Solution to exercise 5.3.13.

\((A7)\) is the principal type schema of the combinator \(S\). The provability of this formula in \(LT_t^L\) means that the consecution \(t \vdash (A \to B \to C) \to (A \to B) \to A \to C\) has a proof. Here is a proof.

\[
\begin{align*}
& B \vdash B, C \vdash C \\
& A \vdash A \quad B \to C; B \vdash C \quad (\to \vdash) \\
& A \vdash A \quad B \to C; (A \to B; A) \vdash C \quad (\to \vdash) \\
& A \to B \to C; A; (A \to B; A) \vdash C \quad (B' \vdash) \\
& A \to B; (A \to B \to C; A); A \vdash C \quad (B' \vdash) \\
& A \to B \to C; A \to B; A; A \vdash C \quad (W \vdash) \\
& t; A \to B; C; A \to B; A \vdash C \quad (t \vdash) \\
& t \vdash (A \to B \to C) \to (A \to B) \to A \to C \quad (\to \vdash), \ 3x
\end{align*}
\]

The proof yields a combinator, namely, \(B(B'B')(B(BW)B')\), which is easily seen to define \(S\).

\((A13)\) is a closely related formula that is the principal type schema of the combinator \(S'\). We need to show that the consecution \(t \vdash (A \to B) \to (A \to B \to C) \to A \to C\) has a proof in \(LT_t^L\). The proof above may be slightly modified to obtain the following proof.

\[
\begin{align*}
& B \vdash B, C \vdash C \\
& A \vdash A \quad B \to C; B \vdash C \quad (\to \vdash) \\
& A \vdash A \quad B \to C; (A \to B; A) \vdash C \quad (\to \vdash) \\
& A \to B \to C; A; (A \to B; A) \vdash C \quad (B' \vdash) \\
& A \to B; (A \to B \to C; A); A \vdash C \quad (B' \vdash) \\
& A \to B; A \to B \to C; A; A \vdash C \quad (W \vdash) \\
& t; A \to B; A \to B \to C; A \vdash C \quad (t \vdash) \\
& t \vdash (A \to B) \to (A \to B \to C) \to A \to C \quad (\to \vdash), \ 3x
\end{align*}
\]

The combinator that we obtain from this proof is \(B(BW)(BBB')\), which—indeed—defines \(S'\).