SUGGESTED NOTATION FOR WAVELET ANALYSIS

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ABSTRACT. We propose a system of mathematical notation and symbols to accommodate almost all aspects of wavelet analysis such as fast wavelet transform, filter design, wavelet systems and wavelet frames, function spaces, refinable functions, subdivision schemes, cascade algorithms, refinable function vectors, multiwavelets, dilation factor, dilation matrix, multiwavelets, high dimensions, periodic multiwavelets, and their mathematical analysis in both the space/time domain and frequency domain. We keep as many as possible commonly used mathematical notation and symbols. The system has a simple one-to-one correspondence between symbols in the space/time domain and symbols in the frequency domain. We also provide their latex macro commands for each notation. The file named `wavelet.sty` containing all these latex macro commands is available at `http://www.ualberta.ca/~bhan/publ.htm`

I. Commonly used notation in mathematics:

(1) Dimension: d. Universal constants: π, e. Imaginary unit: i (\iu, avoid $\imath$).
(2) f_{re}, f^{re}, $\Re(f)$ or $\mathrm{Re}(f)$ and f_{im}, f^{im}, $\Im(f)$ or $\mathrm{Im}(f)$ for real and imaginary parts of complex f.
(3) Complex conjugate: $\bar{f}$ for complex f. The $d \times d$ identity matrix: I_d. Identity mapping: Id (\id).
(4) Dirac sequence: $\boldsymbol{\delta}$ (\td, $\boldsymbol{\delta}(0) = 1$ and $\boldsymbol{\delta}(x) = 0\ \forall\ x \neq 0$). (Right) Kronecker/tensor product: $\otimes$.
(5) Small o notation: $f(x) = o(g(x)), \quad x \to x_0$ means $|f(x)/g(x)| \to 0$ as $x \to x_0$ (command \so)
(6) Big $\mathcal{O}$ notation: $f(x) = \mathcal{O}(g(x)), x \to x_0$ means $|f(x)| \leqslant C|g(x)|$ as $x \to x_0$ (command \bo)
(7) The characteristic function of a set $\mathfrak{S}$ (or X, Y, Z): $\chi_{\mathfrak{S}}$. Small number notation: ε, ϵ.
(8) Partial derivatives: (∂ or) $\partial = (\partial_1, \ldots, \partial_d)^{\mathsf{T}}$ (command \grad). $f^{(n)}$ for nth derivative in 1D.
(9) Commonly used mathematical blackboard fonts:
  - (a) Complex numbers: $\mathbb{C}$ (command \C). Real numbers: $\mathbb{R}$ (command \R).
  - (b) Natural numbers: $\mathbb{N}$ (command \N). Nonnegative $\mathbb{N}$: $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ (command \NN)
  - (c) Rational numbers: $\mathbb{Q}$ (command \Q). Integers: $\mathbb{Z}$ (command \Z) Field: $\mathbb{F}$ (\F).
  - (d) Torus: $\mathbb{T} := \mathbb{R}/[2\pi\mathbb{Z}]$ (command \T). Polynomial ring $\mathbb{P}$ (command \PL).
  - (e) Open unit disk $\mathbb{D}$ (command \D). Set separator $:$ (or use $|$ or $:|:$) (\setsp).
  - (f) For dD, add d in front, e.g., $\mathbb{R}^d$ (command \dR), $\mathbb{N}_0^d$ (command \dNN).

II. Variables, integers, functions, and sequences:

(1) Variables and parameters:
  - (a) Space/time domain variables: x (first choice), y (second choice), t (last choice).
  - (b) Complex domain variables: z (first choice), ς (command \varsigma, last choice).
  - (c) Frequency domain variables: ξ (first choice), ζ (second choice).
  - (d) Polar coordinates: (r, θ) (commands \pcr, \pcth). Identify $\mathsf{r}e^{i\theta}$ with $(\mathsf{r}\cos\theta, \mathsf{r}\sin\theta)$ in 2D.
  - (e) Constants: C, C_1, $C_2, \ldots$. If needed, also $N, N_1, \ldots$, (last choice), as well as $R, R_1, \ldots$.
  - (f) Parameters: $c, c_1, c_2 \ldots$. If needed, $t, t_1, \ldots$, $d, d_1, \ldots$, and λ, ρ, ϱ (\varrho) for real numbers.

(2) Integers, indices, vectors/matrices, and sets:
  - (a) Integers: j, k, ℓ, m, n (scale j, shift k, multiplicity ℓ (\mi for ℓ. Note l for $l_p(\mathbb{Z})$), general m, n.
  - (b) Second choice of integers/constants: $\imath, \jmath, \kappa, \varkappa$ (\varkappa), as well as J, K, L.
  - (c) $\mathsf{k}, \mathsf{m}, \mathsf{n}, \mathsf{j}, \alpha, \beta \in \mathbb{Z}^d$ (command \vk, \vm, \vn, \vj). $\mu, \nu \in \mathbb{N}_0^d$. $\alpha, \beta, \gamma \in \mathbb{N}_0^d$ (last choice).
  - (d) Indices: $\mathscr{I}, \mathscr{J}, \mathscr{K}$ (\ind{I}, \ind{J}, \ind{K}), and $\mathcal{I}, \mathcal{J}, \mathcal{K}$. $\Lambda_n := \{\beta \in \mathbb{N}_0^d : |\beta| < n\}$.
  - (e) Matrices: E, F, G, H and U (unitary). $[U]_{m,n}$ denotes the (m, n)-entry of U. T (\tp) transpose.
  - (f) Shear matrix: $S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (command \sh) and $S^\tau = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}, \tau \in \mathbb{R}$.
  - (g) Vectors: $\mathtt{f}, \mathtt{g}, \mathtt{h}, \mathtt{u}$ or $\vec{\mathtt{f}}, \vec{\mathtt{g}}, \vec{\mathtt{h}}, \vec{\mathtt{u}}$ (\vf, \vg, \vh, \vu). Row vector: $[t_1, t_2, t_3]$. Column: $[t_1; t_2; t_3]$.
  - (h) $\mathbf{0}$: a vector/matrix of 0's (command \0). $\mathbf{1}$: a vector/matrix of 1's (command \1).
  - (i) $\mathbf{0}_n$ and $\mathbf{1}_n$: The $n \times 1$ vectors of 0 and 1's. $\mathbf{0}_{m \times n}$ and $\mathbf{1}_{m \times n}$: The $m \times n$ matrices of 0/1's.
  - (j) (Measurable) sets: $\mathfrak{O}, \mathfrak{S}, \mathfrak{U}$ and X, Y, Z (\sO, \sS, \sU). Measure space $(X, \Omega, \mu$ or $\nu)$, $\omega \in \Omega$.

(3) Functions in space/time and frequency domains (always indexed as superscript like $\phi^{[\ell]}$ or ϕ^ℓ):
  - (a) Space/time domain functions: $f, g, h, \phi, \varphi, \psi, \eta$ (commands \f, \g, \h, \phi, \varphi, \psi, \eta). Use Φ, Ψ, F, G, H for sets of functions (or even functions if unavoidable).
  - (b) Frequency domain functions (add \mathbf to time domain notation): $\mathbf{f}, \mathbf{g}, \mathbf{h}, \boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\eta}$ (commands \ff, \fg, \fh, \fphi, \fpsi, \feta). Sets $\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{F}, \mathbf{G}, \mathbf{H}$ (command \fPhi, \fPsi, \fF, \fG, \fH).
  - (c) Fourier transform (add \hat in front): $\hat{f}, \hat{g}, \hat{h}, \hat{\phi}, \hat{\psi}, \hat{\eta}$ (commands: \hf, \hg, \hh, \hphi, \hpsi, \heta).

(4) Sequences in space/time and frequency domains (usually indexed as subscript like a_ℓ):

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(a) Sequences in time domain: a (\ta, low-pass filter), b (\tb, high-pass filter), u (\tu, data/filter), v (\tv, data/low-pass coefficients), w (\tw, high-pass coefficients), ϑ. Usage: $a = \{a(k)\}_{k\in\mathbb{Z}}$.
(b) Sequences in frequency domain (add \mathbf): $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ (commands \fa, \fb, \fu, \fv, \fw).
(c) Fourier transform of sequences (add \hat): $\widehat{a}, \widehat{b}, \widehat{u}, \widehat{v}, \widehat{w}$ (commands \ha, \hb, \hu, \hv, \hw).
(d) Short-hand notation: $\tilde{\vartheta}$ (\ttth), $\tilde{a}$ (\tta), $\widehat{\tilde{a}}$ (\hta), $\mathring{a}$ (\mra), $\widehat{\mathring{a}}$ (\hma). Similar for b, u, v, w, ϑ.
(e) Laurent polynomials of sequences (add \mathsf to time domain notation): $\mathsf{a}, \mathsf{b}, \mathsf{u}, \mathsf{v}, \mathsf{w}$ (commands \pa, \pb, \pu, \pv, \pw). Or $\breve{a}, \breve{b}, \breve{u}, \breve{v}, \breve{w}$ (commands \bpa, \bpb, \bpu, \bpv, \bpw).
(f) Trigonometric polynomials: $\boldsymbol{\theta}, \mathbf{p}, \mathbf{q}, \boldsymbol{\Theta}, \mathbf{P}, \mathbf{Q}, \mathbf{A}, \mathbf{B}, \mathbf{U}$ (\fth, \fp, \fq, \fTh, \fP, \fQ, \fA, \fB, \fU).
(g) Laurent polynomials: $\mathsf{p}, \mathsf{q}, \Theta, \mathsf{P}, \mathsf{Q}, \mathsf{A}, \mathsf{B}, \mathsf{U}$ (\pp, \pq, \pTh, \pP, \pQ, \pA, \pB, \pU).

III. Function spaces:
(1) C^∞ compactly supported test function space: $\mathscr{D}(\mathbb{R})$ (command \DC), $\mathscr{D}(\mathbb{R}^d)$ (command \dDC)
Fast decaying C^∞ Schwarz function space: $\mathscr{S}(\mathbb{R})$ (command \SC), $\mathscr{S}(\mathbb{R}^d)$ (command \dSC)
(2) Distributions: $\mathscr{D}'(\mathbb{R})$ (\DCpr), $\mathscr{D}'(\mathbb{R}^d)$ (\dDCpr). Tempered: $\mathscr{S}'(\mathbb{R})$ (\SCpr), $\mathscr{S}'(\mathbb{R}^d)$ (\dSCpr).
(3) C^τ or Hölder class: $\mathscr{C}^\tau(\mathbb{R})$ (\CH{τ}). $\mathscr{C}^\tau(\mathbb{R}^d)$ (\dCH{τ}). Weight function: $\mathtt{w}$ (\wf).
(4) Fourier transform: $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dx$. Inverse Fourier transform: $\check{f}(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(\xi) e^{ix\cdot\xi} d\xi$.
(5) For $L_p(\mathbb{R})$ (command \Lp{p}) and ℓ_p: p, q, τ are reserved for L_p, L_q, and smoothness exponent τ.
(a) The space of $1\times r$ row vectors of functions in $L_p(\mathbb{R})$: $(L_p(\mathbb{R}))^r$ (command \Lr{p}{r})
(b) The space of $r\times s$ matrices of functions in $L_p(\mathbb{R})$: $(L_p(\mathbb{R}))^{r\times s}$ (command \Lrs{p}{r}{s})
(c) The space of locally L_p integrable functions: $L_p^{loc}(\mathbb{R})$ (command \lLp{p})
(d) The space of $1\times r$ row vectors of functions in $L_p^{loc}(\mathbb{R})$: $(L_p^{loc}(\mathbb{R}))^r$ (command \lLr{p}{r})
(e) The space of $r\times s$ matrices of functions in $L_p(\mathbb{R})$: $(L_p^{loc}(\mathbb{R}))^{r\times s}$ (command \lLrs{p}{r}{s})
(f) For dD, add d in front: $L_p(\mathbb{R}^d)$ (command \dLp{p}). $(L_p^{loc}(\mathbb{R}^d))^{r\times s}$ (command \dlLrs{p}{r}{s})
(g) For sequences in 1D, replace the first L by l: $l_p(\mathbb{Z})$ (command \lp{p}). $(l_p(\mathbb{Z}))^{r\times s}$ (command \lprs{p}{r}{s}). For dD sequences, further add d in front to 1D version.
(h) Space of all sequences: $l(\mathbb{Z})$ (command \sq). $l(\mathbb{Z}^d)$ (command \dsq).
(6) Sobolev space in L_p: $L_p^\tau(\mathbb{R})$ (command $\backslash SL\{\tau\}\{p\}$). $L_p^\tau(\mathbb{R}^d)$ (command \dSL{τ}{p})
(7) Sobolev space in L_2: $H^\tau(\mathbb{R})$ (command \HH{τ}). $H^\tau(\mathbb{R}^d)$ (command \dHH{τ}).
(8) Besov spaces: $B_{p,q}^\tau(\mathbb{R})$ (command \BS{τ}{p}{q}). $B_{p,q}^\tau(\mathbb{R}^d)$ (command \dBS{τ}{p}{q})
(9) Triebel-Lizorkin spaces: $F_{p,q}^\tau(\mathbb{R})$ (command \TL{τ}{p}{q}). $F_{p,q}^\tau(\mathbb{R}^d)$ (command \dTL{τ}{p}{q}).
(10) For function spaces on $\mathbb{T}$ (2π-periodic), add T in front for 1D. Further add d in front for dD.

IV. Suggested notation for wavelet analysis:
(1) 1D dilation factor: d (first choice), (command \df). $\mathsf{M}, \mathsf{d}_{\mathsf{M}}$, (other choices, but not dimension d).
(2) dD dilation matrix: M (\dm). $\mathsf{d}_{\mathsf{M}} := \det\mathsf{M}$ (\ddm). Extra N, L (\dn, \dl, often $\mathsf{N} = (\mathsf{M}^\mathsf{T})^{-1}$).
(3) Coset group: $\Gamma_{\mathsf{M}} := (\mathsf{M}[0,1)^d)\cap\mathbb{Z}^d$ (\dmc or \dmcg{M}). Use $\gamma\in\Gamma_{\mathsf{M}}$ and $u^{[\gamma]}$ for coset sequence.
(4) Frequency coset group: $\Omega_{\mathsf{M}} := [(\mathsf{M}^\mathsf{T})^{-1}\mathbb{Z}^d]\cap[0,1)^d$ (\dmfc or \dmfcg{M}). Use $\omega\in\Omega_{\mathsf{M}}$.
$\mathsf{N} := (\mathsf{M}^\mathsf{T})^{-1}$ and $\mho_{\mathsf{N}} := [\mathsf{N}\mathbb{Z}^d]\cap[0,1)^d = \Omega_{\mathsf{M}}$. $\L_{\mathsf{N}} := ((\mathsf{N}^\mathsf{T})^{-1}[0,1)^d)\cap\mathbb{Z}^d = \Gamma_{\mathsf{M}}$. $\mho$, Ł (\mho, \L).
(5) Space-based affine system: AS (command \AS). Frequency-based: FAS (command \FAS)
(6) Subdivision operator: $\mathcal{S}$ (command \sd). Transition operator: $\mathcal{T}$ (command \tz).
Cascade operator: $\mathcal{R}$ (command \cd). Symmetry operators: $\mathsf{S}, \mathbb{S}$ (command \sym, \symc).
Analysis operator: $\mathcal{W}$ (command \wa). Synthesis operator: $\mathcal{V}$ (command \ws).
Frame/synthesis operator: $\mathcal{F}$ (command \fr). Fourier transform: $\mathscr{F}$ (command \ft).
Quasi-projection operator $\mathcal{Q}$ (\qp). Projection operator/matrix: $\mathcal{P}$ (command \pr).
Shift-invariant space: S (command \si).
(7) Periodization operator: $\mathscr{P}$ (command \pz). Definition $[\mathscr{P}f](x) := f^{per}(x) := \sum_{\mathsf{k}\in\mathbb{Z}^d} f(\frac{x}{2\pi} - \mathsf{k})$.
(8) Multiplicity for ϕ or low-pass filter $\widehat{a}$: r (command \mphi). Indexed as superscript: $\phi^{[r]}$ or ϕ^r.
Multiplicity for ψ or high-pass filter $\widehat{b}$: s (command \mpsi). Indexed as superscript: $\psi^{[s]}$ or ψ^s.
(9) Polyphase matrix: $\mathbf{P}$ or P (\PP, \pPP). Polyphase matrix for cosets: $\mathbf{Q}$ or Q (\PPC, \pPPC).
(10) Symmetry group: $\mathscr{G}$ (use $E, F, G, H\in\mathscr{G}$ for elements). Special: D_4 (square), D_6 (hexagon).
(11) B-spline of order m: B_m. Box-splines with direction matrix Ξ: M_Ξ (mask a^Ξ).
(12) Backward and forward differences: $\nabla_t f := f - f(\cdot - t)$. $\Delta_t f := f(\cdot + t) - f$ (Δ_t is rarely used).
(13) Sum rule vector for matrix masks: υ (command \vgu, general), Υ (command \vgU, special).
(14) $\operatorname{sm}_p(\widehat{a}, \mathsf{M})$, $\operatorname{sm}_p(\phi)$ (\sm) for smoothness exponent. $\operatorname{sr}(\widehat{a}, \mathsf{M})$ (\sr) for highest order of sum rules.
$\operatorname{vm}(\widehat{a})$ (\vmo) for order of vanishing moments. $\operatorname{lpm}(\widehat{a})$ (\lpm) for order of linear-phase moments.
(15) Index a family of masks as $a_{m,n}^P$ or a_m^P having property P, m sum rules, n linear-phase moments.
(16) \tilde for dual. \hat for Fourier. \breve for Laurent polynomial. \mathring for temporary use.
(17) Multiresolution analysis (MRA) spaces: $\mathscr{V}, \mathscr{W}, \mathscr{U}$ (command \V, \W, \U) and $\tilde{\mathscr{V}}, \tilde{\mathscr{W}}, \tilde{\mathscr{U}}$ (\tV, \tW, \tU).
(18) $[\widehat{f}, \widehat{g}](\xi) := \sum_{k\in\mathbb{Z}} \widehat{f}(\xi + 2\pi k)\overline{\widehat{g}(\xi + 2\pi k)} = \sum_{k\in\mathbb{Z}} e^{ik\xi}\int f(x)\overline{g}(x)dx$, where $\widehat{f}(\xi) := \int f(x)e^{-ix\xi}dx$
(19) Dilation/shift/modulation: $[\![U; \mathsf{k}, \mathsf{n}]\!]\phi = \phi_{U;\mathsf{k},\mathsf{n}} := |\det U|^{1/2} e^{-i\mathsf{n}\cdot Ux}\phi(Ux - \mathsf{k})$. (\lb, \rb).
Duality: $\widehat{\phi_{U;\mathsf{k},\mathsf{n}}}(\xi) = |\det U|^{-1/2} e^{-i\mathsf{k}\cdot((U^T)^{-1}\xi + \mathsf{n})}\hat{\phi}((U^T)^{-1}\xi + \mathsf{n}) = e^{-i\mathsf{k}\cdot\mathsf{n}}\hat{\phi}_{(U^T)^{-1};-\mathsf{n},\mathsf{k}}$.
(20) Convention: $\phi_{\mathsf{k},\mathsf{n}} := \phi_{I_d;\mathsf{k},\mathsf{n}}$. Then $\widehat{\phi_{\mathsf{k},0}}(\xi) = \hat{\phi}_{0,\mathsf{k}}(\xi)$.
Convention (try to avoid): $\phi_{j;\mathsf{k}} := \phi_{\mathsf{M}^j;\mathsf{k},0}$ if M is assumed. Then $\widehat{\phi_{j;\mathsf{k}}}(\xi) = \widehat{\phi_{\mathsf{M}^j;\mathsf{k},0}}(\xi) = \hat{\phi}_{(\mathsf{M}^T)^{-j};0,\mathsf{k}}(\xi)$.