On two alleged counter-examples to my proof of the decidability of MELL

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1 Admissibility of cut

The contraction measure is μ and it is defined using a notion of ancestors to count the number of contractions above the application of the cut rule.

$$\dfrac{\dfrac{\dfrac{\vdash ?\Delta, A}{\vdash ?\Delta, !A_i}\,!\qquad \dfrac{\vdash ?A_j^{\perp}, ?A_k^{\perp}, \Gamma}{\vdash ?A_l^{\perp}, \Gamma}\,?\mathrm{c}}{\vdash ?\Delta, \Gamma}\,\mathrm{cut} \qquad \rightarrow$$

$$\dfrac{\dfrac{\vdash ?\Delta, A}{\vdash ?\Delta, !A_i}\,! \qquad \dfrac{\dfrac{\vdash ?\Delta, A}{\vdash ?\Delta, !A_i}\,! \qquad \vdash ?A_j^{\perp}, ?A_k^{\perp}, \Gamma}{\vdash ?\Delta, ?A_k^{\perp}, \Gamma}\,\mathrm{cut}}{\vdash ?\Delta, ?\Delta, \Gamma}\,\mathrm{cut}}{\vdash ?\Delta, \Gamma}\,?\mathrm{c}\ (n \text{ times})$$

The μ for the cut on the left-hand side is $i + l$ (and $l = j + k + 1$). (The indices are inserted to help seeing where the numbers come from.) The μ for the upper cut on the right-hand side is $i + j$, the μ for the lower cut on the right-hand side is $i + k$. (Of course, $i + j < i + j + k + 1$ and $i + k < i + j + k + 1$.) Note also that there is no increase in the degree of the cut formula.

2 Proof search

The question is if the sequent $\vdash (A^{\perp} \otimes A^{\perp}) \otimes (A^{\perp} \otimes A^{\perp}), A, ?(A^{\perp} \otimes (A \parr A))$ is provable in MELL. The answer is "Yes, it is provable." And here is how my decision procedure finds a proof.

The following is an irredundant proof in (|RMELL|). Note that there are two contractions on ancestors of subformulas of $?(A^{\perp} \otimes (A \parr A))$ (in the labeled steps), which means that the heap number for this formula is (at least) 2. (If there is an irredundant proof with more contractions affecting this formula, then its heap number may be larger. However, 2 contractions on $?(A^{\perp} \otimes (A \parr A))$ suffice for a proof in MELL, hence, we will not bother generating all the irredundant proofs. Neither will we calculate heap numbers that are not used in a MELL proof.)

$$\dfrac{\dfrac{\vdash A^{\perp}, A \qquad \dfrac{\dfrac{\vdash A^{\perp}, A \qquad \vdash A^{\perp}, A}{\vdash A^{\perp} \otimes A^{\perp}, A, A}}{\vdash A^{\perp} \otimes A^{\perp}, A \parr A}}{\vdash A^{\perp} \otimes A^{\perp}, A, A^{\perp} \otimes (A \parr A)} \qquad \dfrac{\vdash A^{\perp}, A \qquad \dfrac{\dfrac{\vdash A^{\perp}, A \qquad \vdash A^{\perp}, A}{\vdash A^{\perp} \otimes A^{\perp}, A, A}}{\vdash A^{\perp} \otimes A^{\perp}, A \parr A}}{\vdash A^{\perp} \otimes A^{\perp}, A, A^{\perp} \otimes (A \parr A)}}{\vdash (A^{\perp} \otimes A^{\perp}) \otimes (A^{\perp} \otimes A^{\perp}), A^{\perp} \otimes (A \parr A), A, A}\,(|\otimes|)$$

$$\dfrac{\dfrac{\vdash (A^{\perp} \otimes A^{\perp}) \otimes (A^{\perp} \otimes A^{\perp}), A^{\perp} \otimes (A \parr A), A, A}{\vdash (A^{\perp} \otimes A^{\perp}) \otimes (A^{\perp} \otimes A^{\perp}), A^{\perp} \otimes (A \parr A), A \parr A} \qquad \vdash A^{\perp}, A}{\vdash (A^{\perp} \otimes A^{\perp}) \otimes (A^{\perp} \otimes A^{\perp}), ?(A^{\perp} \otimes (A \parr A)), A \parr A}$$

$$\dfrac{\vdash (A^{\perp} \otimes A^{\perp}) \otimes (A^{\perp} \otimes A^{\perp}), ?(A^{\perp} \otimes (A \parr A)), A \parr A}{\vdash (A^{\perp} \otimes A^{\perp}) \otimes (A^{\perp} \otimes A^{\perp}), A, ?(A^{\perp} \otimes (A \parr A)), A^{\perp} \otimes (A \parr A)}\,(|?|)$$

$$\dfrac{\vdash (A^{\perp} \otimes A^{\perp}) \otimes (A^{\perp} \otimes A^{\perp}), A, ?(A^{\perp} \otimes (A \parr A)), A^{\perp} \otimes (A \parr A)}{\vdash (A^{\perp} \otimes A^{\perp}) \otimes (A^{\perp} \otimes A^{\perp}), A, ?(A^{\perp} \otimes (A \parr A))}$$

The following proof in MELL has two contractions on $?(A^\perp \otimes (A \parr A))$, hence, this proof will be found in the proof search with heap number 2.

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\dfrac{\dfrac{\dfrac{\vdash A, A^\perp \qquad \dfrac{\dfrac{\vdash A, A^\perp \qquad \dfrac{\vdash A, A^\perp \qquad \dfrac{\dfrac{\dfrac{\vdash A, A^\perp \quad \vdash A, A^\perp}{\vdash A^\perp \otimes A^\perp, A, A}}{\vdash A^\perp \otimes A^\perp, A \parr A} \quad \dfrac{\dfrac{\vdash A, A^\perp \quad \vdash A, A^\perp}{\vdash A^\perp \otimes A^\perp, A, A}}{\vdash A^\perp \otimes A^\perp, A \parr A}}{\vdash (A^\perp \otimes A^\perp) \otimes (A^\perp \otimes A^\perp), A \parr A, A \parr A}}{\vdash (A^\perp \otimes A^\perp) \otimes (A^\perp \otimes A^\perp), A, A \parr A, A^\perp \otimes (A \parr A)}}{\vdash (A^\perp \otimes A^\perp) \otimes (A^\perp \otimes A^\perp), A, A, A^\perp \otimes (A \parr A), A^\perp \otimes (A \parr A)}}{\vdash (A^\perp \otimes A^\perp) \otimes (A^\perp \otimes A^\perp), A \parr A, A^\perp \otimes (A \parr A), A^\perp \otimes (A \parr A)}}{\vdash (A^\perp \otimes A^\perp) \otimes (A^\perp \otimes A^\perp), A, A^\perp \otimes (A \parr A), A^\perp \otimes (A \parr A), A^\perp \otimes (A \parr A)}}{\dfrac{\dfrac{\vdash (A^\perp \otimes A^\perp) \otimes (A^\perp \otimes A^\perp), A, ?(A^\perp \otimes (A \parr A)), ?(A^\perp \otimes (A \parr A)), ?(A^\perp \otimes (A \parr A))}{\vdash (A^\perp \otimes A^\perp) \otimes (A^\perp \otimes A^\perp), A, ?(A^\perp \otimes (A \parr A)), ?(A^\perp \otimes (A \parr A))}}{\vdash (A^\perp \otimes A^\perp) \otimes (A^\perp \otimes A^\perp), A, ?(A^\perp \otimes (A \parr A))}}
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Note that my proof search—like many other proof search methods—does not generate *all possible* proofs or the *most "obvious"* proof; it generates <u>a</u> proof.

MELL is *decidable*, hence, the decidability problem of MELL remains *closed* (as it was since 2015) despite recent rumours to the contrary.