## Solution to exercise 4.3.15.

It is not difficult to see that "contraction" (the principal simple type of W) has no cutfree proof. We take this formula for a counterexample to the eliminability of the cut rule. Notice that there is only one rule that *introduces* a ; and that is the cut rule. Then, the third sequent from the bottom does not leave much choice — beyond getting more copies of the same formulas in various orders — than to apply the cut rule.

$$\begin{array}{c} \underbrace{\vdash \sim \mathcal{A}, \sim (\mathcal{A} \to \mathcal{A} \to \mathcal{B}), \mathcal{A} \to \mathcal{B}}_{\vdash \sim \mathcal{A}, \mathcal{A}, \mathcal{B}, \sim (\mathcal{A} \to \mathcal{B}), \mathcal{B}} \xrightarrow[(\mathrm{cut})]{(\mathrm{cut})} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\vdash \sim \mathcal{A}, \sim (\mathcal{A} \to \mathcal{A} \to \mathcal{B}), \sim \mathcal{A}, \mathcal{B}}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\vdash \sim (\mathcal{A} \to \mathcal{A} \to \mathcal{B}), \mathcal{B}, \sim \mathcal{A}, \sim \mathcal{A}}} & \stackrel{(C)}{\underset{\scriptstyle (W)}{(W)}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\vdash \sim (\mathcal{A} \to \mathcal{A} \to \mathcal{B}), \mathcal{B}, \sim \mathcal{A}, \mathcal{B}}} & \stackrel{(C)}{\underset{\scriptstyle (W)}{(W)}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\vdash \sim (\mathcal{A} \to \mathcal{A} \to \mathcal{B}), \mathcal{A} \to \mathcal{B}}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{(W)}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\vdash \sim (\mathcal{A} \to \mathcal{A} \to \mathcal{B}), \mathcal{A} \to \mathcal{B}}} & \stackrel{(C)}{\underset{\scriptstyle (\to^{*})}{(\to^{*})}} & \stackrel{(C)}{\underset{\scriptstyle (\to^{*})}{\vdash (\mathcal{A} \to \mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{B})}} & \stackrel{(C)}{\underset{\scriptstyle (\to^{*})}{(\to^{*})}} & \stackrel{(C)}{\underset{\scriptstyle (\to^{*})}{\to}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\to}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\to} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\to}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\to}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\to} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\to} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\to}} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\to} & \stackrel{(C)}{\underset{\scriptstyle (\mathrm{cut})}{\to$$