MATRIX SPLITTING WITH SYMMETRY AND SYMMETRIC TIGHT FRAMELET FILTER BANKS WITH TWO HIGH-PASS FILTERS

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Abstract. The oblique extension principle introduced in [3, 5] is a general procedure to construct tight wavelet frames and their associated filter banks. Symmetric tight framelet filter banks with two high-pass filters have been studied in [13, 16, 17]. Tight framelet filter banks with or without symmetry have been constructed in [1]–[21] and references therein. This paper is largely motivated by several results in [11, 13, 17] to further study tight wavelet frames and their associated filter banks with symmetry and two high-pass filters. Our study not only leads to a simpler algorithm for the construction of tight framelet filter banks with symmetry and two high-pass filters, but also allows us to obtain a wider class of tight wavelet frames with symmetry which are not available in the current literature. The key ingredient in our investigation is a complete characterization of splitting positive semi-definite 2×2 matrices of Laurent polynomials with symmetry. Several examples are provided to illustrate the results and algorithms in this paper.

1. Introduction and Motivation

In this paper we study dyadic tight wavelet frames with symmetry which are derived via the oblique extension principle introduced in [3, 5]. To do so, let us recall some basic definitions and notation. By $l_0(\mathbb{Z})$ we denote the linear space of all sequences $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ on $\mathbb{Z}$ such that $\{k \in \mathbb{Z} : u(k) \neq 0\}$ is a finite set. For $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$, its z-transform is a Laurent polynomial defined to be $\mathsf{u}(z) := \sum_{k\in\mathbb{Z}} u(k)z^k$. For $\gamma \in \mathbb{Z}$, the γ-coset of u is defined to be

$$u^{[\gamma]}(k) := u(\gamma + 2k), \qquad k \in \mathbb{Z}, \quad \text{or equivalently,} \quad \mathsf{u}^{[\gamma]}(z) = \sum_{k\in\mathbb{Z}} u(\gamma + 2k)z^k. \tag{1.1}$$

For a matrix $\mathsf{P}(z) := \sum_{k\in\mathbb{Z}} P_k z^k$ of Laurent polynomials, we define $\mathsf{P}^\star(z) := \sum_{k\in\mathbb{Z}} \overline{P_k}^{\mathsf{T}} z^{-k}$, where $\overline{P_k}^{\mathsf{T}}$ denotes the complex conjugate of the transpose of the matrix P_k.

Let $\Theta, a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$. We say that $\{a; b_1, \ldots, b_s\}_\Theta$ is *a tight framelet filter bank* if

$$\begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix} \begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix}^\star = \mathcal{M}_{a,\Theta}(z), \tag{1.2}$$

where

$$\mathcal{M}_{a,\Theta}(z) := \begin{bmatrix} \Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) & -\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) \\ -\Theta(z^2)\mathsf{a}(-z)\mathsf{a}^\star(z) & \Theta(-z) - \Theta(z^2)\mathsf{a}(-z)\mathsf{a}^\star(-z) \end{bmatrix}. \tag{1.3}$$

Using the coset sequences in (1.1) and the relation $\mathsf{b}(z) = \mathsf{b}^{[0]}(z^2) + z\mathsf{b}^{[1]}(z^2)$, one can easily check that the condition in (1.2) for a tight framelet filter bank $\{a; b_1, \ldots, b_s\}_\Theta$ can be equivalently rewritten in terms of polyphase matrices as follows:

$$\begin{bmatrix} \mathsf{b}_1^{[0]}(z) & \cdots & \mathsf{b}_s^{[0]}(z) \\ \mathsf{b}_1^{[1]}(z) & \cdots & \mathsf{b}_s^{[1]}(z) \end{bmatrix} \begin{bmatrix} \mathsf{b}_1^{[0]}(z) & \cdots & \mathsf{b}_s^{[0]}(z) \\ \mathsf{b}_1^{[1]}(z) & \cdots & \mathsf{b}_s^{[1]}(z) \end{bmatrix}^\star = \mathcal{N}_{a,\Theta}(z), \tag{1.4}$$

where

$$\mathcal{N}_{a,\Theta}(z) := \begin{bmatrix} \frac{1}{2}\Theta^{[0]}(z) - \Theta(z)\mathsf{a}^{[0]}(z)(\mathsf{a}^{[0]}(z))^\star & \frac{1}{2}z\Theta^{[1]}(z) - \Theta(z)\mathsf{a}^{[0]}(z)(\mathsf{a}^{[1]}(z))^\star \\ \frac{1}{2}\Theta^{[1]}(z) - \Theta(z)\mathsf{a}^{[1]}(z)(\mathsf{a}^{[0]}(z))^\star & \frac{1}{2}\Theta^{[0]}(z) - \Theta(z)\mathsf{a}^{[1]}(z)(\mathsf{a}^{[1]}(z))^\star \end{bmatrix}. \tag{1.5}$$

If $\mathsf{a}(1) = 1$, then we can define (see [1, 4])

$$\boldsymbol{\varphi}^a(\xi) := \prod_{j=1}^\infty \mathsf{a}(e^{-i2^{-j}\xi}) \qquad \text{and} \qquad \boldsymbol{\psi}^{a,b_\ell}(\xi) := \mathsf{b}_\ell(e^{-i\xi/2})\boldsymbol{\varphi}^a(\xi/2), \qquad \ell = 1, \ldots, s, \;\; \xi \in \mathbb{R}.$$

If $\boldsymbol{\varphi}^a, \boldsymbol{\psi}^{a,b_1}, \ldots, \boldsymbol{\psi}^{a,b_s} \in L_2(\mathbb{R})$, then we can further define functions $\phi^a, \psi^{a,b_1}, \ldots, \psi^{a,b_s}$ in $L_2(\mathbb{R})$ via the inverse Fourier transform by $\widehat{\phi^a}(\xi) := \boldsymbol{\varphi}^a(\xi)$ and $\widehat{\psi^{a,b_\ell}}(\xi) := \boldsymbol{\psi}^{a,b_\ell}(\xi)$ for $\ell = 1, \ldots, s$, where the Fourier transform

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$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$ for $f \in L_1(\mathbb{R})$ and can be naturally extended to $L_2(\mathbb{R})$ functions. We shall use the following notation throughout the paper

$$f_{\lambda;k,n}(x) := [\![\lambda; k, n]\!] f(x) := |\lambda|^{1/2} e^{-in\lambda x} f(\lambda x - k), \qquad \lambda, k, n, x \in \mathbb{R}.$$

In particular, $f_{\lambda;k} := f_{\lambda;k,0} = |\lambda|^{1/2} f(\lambda \cdot -k)$. If $\{a; b_1, \ldots, b_s\}_\Theta$ is a tight framelet filter bank such that $\mathsf{a}(1) = \Theta(1) = 1$ and if $\phi^a, \psi^{a,b_1}, \ldots, \psi^{a,b_s} \in L_2(\mathbb{R})$, then it has been shown in [3, 5] that $\{\psi^{a,b_\ell}_{2^j;k} : j \in \mathbb{Z}, k \in \mathbb{Z}, \ell = 1, \ldots, s\}$ is a (normalized) tight frame for $L_2(\mathbb{R})$, that is,

$$\|f\|^2_{L_2(\mathbb{R})} = \sum_{\ell=1}^{s} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle f, \psi^{a,b_\ell}_{2^j;k} \rangle|^2, \qquad \forall\, f \in L_2(\mathbb{R}).$$

For discussion on tight wavelet frames, see [1]–[21] and many references therein. It is interesting to point out that there is indeed a one-to-one correspondence between filter banks and frequency-based framelets in the distribution space, recently established in [10, Theorem 2]. Hence, we mainly concentrate on filter banks and Laurent polynomials in this paper.

For a given low-pass filter a and a moment correcting filter Θ, to obtain high-pass filters $b_1, \ldots, b_s$ in a tight framelet filter bank, we have to factorize the given matrix $\mathcal{N}_{a,\Theta}$ in (1.5) so that (1.4) holds. To reduce computational complexity in the implementation of a tight framelet filter bank, we often prefer a small number s of high-pass filters. Note that if $s = 1$, then we must have $\det(\mathcal{N}_{a,\Theta}(z)) = 0$ for all $z \in \mathbb{C}\backslash\{0\}$ which is too restrictive to be satisfied by many filters a and Θ. In this paper, we shall consider the case $s = 2$ and tight framelet filter banks having symmetry. Recall that a Laurent polynomial p has *symmetry* if

$$\mathsf{Sp}(z) := \frac{\mathsf{p}(z)}{\mathsf{p}(z^{-1})} = \epsilon z^c \qquad \forall\, z \in \mathbb{C}\backslash\{0\} \quad \text{with} \quad \epsilon \in \{-1, 1\},\ c \in \mathbb{Z}. \tag{1.6}$$

Writing $\mathsf{p}(z) = \sum_{k\in\mathbb{Z}} p_k z^k$, we see that (1.6) is equivalent to saying that $p_k = \epsilon p_{c-k}$ for all $k \in \mathbb{Z}$. Tight wavelet frames and tight framelet filter banks without symmetry have been extensively studied in a lot of papers, to mention only a few here, see [2, 3, 4, 5, 8, 12, 18, 19, 20] and many papers therein. Interesting examples of real-valued tight framelet filter banks with symmetry have been obtained in [3, 5, 6, 13, 14, 15, 16, 17, 18, 21].

The *Dirac sequence* $\delta : \mathbb{Z} \to \mathbb{R}$ is defined to be $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. When $\Theta = \delta$, we shall use the simpler notation $\{a; b_1, \ldots, b_s\}$ instead of $\{a; b_1, \ldots, b_s\}_\delta$. This paper is largely motivated by two negative results presented in [13, 17] on symmetric tight framelet filter banks and a positive result in [11]. [17, Corollary 2] shows that if a is a real-valued interpolatory filter (that is, $a(0) = 1/2$ and $a(2k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$), except the trivial cases $\mathsf{a}(z) = 1/2 + (z^{1-2m} + z^{2m-1})/4$ (which have no more than 2 sum rules) for some $m \in \mathbb{N}$, it is impossible to obtain a real-valued tight framelet filter bank $\{a; b_1, b_2\}$ with symmetry. Later on, a general result on matrix splitting with symmetry has been established in [13, Theorem 2.3]. As an application, a necessary and sufficient condition for a real-valued tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with symmetry has been obtained in [13, Theorem 2.4] and a detailed algorithm is given in [13, Algorithm 2.5] to derive the high-pass filters b_1, b_2 with symmetry from the given filters a and Θ. [13, Theorem 2.4] involves a technical greatest common divisor (GCD) condition and an example has been presented in [13, Example 3.1] to show that all the conditions in [13, Theorem 2.4], except the technical GCD condition, are satisfied. Therefore, the technical GCD condition in [13, Theorem 2.4] cannot be dropped and it is impossible to obtain a real-valued tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with symmetry from [13, Example 3.1]. We are largely motivated by the above two negative results in [13, 17] on real-valued symmetric tight framelet filter banks to further explore this topic.

On the other hand, it is well known that except the trivial variants of the Haar orthogonal wavelet, no real-valued compactly supported orthogonal wavelets can have symmetry (see [4]). However, recently it has been shown in [11] that there is a family of complex-valued compactly supported orthogonal wavelets having symmetry, high smoothness, and arbitrarily high vanishing moments (as well as arbitrarily high linear-phase moments). Since orthogonal wavelets are special cases of tight wavelet frames, this automatically yields a family of compactly supported complex-valued tight wavelet frames with symmetry. This naturally motivates us to explore complex-valued tight wavelet frames with symmetry by extending the results in [13] to the more general setting. In particular, we are interested in whether or not we can derive a complex-valued symmetric tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ from [13, Example 3.1], and whether it is possible to construct a complex-valued symmetric tight framelet filter bank $\{a; b_1, b_2\}$ with a being a nontrivial interpolatory filter.

Since filters and tight framelet filter banks that we consider in this paper are not necessarily real-valued, there is another closely related but different notion of symmetry. We say that a Laurent polynomial p has *complex symmetry* if

$$\mathbb{S}\mathsf{p}(z) := \frac{\mathsf{p}(z)}{\mathsf{p}^\star(z)} = \epsilon z^c \qquad \forall\, z \in \mathbb{C}\backslash\{0\} \quad \text{with} \quad \epsilon \in \{-1, 1\},\ c \in \mathbb{Z}. \tag{1.7}$$

Writing $\mathsf{p}(z) = \sum_{k\in\mathbb{Z}} p_k z^k$, the complex symmetry in (1.7) is equivalent to saying that $\overline{p_k} = \epsilon p_{c-k}$ for all $k \in \mathbb{Z}$. Obviously, for a Laurent polynomial p having real coefficients, there is no difference between symmetry and complex symmetry.

The contributions of this paper are as follows. First, considering Laurent polynomials having complex coefficients, we obtain a complete characterization of splitting positive semi-definite 2×2 matrices of Laurent polynomials with symmetry or complex symmetry. This enables us to have a much better understanding about the technical GCD condition in [13, Theorems 2.3 and 2.4]. Secondly, based on [13] and our result on matrix splitting with symmetry, we establish a necessary and sufficient condition for a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with symmetry or complex symmetry. Moreover, we provide a step-by-step algorithm to derive high-pass filters b_1, b_2 having symmetry or complex symmetry from the given low-pass filter a and the moment correcting filter Θ. As an application, we show that a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with complex symmetry can indeed be derived from the real-valued filters a and Θ given in [13, Example 3.1]. We also provide an example to show that we indeed can construct a tight framelet filter bank $\{a; b_1, b_2\}$ with symmetry such that a is a nontrivial interpolatory filter. Therefore, our results lead to a wider class of tight wavelet frames with symmetry or complex symmetry. Thirdly, comparing with [13, Algorithms 2.5 and 5.1], our algorithm in this paper is simpler and more efficient by solving a much smaller system of linear equations. Fourthly, we mention that (complex-valued) tight framelet filter banks with symmetry have been recently investigated in [16] with filter coefficients from algebraic number fields. However, [16] only considers the special case $\Theta = \delta$ and doesn't study the case of complex symmetry. As a byproduct of our results and algorithms developed in this paper, the main result in [16] on tight framelet filter banks with symmetry over algebraic number fields can be easily recovered and be generalized to the more general setting. Finally, complex-valued tight framelet filter banks with symmetry or complex symmetry have a close relation to our current study of directional tight framelets in high dimensions. We shall address this issue in detail elsewhere.

The following is the structure of this paper. In Section 2, we shall study some basic properties of Laurent polynomials. Such auxiliary results are needed in our study of tight framelet filter banks with symmetry and two high-pass filters. In Section 3, we provide a complete characterization of splitting positive semi-definite 2×2 matrices of Laurent polynomials with symmetry or complex symmetry. In Section 4, based on the results in Section 3, we present a necessary and sufficient condition in terms of the low-pass filter a and the moment correcting filter Θ for a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ having symmetry or complex symmetry. Two special examples related to [17, Corollary 2] and [13, Example 3.1] are provided in Section 5 to show some advantages of results obtained in this paper comparing with [13, 17]. In Section 6, we shall present two algorithms for constructing tight framelet filter banks $\{a; b_1, b_2\}$ having symmetry or complex symmetry. Several additional examples are also provided to illustrate the results and algorithms in this paper.

2. Some Properties of Laurent Polynomials with Symmetry

To study matrix splitting with symmetry and symmetric tight framelet filter banks, we first investigate some basic properties and auxiliary results on Laurent polynomials.

For a Laurent polynomial p and $z_0 \in \mathbb{C}\backslash\{0\}$, by $\mathsf{Z}(\mathsf{p}, z_0)$ we denote the multiplicity of zeros of $\mathsf{p}(z)$ at $z = z_0$. We say that p is a nontrivial Laurent polynomial if it is not identically zero. In this paper we often use the following obvious facts:

$$\mathsf{p}^\star(z) = \overline{\mathsf{p}(\bar{z}^{-1})}, \qquad z \in \mathbb{C}\backslash\{0\}, \tag{2.1}$$

and $z = \bar{z}^{-1}$ if and only if $|z| = 1$, that is, $z \in \mathbb{T} := \{z \in \mathbb{C} \ : \ |z| = 1\}$.

For a Laurent polynomial p which is nonnegative (or nonpositive) on $\mathbb{T}$, we have the following result which is essentially known in the literature. For the convenience of the reader, we provide a proof here.

Lemma 2.1. *Let p be a nontrivial Laurent polynomial. Then either $\mathsf{p}(z) \geqslant 0$ or $\mathsf{p}(z) \leqslant 0$ for all $z \in \mathbb{T}$ if and only if $\mathsf{p}^\star(z) = \mathsf{p}(z)$ and*

$$\mathsf{Z}(\mathsf{p}, z) \ \text{ is an even integer for every } \ z \in \mathbb{T}. \tag{2.2}$$

Proof. Necessity. If $\mathsf{p}(z) \leqslant 0$ for all $z \in \mathbb{T}$, then we consider $-\mathsf{p}$ instead of p. Suppose that $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$. It is obvious that $\mathsf{p}^\star(z) = \mathsf{p}(z)$ for all $z \in \mathbb{T}$. It suffices to prove (2.2). We use proof by contradiction. Suppose not. Then there exists $\xi_0 \in \mathbb{R}$ such that $\mathsf{Z}(\mathsf{p}, e^{-i\xi_0}) = 2n + 1$ for some nonnegative integer n. Now we can write $\mathsf{p}(z) = (z - e^{-i\xi_0})^n (z^{-1} - e^{i\xi_0})^n \mathsf{q}(z)$ for a unique Laurent polynomial q such that $\mathsf{q}(e^{-i\xi_0}) = 0$ but $\mathsf{q}'(e^{-i\xi_0}) \neq 0$. Define a function $f(\xi) := \mathsf{q}(e^{-i\xi}), \xi \in \mathbb{R}$. Then $\mathsf{p}(e^{-i\xi}) = |e^{-i\xi} - e^{-i\xi_0}|^{2n} f(\xi)$. By $\mathsf{p}(e^{-i\xi}) \geqslant 0$ for all $\xi \in \mathbb{R}$, we must have $f(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$. Since $f(\xi_0) = \mathsf{q}(e^{-i\xi_0}) = 0$ and $f(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}$, we see that ξ_0 is a local minimum of the real-valued smooth function f. Therefore, we must have $f'(\xi_0) = 0$. Through the relation $f'(\xi) = -ie^{-i\xi}\mathsf{q}'(e^{-i\xi})$, we conclude that $\mathsf{q}'(e^{-i\xi_0}) = 0$, a contradiction to our assumption on q. Therefore, $\mathsf{Z}(\mathsf{p}, e^{-i\xi_0})$ must be an even integer for all $\xi_0 \in \mathbb{R}$. Hence, (2.2) holds.

Sufficiency. A Laurent polynomial p has finitely many zeros on $\mathbb{T}$. Suppose that $z_1,\ldots,z_r$ are all distinct zeros on $\mathbb{T}$ of p. Since $\overline{z_j}=z_j^{-1}$ by $z_j\in\mathbb{T}$ and $\mathsf{Z}(\mathsf{p},z_j)$ is an even integer for all $j=1,\ldots,r$, we can write

$$\mathsf{p}(z)=\mathsf{q}(z)\prod_{j=1}^{r}[(z-z_j)(z^{-1}-z_j^{-1})]^{\mathsf{Z}(\mathsf{p},z_j)/2}$$

for a unique Laurent polynomial q. Since $\mathsf{p}^\star(z)=\mathsf{p}(z)$ for all $z\in\mathbb{T}$, it is trivially seen that $\mathsf{q}^\star(z)=\mathsf{q}(z)$ for all $z\in\mathbb{T}$. Hence,

$$\mathsf{q}(z)=\mathsf{q}^\star(z)=\overline{\mathsf{q}(\overline{z}^{-1})}=\overline{\mathsf{q}(z)},\qquad z\in\mathbb{T}.$$

The above identity shows that q is a real-valued function on $\mathbb{T}$. Since q does not have any roots on $\mathbb{T}$, we conclude that $\mathsf{q}(z)\in\mathbb{R}\backslash\{0\}$ for all $z\in\mathbb{T}$. Consequently, since q is a continuous real-valued function on $\mathbb{T}$, either $\mathsf{q}(z)>0$ for all $z\in\mathbb{T}$ (which implies $\mathsf{p}(z)\geqslant 0$ for all $z\in\mathbb{T}$) or $\mathsf{q}(z)<0$ for all $z\in\mathbb{T}$ (which implies $\mathsf{p}(z)\leqslant 0$ for all $z\in\mathbb{T}$). ☐

For a Laurent polynomial p, we observe that $\mathsf{S}[\lambda\mathsf{p}](z)=\mathsf{S}\mathsf{p}(z)$ and $\mathbb{S}[\lambda\mathsf{p}](z)=\frac{\lambda}{\overline{\lambda}}\mathbb{S}\mathsf{p}(z)$ for all $\lambda\in\mathbb{C}\backslash\{0\}$. Hence, we say that p has *essential complex symmetry* if (1.7) holds with $\epsilon\in\mathbb{T}$ instead of $\epsilon\in\{-1,1\}$. It is straightforward to see that p has essential complex symmetry if and only if $\lambda\mathsf{p}$ has complex symmetry for some $\lambda\in\mathbb{T}$. Note that p has real coefficients if and only if $\mathsf{p}^\star(z)=\mathsf{p}(z^{-1})$. Therefore, for a Laurent polynomial p having real coefficients, p has complex symmetry $\iff$ p has essential complex symmetry $\iff$ p has symmetry.

We have the following result on symmetry and complex symmetry of Laurent polynomials.

Proposition 2.2. *Let p be a nontrivial Laurent polynomial with complex coefficients.*

(i) *p has symmetry as in (1.6) if and only if*

$$\mathsf{Z}(\mathsf{p},z)=\mathsf{Z}(\mathsf{p},z^{-1})\qquad\forall\, z\in\mathbb{C}\backslash\{0\}. \tag{2.3}$$

(ii) *p has essential complex symmetry if and only if*

$$\mathsf{Z}(\mathsf{p},z)=\mathsf{Z}(\mathsf{p},\overline{z}^{-1})\qquad\forall\, z\in\mathbb{C}\backslash\{0\}. \tag{2.4}$$

(iii) *If both (2.4) and (2.2) hold, then there exist $k\in\mathbb{Z}$ and $\tilde\lambda\in\mathbb{T}$ ($\tilde\lambda\in\{-1,1\}$ if p has real coefficients) such that $\tilde\lambda z^{-k}\mathsf{p}(z)\geqslant 0$ for all $z\in\mathbb{T}$.*

Proof. (i) If p has symmetry, then $\mathsf{p}(z)=\mathsf{p}(z^{-1})\mathsf{S}\mathsf{p}(z)$, from which it is trivial to see that (2.3) holds. Conversely, (2.3) implies that $\mathsf{p}(z)$ and $\mathsf{p}(z^{-1})$ are only different from each other by a multiplicative monomial λz^c for some $\lambda\in\mathbb{C}\backslash\{0\}$ and $c\in\mathbb{Z}$, more precisely, $\mathsf{p}(z)=\lambda z^c\mathsf{p}(z^{-1})$ for all $z\in\mathbb{C}\backslash\{0\}$. Write $\mathsf{p}(z)=(z-1)^{\mathsf{Z}(\mathsf{p},1)}\mathsf{q}(z)$ for some Laurent polynomial q. Then $\mathsf{q}(1)\neq 0$ and

$$\lambda z^c=\frac{\mathsf{p}(z)}{\mathsf{p}(z^{-1})}=\frac{(z-1)^{\mathsf{Z}(\mathsf{p},1)}\mathsf{q}(z)}{(z^{-1}-1)^{\mathsf{Z}(\mathsf{p},1)}\mathsf{q}(z^{-1})}=(-z)^{\mathsf{Z}(\mathsf{p},1)}\frac{\mathsf{q}(z)}{\mathsf{q}(z^{-1})}.$$

Plugging $z=1$ into the above identity, we deduce that $\lambda=(-1)^{\mathsf{Z}(\mathsf{p},1)}$. Hence, $\mathsf{p}(z)=(-1)^{\mathsf{Z}(\mathsf{p},1)}z^c\mathsf{p}(z^{-1})$; that is, p has symmetry.

(ii) The proof is similar to (i). Suppose that (1.7) holds. By the simple fact in (2.1), we see that

$$\mathsf{p}(z)=\epsilon z^c\mathsf{p}^\star(z)=\epsilon z^c\overline{\mathsf{p}(\overline{z}^{-1})},\qquad z\in\mathbb{C}\backslash\{0\}.$$

Therefore, (2.4) is obviously true. Conversely, suppose that (2.4) holds. Then $\mathsf{p}(z)=\lambda z^c\mathsf{p}^\star(z)$ for some $\lambda\in\mathbb{C}$ and $c\in\mathbb{Z}$. We now prove $|\lambda|=1$. In fact, by $\overline{z}^{-1}=z$ for all $z\in\mathbb{T}$,

$$|\mathsf{p}(z)|=|\lambda||\mathsf{p}^\star(z)|=|\lambda||\mathsf{p}(\overline{z}^{-1})|=|\lambda||\mathsf{p}(z)|,\qquad z\in\mathbb{T}.$$

Since p is not identically zero, the Laurent polynomial p cannot vanish at all points $z\in\mathbb{T}$. Hence, we must have $|\lambda|=1$. So, p has essential complex symmetry.

We now prove item (iii). Since (2.4) holds, by item (ii), there exists $\lambda\in\mathbb{T}$ ($\lambda=1$ if p has real coefficients) such that $[\lambda\mathsf{p}](z)=\epsilon z^c[\lambda\mathsf{p}(z)]^\star$ with $\epsilon\in\{-1,1\}$ and $c\in\mathbb{Z}$. Since (2.4) and (2.2) hold, we see that the total number of complex roots of p must be an even integer and therefore, $c=2k$ for some integer k. If $\epsilon=-1$ and p has complex coefficients, by replacing λ with $i\lambda$, then we must have $[\lambda\mathsf{p}](z)=z^c[\lambda\mathsf{p}(z)]^\star$. Therefore, we can assume $\epsilon=1$ for p having complex coefficients.

For p having real coefficients, we can write $\mathsf{p}(z)=(z-1)^{\mathsf{Z}(\mathsf{p},1)}\mathsf{q}(z)$ for a Laurent polynomial q having real coefficients and $\mathsf{q}(1)\neq 0$. Therefore,

$$(z-1)^{\mathsf{Z}(\mathsf{p},1)}\mathsf{q}(z)=\mathsf{p}(z)=\epsilon z^c\mathsf{p}^\star(z)=\epsilon z^c(z^{-1}-1)^{\mathsf{Z}(\mathsf{p},1)}\mathsf{q}^\star(z)=(z-1)^{\mathsf{Z}(\mathsf{p},1)}\epsilon(-1)^{\mathsf{Z}(\mathsf{p},1)}z^{c-\mathsf{Z}(\mathsf{p},1)}\mathsf{q}^\star(z).$$

Thus,

$$\mathsf{q}(z)=\epsilon(-1)^{\mathsf{Z}(\mathsf{p},1)}z^{c-\mathsf{Z}(\mathsf{p},1)}\mathsf{q}^\star(z).$$

Plugging $z = 1$ into the above identity and noting that $\mathsf{Z}(\mathsf{p}, 1)$ is an even integer, we must have $\mathsf{q}(1) = \epsilon \mathsf{q}^\star(1)$. Since $\mathsf{q}(1)$ is a nonzero real number, we must have $\mathsf{q}^\star(1) = \mathsf{q}(1)$ and $\epsilon = 1$.

Thus, by $c = 2k$, we have $[\lambda \mathsf{p}]^\star(z) = z^{-2k}\lambda \mathsf{p}(z)$, that is, $[\lambda z^{-k}\mathsf{p}(z)]^\star = \lambda z^{-k}\mathsf{p}(z)$. By (2.2) and Lemma 2.1, there exists $\epsilon \in \{-1, 1\}$ such that $\epsilon\lambda z^{-k}\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$. So, item (iii) holds with $\tilde{\lambda} = \epsilon\lambda$. ☐

For a Laurent polynomial $\mathsf{u}(z) = \sum_{k\in\mathbb{Z}} u(k)z^k$ such that $u(m)u(n) \neq 0$ and $u(k) = 0$ for all $k \in \mathbb{Z}\backslash[m, n]$, $\mathrm{fsupp}(\mathsf{u}) := \mathrm{fsupp}(u) := [m, n]$ denotes the filter support of u, the shortest interval outside which u vanishes.

We have the following result on Laurent polynomials having both symmetry and complex symmetry.

Lemma 2.3. *If a Laurent polynomial* p *has both symmetry and complex symmetry, then either all the coefficients of* p *are real numbers and* $\mathsf{S}\mathsf{p} = \mathbb{S}\mathsf{p}$ *or all the coefficients of* p *are imaginary numbers and* $\mathsf{S}\mathsf{p} = -\mathbb{S}\mathsf{p}$. *In particular, if* $\mathsf{p}(z) \geqslant 0$ *for all* $z \in \mathbb{T}$ *and* p *has symmetry, then all the coefficients of* p *must be real numbers.*

Proof. Since p has both complex symmetry and symmetry, we have $\mathbb{S}\mathsf{p}(z) = \epsilon z^c$ and $\mathsf{S}\mathsf{p}(z) = \tilde{\epsilon} z^{\tilde{c}}$ for some $\epsilon, \tilde{\epsilon} \in \{-1, 1\}$ and $c, \tilde{c} \in \mathbb{Z}$. Because the symmetry centers $\frac{c}{2}$ and $\frac{\tilde{c}}{2}$ must be the center of the interval $\mathrm{fsupp}(\mathsf{p})$, we deduce that $c = \tilde{c}$. If $\epsilon = \tilde{\epsilon}$, then it is easy to see that all the coefficients of p must be real. If $\epsilon = -\tilde{\epsilon}$, then all the coefficients of p must be imaginary. If $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$, then $\mathbb{S}\mathsf{p}(z) = 1$ and $\mathsf{Z}(\mathsf{p}, 1)$ is an even integer. Write $\mathsf{p}(z) = (z - 1)^{\mathsf{Z}(\mathsf{p},1)}\mathsf{q}(z)$. Then

$$\tilde{\epsilon} z^{\tilde{c}} = \frac{\mathsf{p}(z)}{\mathsf{p}(z^{-1})} = \frac{(z-1)^{\mathsf{Z}(\mathsf{p},1)}\mathsf{q}(z)}{(z^{-1}-1)^{\mathsf{Z}(\mathsf{p},1)}\mathsf{q}(z^{-1})} = (-z)^{\mathsf{Z}(\mathsf{p},1)}\frac{\mathsf{q}(z)}{\mathsf{q}(z^{-1})}.$$

Plugging $z = 1$ into the above identity, we see that $\tilde{\epsilon} = (-1)^{\mathsf{Z}(\mathsf{p},1)} = 1$ since $\mathsf{Z}(\mathsf{p}, 1)$ is even. Hence, $\mathbb{S}\mathsf{p} = \mathsf{S}\mathsf{p} = 1$ and all the coefficients of p must be real numbers. ☐

We are now ready to discuss greatest common divisors of Laurent polynomials. For two nontrivial Laurent polynomials p_1 and p_2, we say that a Laurent polynomial p is a greatest common divisor of p_1 and p_2 if

(i) p is a common divisor/factor of p_1 and p_2, that is, $\mathsf{p} \mid \mathsf{p}_1$ and $\mathsf{p} \mid \mathsf{p}_2$;
(ii) any common divisor/factor of p_1 and p_2 must be a factor of p.

In terms of the notation $\mathsf{Z}(\mathsf{p}, z)$, we see that p is a greatest common divisor of p_1 and p_2 if and only if $\mathsf{Z}(\mathsf{p}, z) = \min(\mathsf{Z}(\mathsf{p}_1, z), \mathsf{Z}(\mathsf{p}_2, z))$ for all $z \in \mathbb{C}\backslash\{0\}$. Up to a multiplicative monomial λz^k for some $\lambda \in \mathbb{C}\backslash\{0\}$ and $k \in \mathbb{Z}$, a greatest common divisor of two Laurent polynomials p_1 and p_2 are unique and can be found by the Euclidean algorithm.

We now normalize a greatest common divisor $\gcd(\mathsf{p}_1, \mathsf{p}_2)$ for any two Laurent polynomials p_1 and p_2. Let p be a great common divisor of p_1 and p_2. If (2.4) fails, then we define $\gcd(\mathsf{p}_1, \mathsf{p}_2) := \mathsf{p}$; otherwise, by Proposition 2.2 the Laurent polynomial p has essential complex symmetry: $[\lambda \mathsf{p}(z)]^\star = \epsilon z^{-c}[\lambda \mathsf{p}(z)]$ with $\lambda \in \mathbb{T}$ and $\epsilon = 1$ for p having complex coefficients or with $\lambda = 1$ for p having real coefficients. We define $\gcd(\mathsf{p}_1, \mathsf{p}_2) := \lambda \mathsf{p}$ if (2.4) holds but (2.2) fails. If both conditions (2.4) and (2.2) are satisfied, by item (iii) of Proposition 2.2, there exist $k \in \mathbb{Z}$ and $\tilde{\lambda} \in \mathbb{T}$ ($\tilde{\lambda} \in \{-1, 1\}$ if p has real coefficients) such that $\tilde{\lambda} z^{-k}\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$. For this case, we define $\gcd(\mathsf{p}_1, \mathsf{p}_2)(z) := \tilde{\lambda} z^{-k}\mathsf{p}(z)$ so that $\gcd(\mathsf{p}_1, \mathsf{p}_2)(z) \geqslant 0$ for all $z \in \mathbb{T}$. For more than two Laurent polynomials $\mathsf{p}_1, \ldots, \mathsf{p}_r$, we can recursively define $\gcd(\mathsf{p}_1, \ldots, \mathsf{p}_r) = \gcd(\mathsf{p}_1, \gcd(\mathsf{p}_2, \ldots, \mathsf{p}_r))$.

Lemma 2.4. *Suppose that* $\mathsf{p}_1, \ldots, \mathsf{p}_r$ *are Laurent polynomials having symmetry (or complex symmetry). Then* $\gcd(\mathsf{p}_1, \ldots, \mathsf{p}_r)$ *also has symmetry (or complex symmetry). If* $\mathsf{p}_j(z) \geqslant 0$ *for all* $z \in \mathbb{T}$ *and* $j = 1, \ldots, r$, *then* $\gcd(\mathsf{p}_1, \ldots, \mathsf{p}_r)(z) \geqslant 0$ *for all* $z \in \mathbb{T}$.

Proof. Define $\mathsf{p} := \gcd(\mathsf{p}_1, \ldots, \mathsf{p}_r)$. For the case of symmetry, we have

$$\mathsf{Z}(\mathsf{p}, z) = \min(\mathsf{Z}(\mathsf{p}_1, z), \ldots, \mathsf{Z}(\mathsf{p}_r, z)) = \min(\mathsf{Z}(\mathsf{p}_1, z^{-1}), \ldots, \mathsf{Z}(\mathsf{p}_r, z^{-1})) = \mathsf{Z}(\mathsf{p}, z^{-1}) \quad \forall\, z \in \mathbb{C}\backslash\{0\}.$$

Therefore, by item (i) of Proposition 2.2, p must have symmetry. For the case of complex symmetry, by (2.1), we deduce that

$$\mathsf{Z}(\mathsf{p}, z) = \min(\mathsf{Z}(\mathsf{p}_1, z), \ldots, \mathsf{Z}(\mathsf{p}_r, z)) = \min(\mathsf{Z}(\mathsf{p}_1, \bar{z}^{-1}), \ldots, \mathsf{Z}(\mathsf{p}_r, \bar{z}^{-1})) = \mathsf{Z}(\mathsf{p}, \bar{z}^{-1}) \quad \forall\, z \in \mathbb{C}\backslash\{0\}.$$

Consequently, by the definition of $\gcd(\mathsf{p}_1, \ldots, \mathsf{p}_r)$ and Proposition 2.2, we conclude that p must have complex symmetry.

Suppose that $\mathsf{p}_j(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $j = 1, \ldots, r$. It follows from Lemma 2.1 that $\mathsf{p}_j^\star(z) = \mathsf{p}_j(z)$ and $\mathsf{Z}(\mathsf{p}_j, z)$ is an even integer for all $z \in \mathbb{T}$. Thus,

$$\mathsf{Z}(\mathsf{p}, z) = \min(\mathsf{Z}(\mathsf{p}_1, z), \ldots, \mathsf{Z}(\mathsf{p}_r, z)) \in 2\mathbb{Z} \qquad \forall z \in \mathbb{T}. \tag{2.5}$$

Since p has complex symmetry and (2.5) holds, it follows directly from item (iii) of Proposition 2.2 and the definition of p that $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$. ☐

To investigate tight framelet filter banks with symmetry or complex symmetry, as discussed in [13], we have to study a closely related problem about how to split a Laurent polynomial into a sum of squares of no more than two Laurent polynomials having symmetry or complex symmetry. The following results will play an indispensable role in our discussion of matrix splitting with symmetry and tight framelet filter banks later.

For $\epsilon \in \{-1,1\}$ and $c \in \mathbb{Z}$, we say that a Laurent polynomial p has *the complex SOS (sum of squares) property with respect to symmetry type* ϵz^c if there exist two Laurent polynomials p_1 and p_2 having complex symmetry and complex coefficients such that

$$\mathsf{p}_1(z)\mathsf{p}_1^\star(z)+\mathsf{p}_2(z)\mathsf{p}_2^\star(z)=\mathsf{p}(z) \quad \text{and} \quad \frac{\mathbb{S}\mathsf{p}_1(z)}{\mathbb{S}\mathsf{p}_2(z)}=\epsilon z^c. \tag{2.6}$$

Similarly, we say that a Laurent polynomial p has *the real SOS property with respect to symmetry type* ϵz^c if there exist two Laurent polynomials p_1 and p_2 having symmetry and complex coefficients such that

$$\mathsf{p}_1(z)\mathsf{p}_1^\star(z)+\mathsf{p}_2(z)\mathsf{p}_2^\star(z)=\mathsf{p}(z) \quad \text{and} \quad \frac{\mathsf{S}\mathsf{p}_1(z)}{\mathsf{S}\mathsf{p}_2(z)}=\epsilon z^c. \tag{2.7}$$

As we shall see in Theorem 2.7, if p has the real SOS property with respect to the symmetry type ϵz^c, then we can always construct two Laurent polynomials $\mathsf{p}_1,\mathsf{p}_2$ having symmetry and *real coefficients* such that (2.7) is satisfied.

As we discussed in [13], the problem of sum of squares of Laurent polynomials is closely related to how to split a special 2×2 matrix $\mathsf{p}I_2$ as $\mathsf{p}I_2=\mathcal{U}\mathcal{U}^\star$, where the 2×2 matrix $\mathcal{U}$ has symmetry or complex symmetry. This link can be seen from the following auxiliary result, which is needed in our study of the SOS property.

Lemma 2.5. *Suppose that* $\mathsf{p}_1,\mathsf{p}_2,\mathsf{p}_3,\mathsf{p}_4$ *are Laurent polynomials satisfying*

$$\mathbb{S}\mathsf{p}_j(z)=\epsilon_j z^{c_j} \quad \text{with} \quad \epsilon_j\in\{-1,1\},\;\; c_j\in\mathbb{Z}, \quad j=1,\dots,4$$

such that

$$\epsilon_1\epsilon_2=\epsilon_3\epsilon_4 \quad \text{and} \quad c_1+c_2+c_3+c_4\in 2\mathbb{Z}. \tag{2.8}$$

Define

$$\mathsf{p}_5(z):=\mathsf{p}_1(z)\mathsf{p}_3(z)-z^{(c_1+c_2+c_3-c_4)/2}\mathsf{p}_2^\star(z)\mathsf{p}_4(z),$$
$$\mathsf{p}_6(z):=\mathsf{p}_2(z)\mathsf{p}_3(z)+z^{(c_1+c_2+c_3-c_4)/2}\mathsf{p}_1^\star(z)\mathsf{p}_4(z).$$

Then p_5 *and* p_6 *satisfy*

$$\mathbb{S}\mathsf{p}_5(z)=\epsilon_1\epsilon_3 z^{c_1+c_3}, \qquad \mathbb{S}\mathsf{p}_6(z)=\epsilon_2\epsilon_3 z^{c_2+c_3}, \qquad \frac{\mathbb{S}\mathsf{p}_5}{\mathbb{S}\mathsf{p}_6}=\frac{\mathbb{S}\mathsf{p}_1}{\mathbb{S}\mathsf{p}_2}, \tag{2.9}$$

and

$$\mathsf{p}_5(z)\mathsf{p}_5^\star(z)+\mathsf{p}_6(z)\mathsf{p}_6^\star(z)=[\mathsf{p}_1(z)\mathsf{p}_1^\star(z)+\mathsf{p}_2(z)\mathsf{p}_2^\star(z)][\mathsf{p}_3(z)\mathsf{p}_3^\star(z)+\mathsf{p}_4(z)\mathsf{p}_4^\star(z)]. \tag{2.10}$$

Moreover, the same conclusion holds if complex symmetry operator $\mathbb{S}$ *is replaced by symmetry operator* S.

Proof. Note that

$$\begin{bmatrix}\mathsf{p}_1(z) & -\mathsf{p}_2^\star(z)\\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z)\end{bmatrix}\begin{bmatrix}\mathsf{p}_1(z) & -\mathsf{p}_2^\star(z)\\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z)\end{bmatrix}^\star=[\mathsf{p}_1(z)\mathsf{p}_1^\star(z)+\mathsf{p}_2(z)\mathsf{p}_2^\star(z)]I_2$$

and

$$\begin{bmatrix}\mathsf{p}_5(z) & -\mathsf{p}_6^\star(z)\\ \mathsf{p}_6(z) & \mathsf{p}_5^\star(z)\end{bmatrix}=\begin{bmatrix}\mathsf{p}_1(z) & -\mathsf{p}_2^\star(z)\\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z)\end{bmatrix}\begin{bmatrix}\mathsf{p}_3(z) & -[z^k\mathsf{p}_4(z)]^\star\\ z^k\mathsf{p}_4(z) & \mathsf{p}_3^\star(z)\end{bmatrix},$$

where $k:=(c_1+c_2+c_3-c_4)/2\in\mathbb{Z}$ by (2.8). Now it is straightforward to verify that (2.10) holds, since

$$\begin{aligned}[\mathsf{p}_5(z)\mathsf{p}_5^\star(z)+\mathsf{p}_6(z)\mathsf{p}_6^\star(z)]I_2&=\begin{bmatrix}\mathsf{p}_5(z) & -\mathsf{p}_6^\star(z)\\ \mathsf{p}_6(z) & \mathsf{p}_5^\star(z)\end{bmatrix}\begin{bmatrix}\mathsf{p}_5(z) & -\mathsf{p}_6^\star(z)\\ \mathsf{p}_6(z) & \mathsf{p}_5^\star(z)\end{bmatrix}^\star\\ &=\begin{bmatrix}\mathsf{p}_1(z) & -\mathsf{p}_2^\star(z)\\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z)\end{bmatrix}\begin{bmatrix}\mathsf{p}_3(z) & -[z^k\mathsf{p}_4(z)]^\star\\ z^k\mathsf{p}_4(z) & \mathsf{p}_3^\star(z)\end{bmatrix}\begin{bmatrix}\mathsf{p}_3(z) & -[z^k\mathsf{p}_4(z)]^\star\\ z^k\mathsf{p}_4(z) & \mathsf{p}_3^\star(z)\end{bmatrix}^\star\begin{bmatrix}\mathsf{p}_1(z) & -\mathsf{p}_2^\star(z)\\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z)\end{bmatrix}^\star\\ &=[\mathsf{p}_1(z)\mathsf{p}_1^\star(z)+\mathsf{p}_2(z)\mathsf{p}_2^\star(z)][\mathsf{p}_3(z)\mathsf{p}_3^\star(z)+\mathsf{p}_4(z)\mathsf{p}_4^\star(z)]I_2.\end{aligned}$$

(2.9) can be directly checked. ☐

For Laurent polynomials having the complex SOS property, we have

Theorem 2.6. *Let* $\epsilon\in\{-1,1\}$ *and* $c\in\mathbb{Z}$. *A Laurent polynomial* p *has the complex SOS property with respect to the symmetry type* ϵz^c *if and only if* $\mathsf{p}(z)\geqslant 0$ *for all* $z\in\mathbb{T}$.

Proof. The necessity part is trivial, since (2.6) obviously implies $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$.

We now prove the sufficiency part. If (2.6) holds, then we have

$$\mathsf{p}_1(z)\mathsf{p}_1^\star(z) + [i\mathsf{p}_2(z)][i\mathsf{p}_2(z)]^\star = \mathsf{p}(z) \quad \text{and} \quad \frac{\mathbb{S}\mathsf{p}_1(z)}{\mathbb{S}(i\mathsf{p}_2(z))} = -\epsilon z^c$$

and for any integer $k \in \mathbb{Z}$,

$$\mathsf{p}_1(z)\mathsf{p}_1^\star(z) + [z^k\mathsf{p}_2(z)][z^k\mathsf{p}_2(z)]^\star = \mathsf{p}(z) \quad \text{and} \quad \frac{\mathbb{S}\mathsf{p}_1(z)}{\mathbb{S}(z^k\mathsf{p}_2(z))} = \epsilon z^{c-2k}.$$

Therefore, it suffices to prove (2.6) for the cases $\epsilon = -1$ and $c \in \{0, 1\}$.

Since $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$, by Lemma 2.1 and Proposition 2.2, the relations in (2.2) and (2.4) must hold. Therefore, for a positive constant λ, $\lambda\mathsf{p}$ is a product of the factors $(z - z_0)(z^{-1} - \overline{z_0})$ with $z_0 \in \mathbb{C}\backslash\{0\}$. Define

$$\tilde{\mathsf{p}}_1(z) := \tfrac{1}{2}z + \tfrac{1}{2}z^{-1} - \operatorname{Re}(z_0), \qquad \tilde{\mathsf{p}}_2(z) := \tfrac{1}{2}z - \tfrac{1}{2}z^{-1} - i\operatorname{Im}(z_0)$$

and

$$\mathring{\mathsf{p}}_1(z) := 1 - |z_0|, \qquad \mathring{\mathsf{p}}_2(z) := \overline{\sqrt{z_0}} - \sqrt{z_0}z^{-1},$$

where $\sqrt{z_0}$ denotes a complex number such that $(\sqrt{z_0})^2 = z_0$ (more precisely, write $z_0 = re^{i\theta}$ with $r \geqslant 0$ and $\theta \in \mathbb{R}$, then $\sqrt{z_0} := \sqrt{r}e^{i\theta/2}$ or $\sqrt{z_0} := -\sqrt{r}e^{i\theta/2}$). Then

$$\mathbb{S}\tilde{\mathsf{p}}_1(z) = 1, \quad \mathbb{S}\tilde{\mathsf{p}}_2(z) = -1, \quad \mathbb{S}\mathring{\mathsf{p}}_1(z) = 1, \quad \mathbb{S}\mathring{\mathsf{p}}_2(z) = -z^{-1}, \quad \frac{\mathbb{S}\tilde{\mathsf{p}}_1}{\mathbb{S}\tilde{\mathsf{p}}_2} = -1, \ \frac{\mathbb{S}\mathring{\mathsf{p}}_1}{\mathbb{S}\mathring{\mathsf{p}}_2} = -z.$$

By direct calculation, we have

$$(z - z_0)(z^{-1} - \overline{z_0}) = \tilde{\mathsf{p}}_1(z)\tilde{\mathsf{p}}_1^\star(z) + \tilde{\mathsf{p}}_2(z)\tilde{\mathsf{p}}_2^\star(z) = \mathring{\mathsf{p}}_1(z)\mathring{\mathsf{p}}_1^\star(z) + \mathring{\mathsf{p}}_2(z)\mathring{\mathsf{p}}_2^\star(z).$$

Now it follows from Lemma 2.5 that the claim in (2.6) holds with $cz^c = -1$ or $-z$. □

For Laurent polynomials having the real SOS property, the situation is more complicated and the following result is essentially [13, Lemma 4.4].

Theorem 2.7. *Let $\epsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$. A Laurent polynomial p has the real SOS property with respect to the symmetry type ϵz^c if and only if*

(i) *p has real coefficients and $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$;*

(ii) *p satisfies the technical condition for the real SOS property with respect to the symmetry type ϵz^c, that is, one of the following four cases must hold:*
   - (a) *if $\epsilon = -1$ and c is an even integer, then there is no condition;*
   - (b) *if $\epsilon = -1$ and c is an odd integer, then $\mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}$ for all $x \in (-1, 0)$;*
   - (c) *if $\epsilon = 1$ and c is an even integer, then $\mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}$ for all $x \in (-1, 0) \cup (0, 1)$;*
   - (d) *if $\epsilon = 1$ and c is an odd integer, then $\mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}$ for all $x \in (0, 1)$.*

Moreover, if items (i) and (ii) are satisfied, then there exist Laurent polynomials p_1 and p_2 with real coefficients and symmetry such that (2.7) *holds.*

Proof. Necessity. By (2.7), it is trivially seen that $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$. Since p_1 and p_2 have symmetry, p must have symmetry. It follows from Lemma 2.3 that p has real coefficients. Therefore, item (i) holds.

On the other hand, by (2.7) and the symmetry of p_1 and p_2, we have $\mathsf{p}_1^\star(z) = \overline{\mathsf{p}_1(\overline{z}^{-1})} = \overline{\mathsf{p}_1(\overline{z})/\mathsf{S}\mathsf{p}_1(\overline{z})} = \mathsf{p}_1(\overline{z})/\mathsf{S}\mathsf{p}_1(z)$ and hence (2.7) becomes

$$\mathsf{p}(x) = [|\mathsf{p}_1(x)|^2 + \epsilon x^c|\mathsf{p}_2(x)|^2]/\mathsf{S}\mathsf{p}_1(x), \qquad x \in \mathbb{R}\backslash\{0\}. \tag{2.11}$$

When $\epsilon = -1$ and c is an odd integer, since $\epsilon x^c > 0$ for all $x < 0$, one can easily deduce from (2.11) that $\mathsf{Z}(\mathsf{p}, x) = 2\min(\mathsf{Z}(\mathsf{p}_1, x), \mathsf{Z}(\mathsf{p}_2, x))$ for all $x < 0$. Hence, item (b) holds. Items (c) and (d) can be proved similarly.

The sufficiency part is the same as the proof of [13, Lemma 4.4]. □

In the rest of this section, let us consider the problem on sum of squares using only one Laurent polynomial with symmetry or complex symmetry. For a complex number $z \in \mathbb{C}$, its sign is defined to be

$$\operatorname{sgn}(z) := \begin{cases} \frac{z}{|z|}, & \text{if } z \in \mathbb{C}\backslash\{0\}; \\ 0, & \text{if } z = 0. \end{cases}$$

For a real number $x \in \mathbb{R}$, we have $\operatorname{sgn}(x) = 1$ if $x > 0$ and $\operatorname{sgn}(x) = -1$ if $x < 0$. Recall that $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For the case of complex symmetry, we have

Theorem 2.8. *Let $\mathsf{p}(z) = p_0 + \sum_{k=1}^N (p_k z^k + \overline{p_k} z^{-k})$ with $p_N \neq 0, p_0 \in \mathbb{R}$ and $p_1, \ldots, p_N \in \mathbb{C}$. Then the following statements are equivalent:*

(1) *There exists a Laurent polynomial* q *having complex symmetry and satisfying* $\mathsf{p}(z)=\mathsf{q}(z)\mathsf{q}^\star(z)$ *for all* $z\in\mathbb{C}\backslash\{0\}$.

(2) $\mathsf{p}(z)\geqslant 0$ *for all* $z\in\mathbb{T}$ *and the following relation holds:*

$$\mathsf{Z}(\mathsf{p},z)\in 2\mathbb{Z}\qquad \forall\, z\in\mathbb{C}\backslash\{0\}. \tag{2.12}$$

(3) *There exists a nonzero number* λ *such that* $\lambda\mathsf{q}$ *has complex symmetry and* $[\lambda\mathsf{q}(z)][\lambda\mathsf{q}(z)]^\star=\mathsf{p}(z)$ *for all* $z\in\mathbb{C}\backslash\{0\}$, *where*

$$\mathsf{q}(z):=\prod_{z_0\in\mathbb{C}\backslash\{0\}}(z-z_0)^{\mathsf{Z}(\mathsf{p},z_0)/2},\qquad z\in\mathbb{C}\backslash\{0\}. \tag{2.13}$$

Moreover, if p *has real coefficients, then* $\lambda\mathsf{q}$ *can also have real coefficients.*

(4) $\mathsf{p}(z)=\mathsf{q}_\mathsf{p}(z)\mathsf{q}_\mathsf{p}^\star(z)$ *for all* $z\in\mathbb{C}\backslash\{0\}$, *where* q_p *is defined by one of the following two cases:*

Case 1: *If* $N=2n$ *for* $n\in\mathbb{N}_0$, *define* $\mathsf{q}_\mathsf{p}(z):=\sqrt{|p_N|}\Big[\frac{e^{i\alpha}t_0+e^{-i\alpha}\overline{t_0}}{2}+\sum_{k=1}^n(e^{i\alpha}t_kz^k+e^{-i\alpha}\overline{t_k}z^{-k})\Big]$.

Case 2: *If* $N=2n+1$ *for* $n\in\mathbb{N}_0$, *define* $\mathsf{q}_\mathsf{p}(z):=\sqrt{|p_N|}\sum_{k=0}^n(e^{i\alpha}t_kz^k+e^{-i\alpha}\overline{t_k}z^{-1-k})$,

where α *is a real number satisfying* $e^{i\alpha}:=sgn(\sqrt{p_N})$, $t_n:=1$ *and*

$$t_{n-j}:=\frac{1}{2}\Big[\frac{p_{N-j}}{p_N}-\sum_{k=n-j+1}^{n-1}t_kt_{2n-j-k}\Big],\qquad j=1,\dots,n. \tag{2.14}$$

If in addition p *has real coefficients, replace* q_p *by* $i\mathsf{q}_\mathsf{p}$ *if* $p_N<0$, *then* q_p *also has real coefficients.*

Proof. (1)$\Longrightarrow$(2). Since q has complex symmetry, we have $\mathsf{p}(z)=\mathsf{q}(z)\mathsf{q}^\star(z)=[\mathsf{q}(z)]^2/\mathbb{S}\mathsf{q}(z)$. Consequently, $\mathsf{Z}(\mathsf{p},z)=2\mathsf{Z}(\mathsf{q},z)\in 2\mathbb{Z}$ for all $z\in\mathbb{C}\backslash\{0\}$ and $\mathsf{p}(z)\geqslant 0$ for all $z\in\mathbb{T}$.

(2)$\Longrightarrow$(3). By (2.12), q in (2.13) is a well-defined (Laurent) polynomial. Since $\mathsf{p}^\star=\mathsf{p}$, we see that (2.4) holds. Consequently, by the definition of q in (2.13), we see that $\mathsf{Z}(\mathsf{q},z)=\mathsf{Z}(\mathsf{q},\overline{z}^{-1})$ for all $z\in\mathbb{C}\backslash\{0\}$. Note that if p has real coefficients, then q also has real coefficients. By Proposition 2.2, there is $\tilde\lambda\in\mathbb{T}$ ($\tilde\lambda=1$ if q has real coefficients) such that $\tilde\lambda\mathsf{q}$ has complex symmetry. It is also easy to see that there exists a positive number ρ such that $\rho[\tilde\lambda\mathsf{q}(z)][\tilde\lambda\mathsf{q}(z)]^\star=\mathsf{p}(z)$ for all $z\in\mathbb{C}\backslash\{0\}$. Define $\lambda:=\tilde\lambda\sqrt{\rho}$. Then item (3) holds.

It is trivially seen that (3)$\Longrightarrow$(1). Item (4) is an explicit way of finding a particular q in item (1) and the solution in (2.14) is obtained by directly comparing the first $n+1$ highest terms in $\mathsf{q}_\mathsf{p}(z)\mathsf{q}_\mathsf{p}^\star(z)$ and $\mathsf{p}(z)$. ☐

By item (3) of Theorem 2.8, up to a multiplicative monomial λz^k with $\lambda\in\{\pm1,\pm i\}$ and $k\in\mathbb{Z}$, a Laurent polynomial q having complex symmetry and satisfying $\mathsf{p}(z)=\mathsf{q}(z)\mathsf{q}^\star(z)$ in Theorem 2.8 is unique.

For a Laurent polynomial p having symmetry, we have

Theorem 2.9. *Let* $\mathsf{p}(z)=p_0+\sum_{k=1}^N p_k(z^k+z^{-k})$ *with* $p_N\neq 0$ *be a Laurent polynomial with complex coefficients. Then the following statements are equivalent:*

(1) $\mathsf{p}(z)=\mathsf{q}(z)\mathsf{q}^\star(z)$ *for some Laurent polynomial* q *having symmetry.*

(2) $\mathsf{p}(z)=\mathsf{q}^{[r]}(z)(\mathsf{q}^{[r]}(z))^\star+\mathsf{q}^{[i]}(z)(\mathsf{q}^{[i]}(z))^\star$ *for some Laurent polynomials* $\mathsf{q}^{[r]}$ *and* $\mathsf{q}^{[i]}$ *having real coefficients and symmetry such that* $\mathsf{S}\mathsf{q}^{[r]}(z)=\mathsf{S}\mathsf{q}^{[i]}(z)$.

(3) *All coefficients* $p_0,\dots,p_N\in\mathbb{R}$, $\mathsf{p}(z)\geqslant 0$ *for all* $z\in\mathbb{T}$, *and* $\mathsf{Z}(\mathsf{p},x)\in 2\mathbb{Z}$ *for all* $x\in(-1,0)\cup(0,1)$ *(the last condition can be replaced by* $\mathsf{Z}(\mathsf{p},x)\in 2\mathbb{Z}$ *for all* $x\in\mathbb{R}\backslash\{0\}$*).*

(4) *Both* $\mathsf{Z}(\mathsf{p},1)$ *and* $\mathsf{Z}(\mathsf{p},-1)$ *are even integers and* $\mathsf{p}(z)=\zeta^{\mathsf{Z}(\mathsf{p},1)/2}(1-\zeta)^{\mathsf{Z}(\mathsf{p},-1)/2}\mathsf{P}(\zeta)$ *with* $\zeta:=\frac12-\frac14z-\frac14z^{-1}$; *that is,* $\mathsf{p}(e^{-i\xi})=\sin^{\mathsf{Z}(\mathsf{p},1)}(\xi/2)\cos^{\mathsf{Z}(\mathsf{p},-1)}(\xi/2)\mathsf{P}(\sin^2(\xi/2))$, *where* P *is a polynomial having real coefficients and satisfying* $\mathsf{P}(x)\geqslant 0$ *for all* $x\in\mathbb{R}$.

(5) *All coefficients* $p_0,\dots,p_N\in\mathbb{R}$ *and* $\mathsf{p}(z)=\mathsf{q}_\mathsf{p}(z)\mathsf{q}_\mathsf{p}^\star(z)$, *where the Laurent polynomial* q_p *is defined by one of the following two cases:*

Case 1: *If* $N=2n$ *for* $n\in\mathbb{N}_0$, *define* $\mathsf{q}_\mathsf{p}(z):=\sqrt{|p_N|}\Big[\frac{1+sgn(p_N)}{2}t_0+\sum_{k=1}^n t_k(z^k+sgn(p_N)z^{-k})\Big]$;

Case 2: *If* $N=2n+1$ *for* $n\in\mathbb{N}_0$, *define* $\mathsf{q}_\mathsf{p}(z):=\sqrt{|p_N|}\sum_{k=0}^n t_k(z^k+sgn(p_N)z^{-1-k})$,

where $t_n:=1$ *and*

$$\mathrm{Re}(t_{n-j}):=\frac12\Big[\frac{p_{N-j}}{p_N}-\sum_{k=n-j+1}^{n-1}\mathrm{Re}(t_k\overline{t_{2n-j-k}})\Big],\qquad j=1,\dots,n. \tag{2.15}$$

Proof. Write $\mathsf{q}=\mathsf{q}^{[r]}+i\mathsf{q}^{[i]}$, where $\mathsf{q}^{[r]}$ and $\mathsf{q}^{[i]}$ are Laurent polynomials having real coefficients. Since q has symmetry, we have $\mathsf{S}\mathsf{q}^{[r]}=\mathsf{S}\mathsf{q}^{[i]}=\mathsf{S}\mathsf{q}$ and $\mathsf{q}^{[r]}(z)(\mathsf{q}^{[r]}(z))^\star+\mathsf{q}^{[i]}(z)(\mathsf{q}^{[i]}(z))^\star=\mathsf{p}(z)$. So, (1)$\Longrightarrow$(2). Conversely, we take $\mathsf{q}=\mathsf{q}^{[r]}+i\mathsf{q}^{[i]}$ and therefore, (2)$\Longrightarrow$(1).

The equivalence between items (2) and (3) is proved in Theorem 2.7. We now prove (3)$\Longrightarrow$(4). Since $\mathsf{S}\mathsf{p}=1$ and p has real coefficients, we can always write $\mathsf{p}(z)=\zeta^{\mathsf{Z}(\mathsf{p},1)/2}(1-\zeta)^{\mathsf{Z}(\mathsf{p},-1)/2}\mathsf{P}(\zeta)$ with $\zeta=\frac12-\frac14z-\frac14z^{-1}$, where P is a polynomial with real coefficients. Consider the map $\eta:\mathbb{C}\backslash\{0\}\to\mathbb{C}$ with $\eta(z)=\frac12-\frac14z-\frac14z^{-1}$.

Then η is one-to-one and onto from $\{z \in \mathbb{T} : \operatorname{Im}(z) > 0\}$ to $(0,1)$ and from $(-1,0) \cup (0,1)$ to $\mathbb{R}\backslash[0,1]$. Note that $\mathsf{Z}(\mathsf{P},0) = \mathsf{Z}(\mathsf{P},1) = 0$. Now it follows directly from item (3) that $\mathsf{Z}(\mathsf{P},x) \in 2\mathbb{Z}$ for all $x \in \mathbb{R}$. Since P has real coefficients, either $\mathsf{P}(x) \geqslant 0$ or $\mathsf{P}(x) \leqslant 0$ for all $x \in \mathbb{R}$. Since $\sin^{\mathsf{Z}(\mathsf{p},1)}(\xi/2)\cos^{\mathsf{Z}(\mathsf{p},-1)}(\xi/2)\mathsf{P}(\sin^2(\xi/2)) = \mathsf{p}(e^{-i\xi}) \geqslant 0$ for all $\xi \in \mathbb{R}$ and since both $\mathsf{Z}(\mathsf{p},1)$ and $\mathsf{Z}(\mathsf{p},-1)$ are even, we must have $\mathsf{P}(\sin^2(\xi/2)) \geqslant 0$ for all $\xi \in \mathbb{R}$ and consequently, $\mathsf{P}(x) \geqslant 0$ for all $x \in \mathbb{R}$.

(4)$\Longrightarrow$(2) follows directly from [11, Proposition 5]. Item (5) is an explicit way of finding a particular q in item (1) and (2.15) is obtained by directly comparing the first $n+1$ highest terms in $\mathsf{q}_\mathsf{p}(z)\mathsf{q}_\mathsf{p}^\star(z)$ and $\mathsf{p}(z)$. $\square$

There are often finitely many but essentially different solutions q having symmetry and satisfying $\mathsf{p}(z) = \mathsf{q}(z)\mathsf{q}^\star(z)$. Nevertheless, up to a factor z^{2k} with $k \in \mathbb{Z}$, its symmetry type $\mathsf{S}\mathsf{q}$ is uniquely determined by p.

3. Splitting Matrices of Laurent Polynomials with Symmetry or Complex Symmetry

In this section we shall study the general problem of splitting a positive semi-definite 2×2 matrix of Laurent polynomials with symmetry or complex symmetry. We shall present a complete characterization on this matrix splitting problem with the symmetry constraint.

Before presenting our general result on factorizing a 2×2 matrix of Laurent polynomials with the symmetry constraint, we need the following auxiliary result, which slightly generalizes a corresponding result which is not explicitly stated in [13].

Lemma 3.1. *Let*

$$\mathcal{N}(z) := \begin{bmatrix} \mathcal{N}_{1,1}(z) & \mathcal{N}_{1,2}(z) \\ \mathcal{N}_{2,1}(z) & \mathcal{N}_{2,2}(z) \end{bmatrix} \tag{3.1}$$

be a 2×2 matrix of Laurent polynomials with complex coefficients. Define

$$\mathsf{p}(z) := \gcd(\mathcal{N}_{1,1}(z), \mathcal{N}_{1,2}(z), \mathcal{N}_{2,1}(z), \mathcal{N}_{2,2}(z)).$$

If $\mathcal{N}(z) \geqslant 0$ for all $z \in \mathbb{T}$, then $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$.

Proof. Since $\mathcal{N}(z) \geqslant 0$ for all $z \in \mathbb{T}$, we have

$$\mathcal{N}^\star(z) = \mathcal{N}(z), \quad \mathcal{N}_{1,1}(z) \geqslant 0, \quad \mathcal{N}_{2,2}(z) \geqslant 0, \quad \det(\mathcal{N}(z)) \geqslant 0, \qquad \forall\, z \in \mathbb{T}. \tag{3.2}$$

By $\mathcal{N}^\star = \mathcal{N}$, we have $\mathcal{N}_{1,2}^\star = \mathcal{N}_{2,1}$. Hence $\mathsf{Z}(\mathcal{N}_{2,1}, z) = \mathsf{Z}(\mathcal{N}_{1,2}, \bar{z}^{-1})$. By

$$\begin{aligned} \mathsf{Z}(\gcd(\mathcal{N}_{1,2}, \mathcal{N}_{2,1}), z) &= \min(\mathsf{Z}(\mathcal{N}_{1,2}, z), \mathsf{Z}(\mathcal{N}_{2,1}, z)) = \min(\mathsf{Z}(\mathcal{N}_{1,2}, z), \mathsf{Z}(\mathcal{N}_{1,2}, \bar{z}^{-1})) \\ &= \min(\mathsf{Z}(\mathcal{N}_{1,2}, \bar{z}^{-1}), \mathsf{Z}(\mathcal{N}_{1,2}, z)) = \mathsf{Z}(\gcd(\mathcal{N}_{1,2}, \mathcal{N}_{2,1}), \bar{z}^{-1}) \end{aligned}$$

and by Proposition 2.2, we see that $\gcd(\mathcal{N}_{1,2}, \mathcal{N}_{2,1})$ must have complex symmetry and therefore, p has complex symmetry. We can also use the relation $\gcd(\mathcal{N}_{1,2}, \mathcal{N}_{2,1}) = \gcd(\mathcal{N}_{1,2}, \mathcal{N}_{1,2}^\star) = \gcd(\mathcal{N}_{1,2} + \mathcal{N}_{1,2}^\star, \mathcal{N}_{1,2} - \mathcal{N}_{1,2}^\star)$ to conclude that $\gcd(\mathcal{N}_{1,2}, \mathcal{N}_{2,1})$ has complex symmetry.

On the other hand, since $\det(\mathcal{N}(z)) \geqslant 0$ for all $z \in \mathbb{T}$, we have $\mathcal{N}_{1,1}(z)\mathcal{N}_{2,2}(z) - \mathcal{N}_{1,2}(z)\mathcal{N}_{2,1}(z) = \det(\mathcal{N}(z)) \geqslant 0$ for all $z \in \mathbb{T}$. Consequently,

$$0 \leqslant \mathcal{N}_{1,2}(z)\mathcal{N}_{1,2}^\star(z) = \mathcal{N}_{1,2}(z)\mathcal{N}_{2,1}(z) \leqslant \mathcal{N}_{1,1}(z)\mathcal{N}_{2,2}(z), \qquad \forall\, z \in \mathbb{T}. \tag{3.3}$$

Since $\mathcal{N}_{2,1}(z) = \mathcal{N}_{1,2}^\star(z) = \overline{\mathcal{N}_{1,2}(\bar{z}^{-1})}$ and $z = \bar{z}^{-1}$ for all $z \in \mathbb{T}$, we must have $\mathsf{Z}(\mathcal{N}_{2,1}, z) = \mathsf{Z}(\mathcal{N}_{1,2}, z)$ for all $z \in \mathbb{T}$. Therefore, it follows from (3.3) that

$$2\mathsf{Z}(\mathcal{N}_{1,2}, z) = \mathsf{Z}(\mathcal{N}_{1,2}, z) + \mathsf{Z}(\mathcal{N}_{2,1}, z) \geqslant \mathsf{Z}(\mathcal{N}_{1,1}, z) + \mathsf{Z}(\mathcal{N}_{2,2}, z) \qquad \forall\, z \in \mathbb{T}. \tag{3.4}$$

By $\mathsf{Z}(\mathsf{p}, z) = \min(\mathsf{Z}(\mathcal{N}_{1,1}, z), \mathsf{Z}(\mathcal{N}_{1,2}, z), \mathsf{Z}(\mathcal{N}_{2,1}, z), \mathsf{Z}(\mathcal{N}_{2,2}, z))$ and Lemma 2.1, it follows from (3.2) and (3.4) that $\mathsf{Z}(\mathsf{p}, z) = \min(\mathsf{Z}(\mathcal{N}_{1,1}, z), \mathsf{Z}(\mathcal{N}_{2,2}, z))$ must be an even integer for all $z \in \mathbb{T}$. Since p has complex symmetry, it follows from item (iii) of Proposition 2.2 and the definition of p that $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$. $\square$

Since a Laurent polynomial or a filter can have symmetry, complex symmetry, real-valued coefficients, or complex-valued coefficients, we have a total of three essentially different types of tight framelet filter banks with symmetry properties: (1) with complex coefficients and symmetry, (2) with complex coefficients and complex symmetry, or (3) with real coefficients and symmetry (for filters or Laurent polynomials having real coefficients, there is no difference between symmetry and complex symmetry). Due to Theorem 2.7 and the fact that the case of symmetry is often more involved than the case of complex symmetry, to avoid redundancy in studying these three essentially different types of tight framelet filter banks with symmetry properties, we often state and prove results in this section for the case of symmetry. For the case of complex symmetry, the common parts/properties, which are shared by both symmetry and complex symmetry, will be ignored, but details, which are different and unique for the case of complex symmetry, will be provided. We sometimes use [complex] symmetry to handle both symmetry and complex symmetry simultaneously.

We now present a general result on factorizing a 2×2 matrix of Laurent polynomials with the symmetry constraint. The following result generalizes [13, Theorem 2.3] to the general case of complex coefficients.

Theorem 3.2. *Let $\mathcal{N}$ be a 2×2 matrix of Laurent polynomials as in* (3.1) *such that all its entries $\mathcal{N}_{1,1},\mathcal{N}_{1,2}$, $\mathcal{N}_{2,1}$, $\mathcal{N}_{2,2}$ have symmetry (and real coefficients) and at least one of them is not identically zero. Then there exist Laurent polynomials $\mathcal{U}_{1,1},\mathcal{U}_{1,2},\mathcal{U}_{2,1},\mathcal{U}_{2,2}$ having symmetry (and real coefficients) such that*

$$\mathcal{U}(z)\mathcal{U}^\star(z)=\mathcal{N}(z) \qquad \text{with} \qquad \mathcal{U}(z):=\begin{bmatrix}\mathcal{U}_{1,1}(z) & \mathcal{U}_{1,2}(z)\\ \mathcal{U}_{2,1}(z) & \mathcal{U}_{2,2}(z)\end{bmatrix}, \qquad \frac{\mathsf{S}\mathcal{U}_{1,1}(z)}{\mathsf{S}\mathcal{U}_{2,1}(z)}=\frac{\mathsf{S}\mathcal{U}_{1,2}(z)}{\mathsf{S}\mathcal{U}_{2,2}(z)}, \qquad z\in\mathbb{C}\backslash\{0\} \tag{3.5}$$

if and only if

(i) $\mathcal{N}(z)\geqslant 0$ *for all* $z\in\mathbb{T}$;
(ii) $\det(\mathcal{N}(z))=\mathsf{d}(z)\mathsf{d}^\star(z)$ *for some Laurent polynomial* d *having symmetry (and real coefficients);*
(iii) $\mathsf{p}(z):=\gcd(\mathcal{N}_{1,1}(z),\mathcal{N}_{1,2}(z),\mathcal{N}_{2,1}(z),\mathcal{N}_{2,2}(z))$ *must satisfy the technical condition for the real SOS property with respect to* $[\mathsf{S}\mathcal{N}_{2,1}][\mathsf{S}\mathsf{d}]$ *(see item (ii) of Theorem 2.7).*

Moreover, if all involved Laurent polynomials (that is, all $\mathcal{N}$,$\mathcal{U}$ and d) have complex symmetry instead of symmetry, replace the symmetry operator S by the complex symmetry operator $\mathbb{S}$, then the same necessary and sufficient condition, with item (iii) removed, still holds.

Proof. We only prove the necessity part, while the sufficiency part will be proved in Algorithm 2.

The proof essentially follows the same line of argument as in the proof of [13, Theorem 2.3]. Suppose that (3.5) is satisfied. It is obvious that item (i) holds. Define $\mathsf{d}(z):=\det(\mathcal{U}(z))$. We see from (3.5) that $\mathsf{d}(z)\mathsf{d}^\star(z)=\det(\mathcal{N}(z))$. Due to the symmetry relation in (3.5), the Laurent polynomial d has symmetry with $\mathsf{S}\mathsf{d}(z)=[\mathsf{S}\mathcal{U}_{1,1}(z)][\mathsf{S}\mathcal{U}_{2,2}(z)]$. Hence, item (ii) must hold.

We now prove item (iii) for the case of symmetry. By item (i), it follows from Lemma 3.1 that $\mathsf{p}(z)\geqslant 0$ for all $z\in\mathbb{T}$. Since p has symmetry, by Lemma 2.3, p must have real coefficients. Since $\mathcal{N}$ is not identically zero, p is not identically zero. Define

$$\breve{\mathcal{N}}(z):=\begin{bmatrix}\breve{\mathcal{N}}_{1,1}(z) & \breve{\mathcal{N}}_{1,2}(z)\\ \breve{\mathcal{N}}_{2,1}(z) & \breve{\mathcal{N}}_{2,2}(z)\end{bmatrix}:=\begin{bmatrix}\frac{\mathcal{N}_{1,1}(z)}{\mathsf{p}(z)} & \frac{\mathcal{N}_{1,2}(z)}{\mathsf{p}(z)}\\ \frac{\mathcal{N}_{2,1}(z)}{\mathsf{p}(z)} & \frac{\mathcal{N}_{2,2}(z)}{\mathsf{p}(z)}\end{bmatrix}=\frac{1}{\mathsf{p}(z)}\mathcal{N}(z). \tag{3.6}$$

Since $\mathsf{p}(z)\geqslant 0$ and $\mathcal{N}(z)\geqslant 0$ for all $z\in\mathbb{T}$, it is not difficult to see that $\breve{\mathcal{N}}(z)\geqslant 0$ for all $z\in\mathbb{T}$. Since all entries of $\mathcal{N}$ have symmetry and $\mathsf{S}\mathsf{p}(z)=1$, all the entries of $\breve{\mathcal{N}}$ have symmetry. We now show that $\det(\breve{\mathcal{N}}(z))=\breve{\mathsf{d}}(z)\breve{\mathsf{d}}^\star(z)$ for some Laurent polynomial $\breve{\mathsf{d}}$ with symmetry. By Theorem 2.9 and $[\mathsf{p}(z)]^2\det(\breve{\mathcal{N}}(z))=\det(\mathcal{N}(z))=\mathsf{d}(z)\mathsf{d}^\star(z)$, it is trivial to see that all the conditions in Theorem 2.9 with p being replaced by $\det(\breve{\mathcal{N}}(z))$ are satisfied. Therefore, there exists a Laurent polynomial $\breve{\mathsf{d}}$ with symmetry such that $\det(\breve{\mathcal{N}}(z))=\breve{\mathsf{d}}(z)\breve{\mathsf{d}}^\star(z)$.

Note that $\gcd(\breve{\mathcal{N}}_{1,1},\breve{\mathcal{N}}_{1,2},\breve{\mathcal{N}}_{2,1},\breve{\mathcal{N}}_{2,2})=1$. Therefore, all three conditions in items (i), (ii), and (iii) in Theorem 3.2 with $\mathcal{N}$ being replaced by $\breve{\mathcal{N}}$ are satisfied. By the sufficiency part of Theorem 3.2 which will be proved in Algorithm 2, there exists a 2×2 matrix $\breve{\mathcal{U}}$ of Laurent polynomials having symmetry (and real coefficients) such that $\breve{\mathcal{U}}(z)\breve{\mathcal{U}}^\star(z)=\breve{\mathcal{N}}(z)$ for all $z\in\mathbb{C}\backslash\{0\}$ and $\frac{\mathsf{S}\breve{\mathcal{U}}_{1,1}(z)}{\mathsf{S}\breve{\mathcal{U}}_{2,1}(z)}=\frac{\mathsf{S}\breve{\mathcal{U}}_{1,2}(z)}{\mathsf{S}\breve{\mathcal{U}}_{2,2}(z)}$, from which and the identity $\breve{\mathcal{N}}_{2,1}(z)=\mathcal{N}_{2,1}(z)/\mathsf{p}(z)$ we deduce that

$$\frac{\mathsf{S}\breve{\mathcal{U}}_{2,1}(z)}{\mathsf{S}\breve{\mathcal{U}}_{1,1}(z)}=\frac{\mathsf{S}\breve{\mathcal{U}}_{2,2}(z)}{\mathsf{S}\breve{\mathcal{U}}_{1,2}(z)}=\mathsf{S}\breve{\mathcal{N}}_{2,1}(z)=\frac{\mathsf{S}\mathcal{N}_{2,1}(z)}{\mathsf{S}\mathsf{p}(z)}=\mathsf{S}\mathcal{N}_{2,1}(z)=\frac{\mathsf{S}\mathcal{U}_{2,1}(z)}{\mathsf{S}\mathcal{U}_{1,1}(z)}=\frac{\mathsf{S}\mathcal{U}_{2,2}(z)}{\mathsf{S}\mathcal{U}_{1,2}(z)}, \tag{3.7}$$

where we used $\mathcal{U}(z)\mathcal{U}^\star(z)=\mathcal{N}(z)$ and (3.5) in the last second identity. Consequently, from $\mathcal{U}(z)\mathcal{U}^\star(z)=\mathcal{N}(z)=\mathsf{p}(z)\breve{\mathcal{N}}(z)=\mathsf{p}(z)\breve{\mathcal{U}}(z)\breve{\mathcal{U}}^\star(z)$, we must have

$$\mathcal{Q}(z)\mathcal{Q}^\star(z)=\breve{\mathsf{d}}(z)\breve{\mathsf{d}}^\star(z)\mathsf{p}(z)I_2, \tag{3.8}$$

where

$$\mathcal{Q}(z):=\begin{bmatrix}\mathsf{q}_1(z) & \mathsf{q}_2(z)\\ \mathsf{q}_3(z) & \mathsf{q}_4(z)\end{bmatrix}:=\operatorname{adj}(\breve{\mathcal{U}}(z))\mathcal{U}(z)=\begin{bmatrix}\breve{\mathcal{U}}_{2,2}(z) & -\breve{\mathcal{U}}_{1,2}(z)\\ -\breve{\mathcal{U}}_{2,1}(z) & \breve{\mathcal{U}}_{1,1}(z)\end{bmatrix}\begin{bmatrix}\mathcal{U}_{1,1}(z) & \mathcal{U}_{1,2}(z)\\ \mathcal{U}_{2,1}(z) & \mathcal{U}_{2,2}(z)\end{bmatrix}.$$

By the symmetry relations in (3.7), we can check that all the entries of $\mathcal{Q}$ have symmetry and

$$\frac{\mathsf{S}\mathsf{q}_1(z)}{\mathsf{S}\mathsf{q}_2(z)}=\frac{\mathsf{S}\breve{\mathcal{U}}_{2,2}(z)\mathsf{S}\mathcal{U}_{1,1}(z)}{\mathsf{S}\breve{\mathcal{U}}_{2,2}(z)\mathsf{S}\mathcal{U}_{1,2}(z)}=\frac{\mathsf{S}\mathcal{U}_{1,1}(z)}{\mathsf{S}\mathcal{U}_{1,2}(z)}=\frac{\mathsf{S}\mathcal{U}_{2,2}(z)}{\mathsf{S}\mathcal{U}_{1,2}(z)}\frac{\mathsf{S}\mathcal{U}_{1,1}}{\mathsf{S}\mathcal{U}_{2,2}}=z^{2k}[\mathsf{S}\mathcal{U}_{2,2}(z)]^{-2}\mathsf{S}\mathcal{N}_{2,1}(z)\mathsf{S}\mathsf{d}(z),$$

where we used the fact that $\mathsf{S}\mathsf{d}(z)=z^{-2k}\mathsf{S}\mathcal{U}_{1,1}(z)\mathsf{S}\mathcal{U}_{2,2}(z)$ for some $k\in\mathbb{Z}$. Note that (3.8) implies $\mathsf{q}_1(z)\mathsf{q}_1^\star(z)+\mathsf{q}_2(z)\mathsf{q}_2^\star(z)=\breve{\mathsf{d}}(z)\breve{\mathsf{d}}^\star(z)\mathsf{p}(z)$. Since q_1 and q_2 have symmetry, by Theorem 2.7, $\breve{\mathsf{d}}(z)\breve{\mathsf{d}}^\star(z)\mathsf{p}(z)$ must satisfy the technical condition for the real SOS property with respect to $z^{2k}[\mathsf{S}\mathcal{U}_{2,2}(z)]^{-2}[\mathsf{S}\mathcal{N}_{2,1}(z)][\mathsf{S}\mathsf{d}(z)]$. Because $\breve{\mathsf{d}}$ has symmetry, we have $\mathsf{Z}(\breve{\mathsf{d}}^\star, x) = \mathsf{Z}(\breve{\mathsf{d}}, \bar{x}^{-1}) = \mathsf{Z}(\breve{\mathsf{d}}, \bar{x}) = \mathsf{Z}(\breve{\mathsf{d}}, x)$ for all $x \in \mathbb{R}\backslash\{0\}$. We conclude that p must satisfy the technical condition for the real SOS property with respect to $[\mathsf{S}\mathcal{N}_{2,1}][\mathsf{Sd}]$. Hence, item (iii) holds. ☐

By Lemma 3.1, for the case of symmetry, item (iii) of Theorem 3.2 can be replaced by a stronger condition:

(iii') $\mathsf{p} = \gcd(\mathcal{N}_{1,1}, \mathcal{N}_{1,2}, \mathcal{N}_{2,1}, \mathcal{N}_{2,2})$ has the real SOS property with respect to $[\mathsf{S}\mathcal{N}_{2,1}][\mathsf{Sd}]$.

Though the condition in item (iii) or item (iii') of Theorem 3.2 for the case of symmetry appears to be technical, [13, Example 3.1] and a slightly modified Example 2 in Section 5 show that item (iii) of Theorem 3.2 cannot be removed. Fortunately, we often have $\mathsf{p} = 1$ and the condition in item (iii) of Theorem 3.2 is often not an issue.

To present an algorithm to prove the sufficiency part of Theorem 3.2, we need the following algorithm.

Algorithm 1. *Let p_1 and p_2 be nontrivial Laurent polynomials such that $\mathsf{p}_1(z) \geqslant 0$ for all $z \in \mathbb{T}$. Define $\mathsf{p}(z) := \gcd(\mathsf{p}_1(z), \mathsf{p}_2(z)\mathsf{p}_2^\star(z))$. Construct a Laurent polynomial q according to the following procedure:*

(S1) *Calculate $\mathsf{q}_1 := \gcd(\mathsf{p}, \mathsf{p}_2)$;*

(S2) *Define $\tilde{\mathsf{p}}(z) := \frac{\mathsf{q}_1(z)\mathsf{q}_1^\star(z)}{\mathsf{p}(z)}$. Then $\tilde{\mathsf{p}}$ is a well-defined Laurent polynomial such that $\tilde{\mathsf{p}}(z) \geqslant 0$ for all $z \in \mathbb{T}$;*

(S3) *Define $\mathsf{q}(z) := \mathsf{q}_1(z)/\mathsf{q}_2(z)$, where q_2 is any Laurent polynomial obtained via the Fejér-Riesz lemma such that $\mathsf{q}_2(z)\mathsf{q}_2^\star(z) = \tilde{\mathsf{p}}(z)$ for all $z \in \mathbb{T}$. Furthermore,*

  (i) *if both p_1 and p_2 have real coefficients, then we can also require that q_2 should have real coefficients so that q also has real coefficients;*

  (ii) *if both p_1 and p_2 have symmetry and $\mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}$ for all $x \in (-1, 0) \cup (0, 1)$, then we can also require that q_2 should have symmetry and real coefficients (such q_2 can be constructed by Theorem 2.9) so that q also has symmetry;*

  (iii) *if both p_1 and p_2 have complex symmetry and $\mathsf{Z}(\mathsf{p}, z) \in 2\mathbb{Z}$ for all $z \in \mathbb{C}\backslash\{0\}$, then we can also require that q_2 should have complex symmetry (such q_2 can be constructed by Theorem 2.8) so that q also has complex symmetry. If in addition both p_1 and p_2 have real coefficients, then q_2 can also have real coefficients and complex symmetry.*

Then q is a well-defined Laurent polynomial satisfying

$$\mathsf{q}(z)\mathsf{q}^\star(z) = \mathsf{p}(z) \qquad \forall\, z \in \mathbb{C}\backslash\{0\} \quad \textit{and} \quad \mathsf{q} \mid \mathsf{p}_2. \tag{3.9}$$

Moreover, for any Laurent polynomial q satisfying (3.9), $\gcd(\mathsf{p}_1/\mathsf{p}, \mathsf{p}_2/\mathsf{q}) = 1$*, that is, the Laurent polynomials p_1/p and p_2/q have no common zeros in $\mathbb{C}\backslash\{0\}$.*

Proof. Since $\mathsf{p}_1(z) \geqslant 0$ and $\mathsf{p}_2(z)\mathsf{p}_2^\star(z) \geqslant 0$ for all $z \in \mathbb{T}$, by Lemma 2.4 we see that $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$. In particular, we have $\mathsf{p}^\star(z) = \mathsf{p}(z)$.

To see that $\tilde{\mathsf{p}}$ is a well-defined Laurent polynomial, it suffices to show that

$$\mathsf{Z}(\mathsf{q}_1, z) + \mathsf{Z}(\mathsf{q}_1^\star, z) \geqslant \mathsf{Z}(\mathsf{p}, z), \qquad \forall\, z \in \mathbb{C}\backslash\{0\}. \tag{3.10}$$

By $\mathsf{q}_1 = \gcd(\mathsf{p}, \mathsf{p}_2)$, $\mathsf{Z}(\mathsf{q}_1, z) = \min(\mathsf{Z}(\mathsf{p}, z), \mathsf{Z}(\mathsf{p}_2, z))$. Hence, by $\mathsf{p}^\star = \mathsf{p}$ and $\mathsf{Z}(\mathsf{p}, z) = \min(\mathsf{Z}(\mathsf{p}_1, z), \mathsf{Z}(\mathsf{p}_2\mathsf{p}_2^\star, z))$,

$$\begin{aligned}\mathsf{Z}(\mathsf{q}_1, z) + \mathsf{Z}(\mathsf{q}_1^\star, z) &= \min(\mathsf{Z}(\mathsf{p}, z), \mathsf{Z}(\mathsf{p}_2, z)) + \min(\mathsf{Z}(\mathsf{p}, z), \mathsf{Z}(\mathsf{p}_2^\star, z))\\ &\geqslant \min(\mathsf{Z}(\mathsf{p}, z), \mathsf{Z}(\mathsf{p}_2, z) + \mathsf{Z}(\mathsf{p}_2^\star, z))\\ &= \min(\min(\mathsf{Z}(\mathsf{p}_1, z), \mathsf{Z}(\mathsf{p}_2\mathsf{p}_2^\star, z)), \mathsf{Z}(\mathsf{p}_2\mathsf{p}_2^\star, z))\\ &= \min(\mathsf{Z}(\mathsf{p}_1, z), \mathsf{Z}(\mathsf{p}_2\mathsf{p}_2^\star, z)) = \mathsf{Z}(\mathsf{p}, z).\end{aligned}$$

Therefore, (3.10) is verified and $\tilde{\mathsf{p}}$ is a well-defined Laurent polynomial. Because $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$, we also have $\tilde{\mathsf{p}}(z) = \mathsf{q}_1(z)\mathsf{q}_1^\star(z)/\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$.

Next, we prove that q is a well-defined Laurent polynomial. By $\mathsf{q}_1 = \gcd(\mathsf{p}, \mathsf{p}_2)$, we have

$$\mathsf{Z}(\mathsf{q}_1^\star, z) = \min(\mathsf{Z}(\mathsf{p}^\star, z), \mathsf{Z}(\mathsf{p}_2^\star, z)) = \min(\mathsf{Z}(\mathsf{p}, z), \mathsf{Z}(\mathsf{p}_2^\star, z)) \leqslant \mathsf{Z}(\mathsf{p}, z).$$

Since $\mathsf{q}_2\mathsf{q}_2^\star = \tilde{\mathsf{p}} = \mathsf{q}_1\mathsf{q}_1^\star/\mathsf{p}$, using the above inequality, we have

$$\mathsf{Z}(\mathsf{q}_2, z) \leqslant \mathsf{Z}(\mathsf{q}_2\mathsf{q}_2^\star, z) = \mathsf{Z}(\tilde{\mathsf{p}}, z) = \mathsf{Z}(\mathsf{q}_1, z) + \mathsf{Z}(\mathsf{q}_1^\star, z) - \mathsf{Z}(\mathsf{p}, z) \leqslant \mathsf{Z}(\mathsf{q}_1, z),$$

from which we conclude that q is a well-defined Laurent polynomial.

We now verify (3.9). By (S2) and (S3),

$$\mathsf{q}(z)\mathsf{q}^\star(z) = \frac{\mathsf{q}_1(z)\mathsf{q}_1^\star(z)}{\mathsf{q}_2(z)\mathsf{q}_2^\star(z)} = \frac{\mathsf{q}_1(z)\mathsf{q}_1^\star(z)}{\tilde{\mathsf{p}}(z)} = \mathsf{p}(z).$$

By definition, we have $\mathsf{q}_1 \mid \mathsf{p}_2$. Since $\mathsf{q} = \mathsf{q}_1/\mathsf{q}_2$, it is obvious that $\mathsf{q} \mid \mathsf{q}_1$ and therefore $\mathsf{q} \mid \mathsf{p}_2$. So, (3.9) is verified.

Suppose that $\gcd(\mathsf{p}_1/\mathsf{p}, \mathsf{p}_2/\mathsf{q})$ is not a monomial. Then there exists $z_0 \in \mathbb{C}\backslash\{0\}$ such that

$$\mathsf{Z}(\mathsf{p}_1, z_0) > \mathsf{Z}(\mathsf{p}, z_0) \quad \text{and} \quad \mathsf{Z}(\mathsf{p}_2, z_0) > \mathsf{Z}(\mathsf{q}, z_0). \tag{3.11}$$

Note that $\mathsf{q} \mid \mathsf{p}_2$ implies $\mathsf{Z}(\mathsf{p}_2^\star, z_0) \geqslant \mathsf{Z}(\mathsf{q}^\star, z_0)$. By definition $\mathsf{p} = \gcd(\mathsf{p}_1, \mathsf{p}_2\mathsf{p}_2^\star)$, it follows from (3.11) and (3.9) that

$$\begin{aligned}\mathsf{Z}(\mathsf{p}, z_0) &= \min(\mathsf{Z}(\mathsf{p}_1, z_0), \mathsf{Z}(\mathsf{p}_2\mathsf{p}_2^\star, z_0)) = \mathsf{Z}(\mathsf{p}_2\mathsf{p}_2^\star, z_0) = \mathsf{Z}(\mathsf{p}_2, z_0) + \mathsf{Z}(\mathsf{p}_2^\star, z_0)\\ &> \mathsf{Z}(\mathsf{q}, z_0) + \mathsf{Z}(\mathsf{q}^\star, z_0) = \mathsf{Z}(\mathsf{q}\mathsf{q}^\star, z_0) = \mathsf{Z}(\mathsf{p}, z_0),\end{aligned}$$

which is a contradiction. Hence, if (3.9) is satisfied, then $\gcd(\mathsf{p}_1/\mathsf{p}, \mathsf{p}_2/\mathsf{q})$ must be a monomial.

If both p_1 and p_2 have real coefficients, then $\tilde{\mathsf{p}}$ also has real coefficients. Therefore, by the Fejér-Riesz lemma, q_2 can have real coefficients.

Suppose that both p_1 and p_2 have symmetry and $\mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}$ for all $x \in (-1, 0) \cup (0, 1)$. By Lemma 2.3, we see that p must have real coefficients and q_1 has symmetry. Therefore, $\mathsf{q}_1\mathsf{q}_1^\star$ must have real coefficients and consequently, $\tilde{\mathsf{p}}$ has real coefficients and

$$\mathsf{Z}(\tilde{\mathsf{p}}, x) = \mathsf{Z}(\mathsf{q}_1\mathsf{q}_1^\star, x) - \mathsf{Z}(\mathsf{p}, x) = 2\mathsf{Z}(\mathsf{q}_1, x) - \mathsf{Z}(\mathsf{p}, x) \in 2\mathbb{Z}, \qquad \forall\, x \in (-1, 0) \cup (0, 1).$$

Note that we already proved $\tilde{\mathsf{p}}(z) \geqslant 0$ for all $z \in \mathbb{T}$. Hence, all the conditions in Theorem 2.9 are satisfied with p being replaced by $\tilde{\mathsf{p}}$. Consequently, there is a Laurent polynomial q_2 with symmetry and real coefficients such that $\tilde{\mathsf{p}}(z) = \mathsf{q}_2(z)\mathsf{q}_2^\star(z)$ holds.

Suppose that both p_1 and p_2 have complex symmetry and $\mathsf{Z}(\mathsf{p}, z) \in 2\mathbb{Z}$ for all $z \in \mathbb{C}\backslash\{0\}$. By Theorem 2.8, we have

$$\mathsf{Z}(\tilde{\mathsf{p}}, z) = \mathsf{Z}(\mathsf{q}_1\mathsf{q}_1^\star, z) - \mathsf{Z}(\mathsf{p}, z) = 2\mathsf{Z}(\mathsf{q}_1, z) - \mathsf{Z}(\mathsf{p}, z) \in 2\mathbb{Z}, \qquad \forall\, z \in \mathbb{C}\backslash\{0\}.$$

Note that we already proved $\tilde{\mathsf{p}}(z) \geqslant 0$ for all $z \in \mathbb{T}$. Hence, all the conditions in Theorem 2.8 are satisfied with p being replaced by $\tilde{\mathsf{p}}$. Consequently, there is a Laurent polynomial q_2 with complex symmetry such that $\tilde{\mathsf{p}}(z) = \mathsf{q}_2(z)\mathsf{q}_2^\star(z)$ holds. This completes the proof. ☐

For an integer m, we define $\text{odd}(m) = 1$ if m is odd, and $\text{odd}(m) = 0$ if m is even. Based on some ideas from [3, 13], we now present a concrete algorithm to prove the sufficiency part of Theorem 3.2. The following algorithm improves and generalizes [13, Section 4].

Algorithm 2. *Let $\mathcal{N}$ be a 2×2 matrix of Laurent polynomials as in (3.1) such that all the entries of $\mathcal{N}$ have [complex] symmetry (and real coefficients) and $\mathcal{N}$ is not identically zero. Assume that all the conditions in items (i), (ii), (iii) of Theorem 3.2 are satisfied [for the case of complex symmetry, remove item (iii) and replace S by $\mathbb{S}$].*

(S1) *Define $\breve{\mathcal{N}}$ as in (3.6) and $\breve{\mathsf{p}}(z) := \gcd(\breve{\mathcal{N}}_{1,1}(z), \breve{\mathcal{N}}_{1,2}(z)\breve{\mathcal{N}}_{1,2}^\star(z))$. Then $\breve{\mathsf{p}}(z) \geqslant 0$ for all $z \in \mathbb{T}$, and for the case of symmetry,*

$$\mathsf{Z}(\breve{\mathsf{p}}, x) \in 2\mathbb{Z}, \qquad \forall\, x \in \mathbb{R}\backslash\{0\}, \tag{3.12}$$

or for the case of complex symmetry (and real coefficients),

$$\mathsf{Z}(\breve{\mathsf{p}}, z) \in 2\mathbb{Z}, \qquad \forall\, z \in \mathbb{C}\backslash\{0\}. \tag{3.13}$$

Consequently, we can always construct a Laurent polynomial q by Algorithm 1 such that q has [complex] symmetry (and real coefficients) satisfying

$$\mathsf{q}(z)\mathsf{q}^\star(z) = \breve{\mathsf{p}}(z), \qquad \mathsf{q} \mid \breve{\mathcal{N}}_{1,2}.$$

Define $\mathring{\mathcal{N}}$ by

$$\mathring{\mathcal{N}}(z) := \begin{bmatrix} \mathring{\mathcal{N}}_{1,1}(z) & \mathring{\mathcal{N}}_{1,2}(z) \\ \mathring{\mathcal{N}}_{2,1}(z) & \mathring{\mathcal{N}}_{2,2}(z) \end{bmatrix} := \begin{bmatrix} \frac{\breve{\mathcal{N}}_{1,1}(z)}{\mathsf{q}(z)\mathsf{q}^\star(z)} & \frac{\breve{\mathcal{N}}_{1,2}(z)}{\mathsf{q}(z)} \\ \frac{\breve{\mathcal{N}}_{2,1}(z)}{\mathsf{q}^\star(z)} & \breve{\mathcal{N}}_{2,2}(z) \end{bmatrix}. \tag{3.14}$$

Then all entries of $\mathring{\mathcal{N}}$ must have [complex] symmetry (and real coefficients), and $\mathring{\mathcal{N}}_{1,1}$ and $\mathring{\mathcal{N}}_{1,2}$ have no common zeros in $\mathbb{C}\backslash\{0\}$;

(S2) *For the case of symmetry, we can always construct a Laurent polynomial $\tilde{\mathsf{d}}$ with symmetry by Theorem 2.9 such that $\tilde{\mathsf{d}}(z)\tilde{\mathsf{d}}^\star(z) = \det(\mathring{\mathcal{N}}(z))$. For the case of complex symmetry, we can directly take $\tilde{\mathsf{d}}(z) := \frac{\mathsf{d}(z)}{\mathsf{p}(z)\mathsf{q}(z)}$ so that $\tilde{\mathsf{d}}$ is a well-defined Laurent polynomial and has complex symmetry (and real coefficients) such that $\tilde{\mathsf{d}}(z)\tilde{\mathsf{d}}^\star(z) = \det(\mathring{\mathcal{N}}(z))$;*

(S3) *Define $[-n, n] := \text{fsupp}(\mathring{\mathcal{N}}_{1,1})$, $\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{c_{\mathring{\mathcal{N}}_{2,1}}} := \mathsf{S}\mathring{\mathcal{N}}_{2,1}(z)$, $\mathring{\epsilon} z^{\mathring{c}} := \mathsf{S}\tilde{\mathsf{d}}(z)$, $c_{\text{odd}} := \text{odd}(c_{\mathring{\mathcal{N}}_{2,1}} - \mathring{c})$, and*

$$\mathring{\mathsf{d}}(z) := z^{n + \frac{c_{\mathring{\mathcal{N}}_{2,1}} - \mathring{c} - c_{\text{odd}}}{2}} \tilde{\mathsf{d}}(z). \tag{3.15}$$

Then $\mathring{\mathsf{d}}$ has [complex] symmetry $\mathsf{S}\mathring{\mathsf{d}}(z) = \mathring{\epsilon} z^{2n + c_{\mathring{\mathcal{N}}_{2,1}} - c_{\text{odd}}}$ (and real coefficients);

(S4) *Write* $\mathring{\mathcal{U}}_{1,1}(z) = \sum_{j=0}^{n-c_{\rm odd}} t_j z^j$ *and* $\mathring{\mathcal{U}}_{1,2}(z) = \sum_{j=0}^{n} \tilde{t}_j z^j$ *(when* $n < c_{\rm odd}$*, set* $\mathring{\mathcal{U}}_{1,1}(z) = 0$*), where* $\{t_0, \dots, t_{n-c_{\rm odd}}, \tilde{t}_0, \dots, \tilde{t}_n\}$ *is a nontrivial solution to the homogeneous system* X *of* $2n$ *linear equations induced by* $\mathring{\mathcal{R}}(z) \equiv 0$*, where* $\mathring{\mathcal{R}}$ *and* $\mathring{\mathcal{U}}_{2,1}$ *are uniquely determined by*

$$\mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{U}}_{1,1}(z) - \mathring{\mathsf{d}}(z)\mathring{\mathcal{U}}^\star_{1,2}(z) = \mathring{\mathcal{N}}_{1,1}(z)\mathring{\mathcal{U}}_{2,1}(z) + \mathring{\mathcal{R}}(z) \quad \textit{with} \quad \mathrm{fsupp}(\mathring{\mathcal{R}}) \subseteq [-n, n-1].$$

Then the space of all solutions to X *has dimension at least one;*

(S5) *For the case of symmetry, replace* $\mathring{\mathcal{U}}_{1,1}$ *and* $\mathring{\mathcal{U}}_{1,2}$ *by*

$$[\mathring{\mathcal{U}}_{1,1}(z) + \epsilon\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{n-c_{\rm odd}} \mathring{\mathcal{U}}_{1,1}(z^{-1})]/2, \qquad [\mathring{\mathcal{U}}_{1,2}(z) + \epsilon\mathring{\epsilon} z^n \mathring{\mathcal{U}}_{1,2}(z^{-1})]/2. \tag{3.16}$$

For the case of complex symmetry, replace $\mathring{\mathcal{U}}_{1,1}$ *and* $\mathring{\mathcal{U}}_{1,2}$ *by*

$$[\mathring{\mathcal{U}}_{1,1}(z) + \epsilon\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{n-c_{\rm odd}} \mathring{\mathcal{U}}^\star_{1,1}(z)]/2, \qquad [\mathring{\mathcal{U}}_{1,2}(z) + \epsilon\mathring{\epsilon} z^n \mathring{\mathcal{U}}^\star_{1,2}(z)]/2, \tag{3.17}$$

where ϵ *is any choice from* $\{-1, 1\}$ *such that the two Laurent polynomials in* (3.16) *or* (3.17) *are not simultaneously identically zero. Then the symmetrized pair* $\{\mathring{\mathcal{U}}_{1,1}, \mathring{\mathcal{U}}_{1,2}\}$ *satisfies*

$$\mathring{\mathcal{N}}_{1,1}(z) \mid [\mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{U}}_{1,1}(z) - \mathring{\mathsf{d}}(z)\mathring{\mathcal{U}}^\star_{1,2}(z)] \tag{3.18}$$

with $\mathsf{S}\mathring{\mathcal{U}}_{1,1}(z) = \epsilon\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{n-c_{\rm odd}}$ *and* $\mathsf{S}\mathring{\mathcal{U}}_{1,2}(z) = \epsilon\mathring{\epsilon} z^n$*;*

(S6) *Normalize* $\{\mathring{\mathcal{U}}_{1,1}, \mathring{\mathcal{U}}_{1,2}\}$ *in (S5) by multiplying them with the well-defined positive number* $\sqrt{\frac{\mathring{\mathcal{N}}_{1,1}(1)}{|\mathring{\mathcal{U}}_{1,1}(1)|^2 + |\mathring{\mathcal{U}}_{1,2}(1)|^2}}$. *Define* $\mathring{\mathcal{U}}_{2,1}$ *and* $\mathring{\mathcal{U}}_{2,2}$ *by*

$$\mathring{\mathcal{U}}_{2,1}(z) := \frac{\mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{U}}_{1,1}(z) - \mathring{\mathsf{d}}(z)\mathring{\mathcal{U}}^\star_{1,2}(z)}{\mathring{\mathcal{N}}_{1,1}(z)} \tag{3.19}$$

and

$$\mathring{\mathcal{U}}_{2,2}(z) := \frac{\mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)\mathring{\mathcal{U}}^\star_{1,1}(z)}{\mathring{\mathcal{N}}_{1,1}(z)}. \tag{3.20}$$

Then $\mathring{\mathcal{U}}_{2,1}$ *and* $\mathring{\mathcal{U}}_{2,2}$ *are well-defined Laurent polynomials having [complex] symmetry (and real coefficients)* $\mathsf{S}\mathring{\mathcal{U}}_{2,1}(z) = \epsilon z^{n+c_{\mathring{\mathcal{N}}_{2,1}}-c_{\rm odd}}$ *and* $\mathsf{S}\mathring{\mathcal{U}}_{2,2}(z) = \epsilon\mathring{\epsilon}\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{n+c_{\mathring{\mathcal{N}}_{2,1}}}$*. Moreover,*

$$\mathring{\mathcal{U}}(z)\mathring{\mathcal{U}}^\star(z) = \mathring{\mathcal{N}}(z) \quad \textit{and} \quad \frac{\mathsf{S}\mathring{\mathcal{U}}_{1,2}(z)}{\mathsf{S}\mathring{\mathcal{U}}_{1,1}(z)} = \frac{\mathsf{S}\mathring{\mathcal{U}}_{2,2}(z)}{\mathsf{S}\mathring{\mathcal{U}}_{2,1}(z)} = \mathring{\epsilon}\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{c_{\rm odd}};$$

(S7) *By Theorem 2.7 and item (iii) for the case of symmetry (or by Theorem 2.6 for the case of complex symmetry), there exist Laurent polynomials* p_1 *and* p_2 *having [complex] symmetry (and real coefficients) such that* (2.7) *(or* (2.6)*) holds with respect to the symmetry type* $\mathring{\epsilon}\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{c_{\rm odd}}$*, which equals* $z^{2k}\mathsf{S}\mathcal{N}_{2,1}(z)\mathsf{S}\mathsf{d}(z)$ *for some* $k \in \mathbb{Z}$*;*

(S8) *Define a* 2×2 *matrix* $\mathcal{U}$ *of Laurent polynomials by*

$$\mathcal{U}(z) := \begin{bmatrix} \mathcal{U}_{1,1}(z) & \mathcal{U}_{1,2}(z) \\ \mathcal{U}_{2,1}(z) & \mathcal{U}_{2,2}(z) \end{bmatrix} := \begin{bmatrix} \mathsf{q}(z) & 0 \\ 0 & 1 \end{bmatrix} \mathring{\mathcal{U}}(z) \begin{bmatrix} \mathsf{p}_1(z) & -\mathsf{p}_2^\star(z) \\ \mathsf{p}_2(z) & \mathsf{p}_1^\star(z) \end{bmatrix}.$$

Then all the entries in $\mathcal{U}$ *have [complex] symmetry (and real coefficients) such that*

$$\mathcal{U}(z)\mathcal{U}^\star(z) = \mathcal{N}(z) \quad \textit{and} \quad \frac{\mathsf{S}\mathcal{U}_{1,1}(z)}{\mathsf{S}\mathcal{U}_{2,1}(z)} = \frac{\mathsf{S}\mathcal{U}_{1,2}(z)}{\mathsf{S}\mathcal{U}_{2,2}(z)}. \tag{3.21}$$

Proof. By item (i) of Theorem 3.2 and Lemma 3.1, we must have $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$. For the case of symmetry, by Lemma 2.3, we also see that p has real coefficients.

Hence, $\gcd(\breve{\mathcal{N}}_{1,1}, \breve{\mathcal{N}}_{1,2}, \breve{\mathcal{N}}_{2,1}, \breve{\mathcal{N}}_{2,2}) = 1$ and $\breve{\mathcal{N}}(z) \geqslant 0$ for all $z \in \mathbb{T}$. It is now trivial to see that $\breve{\mathsf{p}}(z) \geqslant 0$ for all $z \in \mathbb{T}$. We now prove (3.12) for the case of symmetry, and (3.13) for the case of complex symmetry. Note that

$$\mathsf{d}(z)\mathsf{d}^\star(z) = (\mathsf{p}(z))^2[\breve{\mathcal{N}}_{1,1}(z)\breve{\mathcal{N}}_{2,2}(z) - \breve{\mathcal{N}}_{1,2}(z)\breve{\mathcal{N}}_{2,1}(z)]. \tag{3.22}$$

For the case of complex symmetry, since d has complex symmetry, we have $\mathsf{Z}(\mathsf{d}\mathsf{d}^\star, z) \in 2\mathbb{Z}$ for all $z \in \mathbb{C}\backslash\{0\}$. Hence, it follows from (3.22) that

$$\mathsf{Z}(\breve{\mathcal{N}}_{1,1}\breve{\mathcal{N}}_{2,2} - \breve{\mathcal{N}}_{1,2}\breve{\mathcal{N}}_{2,1}, z) \in 2\mathbb{Z} \tag{3.23}$$

for all $z \in \mathbb{C}\backslash\{0\}$. Since $\breve{\mathcal{N}}_{2,1} = \breve{\mathcal{N}}^\star_{1,2} = \breve{\mathcal{N}}_{1,2}/\mathbb{S}\breve{\mathcal{N}}_{1,2}$, we must have $\mathsf{Z}(\breve{\mathcal{N}}_{1,2}\breve{\mathcal{N}}_{2,1}, z) \in 2\mathbb{Z}$. If $\mathsf{Z}(\breve{\mathsf{p}}, z_0) = 2m - 1$ for some $m \in \mathbb{N}$ and $z_0 \in \mathbb{C}\backslash\{0\}$, by $\breve{\mathsf{p}} = \gcd(\breve{\mathcal{N}}_{1,1}, \breve{\mathcal{N}}_{1,2}\breve{\mathcal{N}}^\star_{1,2})$, then $\mathsf{Z}(\breve{\mathcal{N}}_{1,2}\breve{\mathcal{N}}_{2,1}, z_0) \geqslant 2m$ and $\mathsf{Z}(\breve{\mathcal{N}}_{1,1}, z_0) = 2m - 1$. Consequently, it follows from (3.23) that $\mathsf{Z}(\breve{\mathcal{N}}_{2,2}, z_0)$ must be an odd integer. However, this implies $\breve{\mathcal{N}}_{1,1}(z_0) = \breve{\mathcal{N}}_{2,2}(z_0) = \breve{\mathcal{N}}_{1,2}(z_0) = \breve{\mathcal{N}}_{2,1}(z_0) = 0$, which is a contradiction to $\gcd(\breve{\mathcal{N}}_{1,1}, \breve{\mathcal{N}}_{1,2}, \breve{\mathcal{N}}_{2,1}, \breve{\mathcal{N}}_{2,2}) = 1$. This shows that

(3.13) must hold. Therefore, by Theorem 2.8, we have a Laurent polynomial q with complex symmetry such that $\mathsf{q}(z)\mathsf{q}^\star(z) = \breve{\mathsf{p}}(z)$. The relation $\mathsf{q} \mid \breve{\mathcal{N}}_{1,2}$ is automatically true, since

$$2\mathsf{Z}(\mathsf{q}, z) = \mathsf{Z}(\mathsf{q}\mathsf{q}^\star, z) = \mathsf{Z}(\breve{\mathsf{p}}, z) \leqslant \mathsf{Z}(\breve{\mathcal{N}}_{1,2}\breve{\mathcal{N}}_{1,2}^\star, z) = 2\mathsf{Z}(\breve{\mathcal{N}}_{1,2}, z), \qquad \forall\, z \in \mathbb{C}\backslash\{0\}.$$

For the case of symmetry, since d and $\breve{\mathcal{N}}_{1,2}$ have symmetry, by Theorem 2.9, we have $\mathsf{Z}(\mathsf{d}\mathsf{d}^\star, z) \in 2\mathbb{Z}$ and $\mathsf{Z}(\mathsf{d}\mathsf{d}^\star, z) \in 2\mathbb{Z}$ for all $z \in \mathbb{R}\backslash\{0\}$. Therefore, it follows from (3.22) that (3.23) holds for all $z \in \mathbb{R}\backslash\{0\}$. Now using the same proof by contradiction for the case of complex symmetry by considering only $z_0 \in \mathbb{R}\backslash\{0\}$, we conclude that (3.12) must hold. By Lemma 2.3, we also see that $\breve{\mathsf{p}}$ has real coefficients.

For the case of complex symmetry in (S5), since $\{\mathring{\mathcal{U}}_{1,1}, \mathring{\mathcal{U}}_{1,2}\}$ obtained in (S4) is a solution to (3.18), then

$$\epsilon z^{n+c_{\mathring{\mathcal{N}}_{2,1}}-c_{\mathrm{odd}}}[\mathring{\mathcal{N}}_{2,1}^\star(z)\mathring{\mathcal{U}}_{1,1}^\star(z) - \mathring{\mathsf{d}}^\star(z)\mathring{\mathcal{U}}_{1,2}(z)] = \epsilon z^{n+c_{\mathring{\mathcal{N}}_{2,1}}-c_{\mathrm{odd}}}\mathring{\mathcal{N}}_{1,1}^\star(z)\mathring{\mathcal{U}}_{2,1}^\star(z).$$

Note that $\mathring{\mathcal{N}}_{1,1}^\star(z) = \mathring{\mathcal{N}}_{1,1}(z)$, $\mathring{\mathcal{N}}_{2,1}^\star(z) = \epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{-c_{\mathring{\mathcal{N}}_{2,1}}}\mathring{\mathcal{N}}_{2,1}(z)$, and $\mathring{\mathsf{d}}^\star(z) = \mathring{\epsilon} z^{-2n-c_{\mathring{\mathcal{N}}_{2,1}}+c_{\mathrm{odd}}}\mathring{\mathsf{d}}(z)$. Therefore,

$$\mathring{\mathcal{N}}_{2,1}(z)\epsilon\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{n-c_{\mathrm{odd}}}\mathring{\mathcal{U}}_{1,1}^\star(z) - \mathring{\mathsf{d}}(z)\epsilon\mathring{\epsilon} z^{-n}\mathring{\mathcal{U}}_{1,2}(z) = \mathring{\mathcal{N}}_{1,1}(z)\epsilon z^{n+c_{\mathring{\mathcal{N}}_{2,1}}-c_{\mathrm{odd}}}\mathring{\mathcal{U}}_{2,1}^\star(z).$$

Now the above identity and (3.18) together imply that the new pair $\{\mathring{\mathcal{U}}_{1,1}, \mathring{\mathcal{U}}_{1,2}\}$ defined in (3.17) is also a solution to (3.18), and $\operatorname{fsupp}(\mathring{\mathcal{U}}_{1,1}) \subseteq [0, n - c_{\mathrm{odd}}] \subseteq [0, n]$ and $\operatorname{fsupp}(\mathring{\mathcal{U}}_{1,2}) \subseteq [0, n]$.

For the case of symmetry in (S5), since $\{\mathring{\mathcal{U}}_{1,1}, \mathring{\mathcal{U}}_{1,2}\}$ obtained in (S4) is a solution to (3.18), then

$$\epsilon z^{n+c_{\mathring{\mathcal{N}}_{2,1}}-c_{\mathrm{odd}}}[\mathring{\mathcal{N}}_{2,1}(z^{-1})\mathring{\mathcal{U}}_{1,1}(z^{-1}) - \mathring{\mathsf{d}}(z^{-1})\mathring{\mathcal{U}}_{1,2}^\star(z^{-1})] = \epsilon z^{n+c_{\mathring{\mathcal{N}}_{2,1}}-c_{\mathrm{odd}}}\mathring{\mathcal{N}}_{1,1}(z^{-1})\mathring{\mathcal{U}}_{2,1}(z^{-1}).$$

Since $\mathring{\mathcal{N}}_{1,1}$ has symmetry and $\mathring{\mathcal{N}}_{1,1}(z) \geqslant 0$ for all $z \in \mathbb{T}$, by Lemma 2.3, $\mathring{\mathcal{N}}_{1,1}$ must have real coefficients and $\mathsf{S}\mathring{\mathcal{N}}_{1,1}(z) = \mathbb{S}\mathring{\mathcal{N}}_{1,1}(z) = 1$. That is, $\mathring{\mathcal{N}}_{1,1}(z^{-1}) = \mathring{\mathcal{N}}_{1,1}(z)$ holds. Note that $\mathring{\mathcal{N}}_{2,1}(z^{-1}) = \epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{-c_{\mathring{\mathcal{N}}_{2,1}}}\mathring{\mathcal{N}}_{2,1}(z)$ and $\mathring{\mathsf{d}}(z^{-1}) = \mathring{\epsilon} z^{-2n-c_{\mathring{\mathcal{N}}_{2,1}}+c_{\mathrm{odd}}}\mathring{\mathsf{d}}(z)$. Therefore,

$$\mathring{\mathcal{N}}_{2,1}(z)\epsilon\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{n-c_{\mathrm{odd}}}\mathring{\mathcal{U}}_{1,1}(z^{-1}) - \mathring{\mathsf{d}}(z)\epsilon\mathring{\epsilon} z^{-n}\mathring{\mathcal{U}}_{1,2}^\star(z^{-1}) = \mathring{\mathcal{N}}_{1,1}(z)\epsilon z^{n+c_{\mathring{\mathcal{N}}_{2,1}}-c_{\mathrm{odd}}}\mathring{\mathcal{U}}_{2,1}(z^{-1}).$$

Now the above identity and (3.18) together imply that the new pair $\{\mathring{\mathcal{U}}_{1,1}, \mathring{\mathcal{U}}_{1,2}\}$ defined in (3.16) is also a solution to (3.18), and $\operatorname{fsupp}(\mathring{\mathcal{U}}_{1,1}) \subseteq [0, n - c_{\mathrm{odd}}] \subseteq [0, n]$ and $\operatorname{fsupp}(\mathring{\mathcal{U}}_{1,2}) \subseteq [0, n]$.

By $\mathcal{N}_{2,1}(z) = \mathsf{p}(z)\mathsf{q}^\star(z)\mathring{\mathcal{N}}_{2,1}(z)$ and $\mathsf{S}\mathsf{d}(z) = \mathring{\epsilon} z^{\mathring{c}+2j}\mathsf{S}\mathsf{q}(z)$ for some $j \in \mathbb{Z}$, by (3.15) we see that

$$\mathsf{S}\mathcal{N}_{2,1}(z)\mathsf{S}\mathsf{d}(z) = \mathring{\epsilon}\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{c_{\mathring{\mathcal{N}}_{2,1}}+\mathring{c}+2j} = \mathring{\epsilon}\epsilon_{\mathring{\mathcal{N}}_{2,1}} z^{c_{\mathrm{odd}}} z^{2k},$$

where $k := j + (c_{\mathring{\mathcal{N}}_{2,1}} + \mathring{c} - c_{\mathrm{odd}})/2 \in \mathbb{Z}$. Now by Theorem 2.7 or Theorem 2.6, the existence of p_1 and p_2 in (S7) is justified.

By $\operatorname{fsupp}(\mathring{\mathcal{R}}) \subseteq [-n, n-1]$, there are no more than $2n$ homogeneous linear equations in X, while there are $2n + 2 - c_{\mathrm{odd}} \geqslant 2n + 1$ unknowns. Therefore, the space of all solutions to X must have dimension at least one. Therefore, we must have a nontrivial symmetrized solution $\{\mathring{\mathcal{U}}_{1,1}, \mathring{\mathcal{U}}_{1,2}\}$ to X in (S5). By $\mathring{\mathcal{R}}(z) = 0$ and $\mathsf{S}\mathring{\mathcal{N}}_{2,1}(z)\mathsf{S}\mathring{\mathcal{U}}_{1,1}(z) = \mathsf{S}\mathring{\mathsf{d}}(z)\mathsf{S}\mathring{\mathcal{U}}_{1,2}^\star(z)$, we see that $\mathring{\mathcal{U}}_{2,1}$ defined in (3.19) is a well-defined Laurent polynomial with [complex] symmetry.

Next, we show that $\mathring{\mathcal{U}}_{2,2}$ in (3.20) is a well-defined Laurent polynomial with [complex] symmetry. Since $\mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^\star(z) = \det(\mathring{\mathcal{N}}(z)) = \mathring{\mathcal{N}}_{1,1}(z)\mathring{\mathcal{N}}_{2,2}(z) - \mathring{\mathcal{N}}_{1,2}(z)\mathring{\mathcal{N}}_{2,1}(z)$, we have

$$\begin{aligned}
\mathring{\mathcal{N}}_{1,2}(z)[\mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)\mathring{\mathcal{U}}_{1,1}^\star(z)] &= \mathring{\mathcal{N}}_{1,2}(z)\mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)\mathring{\mathcal{N}}_{1,2}(z)\mathring{\mathcal{U}}_{1,1}^\star(z)\\
&= [\mathring{\mathcal{N}}_{1,1}(z)\mathring{\mathcal{N}}_{2,2}(z) - \mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^\star(z)]\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)\mathring{\mathcal{N}}_{1,2}(z)\mathring{\mathcal{U}}_{1,1}^\star(z)\\
&= \mathring{\mathcal{N}}_{1,1}(z)\mathring{\mathcal{N}}_{2,2}(z)\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)[\mathring{\mathcal{N}}_{1,2}(z)\mathring{\mathcal{U}}_{1,1}^\star(z) - \mathring{\mathsf{d}}^\star(z)\mathring{\mathcal{U}}_{1,2}(z)]\\
&= \mathring{\mathcal{N}}_{1,1}(z)\mathring{\mathcal{N}}_{2,2}(z)\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)[\mathring{\mathcal{N}}_{1,2}^\star(z)\mathring{\mathcal{U}}_{1,1}(z) - \mathring{\mathsf{d}}(z)\mathring{\mathcal{U}}_{1,2}^\star(z)]^\star.
\end{aligned}$$

By $\mathring{\mathcal{N}}(z) \geqslant 0$ for all $z \in \mathbb{T}$, we have $\mathring{\mathcal{N}}_{1,1}^\star(z) = \mathring{\mathcal{N}}_{1,1}(z)$ and $\mathring{\mathcal{N}}_{1,2}^\star(z) = \mathring{\mathcal{N}}_{2,1}(z)$. Using (3.19), we deduce from the above identity that

$$\begin{aligned}
\mathring{\mathcal{N}}_{1,2}(z)[\mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)\mathring{\mathcal{U}}_{1,1}^\star(z)] &= \mathring{\mathcal{N}}_{1,1}(z)\mathring{\mathcal{N}}_{2,2}(z)\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)\mathring{\mathcal{N}}_{1,1}^\star(z)\mathring{\mathcal{U}}_{2,1}^\star(z)\\
&= \mathring{\mathcal{N}}_{1,1}(z)[\mathring{\mathcal{N}}_{2,2}(z)\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)\mathring{\mathcal{U}}_{2,1}^\star(z)].
\end{aligned}$$

Since $\mathring{\mathcal{N}}_{1,1}$ and $\mathring{\mathcal{N}}_{1,2}$ have no common zeros in $\mathbb{C}\backslash\{0\}$, we conclude from the above identity that $\mathring{\mathcal{N}}_{1,1}(z) \mid [\mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{U}}_{1,2}(z) + \mathring{\mathsf{d}}(z)\mathring{\mathcal{U}}_{1,1}^\star(z)]$. Also noting that $\mathsf{S}\mathring{\mathcal{N}}_{2,1}(z)\mathsf{S}\mathring{\mathcal{U}}_{1,2}(z) = \mathsf{S}\mathring{\mathsf{d}}(z)\mathsf{S}\mathring{\mathcal{U}}_{1,1}^\star(z)$, we proved that $\mathring{\mathcal{U}}_{2,2}$ in (3.20) is a well-defined Laurent polynomial with [complex] symmetry.

Now (3.19) and (3.20) together imply

$$\begin{bmatrix} \mathring{\mathcal{U}}_{2,2}(z) & -\mathring{\mathcal{U}}_{1,2}(z) \\ -\mathring{\mathcal{U}}_{2,1}(z) & \mathring{\mathcal{U}}_{1,1}(z) \end{bmatrix} \begin{bmatrix} \mathring{\mathcal{N}}_{1,1}(z) \\ \mathring{\mathcal{N}}_{2,1}(z) \end{bmatrix} = \mathring{\mathsf{d}}(z) \begin{bmatrix} \mathring{\mathcal{U}}^{\star}_{1,1}(z) \\ \mathring{\mathcal{U}}^{\star}_{1,2}(z) \end{bmatrix}. \tag{3.24}$$

Multiplying $[-\mathring{\mathcal{U}}^{\star}_{1,2}(z), \mathring{\mathcal{U}}^{\star}_{1,1}(z)]$ from the left on both sides of (3.24), we have

$$-[\mathring{\mathcal{U}}_{2,2}(z)\mathring{\mathcal{U}}^{\star}_{1,2}(z) + \mathring{\mathcal{U}}_{2,1}(z)\mathring{\mathcal{U}}^{\star}_{1,1}(z)]\mathring{\mathcal{N}}_{1,1}(z) + [\mathring{\mathcal{U}}_{1,1}(z)\mathring{\mathcal{U}}^{\star}_{1,1}(z) + \mathring{\mathcal{U}}_{1,2}(z)\mathring{\mathcal{U}}^{\star}_{1,2}(z)]\mathring{\mathcal{N}}_{2,1}(z) = 0.$$

That is,

$$[\mathring{\mathcal{U}}_{1,1}(z)\mathring{\mathcal{U}}^{\star}_{1,1}(z) + \mathring{\mathcal{U}}_{1,2}(z)\mathring{\mathcal{U}}^{\star}_{1,2}(z)]\mathring{\mathcal{N}}_{2,1}(z) = [\mathring{\mathcal{U}}_{2,2}(z)\mathring{\mathcal{U}}^{\star}_{1,2}(z) + \mathring{\mathcal{U}}_{2,1}(z)\mathring{\mathcal{U}}^{\star}_{1,1}(z)]\mathring{\mathcal{N}}_{1,1}(z). \tag{3.25}$$

Since the solution $\{\mathring{\mathcal{U}}_{1,1}, \mathring{\mathcal{U}}_{1,2}\}$ to the system X of linear equations is nontrivial, we see that

$$\mathring{\mathcal{U}}_{1,1}(z)\mathring{\mathcal{U}}^{\star}_{1,1}(z) + \mathring{\mathcal{U}}_{1,2}(z)\mathring{\mathcal{U}}^{\star}_{1,2}(z) \quad \text{must be nontrivial.} \tag{3.26}$$

Because $\mathring{\mathcal{U}}_{1,1}$ and $\mathring{\mathcal{U}}_{1,2}$ are polynomials of degree at most n, the filter support of the Laurent polynomial on the left-hand side of (3.26) is contained inside $[-n, n]$. Note that $\mathring{\mathcal{N}}_{1,1} = \mathring{\mathcal{N}}^{\star}_{1,1}$ and $\mathring{\mathcal{N}}_{2,1} = \mathring{\mathcal{N}}^{\star}_{1,2}$. Since $\mathring{\mathcal{N}}_{1,1}$ and $\mathring{\mathcal{N}}_{1,2}$ have no common zeros in $\mathbb{C}\backslash\{0\}$, we see that $\mathring{\mathcal{N}}_{1,1}$ and $\mathring{\mathcal{N}}_{2,1}$ also have no common zeros in $\mathbb{C}\backslash\{0\}$. By (3.26) and noting that the filter support of the Laurent polynomial in (3.26) is contained inside $[-n, n] = \text{fsupp}(\mathring{\mathcal{N}}_{1,1})$, from (3.25), we must have

$$\mathring{\mathcal{U}}_{1,1}(z)\mathring{\mathcal{U}}^{\star}_{1,1}(z) + \mathring{\mathcal{U}}_{1,2}(z)\mathring{\mathcal{U}}^{\star}_{1,2}(z) = \lambda^{-1}\mathring{\mathcal{N}}_{1,1}(z) \qquad \text{for some } \lambda \in \mathbb{C}\backslash\{0\}. \tag{3.27}$$

We now show that

$$\lambda = \frac{\mathring{\mathcal{N}}_{1,1}(1)}{|\mathring{\mathcal{U}}_{1,1}(1)|^2 + |\mathring{\mathcal{U}}_{1,2}(1)|^2} > 0. \tag{3.28}$$

Since $\mathring{\mathcal{N}}_{1,1}$ and $\mathring{\mathcal{N}}_{2,1}$ have no common zeros in $\mathbb{C}\backslash\{0\}$ and since $\mathring{\mathcal{N}}(z) \geqslant 0$ for all $z \in \mathbb{T}$, we see that $\mathring{\mathcal{N}}_{1,1}(z) > 0$ for all $z \in \mathbb{T}$. In particular, $\mathring{\mathcal{N}}_{1,1}(1) > 0$. Consequently, (3.26) and (3.27) imply that $|\mathring{\mathcal{U}}_{1,1}(1)|^2 + |\mathring{\mathcal{U}}_{1,2}(1)|^2 > 0$. Hence, (3.28) holds. Therefore, normalizing the solution $\{\mathring{\mathcal{U}}_{1,1}, \mathring{\mathcal{U}}_{1,2}\}$ by multiplying them with the factor $\sqrt{\lambda}$, we have

$$\mathring{\mathcal{U}}_{1,1}(z)\mathring{\mathcal{U}}^{\star}_{1,1}(z) + \mathring{\mathcal{U}}_{1,2}(z)\mathring{\mathcal{U}}^{\star}_{1,2}(z) = \mathring{\mathcal{N}}_{1,1}(z). \tag{3.29}$$

Now by (3.25) and (3.29), we further have

$$\mathring{\mathcal{U}}_{2,2}(z)\mathring{\mathcal{U}}^{\star}_{1,2}(z) + \mathring{\mathcal{U}}_{2,1}(z)\mathring{\mathcal{U}}^{\star}_{1,1}(z) = \mathring{\mathcal{N}}_{2,1}(z). \tag{3.30}$$

Multiplying $[\mathring{\mathcal{U}}_{1,1}(z), \mathring{\mathcal{U}}_{1,2}(z)]$ from the left on both sides of (3.24), we have

$$[\mathring{\mathcal{U}}_{1,1}(z)\mathring{\mathcal{U}}_{2,2}(z) - \mathring{\mathcal{U}}_{1,2}(z)\mathring{\mathcal{U}}_{2,1}(z)]\mathring{\mathcal{N}}_{1,1}(z) = \mathring{\mathsf{d}}(z)[\mathring{\mathcal{U}}_{1,1}(z)\mathring{\mathcal{U}}^{\star}_{1,1}(z) + \mathring{\mathcal{U}}_{1,2}(z)\mathring{\mathcal{U}}^{\star}_{1,2}(z)].$$

Combining the above identity with (3.29), we conclude that

$$\det(\mathring{\mathcal{U}}(z)) = \mathring{\mathcal{U}}_{1,1}(z)\mathring{\mathcal{U}}_{2,2}(z) - \mathring{\mathcal{U}}_{1,2}(z)\mathring{\mathcal{U}}_{2,1}(z) = \mathring{\mathsf{d}}(z).$$

Multiplying $[\mathring{\mathcal{U}}^{\star}_{2,2}(z), -\mathring{\mathcal{U}}^{\star}_{2,1}(z)]$ from the left on both sides of (3.24) and by (3.30), we have

$$\begin{aligned}[\mathring{\mathcal{U}}_{2,2}(z)\mathring{\mathcal{U}}^{\star}_{2,2}(z) + \mathring{\mathcal{U}}_{2,1}(z)\mathring{\mathcal{U}}^{\star}_{2,1}(z)]\mathring{\mathcal{N}}_{1,1}(z) - \mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{N}}^{\star}_{2,1}(z) &= \mathring{\mathsf{d}}(z)\mathring{\mathsf{d}}^{\star}(z) = \det(\mathring{\mathcal{N}}(z)) \\ &= \mathring{\mathcal{N}}_{2,2}(z)\mathring{\mathcal{N}}_{1,1}(z) - \mathring{\mathcal{N}}_{2,1}(z)\mathring{\mathcal{N}}_{1,2}(z).\end{aligned}$$

Since $\mathring{\mathcal{N}}^{\star}(z) = \mathring{\mathcal{N}}(z)$, we have $\mathring{\mathcal{N}}_{1,2}(z) = \mathring{\mathcal{N}}^{\star}_{2,1}(z)$ and therefore,

$$[\mathring{\mathcal{U}}_{2,2}(z)\mathring{\mathcal{U}}^{\star}_{2,2}(z) + \mathring{\mathcal{U}}_{2,1}(z)\mathring{\mathcal{U}}^{\star}_{2,1}(z)]\mathring{\mathcal{N}}_{1,1}(z) = \mathring{\mathcal{N}}_{2,2}(z)\mathring{\mathcal{N}}_{1,1}(z),$$

from which we deduce that

$$\mathring{\mathcal{U}}_{2,2}(z)\mathring{\mathcal{U}}^{\star}_{2,2}(z) + \mathring{\mathcal{U}}_{2,1}(z)\mathring{\mathcal{U}}^{\star}_{2,1}(z) = \mathring{\mathcal{N}}_{2,2}(z). \tag{3.31}$$

Now (3.29), (3.30), and (3.31) together imply that $\mathring{\mathcal{U}}(z)\mathring{\mathcal{U}}^{\star}(z) = \mathring{\mathcal{N}}(z)$. The last identity in (3.5) can be directly check. That is, we proved that (3.5) holds. □

4. Tight Framelet Filter Banks with Symmetry and Two High-pass Filters

In this section, we study tight framelet filter banks with symmetry or complex symmetry and with two high-pass filters. We shall provide a necessary and sufficient condition on the low-pass filter a and the moment correcting filter Θ for a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with symmetry or complex symmetry.

The following auxiliary result on [complex] symmetry will be needed later.

Lemma 4.1. *Let $\{a; b_1, b_2\}_\Theta$ be a tight framelet filter bank such that all the filters Θ, a, b_1, b_2 are not identically zero and have symmetry:*

$$\mathsf{S}\Theta(z) = 1, \qquad \mathsf{S}\mathsf{a}(z) = \epsilon z^c, \qquad \mathsf{S}\mathsf{b}_1(z) = \epsilon_1 z^{c_1}, \qquad \mathsf{S}\mathsf{b}_2(z) = \epsilon_2 z^{c_2} \tag{4.1}$$

for some $\epsilon, \epsilon_1, \epsilon_2 \in \{-1, 1\}$ and $c, c_1, c_2 \in \mathbb{Z}$. Then

$$c_1 - c \in 2\mathbb{Z} \quad \text{and} \quad c_2 - c \in 2\mathbb{Z}. \tag{4.2}$$

The same conclusion holds if the symmetry operator S is replaced by the complex symmetry operator $\mathbb{S}$.

Proof. Since $\{a; b_1, b_2\}_\Theta$ is a tight framelet filter bank, it is necessary to have $\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$. Hence, $\mathbb{S}\Theta = 1$. If Θ also has symmetry, then by Lemma 2.3, Θ must have real coefficients and the condition $\mathsf{S}\Theta(z) = 1$ must be satisfied. As part of the perfect reconstruction condition in (1.2) with $s = 2$, we have

$$\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) + \mathsf{b}_1(z)\mathsf{b}_1^\star(-z) + \mathsf{b}_2(z)\mathsf{b}_2^\star(-z) = 0. \tag{4.3}$$

Considering the symmetry type of each term in the above identity, we have

$$\mathsf{S}\big(\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z)\big) = \mathsf{S}(\Theta(z^2))\mathsf{S}(\mathsf{a}(z))\mathsf{S}(\mathsf{a}^\star(-z)) = \epsilon z^c \epsilon(-z)^{-c} = (-1)^c$$

and

$$\mathsf{S}\big(\mathsf{b}_1(z)\mathsf{b}_1^\star(-z)\big) = \mathsf{S}\mathsf{b}_1(z)(\mathsf{S}\mathsf{b}_1(-z))^\star = (-1)^{c_1} \quad \text{and} \quad \mathsf{S}\big(\mathsf{b}_2(z)\mathsf{b}_2^\star(-z)\big) = (-1)^{c_2}.$$

Because all c, c_1, c_2 are integers, at least two of the three terms $(-1)^c, (-1)^{c_1}, (-1)^{c_2}$ must be the same. That is, the symmetry types of at least two terms in (4.3) must be the same. We now conclude that all the symmetry types of the three terms in (4.3) must be the same and therefore, (4.2) holds. Suppose not, say $(-1)^c = (-1)^{c_1} \neq (-1)^{c_2}$. Moving the terms involving b_2 to the right hand of (4.3), we end up with the situation that the term $\mathsf{b}_2(z)\mathsf{b}_2^\star(-z)$ has both the symmetry types $(-1)^{c_2}$ and $(-1)^{c_2+1}$. This can happen if and only if b_2 is identically zero, which is a contradiction to our assumption. □

For a filter u, if $\mathsf{u}(e^{-i\xi}) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$ holds for an integer m but not $m+1$, then we define $\mathrm{vm}(u) := \mathrm{vm}(\mathsf{u}) := m$, the order of vanishing moments of u or u. Similarly, if $\mathsf{u}(-e^{-i\xi}) = \mathscr{O}(|\xi|^n)$ as $\xi \to 0$ holds for an integer n but not $n+1$, $\mathrm{sr}(u) := \mathrm{sr}(\mathsf{u}) := n$ denotes the order of sum rules of u or u.

Let $\{a; b_1, \ldots, b_s\}_\Theta$ be a tight framelet filter bank. If $\Theta(1)\mathsf{a}(1) \neq 0$, then it is well known ([3, 5]) that

$$\min\big(\mathrm{vm}(b_1), \ldots, \mathrm{vm}(b_s)\big) = \min\big(\mathrm{sr}(a), \tfrac{1}{2}\,\mathrm{vm}(\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z))\big). \tag{4.4}$$

Due to the role of Θ to achieve high $\mathrm{vm}(\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z))$, we call Θ *a moment correcting filter* for the filter a. Hence, we often write

$$\mathsf{b}_1(z) = (1 - z^{-1})^{n_b}\mathring{\mathsf{b}}_1(z), \quad \ldots, \quad \mathsf{b}_s(z) = (1 - z^{-1})^{n_b}\mathring{\mathsf{b}}_s(z).$$

Then the prefect reconstruction condition in (1.2) for a tight framelet filter bank $\{a; b_1, \ldots, b_s\}_\Theta$ becomes

$$\begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \cdots & \mathring{\mathsf{b}}_s(z) \\ \mathring{\mathsf{b}}_1(-z) & \cdots & \mathring{\mathsf{b}}_s(-z) \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \cdots & \mathring{\mathsf{b}}_s(z) \\ \mathring{\mathsf{b}}_1(-z) & \cdots & \mathring{\mathsf{b}}_s(-z) \end{bmatrix}^\star = \mathcal{M}_{a,\Theta|n_b}(z), \tag{4.5}$$

where

$$\mathcal{M}_{a,\Theta|n_b}(z) := \begin{bmatrix} \mathsf{A}(z) & \mathsf{B}(-z) \\ \mathsf{B}(z) & \mathsf{A}(-z) \end{bmatrix} \tag{4.6}$$

with

$$\mathsf{A}(z) := \frac{\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z)}{(1-z)^{n_b}(1-z^{-1})^{n_b}}, \qquad \mathsf{B}(z) := -\Theta(z^2)\frac{\mathsf{a}(-z)}{(1-z)^{n_b}}\frac{\mathsf{a}^\star(z)}{(1+z^{-1})^{n_b}}. \tag{4.7}$$

In terms of polyphase matrices and coset sequences, (4.5) can be further rewritten as

$$\begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[1]}(z) \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1^{[0]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[0]}(z) \\ \mathring{\mathsf{b}}_1^{[1]}(z) & \cdots & \mathring{\mathsf{b}}_s^{[1]}(z) \end{bmatrix}^\star = \mathcal{N}_{a,\Theta|n_b}(z), \tag{4.8}$$

where

$$\mathcal{N}_{a,\Theta|n_b}(z) := \frac{1}{2}\begin{bmatrix} \mathsf{A}^{[0]}(z) + \mathsf{B}^{[0]}(z) & z\big(\mathsf{A}^{[1]}(z) + \mathsf{B}^{[1]}(z)\big) \\ \mathsf{A}^{[1]}(z) - \mathsf{B}^{[1]}(z) & \mathsf{A}^{[0]}(z) - \mathsf{B}^{[0]}(z) \end{bmatrix}. \tag{4.9}$$

Note that $\mathcal{M}_{a,\Theta} = \mathcal{M}_{a,\Theta|0}$ and $\mathcal{N}_{a,\Theta} = \mathcal{N}_{a,\Theta|0}$.

As an application of Theorem 3.2, we have the following result on tight framelet filter banks with [complex] symmetry (and real coefficients) and two high-pass filters.

Theorem 4.2. *Let $a, \Theta \in l_0(\mathbb{Z})$ be given filters having symmetry (and real coefficients) such that $\mathsf{S}\Theta(z) = 1$ and $\mathsf{Sa}(z) = \epsilon z^c$. Let n_b be a nonnegative integer satisfying*

$$0 \leqslant n_b \leqslant \min\big(\mathrm{sr}(a), \tfrac{1}{2}\,\mathrm{vm}(\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z))\big). \tag{4.10}$$

Then there exists a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ for some $b_1, b_2 \in l_0(\mathbb{Z})$ having symmetry (and real coefficients) if and only if

(i) *$\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$;*
(ii) *$\det(\mathcal{N}_{a,\Theta|n_b}(z)) = \mathsf{d}_{n_b}(z)\mathsf{d}^\star_{n_b}(z)$ for a Laurent polynomial d_{n_b} having symmetry (and real coefficients), where $\mathcal{N}_{a,\Theta|n_b}$ is defined in (4.9);*
(iii) *the Laurent polynomial p in (4.11) must satisfy the technical condition for the real SOS property with respect to $(-1)^{c+n_b} z^{\mathrm{odd}(c+n_b)-1}\mathsf{Sd}_{n_b}$ (see Theorem 2.7 for detail), where*

$$\mathsf{p}(z) := \gcd([\mathcal{N}_{a,\Theta|n_b}(z)]_{1,1}, [\mathcal{N}_{a,\Theta|n_b}(z)]_{1,2}, [\mathcal{N}_{a,\Theta|n_b}(z)]_{2,1}, [\mathcal{N}_{a,\Theta|n_b}(z)]_{2,2}). \tag{4.11}$$

In case of $\Theta = \delta$, then $\mathsf{p} = 1$ and (iii) is automatically satisfied.

Moreover, if all involved Laurent polynomials and filters have complex symmetry (that is, all filters Θ, a, b_1, b_2 and d_{n_b}) instead of symmetry, replace the symmetry operator S by the complex symmetry operator $\mathbb{S}$, then the same necessary and sufficient condition, with item (iii) removed, still holds.

Proof. We only prove the necessity part. The sufficiency part will be given in Algorithm 3, where we shall present a concrete way of constructing the filters b_1 and b_2 with [complex] symmetry (and real coefficients).

The proof follows the same line as in the proof of [13, Theorem 2.4]. Suppose that there is a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ such that a, b_1, b_2 have [complex] symmetry in (4.1) for some $\epsilon, \epsilon_1, \epsilon_2 \in \{-1, 1\}$ and $c, c_1, c_2 \in \mathbb{Z}$. From the perfect reconstruction condition in (1.4) with $s = 2$, it is straightforward to see that $\mathcal{N}_{a,\Theta}(z) \geqslant 0$ for all $z \in \mathbb{T}$, from which item (i) must hold. In particular, we have $\mathbb{S}\Theta = 1$. By the relation between $\mathcal{N}_{a,\Theta}$ and $\mathcal{N}_{a,\Theta|n_b}$, we also have $\mathcal{N}_{a,\Theta|n_b}(z) \geqslant 0$ for all $z \in \mathbb{T}$ and $\mathcal{N}^\star_{a,\Theta|n_b}(z) = \mathcal{N}_{a,\Theta|n_b}(z)$. Define

$$\mathcal{N}(z) := \begin{bmatrix} \mathcal{N}_{1,1}(z) & \mathcal{N}_{1,2}(z) \\ \mathcal{N}_{2,1}(z) & \mathcal{N}_{2,2}(z) \end{bmatrix} := \begin{cases} \mathcal{N}_{a,\Theta|n_b}(z), & \text{if } c + n_b \text{ is even;} \\ P_{k_a}(z)\mathcal{N}_{a,\Theta|n_b}(z)P^\star_{k_a}(z), & \text{if } c + n_b \text{ is odd,} \end{cases} \tag{4.12}$$

where $\mathcal{N}_{a,\Theta|n_b}$ is defined in (4.9) and

$$P_{k_a}(z) := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & z^{k_a} \\ 1 & -z^{k_a} \end{bmatrix} \tag{4.13}$$

with $k_a := 0$. It is trivial to see that $\det(\mathcal{N}(z)) = \det(\mathcal{N}_{a,\Theta|n_b}(z))$ and $\mathcal{N}^\star = \mathcal{N}$. We now show that all the entries of $\mathcal{N}$ have [complex] symmetry. By the definition of A, B in (4.7), we have

$$\mathsf{SA}(z) = 1, \quad \mathsf{SB}(z) = (-1)^{c+n_b}. \tag{4.14}$$

By (4.14), we see that

$$\mathsf{SA}^{[0]}(z) = 1, \quad \mathsf{SA}^{[1]}(z) = z^{-1}, \quad \mathsf{SB}^{[0]}(z) = (-1)^{c+n_b}, \quad \mathsf{SB}^{[1]}(z) = (-1)^{c+n_b} z^{-1}. \tag{4.15}$$

When $c + n_b$ is even, since $\mathcal{N} = \mathcal{N}_{a,\Theta|n_b}$, by the definition of $\mathcal{N}_{a,\Theta|n_b}(z)$ in (4.9) and by (4.15), we see that $\mathsf{S}\mathcal{N}_{1,1}(z) = \mathsf{S}\mathcal{N}_{2,2}(z) = 1$, $\mathsf{S}\mathcal{N}_{1,2}(z) = z$, and $\mathsf{S}\mathcal{N}_{2,1}(z) = z^{-1}$.

When $c + n_b$ is odd, though some entries of $\mathcal{N}_{a,\Theta|n_b}$ may no longer have any symmetry property, since $\mathcal{N}(z) = P_{k_a}(z)\mathcal{N}_{a,\Theta|n_b}(z)P^\star_{k_a}(z)$, we have

$$\begin{aligned}
[\mathcal{N}(z)]_{1,1} &= \tfrac{1}{2}\big([\mathcal{N}_{a,\Theta|n_b}(z)]_{1,1} + [\mathcal{N}_{a,\Theta|n_b}(z)]_{2,2} + z^{k_a}[\mathcal{N}_{a,\Theta|n_b}(z)]_{2,1} + z^{-k_a}[\mathcal{N}_{a,\Theta|n_b}(z)]_{1,2}\big) \\
&= \tfrac{1}{4}\big(2\mathsf{A}^{[0]}(z) + (z^{1-k_a} + z^{k_a})\mathsf{A}^{[1]}(z) + (z^{1-k_a} - z^{k_a})\mathsf{B}^{[1]}(z)\big), \\
[\mathcal{N}(z)]_{1,2} &= \tfrac{1}{2}\big([\mathcal{N}_{a,\Theta|n_b}(z)]_{1,1} - [\mathcal{N}_{a,\Theta|n_b}(z)]_{2,2} + z^{k_a}[\mathcal{N}_{a,\Theta|n_b}(z)]_{2,1} - z^{-k_a}[\mathcal{N}_{a,\Theta|n_b}(z)]_{1,2}\big) \\
&= \tfrac{1}{4}\big(2\mathsf{B}^{[0]}(z) - (z^{1-k_a} - z^{k_a})\mathsf{A}^{[1]}(z) - (z^{1-k_a} + z^{k_a})\mathsf{B}^{[1]}(z)\big), \\
[\mathcal{N}(z)]_{2,1} &= \tfrac{1}{2}\big([\mathcal{N}_{a,\Theta|n_b}(z)]_{1,1} - [\mathcal{N}_{a,\Theta|n_b}(z)]_{2,2} - z^{k_a}[\mathcal{N}_{a,\Theta|n_b}(z)]_{2,1} + z^{-k_a}[\mathcal{N}_{a,\Theta|n_b}(z)]_{1,2}\big) \\
&= \tfrac{1}{4}\big(2\mathsf{B}^{[0]}(z) + (z^{1-k_a} - z^{k_a})\mathsf{A}^{[1]}(z) + (z^{1-k_a} + z^{k_a})\mathsf{B}^{[1]}(z)\big), \\
[\mathcal{N}(z)]_{2,2} &= \tfrac{1}{2}\big([\mathcal{N}_{a,\Theta|n_b}(z)]_{1,1} + [\mathcal{N}_{a,\Theta|n_b}(z)]_{2,2} - z^{k_a}[\mathcal{N}_{a,\Theta|n_b}(z)]_{2,1} - z^{-k_a}[\mathcal{N}_{a,\Theta|n_b}(z)]_{1,2}\big) \\
&= \tfrac{1}{4}\big(2\mathsf{A}^{[0]}(z) - (z^{1-k_a} + z^{k_a})\mathsf{A}^{[1]}(z) - (z^{1-k_a} - z^{k_a})\mathsf{B}^{[1]}(z)\big).
\end{aligned}$$

Since $c + n_b$ is odd, by (4.15), we have $\mathsf{S}\mathcal{N}_{1,1}(z) = \mathsf{S}\mathcal{N}_{2,2}(z) = 1$, $\mathsf{S}\mathcal{N}_{1,2}(z) = \mathsf{S}\mathcal{N}_{2,1}(z) = -1$.

Therefore, for both cases we always have $\mathsf{S}\mathcal{N}_{2,1}(z) = (-1)^{c+n_b} z^{\mathrm{odd}(c+n_b)-1}$.

By Lemma 4.1, we see that (4.2) holds. Suppose that $c+n_b$ is an even integer. By (4.2) we see that both c_1+n_b and c_2+n_b are even integers. Since $\mathcal{N}=\mathcal{N}_{a,\Theta}$, setting

$$\mathcal{U}(z):=\begin{bmatrix}\mathcal{U}_{1,1}(z) & \mathcal{U}_{1,2}(z)\\ \mathcal{U}_{2,1}(z) & \mathcal{U}_{2,2}(z)\end{bmatrix}:=\begin{bmatrix}\mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z)\\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z)\end{bmatrix},$$

we deduce that all the entries in $\mathcal{U}$ have symmetry, $\mathcal{U}(z)\mathcal{U}^\star(z)=\mathcal{N}(z)$, and

$$\frac{\mathsf{S}\mathcal{U}_{1,1}(z)}{\mathsf{S}\mathcal{U}_{2,1}(z)}=\frac{\mathsf{S}\mathring{\mathsf{b}}_1^{[0]}(z)}{\mathsf{S}\mathring{\mathsf{b}}_1^{[1]}(z)}=\frac{(-1)^{n_b}\epsilon_1 z^{\frac{c_1+n_b}{2}}}{(-1)^{n_b}\epsilon_1 z^{\frac{c_1+n_b}{2}-1}}=z=\frac{(-1)^{n_b}\epsilon_2 z^{\frac{c_2+n_b}{2}}}{(-1)^{n_b}\epsilon_2 z^{\frac{c_2+n_b}{2}-1}}=\frac{\mathsf{S}\mathring{\mathsf{b}}_2^{[0]}(z)}{\mathsf{S}\mathring{\mathsf{b}}_2^{[1]}(z)}=\frac{\mathsf{S}\mathcal{U}_{1,2}(z)}{\mathsf{S}\mathcal{U}_{2,2}(z)}.$$

Suppose that $c+n_b$ is an odd integer. By (4.2) we see that both c_1+n_b and c_2+n_b are odd integers. Define

$$\mathcal{U}(z):=\begin{bmatrix}\mathcal{U}_{1,1}(z) & \mathcal{U}_{1,2}(z)\\ \mathcal{U}_{2,1}(z) & \mathcal{U}_{2,2}(z)\end{bmatrix}:=P_{k_a}(z)\begin{bmatrix}\mathring{\mathsf{b}}_1^{[0]}(z) & \mathring{\mathsf{b}}_2^{[0]}(z)\\ \mathring{\mathsf{b}}_1^{[1]}(z) & \mathring{\mathsf{b}}_2^{[1]}(z)\end{bmatrix}.$$

We can directly verify that all the entries in $\mathcal{U}$ have symmetry, $\mathcal{U}(z)\mathcal{U}^\star(z)=\mathcal{N}(z)$, and

$$\frac{\mathsf{S}\mathcal{U}_{1,1}(z)}{\mathsf{S}\mathcal{U}_{2,1}(z)}=\frac{(-1)^{n_b}\epsilon_1 z^{(c_1+n_b-1)/2}}{(-1)^{n_b+1}\epsilon_1 z^{(c_1+n_b-1)/2}}=-1=\frac{(-1)^{n_b}\epsilon_2 z^{(c_2+n_b-1)/2}}{(-1)^{n_b+1}\epsilon_2 z^{(c_2+n_b-1)/2}}=\frac{\mathsf{S}\mathcal{U}_{1,2}(z)}{\mathsf{S}\mathcal{U}_{2,2}(z)}.$$

Observe that $\mathsf{S}\mathcal{N}_{2,1}(z)\mathsf{S}\mathsf{d}_{n_b}(z)=(-1)^{c+n_b}z^{\text{odd}(c+n_b)-1}\mathsf{S}\mathsf{d}_{n_b}(z)$. Now by Theorem 3.2, we conclude that items (ii) and (iii) must hold. Therefore, we verified the necessity part. The proof for the case of complex symmetry is the same by replacing S with $\mathbb{S}$. $\square$

Note that the Laurent polynomial p in (4.11) is independent of the choice of n_b, and the symmetry type $(-1)^{c+n_b}z^{\text{odd}(c+n_b)-1}\mathsf{S}\mathsf{d}_{n_b}(z)$ only differs by a factor of z^{2k} for some $k\in\mathbb{Z}$. Therefore, the necessary and sufficient condition in Theorem 4.2 for different choices of n_b is almost the same and we can use any choice of n_b in Theorem 4.2. We often take n_b to be the largest integer satisfying (4.10).

We now provide an algorithm to construct the high-pass filters b_1 and b_2 in Theorem 4.2 and prove their existence.

Algorithm 3. *Let $a,\Theta\in l_0(\mathbb{Z})$ such that all the assumptions and items (i)–(iii) of Theorem 4.2 are satisfied [for the case of complex symmetry, remove item (iii) and replace the symmetry operator S by the complex symmetry operator $\mathbb{S}$].*

- (S1) *Choose a nonnegative integer n_b satisfying* (4.10)*. Calculate the 2×2 matrix $\mathcal{N}$ as in* (4.12)*, where P_{k_a} is defined in* (4.13) *and $k_a:=\lfloor\frac{n_++n_-}{2}\rfloor$ with $[n_-,n_+]:=\operatorname{fsupp}([\mathcal{N}_{a,\Theta|n_b}]_{1,2})$ (here k_a is chosen so that the filter supports of all entries in $\mathcal{N}$ are as short as possible).*
- (S2) *Use Algorithm 2 (which implements Theorem 3.2) to factorize $\mathcal{N}$ such that* (3.5) *is satisfied;*
- (S3) *Construct two high-pass filters b_1 and b_2 as follows: If $c+n_b$ is even, define*

$$\mathsf{b}_1(z):=(1-z^{-1})^{n_b}[\mathcal{U}_{1,1}(z^2)+z\mathcal{U}_{2,1}(z^2)],\quad \mathsf{b}_2(z):=(1-z^{-1})^{n_b}[\mathcal{U}_{1,2}(z^2)+z\mathcal{U}_{2,2}(z^2)];\qquad(4.16)$$

If $c+n_b$ is odd, define

$$\begin{aligned}\mathsf{b}_1(z)&:=(1-z^{-1})^{n_b}\left[\frac{1+z^{1-2k_a}}{\sqrt{2}}\mathcal{U}_{1,1}(z^2)+\frac{1-z^{1-2k_a}}{\sqrt{2}}\mathcal{U}_{2,1}(z^2)\right],\\ \mathsf{b}_2(z)&:=(1-z^{-1})^{n_b}\left[\frac{1+z^{1-2k_a}}{\sqrt{2}}\mathcal{U}_{1,2}(z^2)+\frac{1-z^{1-2k_a}}{\sqrt{2}}\mathcal{U}_{2,2}(z^2)\right].\end{aligned}\qquad(4.17)$$

Then $\{a;b_1,b_2\}_\Theta$ is a tight framelet filter bank with [complex] symmetry (and real coefficients).

Proof. We first prove that all the sufficient conditions in Theorem 3.2 are satisfied so that we can carry out step (S2). Note that item (ii) in Theorem 4.2 implies that $\det(\mathcal{N}_{a,\Theta}(z))=|1-z^{-2}|^{2n_b}|\mathsf{d}_{n_b}(z)|^2\geqslant 0$ for all $z\in\mathbb{T}$. Together with item (i) in Theorem 4.2, we must have $\mathcal{N}_{a,\Theta}(z)\geqslant 0$ for all $z\in\mathbb{T}$. Now it is not difficult to see that $\mathcal{N}_{a,\Theta|n_b}(z)\geqslant 0$ for all $z\in\mathbb{T}$. By the definition of $\mathcal{N}$ in (4.12), it is now trivial to see that $\mathcal{N}(z)\geqslant 0$ for all $z\in\mathbb{T}$. Therefore, item (i) of Theorem 3.2 is satisfied. As we argued in the proof of Theorem 4.2, $\mathsf{S}\mathcal{N}_{2,1}(z)=(-1)^{c+n_b}z^{\text{odd}(c+n_b)-1}$. Since $\det(\mathcal{N}_{a,\Theta|n_b}(z))=\det(\mathcal{N}(z))$ and $\gcd(\mathcal{N}_{1,1},\mathcal{N}_{1,2},\mathcal{N}_{2,1},\mathcal{N}_{2,2})=\mathsf{p}$, items (ii) and (iii) of Theorem 3.2 simply correspond to items (ii) and (iii) of Theorem 4.2. Hence, all the sufficient conditions in Theorem 3.2 are satisfied. Thus, the existence of $\mathcal{U}$ in (S2) satisfying (3.5) has been established.

By the definition of b_1 and b_2 in (4.17) and by the definition of $\mathcal{N}$ in (4.12), we can directly check that $\{a;b_1,b_2\}_\Theta$ is a tight framelet filter bank with [complex] symmetry (and real coefficients). $\square$

5. Two Special Examples of Tight Framelet Filter Banks with Complex Symmetry

In this section, we present some special examples to compare with real-valued tight framelet filter banks considered in [13, 17].

The smoothness exponent $\mathrm{sm}(u)$ of a filter u is closely related to the smoothness of its associated refinable function. To define $\mathrm{sm}(u)$, we first write $\widehat{u}(\xi) = (1+e^{-i\xi})^m \widehat{v}(\xi)$ for a nonnegative integer m and a finitely supported sequence v such that $\widehat{v}(\pi) \neq 0$. Then we define the *smoothness exponent* of the filter u ([9]) by

$$\mathrm{sm}(u) := -1/2 - \log_2 \sqrt{\rho(u)},$$

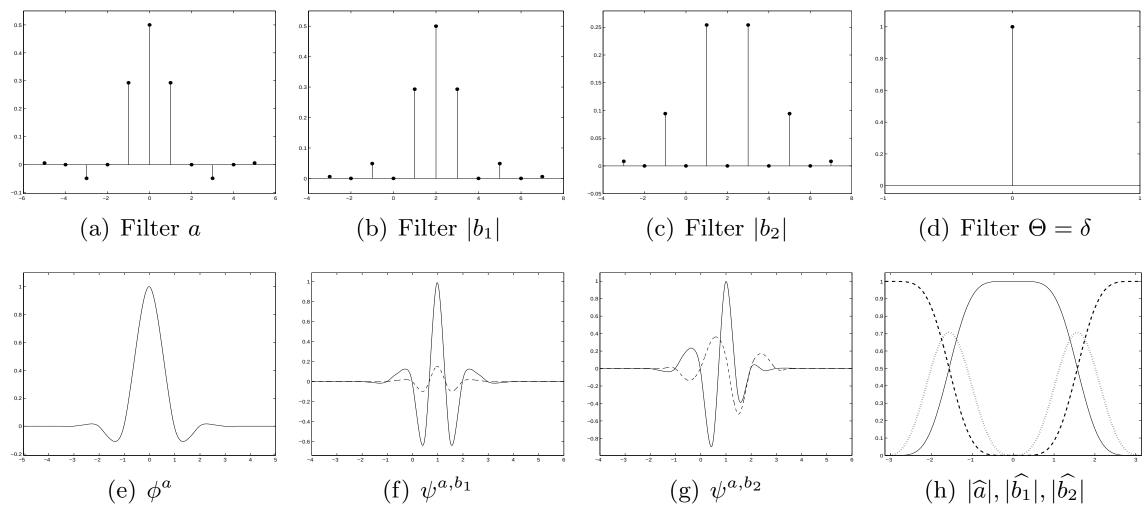
where $\rho(u)$ denotes the spectral radius–the largest of the modulus of all the eigenvalues–of the square matrix $(w(2j-k))_{-K\leqslant j,k\leqslant K}$, where w is determined by $\sum_{k=-K}^{K} w(k)e^{-ik\xi} := |\widehat{v}(\xi)|^2$.

[17, Corollary 2] shows that it is impossible to obtain a real-valued tight framelet filter bank $\{a; b_1, b_2\}$ with symmetry derived from a nontrivial interpolatory filter a. In the following, we provide one example to show that there are (complex-valued) tight framelet filter banks $\{a; b_1, b_2\}$ with symmetry derived from nontrivial real-valued interpolatory filters a.

Example 1. Let $a = \{\frac{3}{512}, 0, -\frac{25}{512}, 0, \frac{75}{256}, \underline{\boldsymbol{\frac{1}{2}}}, \frac{75}{256}, 0, -\frac{25}{512}, 0, \frac{3}{512}\}_{[-5,5]}$. Then a is an interpolatory filter with $\epsilon z^c := \mathsf{S}\mathsf{a}(z) = 1$, $\mathrm{sr}(a) = 6$, and $\mathrm{sm}(a) \approx 3.175132$. Set $n_b = n = 3$ and $\Theta = \delta$ in Algorithms 3. Then $\mathcal{N} = \mathcal{N}_{a,\Theta|2}$. Though items (i) and (iii) of Theorem 4.2 are satisfied, item (ii) of Theorem 4.2 fails for the case of complex symmetry. However, all items (i)—(iii) of Theorem 4.2 are satisfied for the case of symmetry. Take $\mathsf{d}(z) = \frac{\sqrt{2}}{1024} z^2(3-(16-6\sqrt{15}i)z+3z^2)$. Setting $\tilde{t}_3 = 1$ and requiring $\mathsf{b}_2(-1) = 0$, we have $\mathcal{U}_{1,1}(z) = \frac{5\sqrt{5}+\sqrt{3}i}{2048}(z-1)(z-3)(3z-1)$ and $\mathcal{U}_{1,2}(z) = \frac{\sqrt{2}(5\sqrt{5}-\sqrt{3}i)}{2048}(z+1)(3-(16-6\sqrt{15}i)z+3z^2)$. Then

$$\mathsf{b}_1(z) = -\tfrac{\sqrt{2}(5\sqrt{5}+\sqrt{3}i)}{24576} z^{-3}(z-1)^6[9(1+z^4)+54(z+z^3)+114z^2],$$
$$\mathsf{b}_2(z) = \tfrac{5\sqrt{5}-\sqrt{3}i}{12288}(1-z^{-1})^3(1+z)^3[9(1+z^4)-(48-18\sqrt{15}i)z^2].$$

Note that $\mathsf{S}\mathsf{a}(z) = 1, \mathsf{S}\mathsf{b}_1(z) = z^4, \mathsf{S}\mathsf{b}_2(z) = -z^4$ and $\mathrm{vm}(b_1) = 6, \mathrm{vm}(b_2) = 3$. Hence, $\{a; b_1, b_2\}$ is a tight framelet filter bank with symmetry and 3 vanishing moments.



(a) Filter a (b) Filter $|b_1|$ (c) Filter $|b_2|$ (d) Filter $\Theta = \delta$

(e) ϕ^a (f) ψ^{a,b_1} (g) ψ^{a,b_2} (h) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

Figure 5.1. The tight framelet filter bank $\{a; b_1, b_2\}$ with symmetry and 3 vanishing moments constructed in Example 1. (a), (b), (c), (d) are the graphs of the filters $a, b_1, b_2, \Theta = \delta$, respectively. (e), (f), (g) are the graphs of the refinable function ϕ^a and the framelet functions ψ^{a,b_1}, ψ^{a,b_2} (real part in solid line and imaginary part in dotted line), respectively. (h) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), and $\widehat{b_2}$ (in dotted line) on the interval $[-\pi, \pi]$.

Using Algorithm 3, we now show that a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with complex symmetry can indeed be derived from the real-valued filters a and Θ given in [13, Example 3.1].

Example 2. Let $t \approx 0.07390661889361312$ be a real root of

$$t^8 + 8t^7 + 35t^6 + 58t^5 - 10t^4 - 72t^3 - t^2 + 14t - 1 = 0$$

and define

$$\lambda_2 := -\tfrac{1}{128}\sqrt{-96t^7-557t^6-842t^5+255t^4+988t^3-31t^2-178t+13} \approx 0.06116245919177508,$$
$$\lambda_1 := \tfrac{\lambda_2}{4}(42\,t^7+353\,t^6+1614\,t^5+3094\,t^4+844\,t^3-2728\,t^2-1262\,t+75) \approx 0.01806084935914014,$$
$$\lambda_0 := \lambda_2(t^7+10\,t^6+54\,t^5+152\,t^4+216\,t^3+116\,t^2-33\,t-26) \approx -0.002206959294042286.$$

The real-valued low-pass filter a and the moment correcting filter Θ in [13, Example 3.1] are given by

$$\mathsf{a}(z)=z^{-1}(1+z)^2(\tfrac{1}{4}+\tfrac{t}{8}(2-z-z^{-1}), \qquad \Theta(z):=\boldsymbol{\theta}(z)\boldsymbol{\theta}(z^2) \quad \text{with} \quad \boldsymbol{\theta}(z):=1+\tfrac{t^2+2t-1}{4}(2-z-z^{-1}). \quad (5.1)$$

Then $\mathrm{sm}(a)\approx 1.395530$, $\mathrm{sr}(a)=2$, and $\mathrm{fsupp}(a)=[-2,2]$. Since $\boldsymbol{\theta}(e^{-i\xi})=1+(t^2+2t-1)\sin^2(\xi/2) > 1-|t^2+2t-1| = 2t+t^2>0$ for all $\xi\in\mathbb{R}$, we have $\Theta(z)>0$ for all $z\in\mathbb{T}$. Moreover, we have $\det(\mathcal{N}_{a,\Theta|1}(z)) = \mathsf{d}_1(z)\mathsf{d}_1^\star(z)$, where $\mathsf{d}_1(z)=(1+z)[\lambda_0+\lambda_1(z+z^{-1})+\lambda_2(z^2+z^{-2})]$. Hence, the conditions in items (i) and (ii) of Theorem 4.2 are satisfied. As demonstrated in [13, Example 3.1], the item (iii) of Theorem 4.2 fails, since $\mathsf{p}=\boldsymbol{\theta}$ and $\mathsf{Z}(\mathsf{p},x_1)=1$ with $-1<x_1:=\frac{t^2+2t+1-2\sqrt{t^2+2t}}{t^2+2t-1}\approx -0.437293690197<0$. Hence, it is impossible to derive a complex-valued tight framelet filter with symmetry from a and Θ. Nevertheless, since items (i) and (ii) of Theorem 4.2 are indeed satisfied for the case of complex symmetry, according to Theorem 4.2 and Algorithm 3, we can indeed build a complex-valued tight framelet filter bank $\{a;b_1,b_2\}_\Theta$ with complex symmetry from the real-valued low-pass filter a and the real-valued moment correcting filter Θ in (5.1). Since $\mathrm{vm}(\Theta(z)-\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z))=2$, it follows from (4.4) that the high-pass filters have no more than one vanishing moment. For simplicity of presentation, we do not list the constructed high-pass filters b_1,b_2 here. Instead, based on the same idea as in [13, Example 3.1], we present a similar but slightly better example here with two vanishing moments.

Let $t\approx -0.7024971974920165$ be a real root of

$$4\,t^7-65\,t^6+252\,t^5+402\,t^4-2420\,t^3-225\,t^2+872\,t-162=0$$

and define

$$\lambda_1 := -\tfrac{1}{240}\sqrt{4\,t^4-42\,t^3+24\,t^2-2\,t} \approx 0.1869577611993216,$$
$$\lambda_0 := \tfrac{\lambda_1}{4275}(1184\,t^6-18928\,t^5+69558\,t^4+138111\,t^3-683032\,t^2-251541\,t+223684) \approx 0.02235445039673207.$$

Define the real-valued low-pass filter a and the real-valued moment correcting filter Θ by

$$\mathsf{a}(z)=z^{-1}(1+z)^2(-tz^{-1}+2t+2-tz)/8, \qquad \Theta(z):=\boldsymbol{\theta}(z)\boldsymbol{\theta}(z^2) \quad \text{with} \quad \boldsymbol{\theta}(z):=1+\tfrac{1-2t}{30}(2-z-z^{-1}).$$

Then $\mathrm{sm}(a)\approx 1.988792$ and $\mathrm{sr}(a)=2$. Since $\boldsymbol{\theta}(e^{-i\xi})=1+\frac{2-4t}{15}\sin^2(\xi/2)\geqslant \frac{17-4t}{15}>0$ for all $\xi\in\mathbb{R}$, we have $\Theta(z)>0$ for all $z\in\mathbb{T}$. Moreover, we have $\det(\mathcal{N}_{a,\Theta|1}(z))=\mathsf{d}_1(z)\mathsf{d}_1^\star(z)$ with $\mathsf{d}_1(z)=\boldsymbol{\theta}(z)[\lambda_0+\lambda_1(z+z^{-1})]$. Hence, the conditions in items (i) and (ii) of Theorem 4.2 are satisfied. We now check item (iii) of Theorem 4.2. Note that $\mathsf{Sa}(z)=1$ and $\mathsf{Sd}(z)=1$. Hence, by $c=0$ and $n_b=2$, we have $(-1)^{c+n_b}z^{\mathrm{odd}(c+n_b)-1}\mathsf{Sd}_1(z)=z^{-1}$. By calculation, it is not difficult to check that $\mathsf{p}=\boldsymbol{\theta}$, where p is the Laurent polynomial defined in (4.11). By calculation, we see that all the roots of p are $x_1:=\frac{2t-16+\sqrt{255-60t}}{2t-1}\approx 0.06942217452877067$ and $x_2:=\frac{2t-16+\sqrt{255-60t}}{2t-1}\approx 14.40461937108538$. Thus, $\mathsf{Z}(\mathsf{p},x_1)=1$ and $0<x_1<1$. Hence, p does not satisfy the technical condition for the real SOS property with respect to the symmetry type z^{-1}. Therefore, item (iii) of Theorem 4.2 fails and it is impossible to derive a complex-valued tight framelet filter bank with symmetry from a and Θ. These two examples show that item (iii) of Theorem 4.2 cannot be dropped for the case of symmetry (and real coefficients). Nevertheless, according to Theorem 4.2, we can still build a complex-valued tight framelet filter bank with *complex symmetry* from a and Θ.

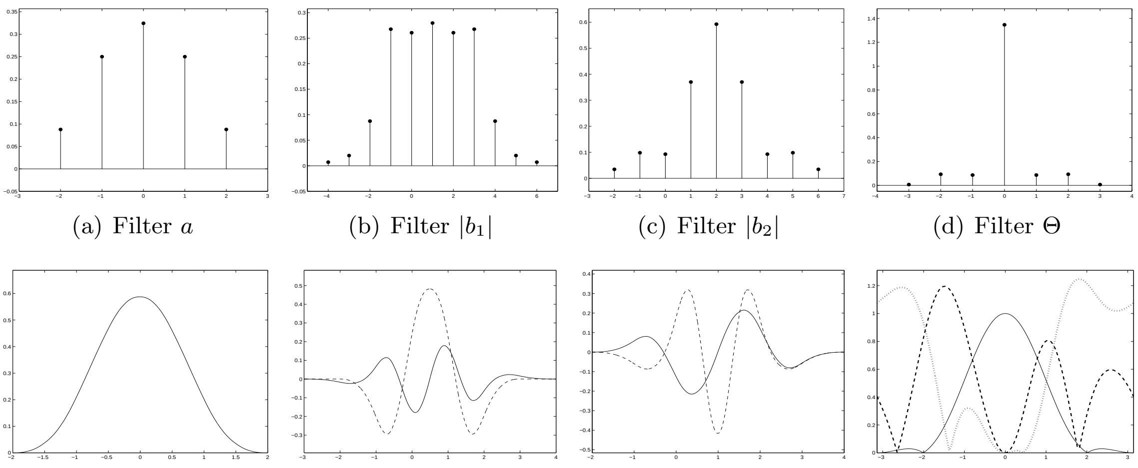
Take $n_b=2$. By calculation, we have $c_{odd}=1$ and

$$\mathring{\mathcal{U}}_{1,1}(z):=-0.550179528966z-0.0846681139556(z^2+1),$$
$$\mathring{\mathcal{U}}_{1,2}(z):=0.0248628560750(1+z^3)+0.163835070534(z+z^2),$$
$$\mathsf{p}_1(z):=i, \qquad \mathsf{p}_2(z):=\sqrt{\tfrac{1-2t}{30}}(1-z^{-1})$$

such that $\boldsymbol{\theta}(z) = \mathsf{p}_1(z)\mathsf{p}_1^\star(z) + \mathsf{p}_2(z)\mathsf{p}_2^\star(z)$. We have

$$\begin{aligned}
\mathsf{b}_1(z) =&(1-z^{-1})^2[-0.007039590926917880\,z^{-2} - 0.03412080181203603\,z^{-1}\\
&- (0.03934815602229675 + 0.08466811395561048\,i)\\
&+ (0.07163370442688816 - 0.4103852007978996\,i)z\\
&- 0.5501795289658676\,iz^2 - (0.07163370442688816 + 0.4103852007978996\,i)z^3\\
&+ (0.03934815602229675 - 0.08466811395561048\,i)z^4\\
&+ 0.03412080181203603z^5 + 0.007039590926917880z^6],\\
\mathsf{b}_2(z) =&(1-z^{-1})^2[(0.02397266367963404 - 0.02486285607500181\,i)\\
&+ (0.1161951759429173 - 0.1205099264180879\,i)z\\
&+ (0.1318034389773006 - 0.1638350705339759\,i)z^2 + 0.1324903546741996\,iz^3\\
&- (0.1318034389773006 + 0.1638350705339759\,i)z^4\\
&- (0.1161951759429173 + 0.1205099264180879\,i)z^5\\
&- (0.02397266367963404 + 0.02486285607500181\,i)z^6].
\end{aligned}$$

with $\mathbb{S}\mathsf{b}_1(z) = -z^2$, $\mathbb{S}\mathsf{b}_2(z) = -z^4$ and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 2$. Hence, $\{a; b_1, b_2\}_\Theta$ is a tight framelet filter bank with complex symmetry and 2 vanishing moments.



(a) Filter a (b) Filter $|b_1|$ (c) Filter $|b_2|$ (d) Filter Θ

(e) ϕ^a (f) ψ^{a,b_1} (g) ψ^{a,b_2} (h) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

FIGURE 5.2. The tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with complex symmetry and 2 vanishing moments is constructed in Example 2. (a), (b), (c), (d) are the graphs of the filters $a, |b_1|, |b_2|, \Theta$, respectively. (e), (f), (g) are the graphs of the refinable function ϕ^a and the framelet functions ψ^{a,b_1}, ψ^{a,b_2} (real part in solid line and imaginary part in dotted line), respectively. (h) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), and $\widehat{b_2}$ (in dotted line) on the interval $[-\pi, \pi]$.

6. Examples of Tight Framelet Filter Banks with Symmetry and Two High-pass Filters

To construct a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with [complex] symmetry, we must obtain a low-pass filter a and a moment correcting filter Θ such that all the conditions in items (i)–(iii) of Theorem 4.2 must be satisfied. We can fix one of the filters a and Θ in advance while designing the other one so that all the conditions in Theorem 4.2 are satisfied. Designing a moment correcting filter Θ for a given low-pass filter a has been addressed in [13]. Here we only consider fixing $\Theta = \delta$ to construct tight framelet filter banks $\{a; b_1, b_2\}$ with [complex] symmetry (and real coefficients). To do so, we need an auxiliary result.

For a filter $u = \{u(k)\}_{k\in\mathbb{Z}}$, if $\mathsf{u}(e^{-i\xi}) = e^{-ic\xi} + \mathcal{O}(|\xi|^m)$ as $\xi \to 0$ holds for an integer m but not $m+1$ with phase $c := \mathrm{Re}(\sum_{k\in\mathbb{Z}} u(k)k)$, then $\mathrm{lpm}(u) := \mathrm{lpm}(\mathsf{u}) := m$ denotes its highest order of linear-phase moments (see [11, 12]).

Proposition 6.1. *If $\{a; b_1, \dots, b_s\}$ is a tight framelet filter bank such that $\mathsf{a}(1) = 1$ and a has complex symmetry, then* $\mathrm{lpm}(\mathsf{a}\mathsf{a}^\star) = \mathrm{lpm}(a)$ *and*

$$\min\big(\mathrm{vm}(b_1), \dots, \mathrm{vm}(b_s)\big) = \min\big(\mathrm{sr}(a), \tfrac{1}{2}\,\mathrm{lpm}(a)\big). \tag{6.1}$$

Proof. Because $\mathsf{a}(1) \neq 0$ and a has complex symmetry $\mathbb{S}\mathsf{a}(e^{-i\xi}) = e^{-i2c\xi}$ for some $c \in \frac{1}{2}\mathbb{Z}$, we deduce that $e^{ic\xi}\mathsf{a}(e^{-i\xi}) = e^{-ic\xi}\overline{\mathsf{a}(e^{-i\xi})} = \overline{e^{ic\xi}\mathsf{a}(e^{-i\xi})}$. Hence,

$$|\mathsf{a}(e^{-i\xi})|^2 = |e^{ic\xi}\mathsf{a}(e^{-i\xi})|^2 = [e^{ic\xi}\mathsf{a}(e^{-i\xi})]^2. \tag{6.2}$$

Denote $n := \operatorname{lpm}(a)$. It follows from $\mathsf{a}(e^{-i\xi}) = e^{-ic\xi} + \mathcal{O}(|\xi|^n)$ as $\xi \to 0$ that $\operatorname{lpm}(\mathsf{a}\mathsf{a}^\star) \geqslant \operatorname{lpm}(a)$.

Conversely, denote $\tilde{n} := \operatorname{lpm}(\mathsf{a}\mathsf{a}^\star)$. Since $\mathsf{a}\mathsf{a}^\star$ is complex symmetric about the origin, its phase must be 0. Now it follows from (6.2) that

$$1 - [e^{ic\xi}\mathsf{a}(e^{-i\xi})]^2 = \mathcal{O}(|\xi|^{\tilde{n}}), \qquad \xi \to 0.$$

Note that

$$1 - [e^{ic\xi}\mathsf{a}(e^{-i\xi})]^2 = [1 - e^{ic\xi}\mathsf{a}(e^{-i\xi})][1 + e^{ic\xi}\mathsf{a}(e^{-i\xi})] = \mathcal{O}(|\xi|^{\tilde{n}}), \qquad \xi \to 0.$$

Since $\mathsf{a}(1) = 1$, we have $[1 + e^{ic\xi}\mathsf{a}(e^{-i\xi})]|_{\xi=0} = 2 \neq 0$. Consequently, we deduce from the above relation that $1 - e^{ic\xi}\mathsf{a}(e^{-i\xi}) = \mathcal{O}(|\xi|^{\tilde{n}})$ as $\xi \to 0$. That is, by $\tilde{n} = \operatorname{lpm}(\mathsf{a}\mathsf{a}^\star)$, we have $\operatorname{lpm}(a) \geqslant \operatorname{lpm}(\mathsf{a}\mathsf{a}^\star)$. Therefore, $\operatorname{lpm}(\mathsf{a}\mathsf{a}^\star) = \operatorname{lpm}(a)$. Since $\Theta = 1$, it follows directly from (4.4) that (6.1) holds. □

By Proposition 6.1 and Theorem 4.2, we see that the problem of constructing a tight framelet filter bank $\{a; b_1, b_2\}$, with complex symmetry (and real coefficients) such that the low-pass filter a has at least m sum rules and the high-pass filters b_1, b_2 have at least n vanishing moments, is equivalent to finding a low-pass filter a such that

(i) Filter a has complex symmetry (and real coefficients), m sum rules, and $2n$ linear-phase moments;
(ii) There exists a Laurent polynomial d_a with complex symmetry (and real coefficients) such that

$$1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z) = \mathsf{d}_\mathsf{a}(z^2)\mathsf{d}_\mathsf{a}^\star(z^2). \tag{6.3}$$

We now present an algorithm to construct all possible such low-pass filters a satisfying the above items (i) and (ii) so that all possible tight framelet filter banks $\{a; b_1, b_2\}$ with complex symmetry (and real coefficients) can be obtained.

Algorithm 4. *Let m and n be positive integers such that $n \leqslant m$.*

(S1) *Calculate the filter $a_{m,2n}$ defined by*

$$\mathsf{a}_{m,2n}(z) := 2^{-m} z^{-\lfloor \frac{m}{2} \rfloor}(z+1)^m \mathsf{P}_{m/2,n}(\tfrac{1}{2} - \tfrac{z+z^{-1}}{4}), \tag{6.4}$$

where

$$\mathsf{P}_{\alpha,n}(x) := \sum_{j=0}^{n-1} \binom{\alpha+j-1}{j} x^j, \qquad \alpha \in \mathbb{R}, n \in \mathbb{N} \cup \{0\}. \tag{6.5}$$

(S2) *Parameterize the low-pass filter a by*

$$\mathsf{a}(z) := \mathsf{a}_{m,2n}(z) + z^{-n-\lfloor \frac{m}{2} \rfloor}(z+1)^m (z-1)^{2n} \boldsymbol{\theta}(z) \quad \text{with} \quad \boldsymbol{\theta}(z) := \lambda_0 + \sum_{j=1}^{\ell-1} (\lambda_j z^j + \overline{\lambda_j} z^{-j}),$$

where $\lambda_0 \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_{\ell-1} \in \mathbb{C}$ (or $\lambda_0, \ldots, \lambda_{\ell-1} \in \mathbb{R}$ for the case of real coefficients) are unknown numbers to be determined later.

(S3) *Calculate the Laurent polynomial p by*

$$\mathsf{p}(z^2) := \frac{1 - \mathsf{a}(z)\mathsf{a}^\star(z) - \mathsf{a}(-z)\mathsf{a}^\star(-z)}{4(2 - z^{-2} - z^2)^{2n}}.$$

Apply Theorem 2.8 to derive a unique Laurent polynomial q_p with complex symmetry (and real coefficients) from p. Let X denote the set of all equations by taking all the coefficients in $\mathsf{p}(z) - \mathsf{q}_\mathsf{p}(z)\mathsf{q}_\mathsf{p}^\star(z)$ to be zero. Use Gröbner basis method to solve equations in X to determine the unknowns $\lambda_0, \ldots, \lambda_{\ell-1}$.

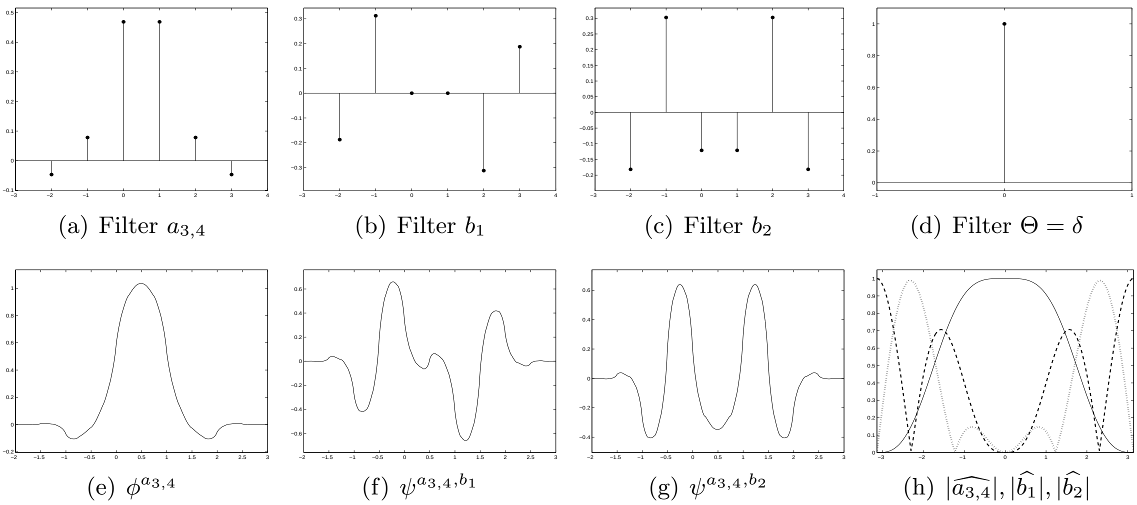
Assume that there is a solution $\{\lambda_0, \ldots, \lambda_{\ell-1}\}$ (of real numbers) to X in (S3). Then the (real-valued) filter a is complex symmetric about the point $\frac{1-(-1)^m}{2}$, has m sum rules and $2n$ linear-phase moments, and satisfies $\det(\mathcal{N}_{a,\delta|n}(z)) = \mathsf{q}_\mathsf{p}(z)\mathsf{q}_\mathsf{p}^\star(z)$. Since all the conditions in Theorem 4.2 are satisfied with $\Theta = \delta$, apply Algorithm 3 using the filter a and $\Theta = \delta$ to construct high-pass filters b_1, b_2 so that $\{a; b_1, b_2\}$ is a tight framelet filter bank with complex symmetry (and real coefficients) such that $\operatorname{sr}(a) \geqslant m$ and $\min(\operatorname{vm}(b_1), \operatorname{vm}(b_2)) \geqslant n$.

Note that the real-valued filter $a_{m,2n}$ is the shortest filter which is symmetric about the point $\frac{1-(-1)^m}{4}$, has m sum rules, and $2n$ linear-phase moments. We now provide some additional examples of tight framelet filter banks having [complex] symmetry (and real coefficients). For simplicity of presentation, we omit the intermediate steps and only provide the final tight framelet filter banks here.

Example 3. Let $a = a_{3,4} = \{-\frac{3}{64}, \frac{5}{64}, \underline{\mathbf{\frac{15}{32}}}, \frac{15}{32}, \frac{5}{64}, -\frac{3}{64}\}_{[-2,3]}$ be the filter given in (6.4). Then $\epsilon z^c := \mathbb{S}\mathsf{a}(z) = z$. By calculation, we have $\mathrm{fsupp}(a_{3,4}) = [-2,3]$, $\mathrm{sr}(a_{3,4}) = 3$, $\mathrm{lpm}(a_{3,4}) = 4$, $\mathrm{sm}(a_{3,4}) \approx 1.646884$. Set $n_b = n = 2$ and $\Theta = \delta$ in Algorithm 3. Then $\mathcal{N} = \mathcal{N}_{a,\Theta|2}$. Take $\mathsf{d}(z) = \frac{\sqrt{15}}{64}z$. Then $\mathcal{U}_{1,1}(z) = \frac{\sqrt{2}}{8}(z-1)$ and $\mathcal{U}_{1,2}(z) = \frac{\sqrt{30}}{32}(z+1)$. Therefore,

$$\mathsf{b}_1(z) = \tfrac{1}{16}z^{-2}(z-1)^3(3+4z+3z^2) = \tfrac{1}{16}\{-3, 5, \underline{\mathbf{0}}, 0, -5, 3\}_{[-2,3]},$$
$$\mathsf{b}_2(z) = -\tfrac{\sqrt{15}}{64}(1-z^{-1})^2(1+z)(3-2z+3z^2) = \tfrac{\sqrt{15}}{64}\{-3, 5, \underline{\mathbf{-2}}, -2, 5, -3\}_{[-2,3]}.$$

Note that $\mathsf{Sa}(z) = z, \mathsf{Sb}_1(z) = -z, \mathsf{Sb}_2(z) = z$ and $\mathrm{vm}(b_1) = 3, \mathrm{vm}(b_2) = 2$.



(a) Filter $a_{3,4}$ (b) Filter b_1 (c) Filter b_2 (d) Filter $\Theta = \delta$

(e) $\phi^{a_{3,4}}$ (f) $\psi^{a_{3,4},b_1}$ (g) $\psi^{a_{3,4},b_2}$ (h) $|\widehat{a_{3,4}}|, |\widehat{b_1}|, |\widehat{b_2}|$

FIGURE 6.1. The tight framelet filter bank $\{a_{3,4}; b_1, b_2\}$ with symmetry and 2 vanishing moments is constructed in Example 3. (a), (b), (c), (d) are the graphs of the filters $a_{3,4}, b_1, b_2, \Theta = \delta$, respectively. (e), (f), (g) are the graphs of the refinable function $\phi^{a_{3,4}}$ and the framelet functions $\psi^{a_{3,4},b_1}$, $\psi^{a_{3,4},b_2}$, respectively. (h) is the magnitudes of $\widehat{a_{3,4}}$ (in solid line), $\widehat{b_1}$ (in dashed line), and $\widehat{b_2}$ (in dotted line) on the interval $[-\pi, \pi]$.

Example 4. Let $m = 2, n = 2, \ell = 1$ in Algorithm 4. Then $\lambda_0 = -\frac{3+t}{64}$ with $t = \pm\sqrt{7}$, $\mathsf{d}(z) = \frac{\sqrt{2}}{2}\lambda_0(1-z)$, and the low-pass filter $a^{\mathsf{SF}}_{2,2,1}$ is given by

$$a^{\mathsf{SF}}_{2,2,1} = \tfrac{1}{64}\{-3-t, 2+2t, 19+t, 28-4t, 19+t, -3-t\}_{[-3,3]}.$$

Setting $t_1 = 0$ and $\tilde{t}_1 = 1$, we have $\mathcal{U}_{1,1}(z) = \frac{t}{16}(z+1)$ and $\mathcal{U}_{1,2}(z) = \frac{\sqrt{2}}{8}(z-1)$. Hence,

$$\mathsf{b}_1(z) = \tfrac{t}{448}(1-z^{-1})^2[(21+7t)(z^{-1}+z^3) + 28(1+z^2) + (14+2t)z],$$
$$\mathsf{b}_2(z) = \tfrac{\sqrt{2}}{32}(1-z^{-1})^2(1-z^{-1})(1+z)[(3+t)(1+z^2) + 4z].$$

Note that $\mathrm{sr}(a^{\mathsf{SF}}_{2,2,1}) = 2, \mathrm{lpm}(a^{\mathsf{SF}}_{2,2,1}) = 4, \mathsf{Sa}^{\mathsf{SF}}_{2,2,1}(z) = 1, \mathsf{Sb}_1(z) = 1, \mathsf{Sb}_2(z) = -1$ and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 2$.

When $t = \sqrt{7}$, $\mathrm{sm}(a^{\mathsf{SF}}_{2,2,1}) \approx 1.220628$ and when $t = -\sqrt{7}$, $\mathrm{sm}(a^{\mathsf{SF}}_{2,2,1}) \approx 1.023927$.

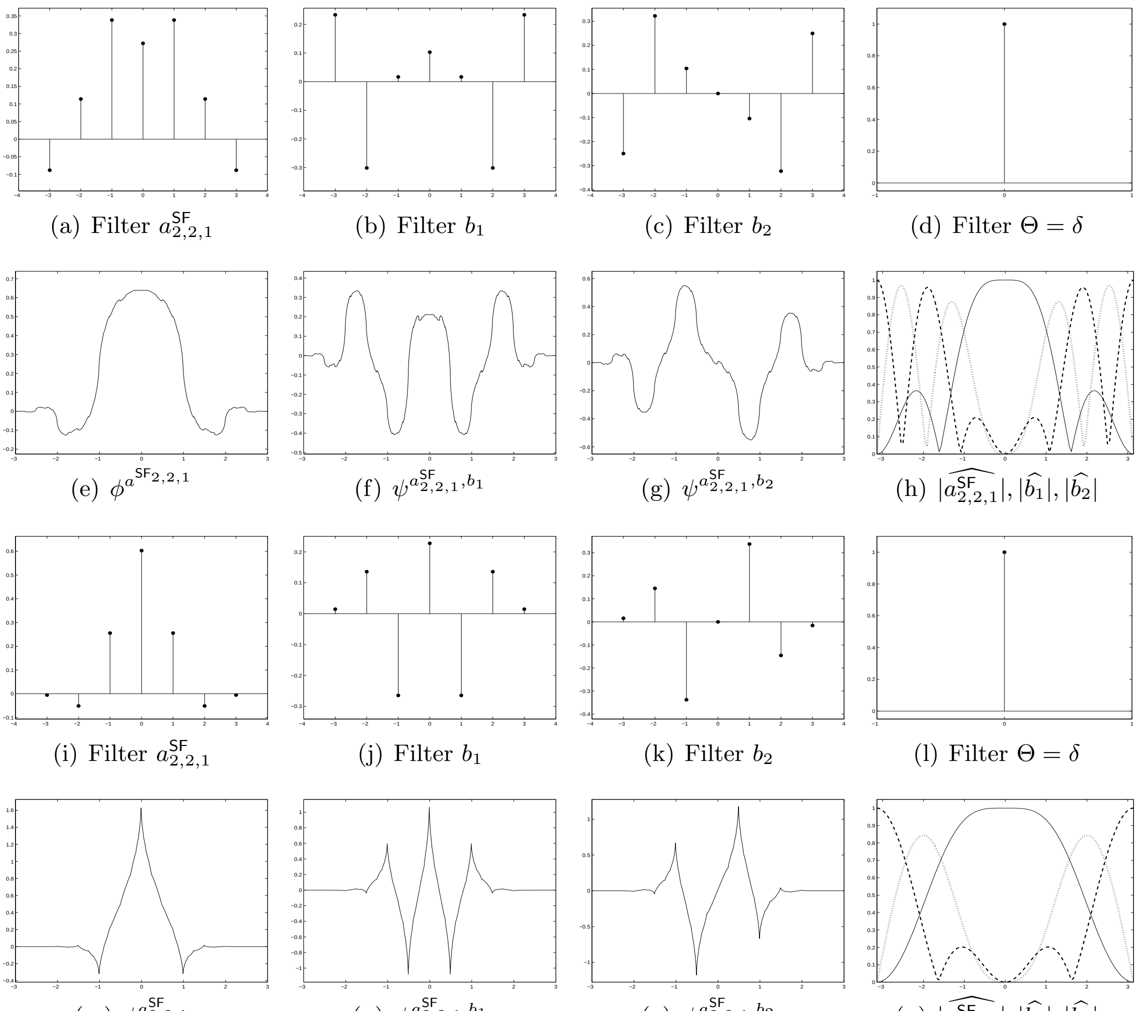
Example 5. Let $m = 4, n = 4, \ell = 2$ in Algorithm 4. Then λ_0 can be expressed as a polynomial of λ_1 of degree 7 and λ_1 is a root of polynomial of degree 8. Numerically,

$$\lambda_0 \approx -0.00469314087521906449, \qquad \lambda_1 \approx -0.000242116619291697624$$

and $a_{4,8} = \frac{1}{256}\{-1, 5, -5, -20, 70, \underline{\mathbf{158}}, 70, -20, -5, 5, -1\}_{[-5,5]}$. Then $\mathrm{sr}(a^{\mathsf{SF}}_{4,4,2}) = 4, \mathrm{lpm}(a^{\mathsf{SF}}_{4,4,2}) = 8, \mathrm{sm}(a^{\mathsf{SF}}_{4,4,2}) \approx 2.37449371$. Then

$$\begin{aligned}\mathsf{b}_1(z) = (1-z^{-1})^4[&0.005059455288(z^7 - z^{-3}) + 0.098071485084(z^6 - z^{-2}) + 0.092589504026(z^5 - z^{-1})\\ &+ 0.032775997542(z^4 - 1) - 0.000858298910(z^3 - z)],\\ \mathsf{b}_2(z) = (1-z^{-1})^4[&0.005053658808(z^7 + z^{-3}) + 0.097959127269(z^6 + z^{-2}) + 0.092543356566(z^5 + z^{-1})\\ &+ 0.033900114397(z^4 + 1) - 0.000101926629(z^3 + z) - 0.006228305842z^2].\end{aligned}$$

Note that $\mathsf{Sa}^{\mathsf{SF}}_{4,4,2}(z) = 1, \mathsf{Sb}_1(z) = z^{-1}, \mathsf{Sb}_2(z) = -z$ and $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 4$.



(a) Filter $a^{\mathsf{SF}}_{2,2,1}$ (b) Filter b_1 (c) Filter b_2 (d) Filter $\Theta = \delta$

(e) $\phi^{a^{\mathsf{SF}}_{2,2,1}}$ (f) $\psi^{a^{\mathsf{SF}}_{2,2,1},b_1}$ (g) $\psi^{a^{\mathsf{SF}}_{2,2,1},b_2}$ (h) $|\widehat{a^{\mathsf{SF}}_{2,2,1}}|, |\widehat{b_1}|, |\widehat{b_2}|$

(i) Filter $a^{\mathsf{SF}}_{2,2,1}$ (j) Filter b_1 (k) Filter b_2 (l) Filter $\Theta = \delta$

(m) $\phi^{a^{\mathsf{SF}}_{2,2,1}}$ (n) $\psi^{a^{\mathsf{SF}}_{2,2,1},b_1}$ (o) $\psi^{a^{\mathsf{SF}}_{2,2,1},b_2}$ (p) $|\widehat{a^{\mathsf{SF}}_{2,2,1}}|, |\widehat{b_1}|, |\widehat{b_2}|$

FIGURE 6.2. The first two rows are for the tight framelet filter bank $\{a^{\mathsf{SF}}_{2,2,1}; b_1, b_2\}$ with symmetry and 2 vanishing moments constructed in Example 4 with the choice $t = \sqrt{7}$. (a), (b), (c), (d) are the graphs of the filters $a^{\mathsf{SF}}_{2,2,1}, b_1, b_2, \Theta = \delta$, respectively. (e), (f), (g) are the graphs of the refinable function $\phi^{a^{\mathsf{SF}}_{2,2,1}}$ and the framelet functions $\psi^{a^{\mathsf{SF}}_{2,2,1},b_1}$, $\psi^{a^{\mathsf{SF}}_{2,2,1},b_2}$, respectively. (h) is the magnitudes of $\widehat{a^{\mathsf{SF}}_{2,2,1}}$ (in solid line), $\widehat{b_1}$ (in dashed line), and $\widehat{b_2}$ (in dotted line) on the interval $[-\pi, \pi]$. The last two rows are for the tight framelet filter bank $\{a^{\mathsf{SF}}_{2,2,1}; b_1, b_2\}$ with symmetry and 2 vanishing moments constructed in Example 4 with the choice $t = -\sqrt{7}$.

However, (6.1) (more precisely, $\operatorname{lpm}(\mathsf{a}\mathsf{a}^\star) = \operatorname{lpm}(a)$) in Proposition 6.1 generally does not hold if complex symmetry is replaced by symmetry. Nevertheless, the problem of constructing a tight framelet filter bank $\{a; b_1, b_2\}$, with symmetry such that the low-pass filter a has at least m sum rules and the high-pass filters b_1, b_2 have at least n vanishing moments, is equivalent to finding a low-pass filter a such that

(1) Filter a has symmetry, m sum rules, and $1 - \mathsf{a}(z)\mathsf{a}^\star(z) = \mathcal{O}(|z-1|^{2n})$ as $z \to 1$;
(2) There exists a Laurent polynomial $\mathsf{d_a}$ with symmetry such that (6.3) holds.

All such filters a satisfying above items (1) and (2) can be constructed by the following algorithm.

Algorithm 5. *Let m, n, ℓ be positive integers (we often require $n+\ell$ to be an odd integer and $\lambda_{\ell-1} > 0$ so that $\mathsf{P}(x) \geqslant 0$ can be true).*

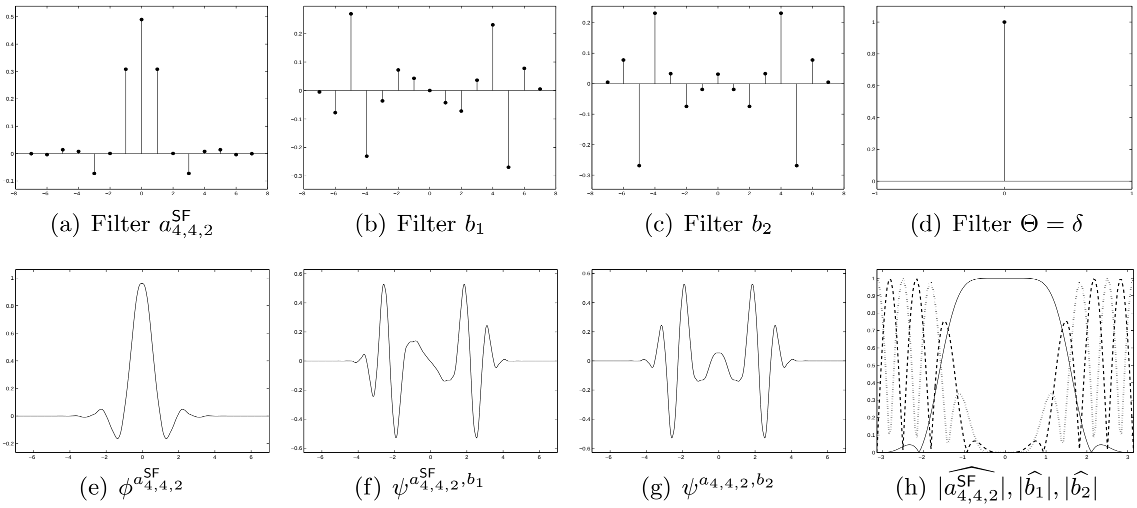
(S1) *Define $\mathsf{P}(x) := \mathsf{P}_{m,n}(x) + \sum_{j=0}^{\ell-1} \lambda_j x^{n+j}$, where $\lambda_0, \ldots, \lambda_{\ell-1} \in \mathbb{R}$ are unknown real numbers to be determined later and $\mathsf{P}_{m,n}$ is defined in (6.5).*

(S2) *Calculate the polynomial R which is uniquely determined by*

$$1 - (1-x)^m \mathsf{P}(x) - x^m \mathsf{P}(1-x) = \mathsf{R}(x(1-x)).$$

Write $\mathsf{P}(x) = (1-x)^{\mathsf{Z}(\mathsf{P},1)} \tilde{\mathsf{P}}(x)$ and $\mathsf{R}(x) = x^{\mathsf{Z}(\mathsf{R},0)}(1-x)^{\mathsf{Z}(\mathsf{R},1)} \tilde{\mathsf{R}}(x)$ for unique polynomials $\tilde{\mathsf{P}}$ and $\tilde{\mathsf{R}}$.

(S3) *Choose the unknown real numbers $\lambda_0, \ldots, \lambda_{\ell-1}$ so that $\tilde{\mathsf{P}}(x) \geqslant 0$ and $\tilde{\mathsf{R}}(x) \geqslant 0$ for all $x \in \mathbb{R}$. [We can also obtain the unknowns $\lambda_0, \ldots, \lambda_{\ell-1}$ by solving a system of algebraic equations using Theorem 2.9.]*



(a) Filter $a^{\mathsf{SF}}_{4,4,2}$ (b) Filter b_1 (c) Filter b_2 (d) Filter $\Theta = \delta$

(e) $\phi^{a^{\mathsf{SF}}_{4,4,2}}$ (f) $\psi^{a^{\mathsf{SF}}_{4,4,2}, b_1}$ (g) $\psi^{a_{4,4,2}, b_2}$ (h) $|\widehat{a^{\mathsf{SF}}_{4,4,2}}|, |\widehat{b_1}|, |\widehat{b_2}|$

FIGURE 6.3. The tight framelet filter bank $\{a^{\mathsf{SF}}_{4,4,2}; b_1, b_2\}$ with symmetry and 2 vanishing moments is constructed in Example 5. (a), (b), (c), (d) are the graphs of the filters $a^{\mathsf{SF}}_{4,4,2}, b_1, b_2, \Theta = \delta$, respectively. (e), (f), (g) are the graphs of the refinable function $\phi^{a^{\mathsf{SF}}_{4,4,2}}$ and the framelet functions $\psi^{a^{\mathsf{SF}}_{4,4,2}, b_1}$, $\psi^{a^{\mathsf{SF}}_{4,4,2}, b_2}$, respectively. (h) is the magnitudes of $\widehat{a^{\mathsf{SF}}_{4,4,2}}$ (in solid line), $\widehat{b_1}$ (in dashed line), and $\widehat{b_2}$ (in dotted line) on the interval $[-\pi, \pi]$.

(S4) *Derive a filter a with symmetry such that* $\mathsf{a}(1) = 1$ *and* $\mathsf{a}(z)\mathsf{a}^\star(z) = (\frac{1+z}{2})^{\mathsf{Z}(\mathsf{P},1)}\mathsf{Q}(\frac{1}{2} - \frac{z+z^{-1}}{4})$*, where* Q *is a polynomial with complex coefficients satisfying* $\tilde{\mathsf{P}}(x) = |\mathsf{Q}(x)|^2$ *for all* $x \in \mathbb{R}$ *and* Q *can be derived from* $\tilde{\mathsf{P}}$ *as discussed in* [11].

Since all the conditions in Theorem 4.2 are satisfied with $\Theta = \delta$*, apply Algorithm 3 using the filter a and* $\Theta = \delta$ *to construct high-pass filters* b_1, b_2 *so that* $\{a; b_1, b_2\}$ *is a tight framelet filter bank with symmetry such that* $\operatorname{sr}(a) \geqslant m$ *and* $\min(\operatorname{vm}(b_1), \operatorname{vm}(b_2)) \geqslant \min(m, n)$.

In the following, we provide two examples to illustrate Algorithm 5.

Example 6. Let $m = 3, n = 2, \ell = 1$ in Algorithm 5. Then $\mathsf{P}_{3,2}(x) = 1 + 3x$, $\mathsf{P}(x) = 1 + 3x + \lambda_0 x^2$, and $\mathsf{R}(x) = (6 - \lambda_0)x^2$. Hence, $\tilde{\mathsf{R}} = 6 - \lambda_0$. The necessary and sufficient condition for $\mathsf{P}(x) \geqslant 0$ and $\tilde{\mathsf{R}}(x) \geqslant 0$ for all $x \in \mathbb{R}$ is $\frac{9}{4} \leqslant \lambda_0 \leqslant 6$. Then

$$\mathsf{a}(z) = z^{-1}(1+z)^3(\tfrac{7-it}{32} + \tfrac{-3+it}{64}(z+z^{-1})) = \tfrac{1}{64}\{-3+ti, 5+ti, \underline{\mathbf{30-2ti}}, 30-2ti, 5+ti, -3+ti\}_{[-2,3]},$$

where $t := \sqrt{4\lambda_0 - 9}$. Then $\mathsf{P}(x) = |1 + \frac{3-ti}{2}x|^2$, $\mathsf{d}(z) = \frac{\sqrt{6-\lambda_0}}{32}z^2$. The case $\lambda_0 = 6$ corresponds to the complex-valued orthogonal filter with symmetry obtained in [11]. The case $\lambda_0 = \frac{9}{4}$ corresponds to Example 3.

Example 7. Let $m = 5, n = 4, \ell = 1$ in Algorithm 5. Then $\mathsf{P}_{5,4}(x) = 1+5x+15x^2+35x^3$, $\mathsf{P}(x) = \mathsf{P}_{5,4}(x)+\lambda_0 x^4$, and $\mathsf{R}(x) = (70 - \lambda_0)x^4$. Hence, $\tilde{\mathsf{R}} = 70 - \lambda_0$. The necessary and sufficient condition for $\mathsf{P}(x) \geqslant 0$ and $\tilde{\mathsf{R}}(x) \geqslant 0$ for all $x \in \mathbb{R}$ is

$$34.2368 \approx \frac{1225(17(15)^{2/3} - (15)^{1/3} - 23)}{790 + 509(15)^{2/3} - 445(15)^{1/3}} \leqslant \lambda_0 \leqslant 70.$$

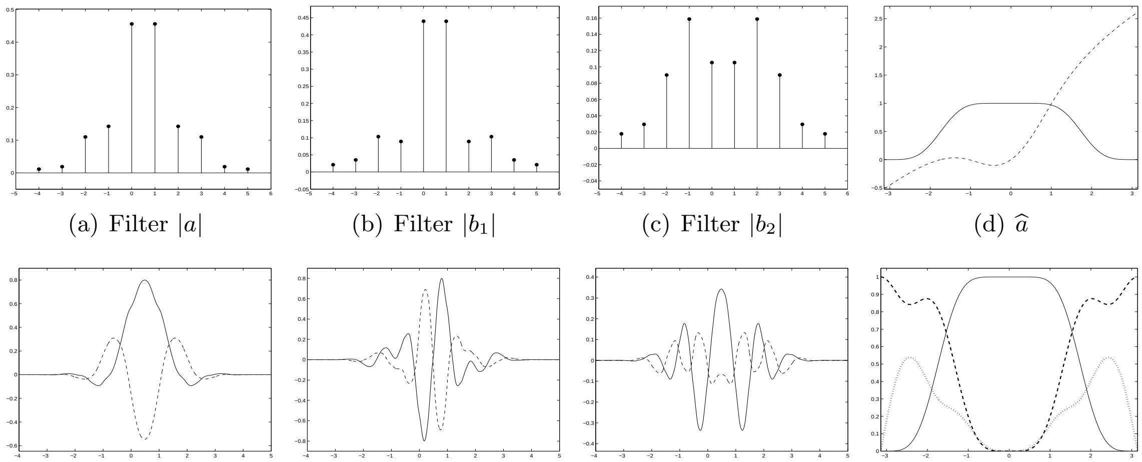
The case $\lambda_0 = 70$ corresponds to the complex-valued orthogonal filter with symmetry obtained in [11]. Taking $\lambda_0 = \frac{277}{8}$, we have

$$\begin{aligned}\mathsf{a}(z) =& 2^{-5}z^{-2}(1+z)^5(1 + \tfrac{5-5i}{2}\zeta + \tfrac{5-23i}{4}\zeta^2) = \tfrac{1}{2048}\{5-23i, -35+17i, -76+212i, 260+132i,\\ & \underline{\mathbf{870-338i}}, 870-338i, 260+132i, -76+212i, -35+17i, 5-23i\}_{[-4,5]},\end{aligned}$$

where $\zeta := 1/2 - \frac{z+z^{-1}}{4}$. We have $\operatorname{sm}(a) \approx 2.261446$ and $\mathsf{d}(z) = \frac{\sqrt{566}}{2048}z^2$. Then

$$\begin{aligned}\mathsf{b}_1(z) =& \tfrac{3553793+1369306i}{805914880\sqrt{5588724041}}(1-z^{-1})^4(1-z)\\ & [-336001(z^4+1) + (-1336726+436680i)(z^3+z) + (-248786+783552i)z^2],\\ \mathsf{b}_2(z) =& \tfrac{283(-1198870873+518956616i)}{312694973440\sqrt{3163217807206}}(1-z^{-1})^4(1+z)\\ & [-26869(z^4+1) + (-53156+34920i)(z^3+z) + (-116734-20288i)z^2].\end{aligned}$$

Note that $\mathsf{S}\mathsf{b}_1(z) = -z$, $\mathsf{S}\mathsf{b}_2(z) = z$ and $\operatorname{vm}(b_1) = 5$ and $\operatorname{vm}(b_2) = 4$.



(a) Filter $|a|$ (b) Filter $|b_1|$ (c) Filter $|b_2|$ (d) $\widehat{a}$

(e) ϕ^a (f) ψ^{a,b_1} (g) ψ^{a,b_2} (h) $|\widehat{a}|, |\widehat{b_1}|, |\widehat{b_2}|$

FIGURE 6.4. The tight framelet filter bank $\{a; b_1, b_2\}$ with symmetry and 4 vanishing moments is constructed in Example 7. (a), (b), (c) are the graphs of the filters a, b_1, b_2, respectively. (d) is the magnitude and the phase of $\widehat{a}$. (e), (f), (g) are the graphs of the refinable function ϕ^a and the framelet functions ψ^{a,b_1}, ψ^{a,b_2}, respectively. (h) is the magnitudes of $\widehat{a}$ (in solid line), $\widehat{b_1}$ (in dashed line), and $\widehat{b_2}$ (in dotted line) on the interval $[-\pi, \pi]$.

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