Solution to exercise 5.3.13.

(A7) is the principal type schema of the combinator S. The provability of this formula in $LT^{\boldsymbol{t}}_{\rightarrow}$ means that the consecution $\boldsymbol{t} \vdash (\mathcal{A} \to \mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{B}) \to \mathcal{A} \to \mathcal{C}$ has a proof. Here is a proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{A} \vdash \mathcal{A} \quad \dfrac{\mathcal{A} \vdash \mathcal{A} \quad \dfrac{\mathcal{B} \vdash \mathcal{B} \quad \mathcal{C} \vdash \mathcal{C}}{\mathcal{B} \to \mathcal{C}; \mathcal{B} \vdash \mathcal{C}}\,(\to\vdash)}{\mathcal{B} \to \mathcal{C}; (\mathcal{A} \to \mathcal{B}; \mathcal{A}) \vdash \mathcal{C}}\,(\to\vdash)}{\mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A}; (\mathcal{A} \to \mathcal{B}; \mathcal{A}) \vdash \mathcal{C}}\,(\to\vdash)}{\mathcal{A} \to \mathcal{B}; (\mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A}); \mathcal{A} \vdash \mathcal{C}}\,(\mathsf{B}'\vdash)}{\mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A} \to \mathcal{B}; \mathcal{A}; \mathcal{A} \vdash \mathcal{C}}\,(\mathsf{B}'\vdash)}{\mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A} \to \mathcal{B}; \mathcal{A} \vdash \mathcal{C}}\,(\mathsf{W}\vdash)}{\boldsymbol{t}; \mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A} \to \mathcal{B}; \mathcal{A} \vdash \mathcal{C}}\,(\boldsymbol{t}\vdash)}{\boldsymbol{t} \vdash (\mathcal{A} \to \mathcal{B} \to \mathcal{C}) \to (\mathcal{A} \to \mathcal{B}) \to \mathcal{A} \to \mathcal{C}}\,(\vdash\to),\ 3\times$$

The proof yields a combinator, namely, $\mathsf{B}(\mathsf{B}'\mathsf{B}')(\mathsf{B}(\mathsf{B}\mathsf{W})\mathsf{B}')$, which is easily seen to define S.

(A13) is a closely related formula that is the principal type schema of the combinator S'. We need to show that the consecution $\boldsymbol{t} \vdash (\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{B} \to \mathcal{C}) \to \mathcal{A} \to \mathcal{C}$ has a proof in $LT^{\boldsymbol{t}}_{\rightarrow}$. The proof above may be slightly modified to obtain the following proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{A} \vdash \mathcal{A} \quad \dfrac{\mathcal{A} \vdash \mathcal{A} \quad \dfrac{\mathcal{B} \vdash \mathcal{B} \quad \mathcal{C} \vdash \mathcal{C}}{\mathcal{B} \to \mathcal{C}; \mathcal{B} \vdash \mathcal{C}}\,(\to\vdash)}{\mathcal{B} \to \mathcal{C}; (\mathcal{A} \to \mathcal{B}; \mathcal{A}) \vdash \mathcal{C}}\,(\to\vdash)}{\mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A}; (\mathcal{A} \to \mathcal{B}; \mathcal{A}) \vdash \mathcal{C}}\,(\to\vdash)}{\mathcal{A} \to \mathcal{B}; (\mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A}); \mathcal{A} \vdash \mathcal{C}}\,(\mathsf{B}'\vdash)}{\mathcal{A} \to \mathcal{B}; \mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A}; \mathcal{A} \vdash \mathcal{C}}\,(\mathsf{B}\vdash)}{\mathcal{A} \to \mathcal{B}; \mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A} \vdash \mathcal{C}}\,(\mathsf{W}\vdash)}{\boldsymbol{t}; \mathcal{A} \to \mathcal{B}; \mathcal{A} \to \mathcal{B} \to \mathcal{C}; \mathcal{A} \vdash \mathcal{C}}\,(\boldsymbol{t}\vdash)}{\boldsymbol{t} \vdash (\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{B} \to \mathcal{C}) \to \mathcal{A} \to \mathcal{C}}\,(\vdash\to),\ 3\times$$

The combinator that we obtain from this proof is $\mathsf{B}(\mathsf{B}\mathsf{W})(\mathsf{B}\mathsf{B}\mathsf{B}')$, which—indeed—defines S'.