Solution to exercise 9.2.3.

There are three formulas to prove: reflexivity, transitivity and dichotomy of $\leq$. In two cases, we take the universally quantified versions of the formulas. (Having or not having the $\forall$'s simply illustrates how this sequent calculus differs from LK, for example.)

(1) $\forall x\, x \leq x$

$$\dfrac{\dfrac{x \leq x \vdash \forall x\, x \leq x, x \leq x}{\vdash \forall x\, x \leq x, x \leq x}\;(\text{pre}_1)}{\vdash \forall x\, x \leq x}\;(\vdash\forall)$$

(2) $x \leq y \supset (y \leq z \supset x \leq z)$

$$\dfrac{\dfrac{\dfrac{x \leq z, x \leq y, y \leq z \vdash x \leq y \supset (y \leq z \supset x \leq z), y \leq z \supset x \leq z, x \leq z}{x \leq y, y \leq z \vdash x \leq y \supset (y \leq z \supset x \leq z), y \leq z \supset x \leq z, x \leq z}\;(\text{pre}_2)}{x \leq y \vdash x \leq y \supset (y \leq z \supset x \leq z), y \leq z \supset x \leq z}\;(\vdash\supset)}{\vdash x \leq y \supset (y \leq z \supset x \leq z)}\;(\vdash\supset)$$

(3) $\forall x\, \forall y\, (x \leq y \vee y \leq x)$ (We abbreviate this formula as $\mathcal{A}$ and $\forall y\, (x \leq y \vee y \leq x)$ as $\mathcal{B}$ in most of the proof — to save space and to make the proof more transparent.)

$$\dfrac{\dfrac{\dfrac{\dfrac{x \leq y \vdash \mathcal{A}, \mathcal{B}, x \leq y \vee y \leq x, x \leq y, y \leq x \qquad y \leq x \vdash \mathcal{A}, \mathcal{B}, x \leq y \vee y \leq x, x \leq y, y \leq x}{\vdash \mathcal{A}, \mathcal{B}, x \leq y \vee y \leq x, x \leq y, y \leq x}\;(\text{lin})}{\vdash \mathcal{A}, \forall y\, (x \leq y \vee y \leq x), x \leq y \vee y \leq x}\;(\vdash\vee)}{\vdash \forall x\, \forall y\, (x \leq y \vee y \leq x), \forall y\, (x \leq y \vee y \leq x)}\;(\vdash\forall)}{\vdash \forall x\, \forall y\, (x \leq y \vee y \leq x)}\;(\vdash\forall)$$