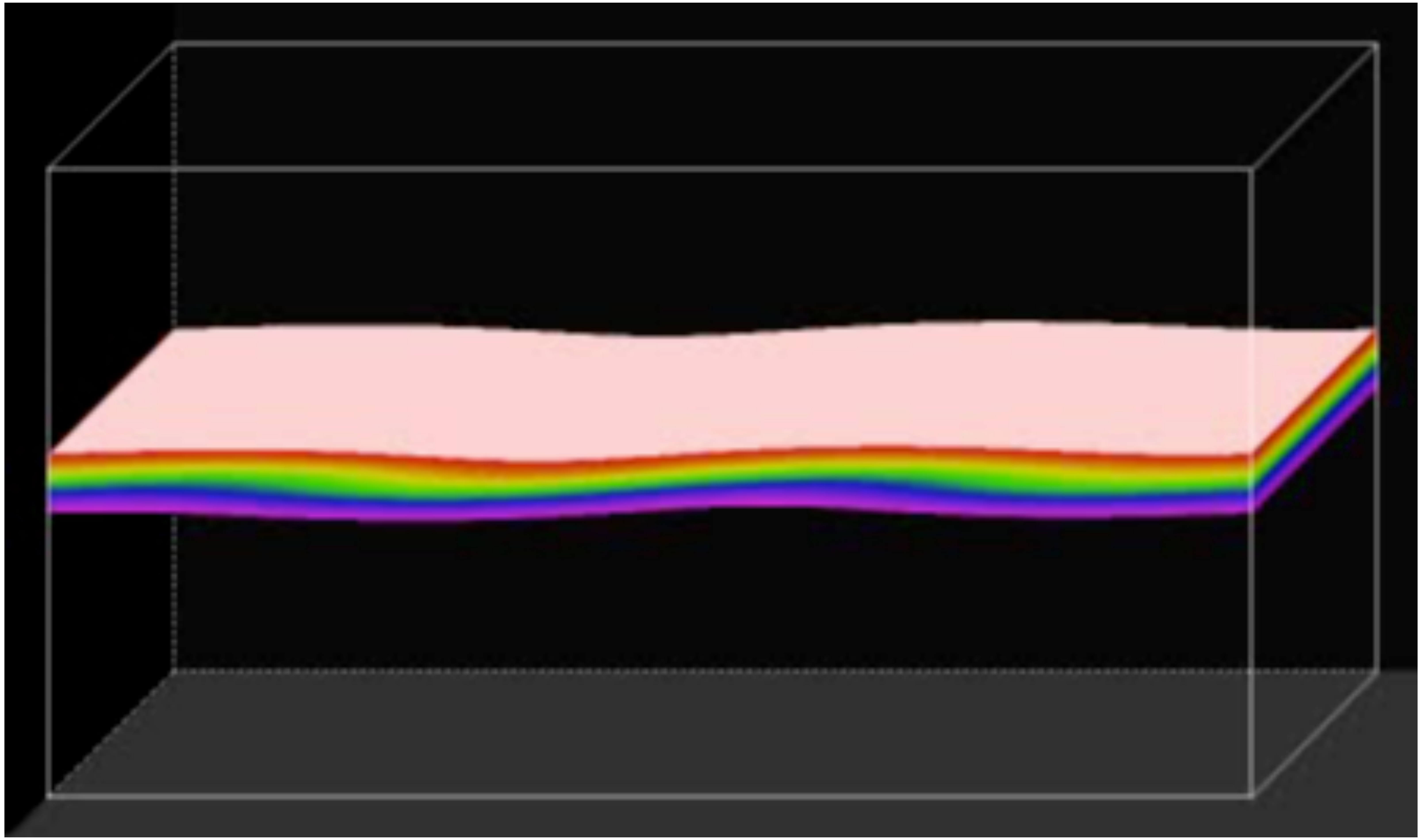
- And so we expect disturbances in background noise that grow fastest have wavenumber $k^\star \simeq 0.398/H$

$$\Rightarrow \quad \lambda^\star = 2\pi/k^\star \simeq 16H$$

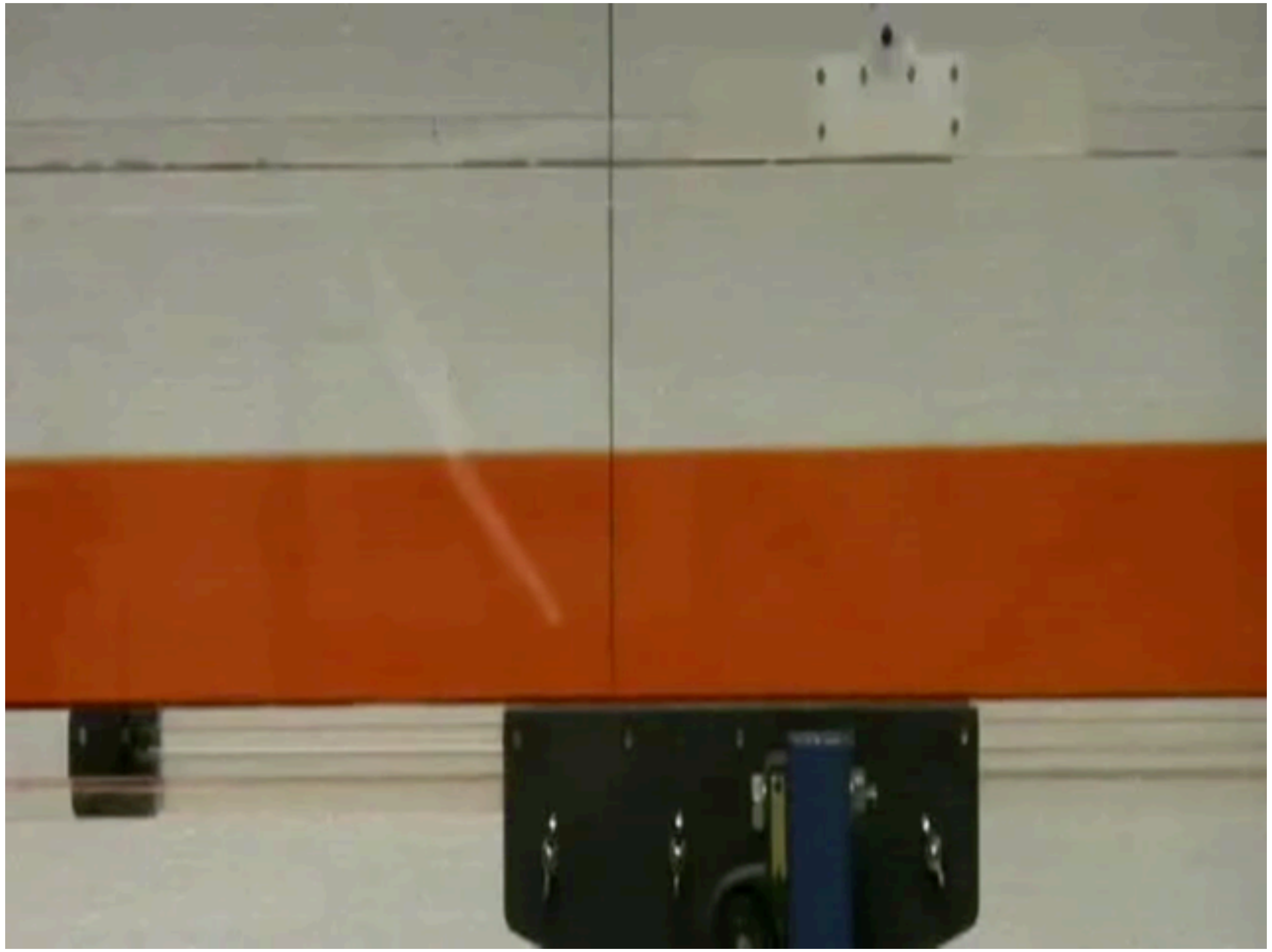
Kelvin-Helmholtz instability: simulation

- The "most unstable mode" with $\lambda^\star \simeq 16H$ is what you see grow out of a shear flow.
- As the waves continue to grow, nonlinear effects kick in, and the waves wrap up into vortices.



6.5] Kelvin-Helmholtz Instability in Stratified Fluid

- Previously we examined instability of a uniform density shear layer.
- Often shear develops across a density interface:



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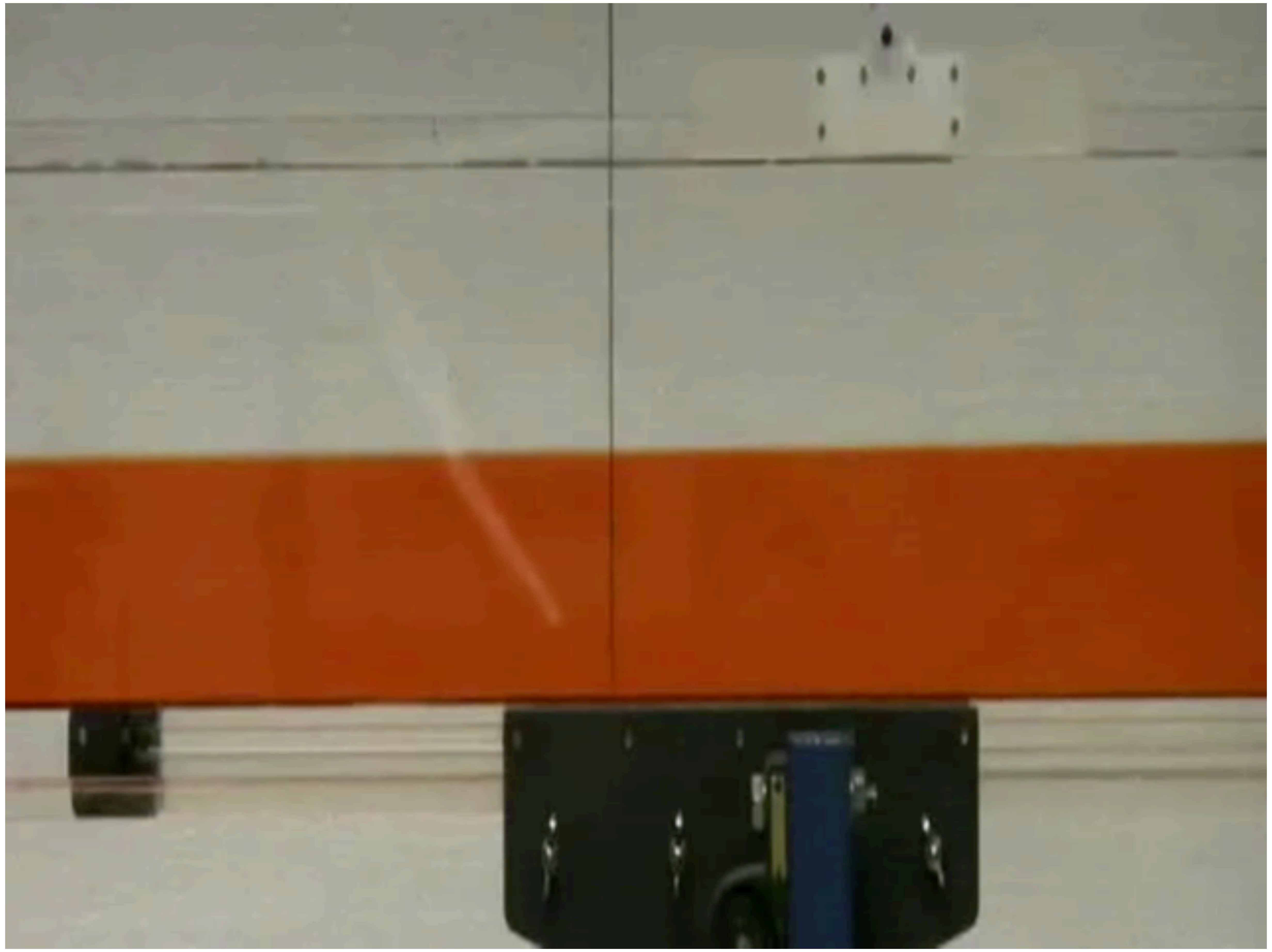

- Here, so we can find analytic solutions, we will assume the density is that of a 3-layer fluid.

Kelvin-Helmholtz Instability in a 3-layer fluid

- This results from resonantly coupled Rayleigh Waves in a shear layer:
  - $\bar{U}(z) = \quad \{-U_0,\ z > H;\ -s_0 z,\ |z| < H;\ U_0,\ z < -H\}$

  where $s_0 \equiv U_0/H$
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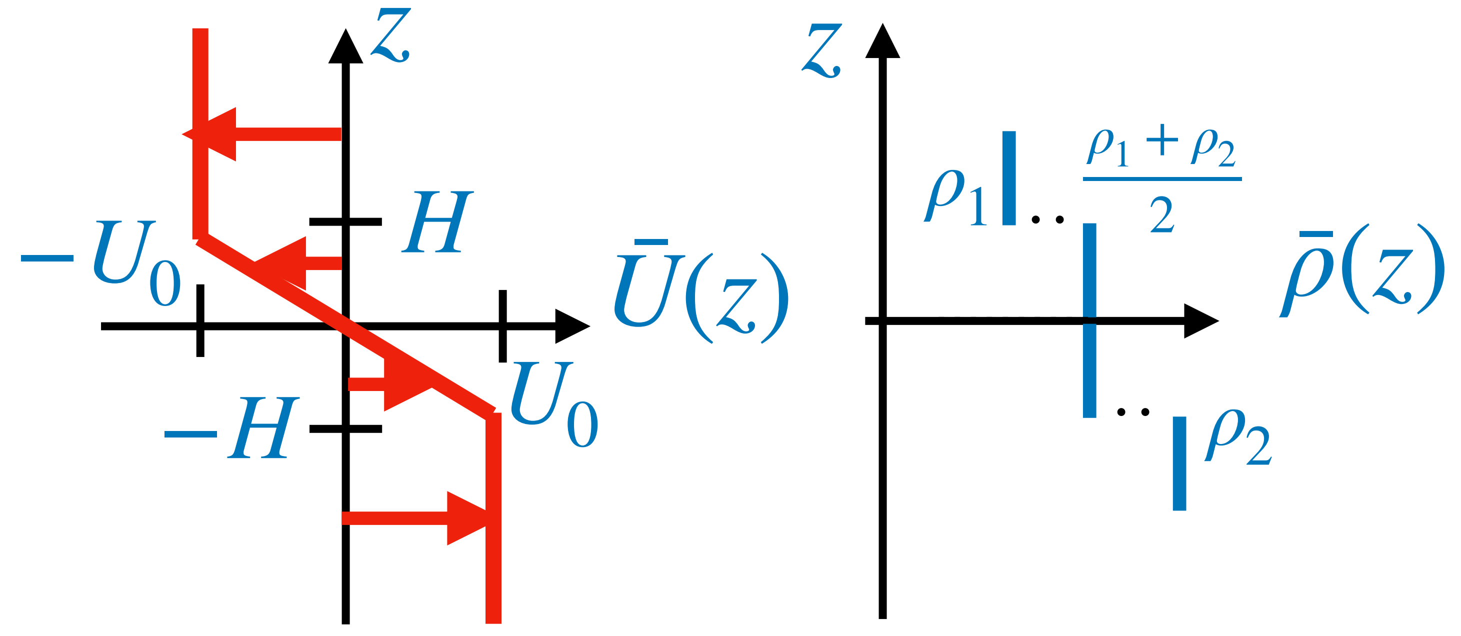

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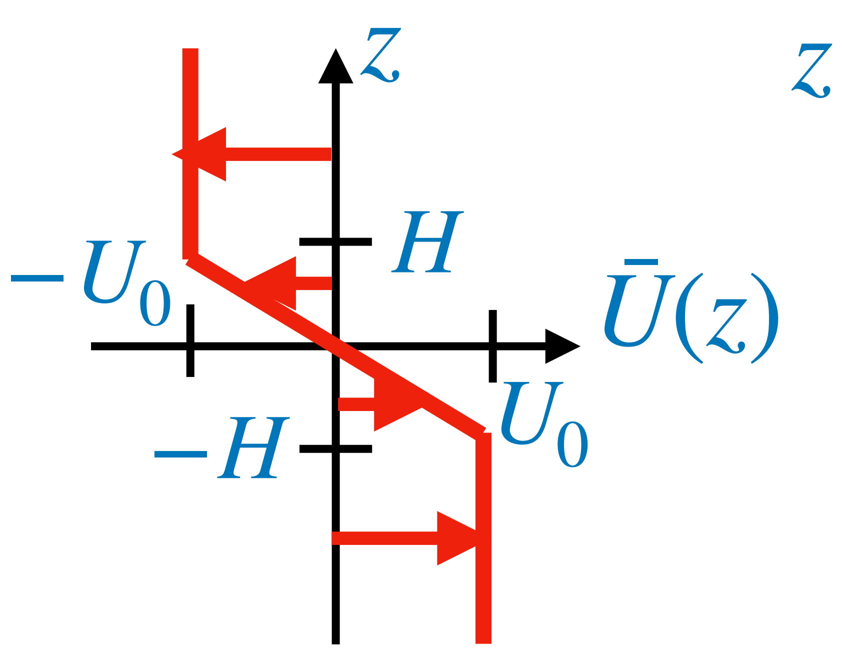

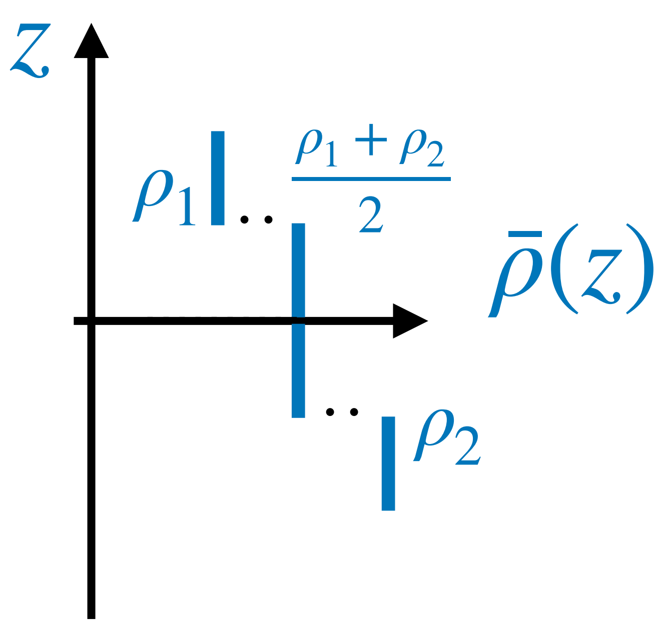

- As before, the vertical structure of the streamfunction is given generally by

$$\hat{\psi} = \begin{cases} \mathscr{A} e^{-kz} & z > H \\ \mathscr{B}_1 e^{kz} + \mathscr{B}_2 e^{-kz} & |z| < H \\ \mathscr{C} e^{kz} & z < -H \end{cases}$$

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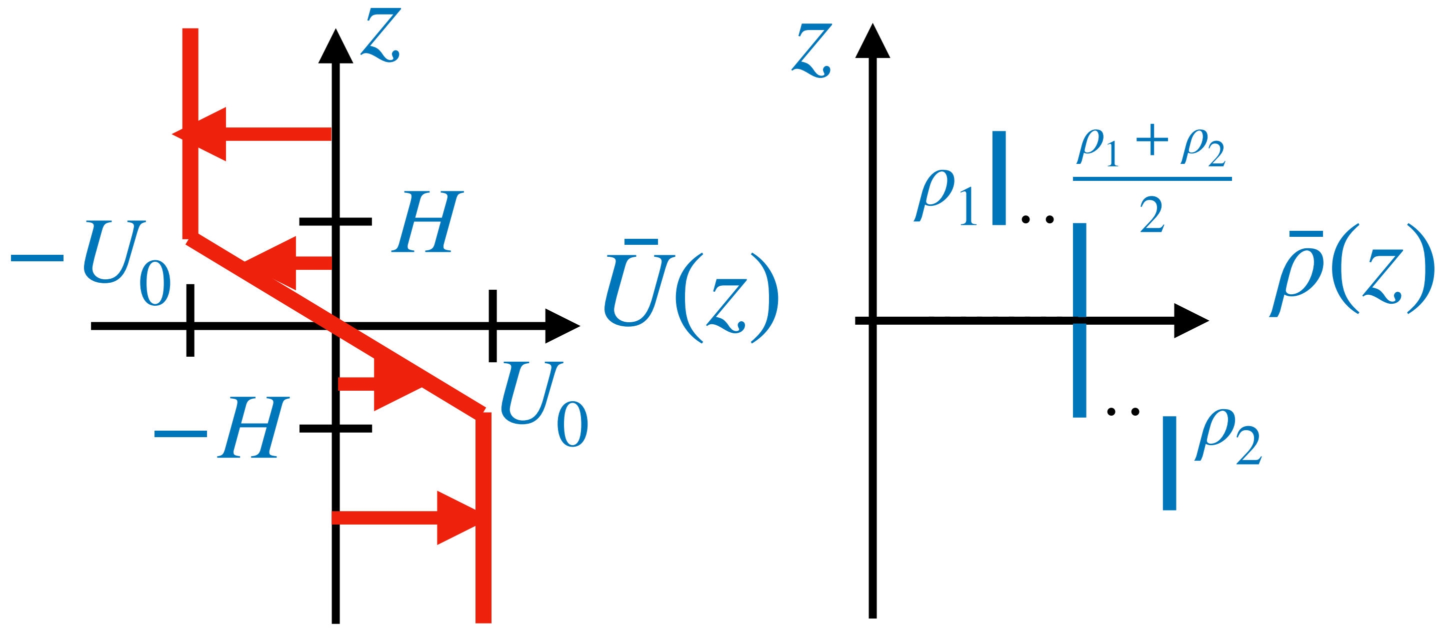

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  The interface conditions, accounting for the density jumps, lead to a more complicated dispersion relation:

$$\tilde{\omega}^4 - \frac{1}{4}\left[8\tilde{k}^2 - 4(1-\mathrm{Ri}_b)\tilde{k} + 1 - e^{-4\tilde{k}}\right]\tilde{\omega}^2 + \frac{1}{4}\tilde{k}^2\left[(2\tilde{k}-1-\mathrm{Ri}_b)^2 - (1+\mathrm{Ri}_b)^2 e^{-4\tilde{k}}\right] = 0$$

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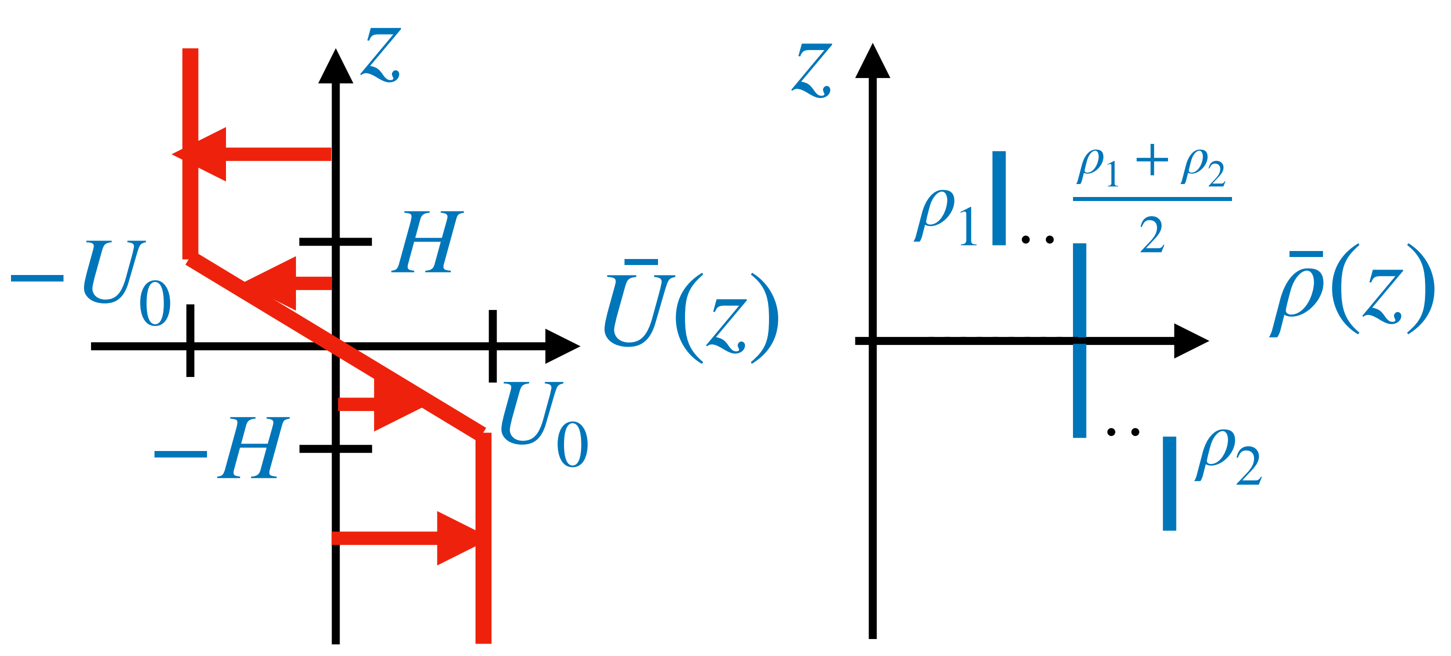

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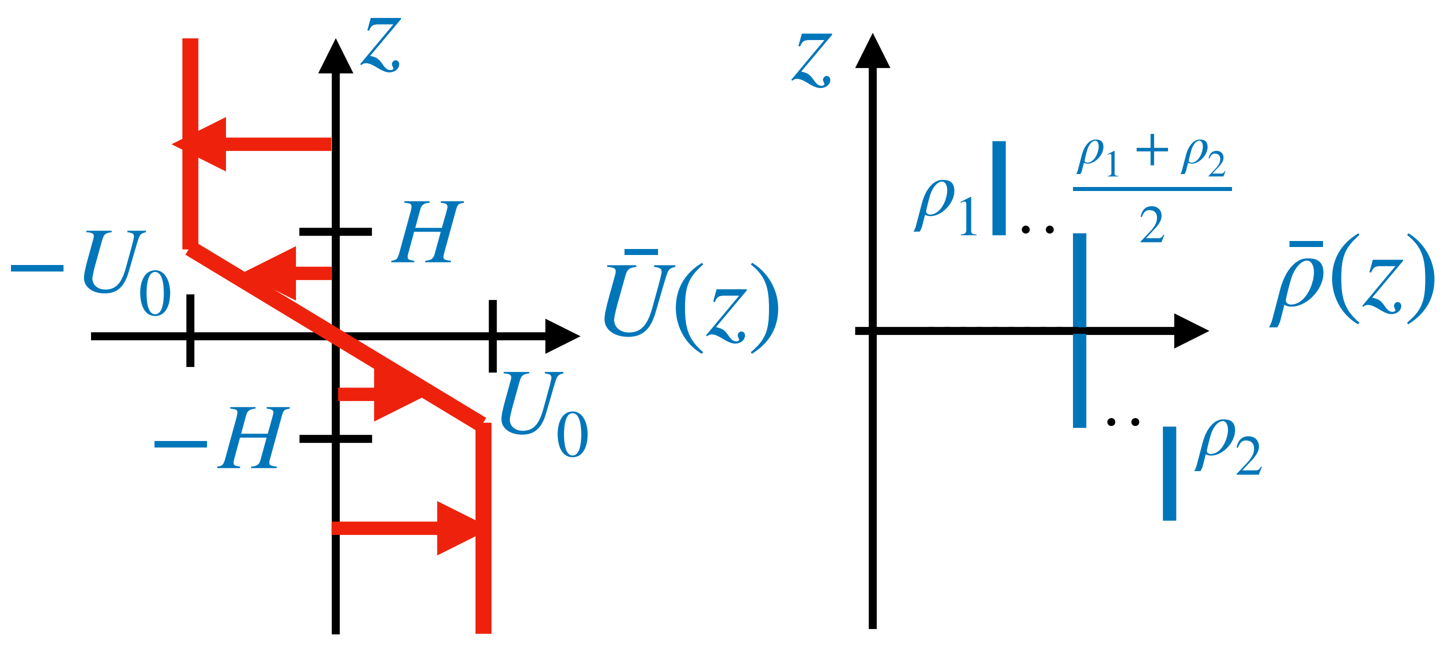

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$$\hat{\psi} = \begin{cases} \mathscr{A} e^{-kz} & z > H \\ \mathscr{B}_1 e^{kz} + \mathscr{B}_2 e^{-kz} & |z| < H \\ \mathscr{C} e^{kz} & z < -H \end{cases}$$

The interface conditions, accounting for the density jumps, lead to a more complicated dispersion relation:

$$\tilde{\omega}^4 - \frac{1}{4}\left[8\tilde{k}^2 - 4(1 - \mathrm{Ri}_b)\tilde{k} + 1 - e^{-4\tilde{k}}\right]\tilde{\omega}^2 + \frac{1}{4}\tilde{k}^2\left[(2\tilde{k} - 1 - \mathrm{Ri}_b)^2 - (1 + \mathrm{Ri}_b)^2\, e^{-4\tilde{k}}\right] = 0$$

with $\tilde{\omega} = \omega/s_0,\ \ \tilde{k} = kH,\ \ \mathrm{Ri}_b \equiv (g'/H)/s_0^2$, and $g' = g(\rho_2 - \rho_1)/\left[(\rho_2 + \rho_1)/2\right]$

The “Bulk Richardson Number”

Kelvin-Helmholtz Instability in a 3-layer fluid

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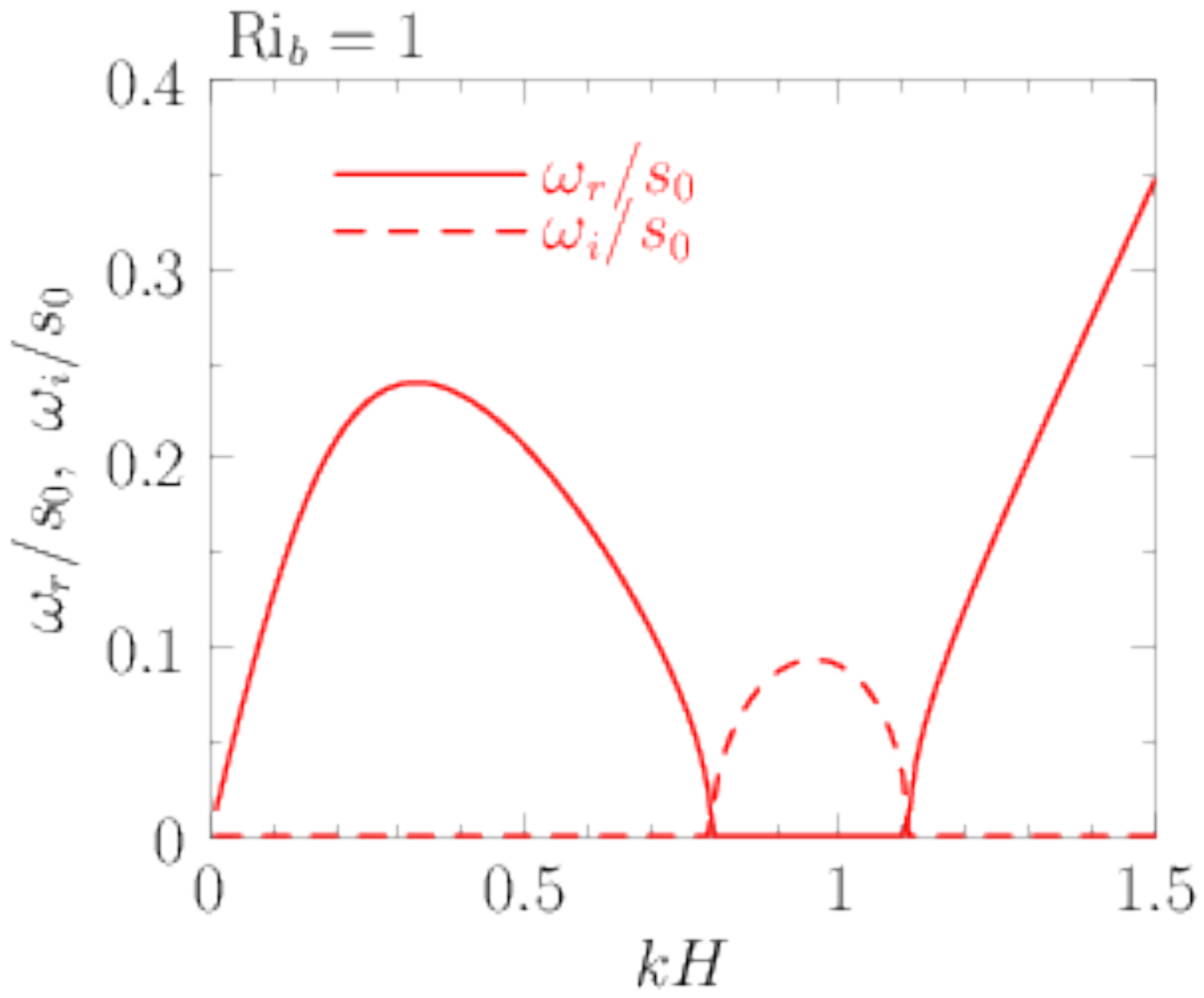

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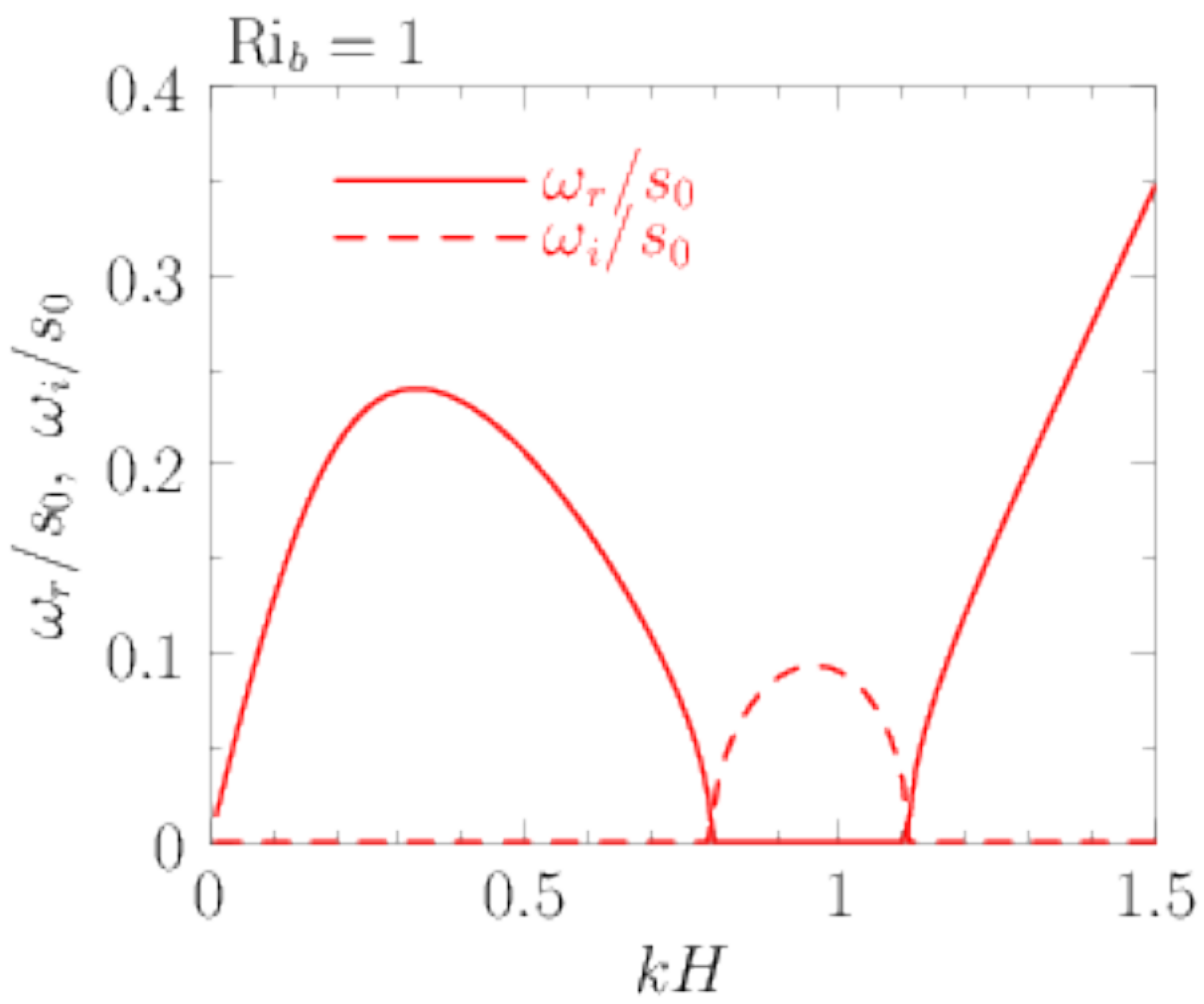
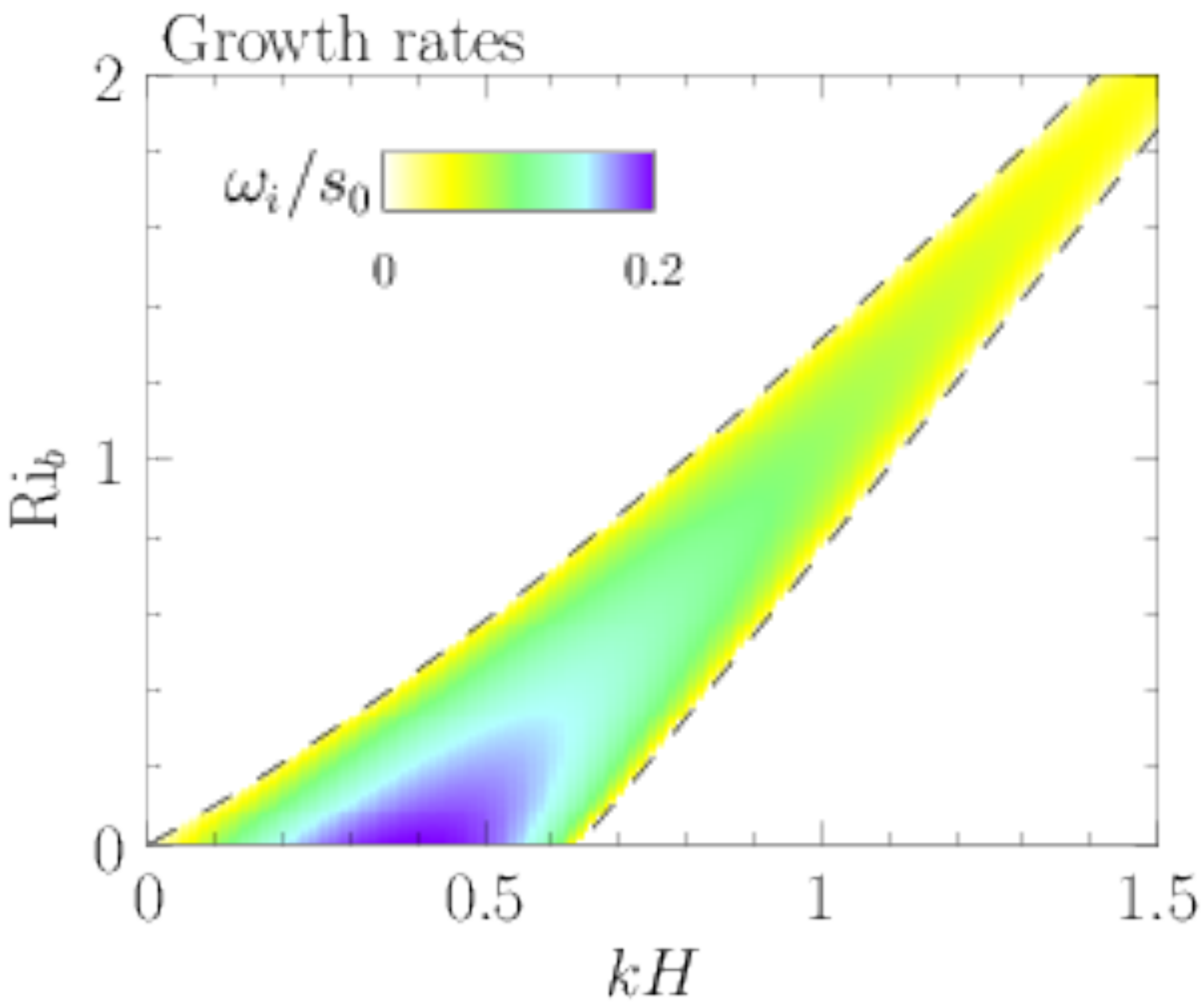
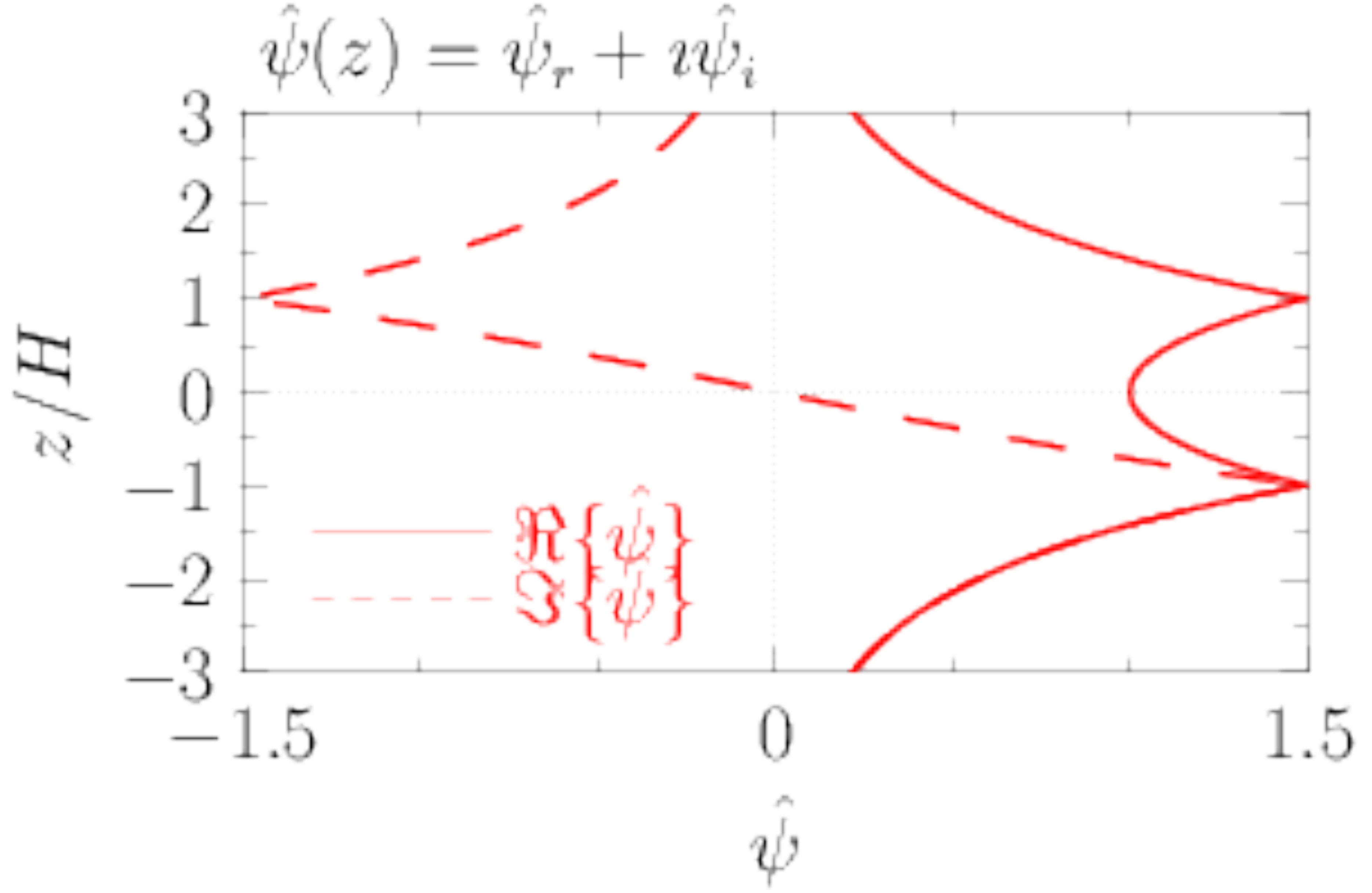




Kelvin-Helmholtz Instability in a 3-layer fluid

- Now examine the eigenfunction of the most unstable mode with $\mathrm{Ri}_b = 1$



Kelvin-Helmholtz Instability in a 3-layer fluid

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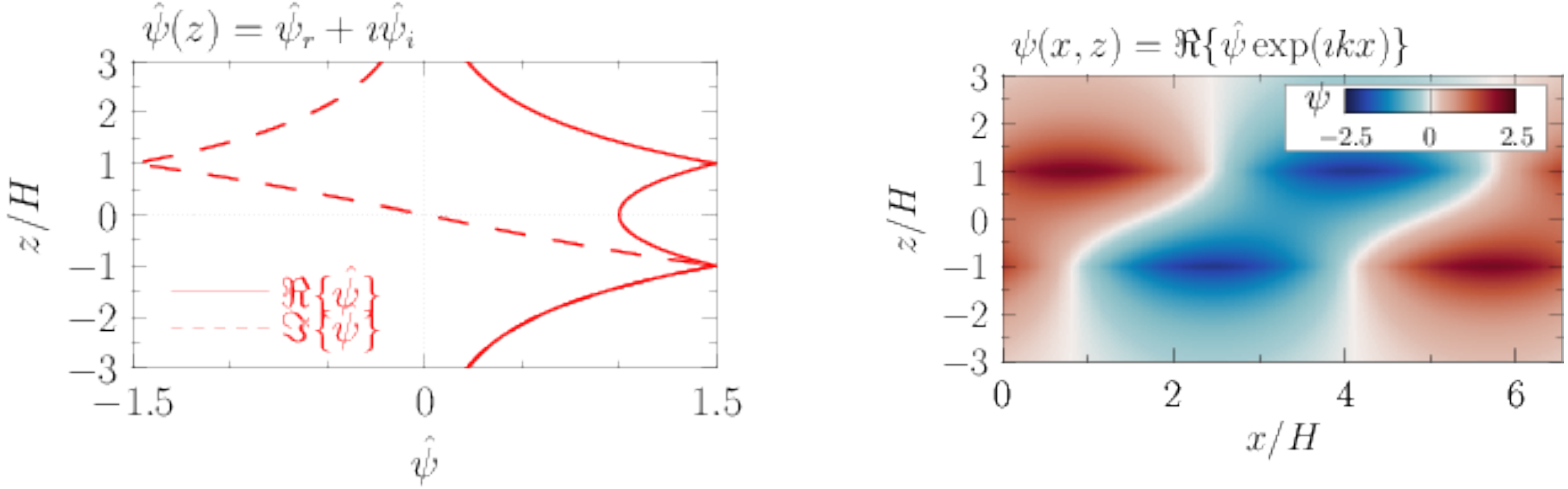

Kelvin-Helmholtz Instability in a 3-layer fluid

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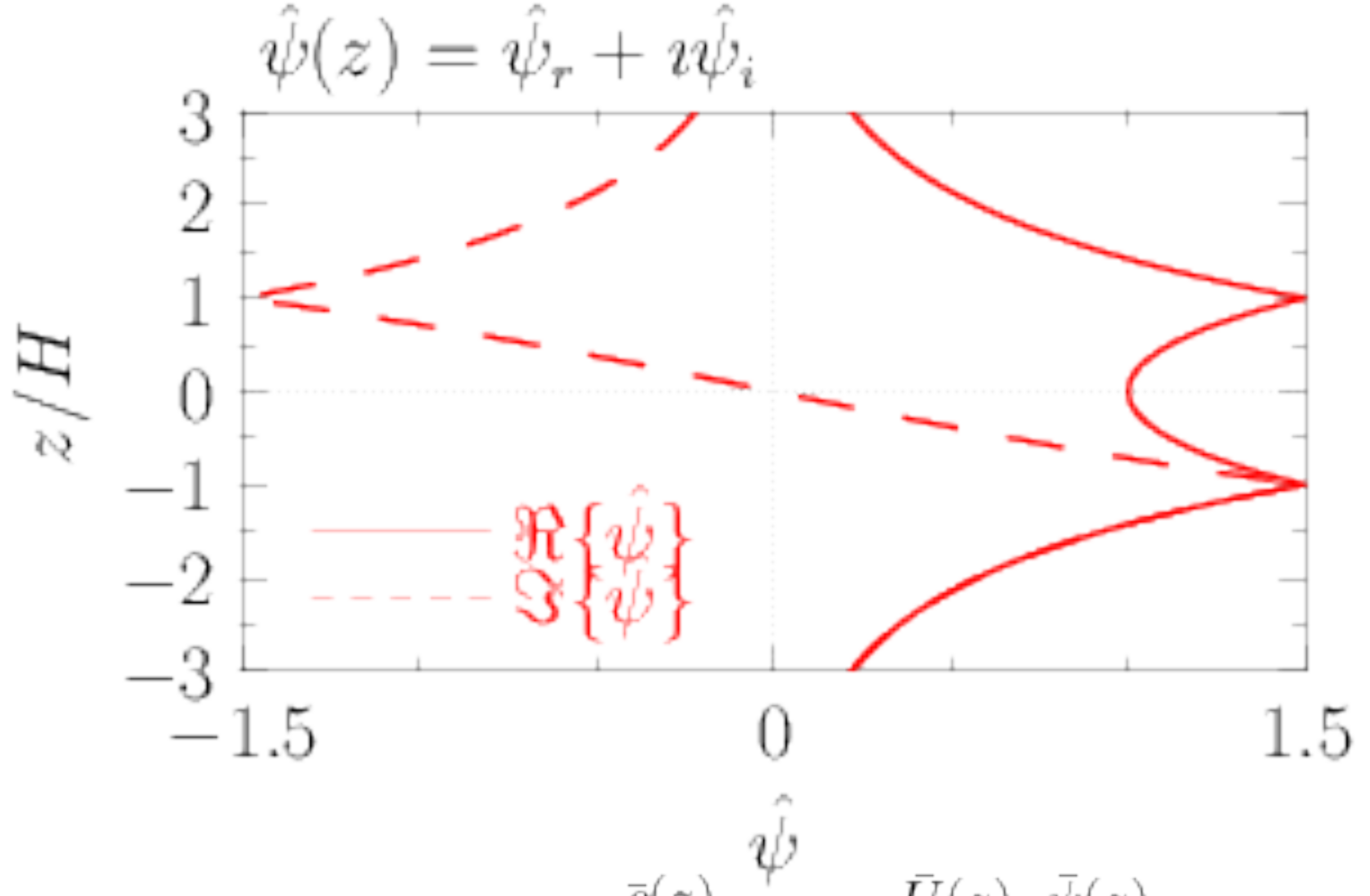

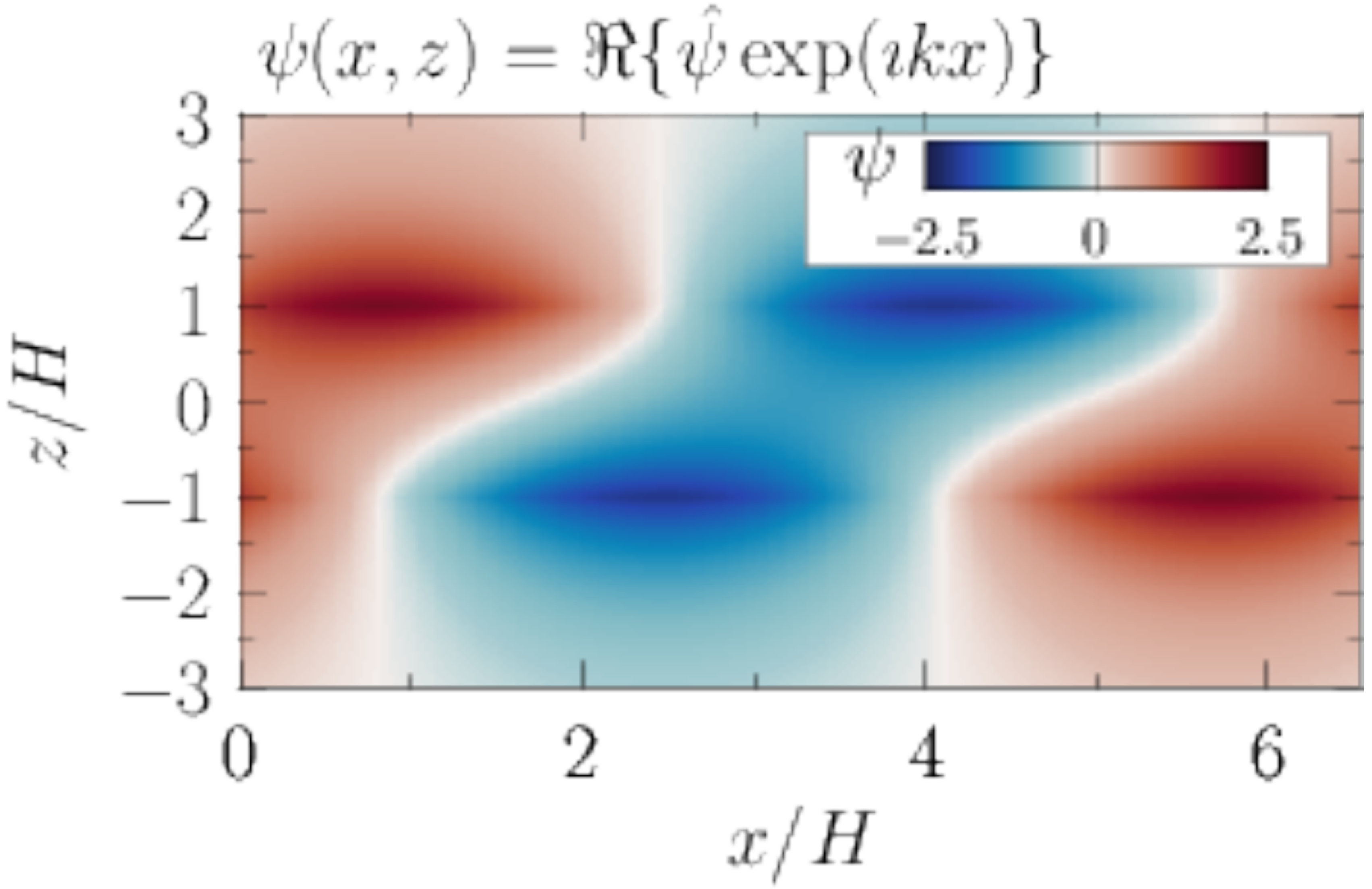

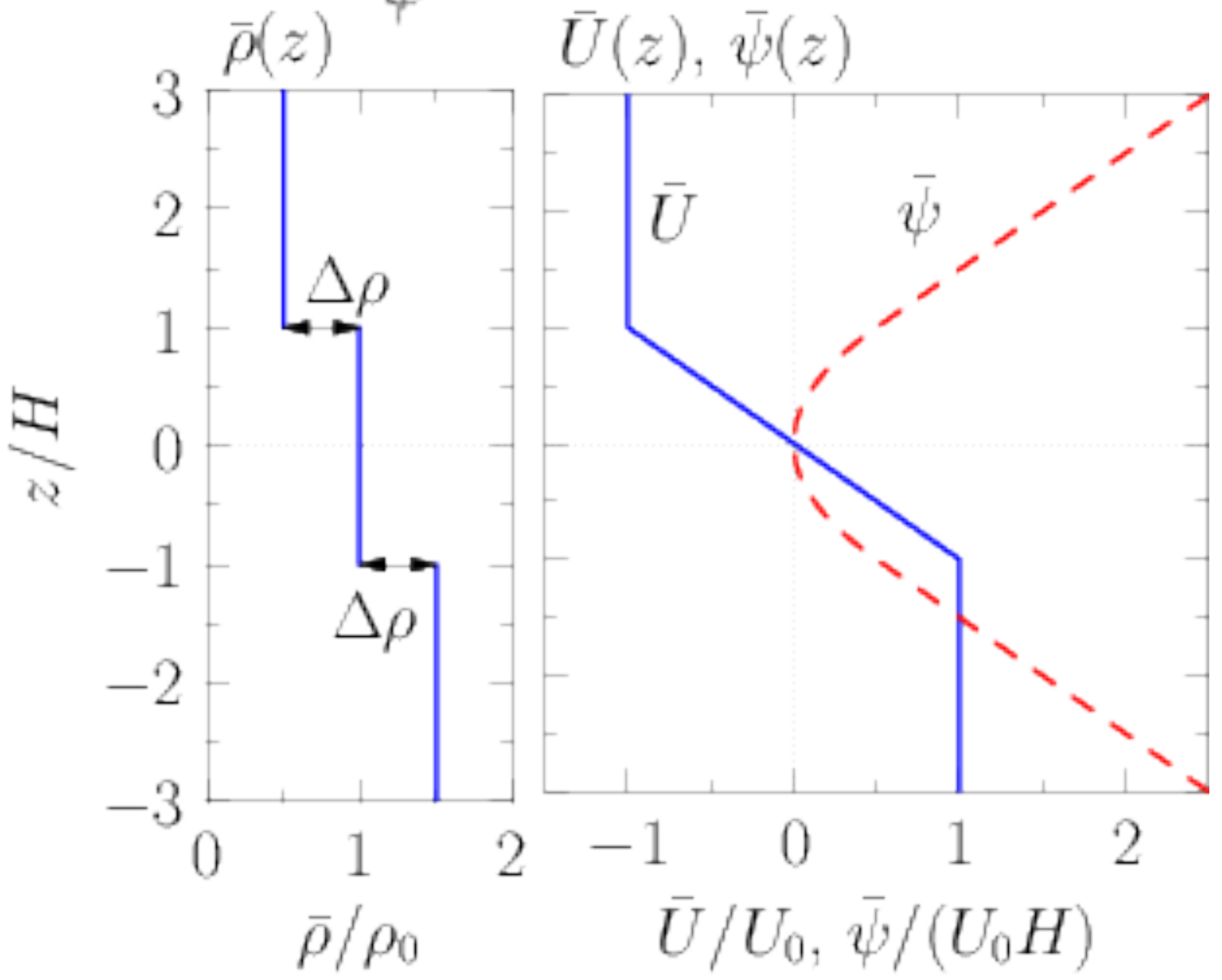

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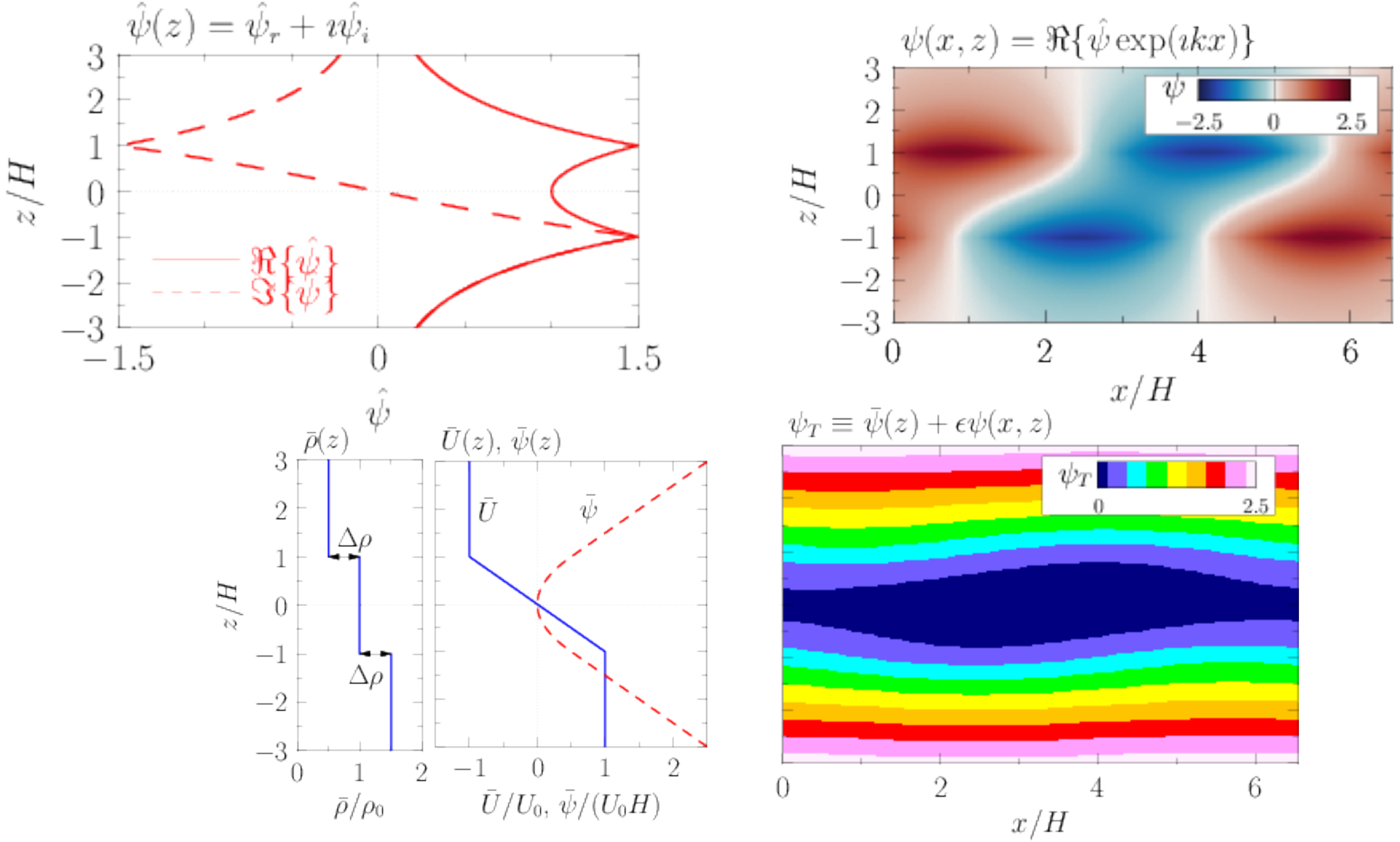
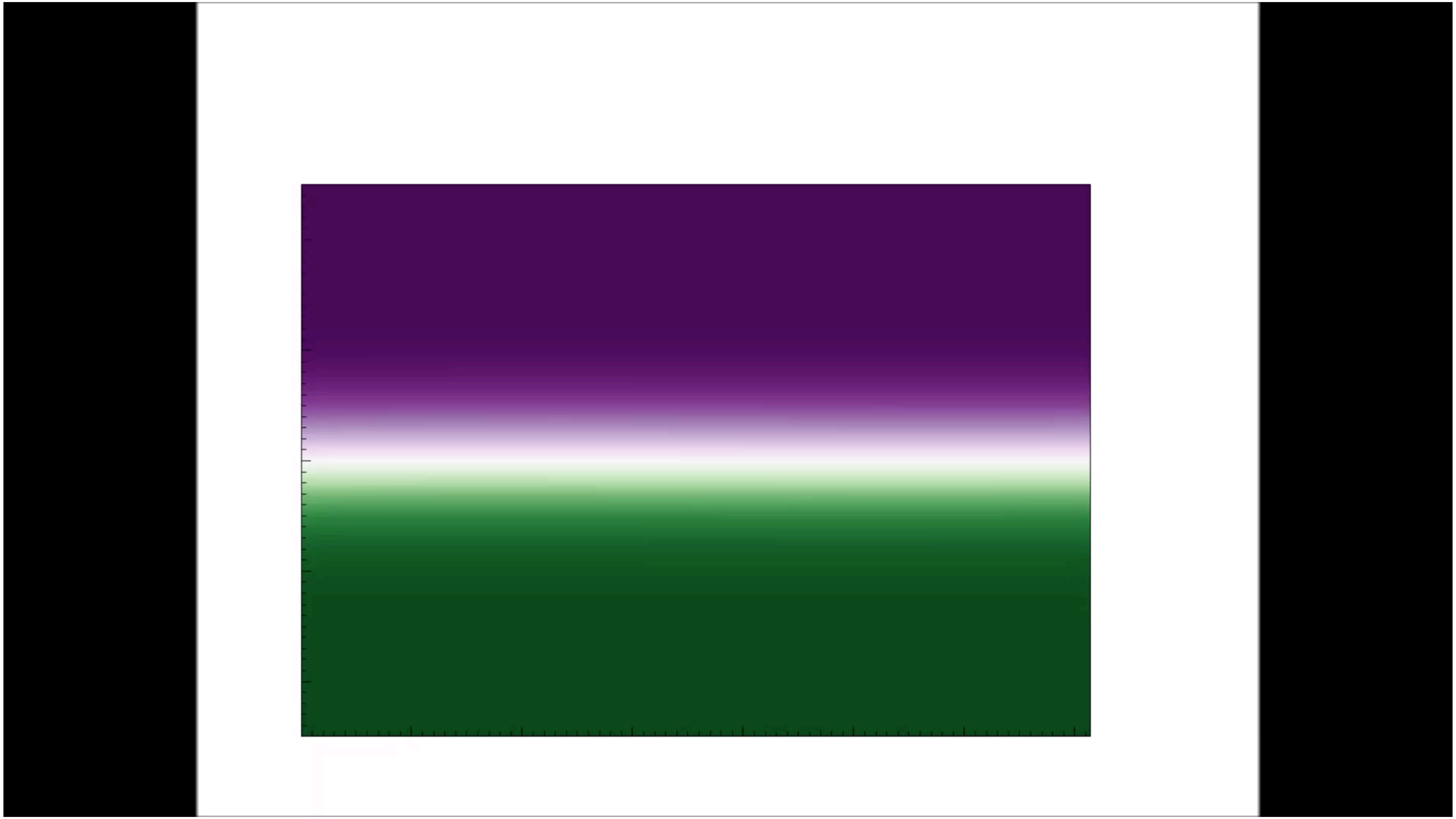


KH Instability: High-Resolution Simulation



[Courtesy of Hesam Salehipour, Woods Hole Oceanographic Institute]