Solution to exercise 7.6.9.

3. If $\mathcal{A}$ is $\neg\mathcal{B}$ and $v(\mathcal{A})$ is T, then $v(\mathcal{B})$ is F, by clause (2) in Definition 7.40. By the hypothesis of induction, $\neg\mathcal{B} \in \mathfrak{T}$. If $v(\mathcal{A})$ is F, then $v(\mathcal{B})$ is T, by clause (2). By inductive hypothesis, $\mathcal{B} \in \mathfrak{T}$. However, from $\mathcal{B}$, we get $\neg\neg\mathcal{B}$ by the rule DN, hence, $\neg\mathcal{A} \in \mathfrak{T}$.
4. If $\mathcal{A}$ is $\mathcal{B} \vee \mathcal{C}$, and $v(\mathcal{A})$ is T, then $v(\mathcal{B})$ is T or $v(\mathcal{C})$ is T. By inductive hypothesis, $\mathcal{B} \in \mathfrak{T}$ or $\mathcal{C} \in \mathfrak{T}$. If one of the two latter formulas is not an element of $\mathfrak{T}$, then the missing formula may be added by an application of the rule K. Then, by rule $\mathcal{B} \vee \mathcal{C}$ follows, hence, $\mathcal{B} \vee \mathcal{C} \in \mathfrak{T}$.

If $v(\mathcal{A})$ is F, then $v(\mathcal{B})$ and $v(\mathcal{C})$ are F, by (3). The hypothesis of the induction gives that $\neg\mathcal{B} \in \mathfrak{T}$ and $\neg\mathcal{C} \in \mathfrak{T}$. From the latter, $\neg(\mathcal{B} \vee \mathcal{C})$ follows by DM; that is, $\neg\mathcal{A} \in \mathfrak{T}$, as we wanted to prove.