

**Solution to exercise 6.4.2.**

We give proofs in  $H_{S5}$  for the three formulas.

(1)  $\Box(A \supset B) \supset (\Box A \supset \Box B)$

$$\frac{\frac{\frac{\frac{\mathcal{A} \vdash \mathcal{A} \quad \mathcal{B} \vdash \mathcal{B}}{\mathcal{A} \supset \mathcal{B}, \mathcal{A} \vdash \mathcal{B}} (\supset\vdash)}{\Box(\mathcal{A} \supset \mathcal{B}), \mathcal{A} \vdash \mathcal{B}} (\Box\vdash)}{\mathcal{A}, \Box(\mathcal{A} \supset \mathcal{B}) \vdash \mathcal{B}} (C\vdash)}{\Box \mathcal{A}, \Box(\mathcal{A} \supset \mathcal{B}) \vdash \mathcal{B}} (\Box\vdash)}{\frac{\mathcal{A} \supset \mathcal{B}, \Box \mathcal{A}, \Box(\mathcal{A} \supset \mathcal{B}) \vdash \mathcal{B}}{\mathcal{A}, \mathcal{A} \supset \mathcal{B}, \Box \mathcal{A}, \Box(\mathcal{A} \supset \mathcal{B}) \vdash \mathcal{B}} (K\vdash)}{\Box \mathcal{A}, \Box(\mathcal{A} \supset \mathcal{B}) \vdash \Box \mathcal{B}} (\vdash\Box)}{\frac{\Box(\mathcal{A} \supset \mathcal{B}) \vdash \Box \mathcal{A} \supset \Box \mathcal{B}}{\vdash \Box(\mathcal{A} \supset \mathcal{B}) \supset (\Box \mathcal{A} \supset \Box \mathcal{B})} (\vdash\supset)}$$

(2)  $\Box A \supset A$

$$\frac{\frac{\mathcal{A} \vdash \mathcal{A}}{\Box \mathcal{A} \vdash \mathcal{A}} (\Box\vdash)}{\vdash \Box \mathcal{A} \supset \mathcal{A}} (\vdash\supset)$$

(3)  $\mathcal{A} \supset \Box \Diamond \mathcal{A}$

$$\frac{\frac{\frac{\frac{\mathcal{A} \vdash \mathcal{A}}{\mathcal{A} \vdash \Diamond \mathcal{A}} (\vdash\Diamond)}{\vdash \Diamond \mathcal{A} \mid \mathcal{A} \vdash} (\text{split})}{\vdash \Box \Diamond \mathcal{A} \mid \mathcal{A} \vdash} (\vdash\Box)}{\frac{\mathcal{A} \vdash \Box \Diamond \mathcal{A} \mid \mathcal{A} \vdash \Box \Diamond \mathcal{A}}{\mathcal{A} \vdash \Box \Diamond \mathcal{A}} (K\vdash)(\vdash K)}{\vdash \mathcal{A} \supset \Box \Diamond \mathcal{A}} (M)$$