The Kac-Wakimoto Character Formula

The complete & detailed proof.

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→ The goal of this document is to present an intelligible proof of the character formula that goes by the names of V.Kac and M.Wakimoto. Considering a Lie algebra with a triangular decomposition, a slightly modified Weyl-Kac character formula still holds in the case of some $L(\lambda)$’s (some irreducible Verma quotients) for certain specific weights $\lambda$’s. This formula is:

$$\text{ch} \, L(\lambda) = \sum_{w \in W^\lambda} \varepsilon(w) \left( \text{ch} \, M(w.\lambda) \right).$$

→ The level of detail is much, much higher than what is found in the three paragraphs of the article by V.Kac and M.Wakimoto.

→ The proof goes in 4 Steps. Throughout, I write remarks and refer to sources whenever an intelligible & already proven result is invoked.

The steps go as follow:

- **Step 1** = A few small results dealing with roots and weights.
- **Step 2** = A few categorical results that are needed.
- **Step 3** = Proof that $T : M(w.\lambda) \mapsto M(w.\mu)$.
- **Step 4** = Proof of the Kac-Wakimoto formula itself.
Notation

Throughout this document, the following notation will be used:

• $a \geq 0 \ (a \in \mathbb{C}) \iff \text{Re} \ a > 0 \ or \ a \in \mathbb{R}_{\geq 0}$

• $L$ is a Lie algebra with triangular decomposition.

• $\mathfrak{T} \subseteq L$ is its Cartan subalgebra.

• $P_+ \subseteq \mathfrak{T}^*$ is the set of dominant integral weights.

• $R = \{ \text{Real coroots} \} \subseteq \mathfrak{T}$.

• $\rho \in \mathfrak{T}^*$ is an element such that $\rho(s) = 1$ for all simple coroots $s$.

• $(-|-) = \text{The symmetric, invariant & non-degenerate bilinear form on } L$.

• $(-, -) = \text{The induced bilinear form on } \mathfrak{T}^*$.

• $K = \left\{ \lambda \in \mathfrak{T}^* \mid \#\{ r \in R_+ \mid \lambda(r) < 0 \} < \infty \right\} \subseteq \mathfrak{T}^*$.

• $K^L = -\rho + K$.

• $O_L = \text{The subcategory of the category } O \text{ formed of objects whose irreducible constituents are all } L(\lambda)\'s \text{ for } \lambda \in -\rho + K$.

• $C = \left\{ \lambda \in \mathfrak{T}^* \mid r \in R_+ \Rightarrow \lambda(r) \geq 0 \right\} \subseteq K \subseteq \mathfrak{T}^*$.

• $R^\lambda = \{ r \in R \mid \lambda(r) \in \mathbb{Z} \}$ for a given $\lambda \in \mathfrak{T}^*$.

• $\Pi^\lambda = \{ \text{Simple elements of } R^\lambda \}$ for a given $\lambda \in \mathfrak{T}^*$.

• $W^\lambda = \langle \sigma_r \mid r \in R^\lambda \rangle = \langle \sigma_s \mid s \in \Pi^\lambda \rangle \leq W$ where $W$ is the Weyl group of $L$.

Here are a few remarks:

→ Both the sets $\{ a \in \mathbb{C} \mid a \geq \mathbb{C} \}$ and its complement are closed under addition.

→ $K$ is $W$-invariant under the standard Weyl action on $\mathfrak{T}^*$.

→ $K^L$ is $W$-invariant under the dot action on $\mathfrak{T}^*$.

→ $C \subseteq K$ and in each $W$-orbit in $K$, there exists a unique element of $C$ in that orbit.
Step 1: Needed results related to the setting

In this first step, I present some relevant and needed results for use in the proof of the Kac-Wakimoto character formula. These results all are related to either weights/roots and Weyl group properties/computations or to some basic properties of the category $O$.

**Result 1.1** There exist a short exact sequence of $L$-modules $0 \to E \to M(\lambda) \to L(\mu) \to 0$ i.e. $L(\mu)$ is a constituent of $M(\lambda)$

\[
\iff \exists \text{ positive roots } \beta_1, \ldots, \beta_N \text{ and } n_1, \ldots, n_N \in \mathbb{N} \text{ such that :}
\]

\[
\begin{align*}
(1) \quad & \mu = \lambda - \sum_{\ell=1}^{N} n_{\ell} \beta_{\ell}; \\
(2) \quad & 2 \left( (\lambda - \sum_{\ell=1}^{j-1} n_{\ell} \beta_{\ell}) + \rho, \beta_{j} \right) = n_{j} \left( \beta_{j}, \beta_{j} \right) \text{ for all } j \in \{0, \ldots, N\}.
\end{align*}
\]

**Proof:** (Can be found in [KK79], see Theorem 2.)

**Result 1.2** Let $\lambda \in K^L$. Then

$L(\mu)$ is a constituent of $M(\lambda)$ $\implies$ $\mu \in K^L$ too.

**Proof:** By the result 1.1, we just need to prove that $\mu' = \lambda - n_1 \beta_1 \in K^L = -\rho + K$. We know that $2(\lambda + \rho, \beta_1) = n_1(\beta_1, \beta_1)$ where $n_1 \in \mathbb{N}\{0\}$ to avoid tackling a triviality. Then, since $\lambda + \rho \in K^L$, the root $\beta_1$ cannot be isotropic meaning that $(\beta, \beta) \neq 0$. Next, we can absolutely write

\[
\begin{align*}
\mu' &= \lambda - n_1 \beta_1 \\
\mu' &= \lambda - 2 \frac{(\lambda + \rho, \beta_1)}{(\beta_1, \beta_1)} \beta_1 \\
\mu' &= (\lambda + \rho) - 2 \frac{(\lambda + \rho, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \rho.
\end{align*}
\]

It means that if $\beta_1$ were a real root, we could write $\mu' = \sigma_{\beta_1} \lambda$ and conclude that $\mu' \in -\rho + K$ by the $W$-invariance of the set $K^L$ under the dot action.

Suppose then that $\beta_1$ is an imaginary root i.e. that $(\beta_1, \beta_1) < 0$. In this case $k\beta_1$ is also a positive root for each $k \in \mathbb{N}\{0\}$ and this leads to having

\[
(\lambda + \rho, k\beta_1) = k(\lambda + \rho, \beta_1) = kn_1(\beta_1, \beta_1) < 0 \quad \text{for each } k \in \mathbb{N}\{0\}.
\]

However since $\lambda + \rho \in K$, this situation cannot occur. Therefore $\beta_1$ is a real root, $\sigma_{\beta_1} \in W$ and so $\mu' = \sigma_{\beta_1} \lambda \in K^L$ since $K^L$ is $W$-invariant under the dot action.

Q.E.D.
Definition 1.3 Define an equivalence relation \( \sim \) on \( K^L \) as follows:

\[
\lambda \sim \mu \iff \exists \lambda = \lambda_0, \lambda_1, \ldots, \lambda_{n-1}, \lambda_n = \mu \text{ such that at each step, we have either...}
\]

\[
\ldots \text{ an SES } 0 \to E \to M(\lambda_i) \to L(\lambda_{i+1}) \to 0 ,
\]

\[
\text{OR}
\]

\[
\ldots \text{ an SES } 0 \to E \to M(\lambda_{i+1}) \to L(\lambda_i) \to 0 .
\]

\( \rightarrow \) This \( \sim \) is not, a priori, the relation defining the extension blocks of the category \( O^L \) although it probably is.

Definition 1.4 The category \( O^L \) is formed of objects whose constituents are all \( L(\mu) \)'s for \( \mu \)'s all in the set \( K^L \). This is a full subcategory of the category \( O \).

Definition 1.5 For a given \( \lambda \in K^L \), we define a category \( O^L_{[\lambda]} \) formed of the objects whose constituents are all \( L(\mu) \)'s for \( \mu \)'s all in the set \( [\lambda] \subseteq K^L \). These are full subcategories of the category \( O^L \).

Result 1.6 The category \( O^L \) "decomposes" as:

\[
O^L = \bigoplus_{\lambda \in K^L} O^L_{[\lambda]}.
\]

Proof: (Can be found in [DGK82], see Theorem 5.7.)

\( \rightarrow \) Moreover, the result WTF shows that \( O^L \) contains all the Verma modules \( M(\lambda) \)'s where \( \lambda \in K^L \).

\( \rightarrow \) In fact, given \( \lambda, \mu \in K^L \), Theorem 4.5 of [DGK82] states that

\[
V \in \text{ob } O^L_{[\lambda]} \quad W \in \text{ob } O^L_{[\mu]} \quad \Rightarrow \quad \text{Ext}^1_L(V, W) = 0 .
\]

This is a property of the decomposition of \( O^L \) given in Result 1.6 that an with an extension block decomposition of \( O^L \) also would share.

Focusing on the category \( O^L \) rather than \( O \) allows one to get a precise description of the equivalence relation \( \sim \) in terms of the Weyl group and some of its subgroups \( W^\eta \)'s for some \( \eta \in \Xi^* \). Here’s a useful thing we can say about these subgroups:
Result 1.7 Let $\eta \in \mathfrak{S}^*$ and let $w \in W$. The following statements are true:

1. $w(R^\eta) = R^{w^\eta}$;
2. $wW^\eta w^{-1} = W^{w^\eta}$. In particular, if $g \in W^\eta$, then $W^\eta = W^{g^\eta}$.

**Proof:** To prove (1), let $r \in R^\eta$. Then

$$2\frac{(\eta, r)}{(r, r)} = 2\frac{(w^\eta(\eta), w^\eta(r))}{(w^\eta(r), w^\eta(r))} \in \mathbb{Z}.$$ 

So we obtain $r \in R^{w^\eta}$. Finally, since the previous line is an equality and since $W$ is a group, the proof of (1) is complete.

To prove of (2), let $g \in W^\eta$, then $g = \Pi_i \sigma_{r_i}$ where $r_i \in R^\eta$ for all $i$'s. Next, we can write

$$wgw^{-1} = \Pi_i (w\sigma_{r_i}w^{-1}) = \Pi_i \sigma_{w(r_i)}.$$ 

By part (1), $w(r_i) \in R^{w^\eta}$ for all $i$'s so by definition, $wgw^{-1} \in W^{w^\eta}$. Again, since $W$ is a group, the proof is complete. Q.E.D.

Result 1.8 The following statements are true:

1. For $\lambda \in K^L$, we have an equality

$$[\lambda] = W^{\lambda + \rho}\lambda.$$ 

2. For $\lambda \in K^L$ and $w \in W$, we have

$$[w.\lambda] = (wW^{\lambda + \rho}).\lambda.$$ 

**Proof:** The property (2) is easy to derive from (1). Let’s start by proving (1)...

If $\mu \in K^L$ is such that $[\mu] = [\lambda]$, then by the definition of $\sim$, we may assume (without loss of generality), that $\mu = \lambda - n\alpha$ where $n \in \mathbb{N}$ and $\alpha$ is a positive root and

$$2(\lambda + \rho, \alpha) = n(\alpha, \alpha).$$ 

By assumption, $\alpha$ is a positive root, but it is also a real one for the same reason as in the proof of result 1.2. With this key information at hand, we can write

$$\mu = \lambda - n\alpha$$

$$= (\lambda + \rho) - 2\frac{(\lambda + \rho, \alpha)}{(\alpha, \alpha)} - \rho$$

$$= \sigma_\alpha \lambda.$$ 

As $\alpha^\vee \in R^{\lambda + \rho}$, we know that $\sigma_\alpha \in W^{\lambda + \rho}$ and this concludes this part of the proof.
We now need to prove that any element of $W^{\lambda+\rho}.\lambda$ is equivalent to $\lambda$ under $\sim$. As $W^{\lambda+\rho}$ is generated by the reflections $\sigma_r$ where $r \in \Pi^{\lambda+\rho} \subseteq R^{\lambda+\rho}_+$, it will be sufficient to prove that $[\sigma_r.\lambda] = [\lambda]$ for a fixed $r \in \Pi^{\lambda+\rho}$.

Write $\beta$ for the positive real root corresponding to the positive real coroot $r \in \Pi^{\lambda+\rho}$ and let $n \in \mathbb{Z}$ be such that $(\lambda + \rho)(r) = n$. Then we can write

$$\sigma_r.\lambda = \lambda - 2(\lambda + \rho)(r)\beta$$

$$= \lambda - n\beta$$

where $\beta$ is a root with $2(\lambda + \rho, \beta) = n(\beta, \beta)$ for a $n \in \mathbb{Z}$. Note here that we do have $$(\lambda + \rho)(r) = n,$$

If $n \in \mathbb{N}$, then setting $\lambda' = \lambda$ and $\mu' = \lambda - ns$, we see by the result 1.1, that $L(\mu')$ is a constituent of $M(\lambda')$. Else, if $-n \in \mathbb{N}$, we see that $L(\lambda')$ is a constituent of $M(\mu')$. In both cases, the definition of $\sim$ allows to conclude that indeed, $[\sigma_r.\lambda] = [\lambda]$ and... finishing the proof of (1).

To prove (2), we use part (1) that tells us we have $[\lambda] = W^{\lambda+\rho}.\lambda$ for any $\lambda \in K_L$. Let $w \in W$, then we can write

$$[w.\lambda] = \left(W^{w.(\lambda)+\rho}\right).(w.\lambda)$$

$$= \left(W^{w.(\lambda)+\rho}w\right).\lambda$$

$$= \left(W^{w(\lambda)+\rho-w}\right).\lambda$$

$$= \left(W^{w(\lambda)+\rho}w^{-1}\right).\lambda \quad \text{by Result 1.7}$$

$$= \left(ww^{\lambda+\rho}\right).\lambda .$$

Q.E.D.

**Corollary 1.9** If $\lambda, \mu \in K_L$ are such that $[\mu] = [\lambda]$, then we have

$$W^{\mu+\rho} = W^{\lambda+\rho}.$$

**Proof**: The hypothesis $[\mu] = [\lambda]$ gives $\mu = g.\lambda$ for a certain $g \in W^{\lambda+\rho}$ by part (1) of the result 1.8. Thus, we know that

$$W^{\mu+\rho} = W^{g.\lambda+\rho} = W^{g(\lambda+\rho)} .$$

Then, using part (2) of corollary 1.7, we get $W^{\lambda+\rho} = W^{g(\lambda+\rho)}$ because $g \in W^{\lambda+\rho}$... which give $W^{\lambda+\rho} = W^{\mu+\rho}$ with the above equality and the proof is complete.

Q.E.D.
The corollary 1.9 explains how a thing such as the result 1.8 (1) is possible at all, knowing that \( \mu \sim \lambda \iff \lambda \sim \mu \) (because \( \sim \) is an equivalence relation, after all).

To conclude **Step 1** here’s a relevant result about an hypothesis that comes back over and over again in the remaining of the document:

**Result 1.10** Let \( \lambda, \mu \in \mathfrak{T}^* \) be such that \( (W(\mu - \lambda)) \cap P_+ \neq \emptyset \), then

\[
W^\lambda = W^\mu .
\]

**Proof:** First note that \( (W(\mu - \lambda)) \cap P_+ \neq \emptyset \) implies that it contains precisely one element. This is because there is at most one dominant integral element in any given \( W \)-orbit inside of \( \mathfrak{T}^* \).

Let \( \theta \) be the dominant integral element in \( (W(\mu - \lambda)) \cap P_+ \neq \emptyset \) and let \( \omega \in W \) be such that \( \omega(\mu - \lambda) = \theta \). Then

\[
\mu = \lambda + \omega^{-1}(\theta) ,
\]

\[
\mu + \rho = \lambda + \rho + \omega^{-1}(\theta) .
\]

We are almost ready to get started. Recall that \( W^{\mu + \rho} = \langle \sigma_r \mid r \in R^{\mu + \rho} \rangle \). Fix any \( r \in R^{\mu + \rho} \). In what follows, we’ll justify that this \( r \) is also in \( R^{\lambda + \rho} \).

Let \( s \) be the real root corresponding to the real coroot \( r \). As \( s \) is a real root, it is in the \( W \)-orbit of a simple root, say \( \alpha \)... so let’s write \( r = w(\alpha) \) for some \( w \in W \).

Next, we know that \( w^{-1}\omega^{-1}(\theta) \) is a weight of the module \( L(\theta) \), so \( w^{-1}\omega^{-1}(\theta) \in \theta - Q_+ \). Therefore, we can write

\[
w^{-1}\omega^{-1}(\theta) = \theta - \sum_{\text{finite}} c_i \alpha_i .
\]

where \( c_i \in \mathbb{N} \) and \( \alpha_i \) is a simple root for all \( i \)'s. Then, we can write

\[
(\mu + \rho)(r) = (\lambda + \rho)(r) + (\omega^{-1}(\theta))(r)
\]

\[
= (\lambda + \rho)(r) + 2 \frac{(\omega^{-1}(\theta), s)}{(s, s)}
\]

\[
= (\lambda + \rho)(r) + 2 \frac{(w^{-1}\omega^{-1}(\theta), \alpha)}{(\alpha, \alpha)}
\]

\[
= (\lambda + \rho)(r) + 2 \frac{(\theta, \alpha)}{(\alpha, \alpha)} - \sum_{\text{finite}} c_i \cdot 2 \frac{(\alpha_i, \alpha)}{(\alpha, \alpha)} . \tag{1.11}
\]

Now, since \( r \in R^{\lambda + \rho} \), the first term in the previous line is an integer. Since \( \theta \in P_+ \) and since \( \alpha \) is a simple root, the second term is also an integer. Also, each term of the sum really is "an integer \( c_i \) times a Cartan integer" (because the \( \alpha_i \)'s are simple roots, just as \( \alpha \) is).
Put together, this tells us that information \((\mu + \rho)(r) \in \mathbb{Z}\). As we started with an arbitrary \(r \in \lambda^{\lambda+\rho}\), this gives
\[
R^{\lambda+\rho} \subseteq R^{\mu+\rho}.
\]
Finally, as we can reproduce another big formula just like (1.11) starting from any \(r \in \lambda^{\mu+\rho}\) instead, we get the other inclusion and thus
\[
R^{\lambda+\rho} = R^{\mu+\rho}.
\]
As \(W^{\lambda+\rho}\) and \(W^{\mu+\rho}\) are generated by the same set, they are equal... and so, the proof is now complete.

Q.E.D.
Step 2 : Translation functors and a weak composition series

In this second step, I present the key elements (in my humble opinion) one must have to prove of the Kac-Wakimoto character formula. These key elements are some translation functors and a specific weak composition series. The results of this section are only related to properties of the category $\mathcal{O}$... and of the category $\mathcal{O}^L$ to be most accurate.

The mathematician J.C.Jantzen has defined some translation functors that will be useful for our purpose. Without knowing the exact and/or setting in which he introduced them, here is an overview of the idea that works for our purpose of proving the Kac-Wakimoto character formula.

Let $\mathcal{C}$ be a category that has a "extension block"-like decomposition $\mathcal{C} = \bigoplus_{\text{ext-block}} \mathcal{C}_b$. Fix a $\theta \in P_+$ for which you know that $C \otimes L(\theta) \in \text{ob} \mathcal{C}$ for all $C \in \text{ob} \mathcal{C}$. Then you can define a "translation functor" from the "ext-block" $b_1$ to another "ext-block" $b_2$ by setting :

$$T(\theta)^{b_2}_{b_1} : \mathcal{C}_{b_1} \to \mathcal{C}_{b_2} \quad (2.1)$$

$$M \mapsto \left[ M \otimes L(\theta) \right]_{b_2-\text{comp}}$$

where $[C]_{b_2-\text{comp}}$ means the $b_2$-component of $C$ in its decomposition as a direct sum of submodules with respect to the 'ext-block' decomposition $\mathcal{C} = \bigoplus_{\text{ext-block}} \mathcal{C}_b$.

→ For the original setting in which such functors appear, see the book [?] (in german) by J.C.Jantzen.

→ In [DGK82], the authors mention that J.C.Jantzen uses such functors in finite dimensional Lie theory... in his book.

→ In our setting, we will use some translation functors with the category $\mathcal{O}^L$ and its decomposition given by the result 1.6.

What follows are some relevant results for later use :

**Result 2.2** At least in our setting, the translation functors (2.1) are additive & exact ones. Without these two properties, we could not use them like we will in Steps 3 & 4.

**Result 2.3** Let $\theta \in P_+$ and let $M \in \text{ob} \mathcal{O}^L$. Then $M \otimes L(\theta) \in \text{ob} \mathcal{O}^L$.

**Proof :** (Can be found in [DGK82], see Proposition 5.9)

**Definition 2.4** For a module in $C \in \text{ob} \mathcal{O}$, a **weak composition series** is an increasing filtration of some of its submodules

$$\{0\} = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$$

such that
(1) $\bigcup_i P_i = C$;

(2) $P_{i+1}/P_i$ is a highest weight module for any given $i$;

(3) If the highest weight of $P_{i+1}/P_i$ is greater than that of $P_{j+1}/P_j$, then $i < j$; (i.e. highest weights of the quotients increase along with the indices)

(4) For any weight $\eta$ of $C$, there exist an index $i_\eta$ such that $(C/P_{i_\eta})_{i_\eta} = 0$.

→ It is called a weak composition series partly because the successive quotients aren’t required to be irreducible modules... and maybe also because the series is not required to be finite.

**Result 2.5** Let $\theta \in P_+$ and let $V$ be a highest weight module of highest weight $\lambda \in \mathfrak{T}^*$, then $V \otimes L(\theta)$ has a weak composition series

$$\{0\} = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots,$$

such that for any $i$, $P_{i+1}/P_i$ is a highest weight module of highest weight $\lambda + \nu_i$ where the $\nu_i$ is a weight of $L(\theta)$.

As part of coming from a weak composition series, the set of weight $\{\nu_i\}_i$ satisfy

$$\nu_i > \nu_j \implies i < j.$$

If in addition, we assume that $V = M(\lambda)$, then, in that case, $P_{i+1}/P_i \cong M(\lambda + \nu_i)$ where the $\nu_i$'s are just as above.

**Proof:** (Can be found in [DGK82], see lemma 5.8.)

→ Later on, the (only) properties of the result 2.5 we’ll really be using are the following:

- $\bigcup_i P_i = M \otimes L(\theta)$;
- $P_{i+1}/P_i \cong M(\lambda + \nu_i)$ for all $i$’s when $V = M(\lambda)$ to start with.

→ Result 2.5 holds for any $\lambda \in \mathfrak{T}^*$.

→ Result 2.5 is in fact Lemma 5.8 of [DGK82]. In the article, it can seem, at first glance, to be a little more precise than the present version, but it is not (I’m still pretty sure about that). The only difference is that in this document, I don’t care about fixing a notation about weights of $L(\theta)$ that we won’t be making use of at all.

Coming back to the category $\mathcal{O}^L = \bigoplus_{\lambda \in \mathfrak{L}} \mathcal{O}_{[\lambda]}^L$, here is a corollary to the above:
Corollary 2.6 Let $\theta \in P_+$, let $V$ be a highest weight module of highest weight $\lambda \in K^L$ (so that $V \in \text{ob} \, O^L$) and fix $\eta \in K^L$.

Then the module $[V \otimes L(\theta)]_{[\eta]}$-comp has a weak composition series

$$\{0\} = \tilde{P}_0 \subseteq \tilde{P}_1 \subseteq \tilde{P}_2 \subseteq \cdots,$$

such that $\tilde{P}_{i+1}/\tilde{P}_i$ is a highest weight module with highest weight $\lambda + \tilde{\nu}_i \in [\eta]$ where $\tilde{\nu}_i$ is a weight of $L(\theta)$.

As part of coming from a weak composition series, the set of weight $\{\tilde{\nu}_i\}_i$ satisfy

$$\tilde{\nu}_i > \tilde{\nu}_j \implies i < j.$$

If in addition, we assume that $V = M(\lambda)$, then, in that case, $\tilde{P}_{i+1}/\tilde{P}_i \cong M(\lambda + \tilde{\nu}_i)$ where the $\tilde{\nu}_i$’s are just as above.

Proof: Take the weak composition series for $V \otimes L(\theta)$ given by Result 2.5 and keep only the $P_i$’s such that after relabeling, the highest weights of any successive quotient is in the correct equivalence class of $\sim$, namely $[\eta]$.

Q.E.D.

→ For more details on Corollary 2.6 and/or on its justification, see Remark 5.11 in [DGK82].

To conclude Step 2, a word on translation functors in the case of $O^L$. We define them just as the functor (2.1). However, there will be a more relevant one to care about...

Let $\lambda, \mu \in K^L$ be such that $(W(\mu - \lambda)) \cap P_+ = \{\theta\}$. Then consider the specific translation functor

$$\mathbb{T}(\theta)_{[\lambda]}^{[\mu]} : O_{[\lambda]}^L \rightarrow O_{[\mu]}^L. \quad (2.7)$$

This one will be the more interesting translation functor. It is also such a specific functor that will allow us to prove the Kac-Wakimoto character formula.

→ My guess of a reason why the specific functor (2.7) is the more relevant one is that the assumption $(W(\mu - \lambda)) \cap P_+ = \{\theta\}$ carries an "existence and unicity" aura (i.e. an "$\neq \emptyset$ and cardinality one" aura)... !

→ Combined with Corollary 2.6, the functor (2.7) is wonderful.
Step 3: \( T(M(w.\lambda)) = M(w.\mu) \) ... under a few assumptions

In this third step, I present a detailed proof of the lemma 1 from [KW88]. This is the last step towards getting the character formula proved. The goal of this lemma is to prove that the specific translation functor (2.7) is indeed wonderful.

Lemma 3.1 Let \( \lambda, \mu \in K^L \) be such that \( (W(\mu - \lambda)) \cap P_+ \neq \emptyset \). Suppose that \( (\lambda + \rho)(r) \neq 0 \) for all \( r \in R^{\lambda+\rho} \). Further, assume that

\[
(\lambda + \rho)(r) > 0 \quad \Rightarrow \quad (\mu + \rho)(r) \geq 0,
\]
\[
(\lambda + \rho)(r) < 0 \quad \Rightarrow \quad (\mu + \rho)(r) \leq 0.
\]

Then for any \( g \in W^{\lambda+\rho} = W^{\mu+\rho} \), we have

\[
T(\theta)_{[\mu]}^{[\lambda]}(M(g.\lambda)) = M(g.\mu). \tag{3.2}
\]

Proof: The corollary 2.6 from Step 2 describes a weak composition series for the translated Verma module of the left handside of equation (3.2).

In order to figure anything out about such a specific weak composition series, we need to invesigate the possibilities of having \( g.\lambda + \nu \in [\mu] \) where \( \nu \) is a weight of \( L(\theta) \). Once we have some information, we’ll proceed to moving on.

Firstly, the result 1.8 (1) gives \( [\mu] = W^{\mu+\rho},.\mu = W^{\lambda+\rho},.\mu \) so we’ll need to solve any possibilities of

\[
g.\lambda + \nu = \tilde{g}.\mu \tag{3.3}
\]

where \( \nu \) is a weight of \( L(\theta) \) and \( \tilde{g} \in W^{\mu+\rho} = W^{\lambda+\rho} \).

Let’s first derive an equivalent formula to solve. We can write

\[
\text{Equation (3.3)} \quad \iff \quad g^{-1}(g.\lambda + \nu) = g^{-1}\tilde{g}.\mu
\]
\[
\iff \quad g^{-1}(g(\lambda + \rho) - \rho + \nu + \rho) - \rho = (g^{-1}\tilde{g}).\mu
\]
\[
\iff \quad (\lambda + \rho) + g^{-1}(\nu) - \rho = (g^{-1}\tilde{g}).\mu
\]
\[
\iff \quad \lambda + g^{-1}(\nu) = (g^{-1}\tilde{g}).\mu
\]
\[
\iff \quad \lambda + \psi = h.\mu \tag{3.4}
\]

where \( \psi \) is a weight of \( L(\theta) \) and \( h \in W^{\mu+\rho} = W^{\lambda+\rho} \).

Next, we’ll transform further (3.4) in order to be able to make relevant assumptions. Let \( w \in W \) be the unique element such that

\[
w.\lambda \in \rho + C.
\]
Then we’ll apply it to both sides of (3.4):

Equation (3.4) \[ \iff w.(\lambda + \psi) = w.h.\mu \]
\[ \iff w(\lambda + \psi + \rho) - \rho = (wh).\mu \]
\[ \iff w(\lambda + \rho) + w(\psi) - \rho = (wh).\mu \]
\[ \iff w(\lambda + \rho) + w(\psi) = (wh)(\mu + \rho) \]
\[ \iff w(\lambda + \rho) + w(\psi) = \left(whw^{-1}\right)(w(\mu + \rho)) \]
\[ \iff w(\lambda + \rho) + \bar{\psi} = \bar{g}(w(\mu + \rho)) \]

(3.5)

where \( \bar{\psi} \) is a weight of \( L(\theta) \) and \( \bar{g} \in W^{w(\mu+\rho)} = W^{w(\lambda+\rho)} \) using result 1.7. Also, note that \( w(\lambda + \rho) \in C \) in (3.5).

The next step to prove this lemma is to show that the problems (3.3) \( \iff \) (3.4) \( \iff \) (3.5) admit precisely one solution. Remember that \( w, \lambda \) and \( \mu \) are fixed.

**Existence...**
By assumption, \( (W(\mu - \lambda)) \cap P_+ = \{\theta\} \). Let’s call \( \omega \in W \) an element so that \( \omega(\mu - \lambda) = \theta \). This leads to the equation

\[ \lambda + \omega^{-1}(\theta) = \mu . \quad (3.6) \]

Since \( \omega^{-1}(\theta) \) is a weight of \( L(\theta) \) and since \( \mu \in [\mu] = W^{\mu+\rho}.\mu \), the line (3.6) represents a solution for equation (3.4) \( \iff \) (3.3) \( \iff \) (3.5).

**Unicity...**
Assume that \( \bar{\psi} \in P(L(\theta)) \) and \( \bar{g} \in W^{w(\mu + \rho)} = W^{w(\lambda + \rho)} \) provides a solution of (3.5), i.e. that

\[ w(\lambda + \rho) + \bar{\psi} = \bar{g}(w(\mu + \rho)) . \]

Then as \( w(\lambda + \rho) \in C \), we get

\[ (w(\lambda + \rho))(r) > 0 \quad \text{for all } r \in R_+^{w(\lambda + \rho)} = R_+^{w(\mu + \rho)} . \quad (3.7) \]

The two assumptions of the lemma then give

\[ (w(\mu + \rho))(r) \geq 0 \quad \text{for all } r \in R_+^{w(\mu + \rho)} = R_+^{w(\mu + \rho)} . \quad (3.8) \]

Let’s set \( \tilde{\lambda} = w(\lambda + \rho) \) and \( \tilde{\mu} = w(\mu + \rho) \) so that we can write

\[ \tilde{\lambda} + \bar{\psi} = \bar{w}(\tilde{\mu}) \]

(3.9)

where \( \bar{\psi} \in P(L(\theta)) \) and \( \bar{w} \in W^{\tilde{\lambda}} = W^{\tilde{\mu}} \). We’ll end up showing the equation (3.9) represents the same solution to (3.4) as the one given in the unicity part.
We can make use of the form \((-,-)\) on \(\mathfrak{g}^* \times \mathfrak{g}^*\) and write
\[
(\bar{\psi},\bar{\psi}) = (\bar{w}(\bar{\mu}) - \bar{\lambda},\bar{w}(\bar{\mu}) - \bar{\lambda})
\]
\[
= (\bar{w}(\bar{\mu}),\bar{w}(\bar{\mu})) + (\bar{\lambda},\bar{\lambda}) - 2(\bar{w}(\bar{\mu}),\bar{\lambda})
\]
\[
= (\bar{\mu},\bar{\mu}) + (\bar{\lambda},\bar{\lambda}) - 2(\bar{w}(\bar{\mu}),\bar{\lambda}) .
\] (3.10)

By the proposition 3 (i) of [MP95], \(R^\tilde{\lambda} = R^\bar{\mu}\) is a subroot system of the whole affine root system and its Weyl group is \(W^{\tilde{\lambda}} = W^\bar{\mu}\). This last fact is really easy to check. We then use the exercise 3.12 of [Kac90] to get
\[
\bar{w}(\bar{\mu}) = \bar{\mu} - \sum_{\text{finite}} \bar{\mu}(s_i)\beta_i .
\] (3.11)

where for any \(i, s_i \in \Pi^\bar{\mu}\) and \(\beta_i\) is a positive real root corresponding to some real coroot \(r_i\) of \(R^\bar{\mu}_+\). Note that the coefficients in the sum of (3.11) are all in \(\mathbb{N}\) by the property (3.8).

Let’s then use (3.11) to rewrite \((\bar{w}(\bar{\mu}),\bar{\lambda})\)...
\[
(\bar{w}(\bar{\mu}),\bar{\lambda}) = (\bar{\mu},\bar{\lambda}) - \sum_{\text{finite}} \bar{\mu}(s_i)(\beta_i,\bar{\lambda})
\]
\[
= (\bar{\mu},\bar{\lambda}) - \sum_{\text{finite}} \bar{\mu}(s_i)\bar{\lambda}(r_i)
\]
\[
\in (\bar{\mu},\bar{\lambda}) - \mathbb{N} \quad \text{by the paragraph just above and (3.7)}.
\]

Therefore, we get \((\bar{w}(\bar{\mu}),\bar{\lambda}) \leq (\bar{\mu},\bar{\lambda})\) with equality \(\iff\) \(\bar{w}(\bar{\mu}) = \bar{\mu}\). Combined to the line (3.10), this gives
\[
(\bar{\psi},\bar{\psi}) \geq (\bar{\mu} - \bar{\lambda},\bar{\mu} - \bar{\lambda}) = (\theta,\theta) \quad \text{with equality } \iff \bar{w}(\bar{\mu}) = \bar{\mu} .
\] (3.12)

On the other hand, the proposition 11.4 a) of [Kac90] shows that we do always have
\[
(\bar{\psi},\bar{\psi}) \leq (\theta,\theta) \quad \text{with equality } \iff \bar{\psi} \in W(\theta) .
\] (3.13)

Both the inequalities (3.12) and (3.13) being true, we are forced to admit that there is equality and so we have
\[
\bar{w}(\bar{\mu}) = \bar{\mu} \quad \text{and} \quad \bar{\psi} = f(\theta) \quad \text{for some } f \in W .
\]

It follows that the equation (3.9) can be rewritten as
\[
\bar{\lambda} + f(\theta) = \bar{\mu} .
\] (3.14)

From the previous equality, we can deduce that
\[
f(\theta) = \bar{\mu} - \bar{\lambda}
\]
\[
= w(\mu + \rho) - w(\lambda + \rho)
\]
\[
= w(\mu - \lambda)
\]
\[
= w\omega^{-1}(\theta) .
\] (3.15)
The equation (3.14) can then be rewritten as
\[ \bar{\lambda} + f(\theta) = \bar{\mu} \]
\[ w(\lambda + \rho) + w\omega^{-1}(\theta) = w(\mu + \rho) \]
\[ \lambda + \rho + \omega^{-1}(\theta) = \mu + \rho \]
\[ \lambda + \omega^{-1}(\theta) = \mu. \]  
(3.16)

As the equations (3.16) and (3.6) are the same, we conclude unicity of the existing solution of the problems (3.3) ⇔ (3.4) ⇔ (3.5).

Finally, the problem (3.3) of solving \( g.\lambda + \nu \in [\mu] \) where \( \nu \) is a weight of \( L(\theta) \) admits precisely one solution in the current setting. This solution is given by applying the \( g \) dot action on both sides of (3.6):
\[ g.(\lambda + \omega^{-1}(\theta)) = g.\mu \]
\[ g(\lambda + \rho + \omega^{-1}(\theta)) - \rho = g.\mu \]
\[ g(\lambda + \rho) + g\omega^{-1}(\theta) - \rho = g.\mu \]
\[ g(\lambda + \rho) - \rho + g\omega^{-1}(\theta) = g.\mu \]
\[ g.\lambda + g\omega^{-1}(\theta) = g.\mu. \]

We then conclude from the corollary 2.6 that the module
\[ \mathbf{T}(\theta)[\mu\lambda](M(g.\lambda)) = [M(g.\lambda) \otimes L(\theta)]_{[\mu]} \text{-comp}, \]
has a weak composition series
\[ \{0\} = \tilde{P}_0 \subseteq \tilde{P}_1 \]  
(3.17)

where \( \tilde{P}_1/\tilde{P}_0 \cong \tilde{P}_1 \cong M(g.\lambda + g\omega^{-1}(\theta)) = M(g.\mu). \)

By the definition of a weak composition series, we must have
\[ \mathbf{T}(\theta)[\mu\lambda](M(g.\lambda)) = \bigcup_i \tilde{P}_i = \tilde{P}_1 \]
\[ \cong M(g.\mu). \]

This concludes **Step 3.**

Q.E.D.
Step 4: The KW formula... i.e. \( \text{ch} \, L(\lambda) = \sum_{w} \varepsilon(w) \, \text{ch} \, M(w.\lambda) \)

In this fourth and last step, I present a detailed proof of the theorem 1 from [KW88]. This is the proof of the Kac-Wakimoto formula. We basically just use general properties about the category \( \mathcal{O} \), a topological lemma about Weyl chambers from [MP95] and some translation functors from Step 3.

**Lemma 4.1** Let \( \lambda \in K^L \) for which \( R^{\lambda+\rho} \neq \emptyset \). Assume that \( \lambda(t) > 0 \) for all \( t \in \Pi_+^{\lambda+\rho} \).

Then for any fixed \( s \in \Pi^{\lambda+\rho} \), there exists a \( \mu \in \Sigma^* \) such that

(i) \( (W(\mu - \lambda)) \cap P_+ \neq \emptyset \)
(ii) \( (\mu + \rho)(s) = 0 \)
(iii) \( (\lambda + \rho)(t) > 0 \) for all \( t \in \Pi^{\mu+\rho}\{s\} = \Pi^\lambda\{s\} \)

**Proof:** (Can be found in [MP95], see lemma 6.8.6.)

→ The proof seems like a topological one involving \( K \), Weyl chambers and hyperplanes orthogonal to a given root. It seems understandable.

**Theorem 4.2** Let \( \lambda \in K^L \) for which \( \lambda(t) > 0 \) for all \( t \in \Pi_+^{\lambda+\rho} \). Then

\[
\text{ch} \, L(\lambda) = \sum_{w \in W^{\lambda+\rho}} \varepsilon(w) \, \text{ch} \, M(w.\lambda)
\]  

(4.3)

where \( \varepsilon(w) \) is the sign of the Weyl group element \( w \).

**Proof:** Since we work in the category \( \mathcal{O} \), any module \( V \) has a character given in terms of a sum of characters \( \text{ch} \, L(\eta) \)’s for certain \( \eta \)’s in \( \Sigma^* \).

Because \( M(\lambda) \) is in \( \mathcal{O}^L \), we have (by result 1.2) that \( \text{ch} \, M(\lambda) \) is given in terms of a sum of \( \text{ch} \, L(\eta) \)’s for certain \( \eta \)’s in \( K^L \).

In the case of \( M(\lambda) \), we have that \( P(M(\lambda)) \subseteq \lambda - \mathbb{N}Q_+ \). Consider the set

\[
P = \{ \lambda = p_0, p_1, p_2, \ldots \} = [\lambda] \cap (\lambda - \mathbb{N}Q_+) \]  

(4.4)

for which \( p_i \in p_j - \mathbb{N}Q_+ \) for all \( i > j \).

Since \( \lambda \in K^L \) and by the result 1.1, the we have \( [M(\lambda) : L(\eta)] \neq 0 \) \( \Rightarrow [\eta] = [\lambda] \) ... and so we have

\[
[M(\lambda) : L(\eta)] \neq 0 \quad \Rightarrow \quad \eta = p_j \text{ for some } j
\]
For any fixed $i \in \mathbb{N}$, we then get

$$\text{ch } M(p_i) = \sum_{j \geq i} [M(p_i) : L(p_j)] \text{ ch } L(p_j) \quad (4.5)$$

Also, note that $[M(p_j) : L(p_j)] = 1$ for any given $j \in \mathbb{N}$ because of the basic properties following from the definition of a Verma module.

Next, we can view the set of equations (4.5) as an "infinite" linear system

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
\ast & 1 & 0 & 0 \\
\ast & \ast & 1 & 0 \\
\vdots & \vdots & \ddots & 1
\end{pmatrix}
\begin{pmatrix}
L(\lambda) \\
L(p_1) \\
L(p_2) \\
\vdots
\end{pmatrix}
= 
\begin{pmatrix}
M(\lambda) \\
M(p_1) \\
M(p_2) \\
\vdots
\end{pmatrix} \quad (4.6)
$$

Formally inverting the "linear system" (4.6) and gets us the formula :

$$\text{ch } L(\lambda) = \sum_{w \in W^{\lambda+\rho}} m(w, \lambda) \text{ ch } M(w.\lambda) \quad (4.7)$$

for some integers $m(w, \lambda)$.

The only differences between the formulas (4.7) and (4.3) are the coefficients of the corresponding sums. Understandably, the last objective will be to justify that $m(w, \lambda) = \varepsilon(w)$ for any given $w \in W^{\lambda+\rho}$. In order to achieve that, let’s first fix an $s \in \Pi^{\lambda+\rho} \neq \emptyset$.

With $s \in \Pi^{\lambda+\rho}$ fixed, the lemma 4.1 ensures the existence of a $\mu \in K^L$ such that

$$\begin{align*}
(W(\mu - \lambda)) \cap P_+ &= \{\theta\} \quad \text{(fixing the notation)} \\
(\mu + \rho)(s) &= 0 \\
(\mu + \rho)(t) &> 0 \quad \text{for all } t \in \Pi^{\mu+\rho} \setminus \{s\} = \Pi^{\lambda+\rho} \setminus \{s\} 
\end{align*} \quad (4.8)$$

The pair $\lambda \& \mu$ does satisfy the conditions to apply the lemma 3.1 and so the corresponding translation functor $T(\theta)_{[\mu]_{[\lambda]}}$ does map $M(w.\lambda)$ to $M(w.\mu)$. Note that this functor was also both exact and additive.

Let $\beta_s$ be the positive root corresponding to the positive coroot $s$. The assumption from the theorem together with the choice of $s \in \Pi^{\lambda+\rho}$ give $(\lambda + \rho)(s) \in \mathbb{N} \setminus \{0\}$. In fact, the result 1.1 then gives that

$$[M(\lambda) : L(\sigma_s.\lambda)] \neq 0$$

Since this multiplicity is non-zero, $M(\lambda)$ must contain a singular vector of weight $\sigma_s.\lambda$, and so $M(\lambda)$ must contain a copy of $M(\sigma_s.\lambda)$. Identify $M(\sigma_s.\lambda)$ with one of its copies inside of
\( M(\lambda) \). Since \((\lambda + \rho)(s) \in \mathbb{N}\setminus\{0\}\), we have \( \sigma_s.\lambda \in \left( \lambda - NQ_+ \right)\setminus\{\lambda\} \). Thus, the highest weight vector of \( M(\lambda) \) is not in \( M(\sigma_s.\lambda) \), we conclude that

\[
M(\sigma_s.\lambda) \subseteq N(\lambda)
\]

where \( N(\lambda) \) is the submodule of \( M(\lambda) \) for which the quotient is \( L(\lambda) \). It follows from the line (4.9) that there is a surjection

\[
\frac{M(\lambda)}{M(\sigma_s.\lambda)} \to \frac{M(\lambda)}{N(\lambda)} \cong L(\lambda)
\]

Equivalently, there is an exact sequence

\[
\frac{M(\lambda)}{M(\sigma_s.\lambda)} \to L(\lambda) \to 0
\]

Let’s apply the exact functor \( T(\theta)[\mu]_{[\lambda]} \) to the sequence (4.10). Using the description of the effect of the functor on Verma modules given at the line (3.2) from Step 3, we obtain the exact sequence

\[
\frac{M(\mu)}{M(\sigma_s.\mu)} \to T(\theta)[\mu]_{[\lambda]}(L(\lambda)) \to 0
\]

As presented at the line 4.8, the choice of \( \mu \), gives

\[
\sigma_s.\mu = \mu + \sigma - (\mu + \rho)(s)\beta_s - \rho = \mu - (\mu + \rho)(s)\beta_s = \mu
\]

Therefore, \( M(\sigma_s.\mu) = M(\mu) \) and the exact sequence (4.11) really is

\[
0 \to T(\theta)[\mu]_{[\lambda]}(L(\lambda)) \to 0
\]

... meaning that \( T(\theta)[\mu]_{[\lambda]}(L(\lambda)) \cong \{0\} \) itself.

Now, since the translation functors are additive, their application on modules commutes with taking characters. From this relevant fact, we rewrite the formula (4.7) as we apply the functor \( T(\theta)[\mu]_{[\lambda]} \). The result is

\[
0 = \text{ch} T(\theta)[\mu]_{[\lambda]}(L(\lambda)) = \sum_{w \in W^{\lambda + \rho}} m(w, \lambda) \text{ ch } M(w.\mu)
\]

Thus, the coefficient of any Verma module of the right handside of (4.13) will be zero. Let’s focus on the coefficient of \( M(w.\mu) \) in the right handside of (4.13) where \( w \in W^{\lambda + \rho} = W^{\mu + \rho} \) is any fixed group element.

To get the coefficient of \( M(w.\mu) \) in (4.13), we must find all the elements \( g \in W^{\lambda + \rho} = W^{\mu + \rho} \) such that

\[
g.\mu = w.\mu \iff g^{-1}w \in \text{Stab}_{W^{\lambda + \rho}}\{\mu\}
\]
Let’s then proceed to find \( \text{Stab}_{W^{\lambda+\rho}} \{ \mu \} \). First, recall that

\[
W^{\lambda+\rho} = \{ \sigma_t \mid t \in \Pi^{\lambda+\rho} \}
\tag{4.15}
\]

With that in mind, looking back at the properties of \( \mu \) at the line (4.8), leads to

\[
\text{Stab}_{W^{\lambda+\rho}} \{ \mu \} = \{ \text{Id}, \sigma_s \}
\]

We deduce that the elements \( g \) from the line (4.14) we are looking for are \( w \) and \( w\sigma_s \).

Finally, we can write that the coefficient of \( M(w, \mu) \) in the right handside of the equation (4.13) is \( m(w, \lambda) + m(w\sigma_s, \lambda) \). From this same equation (4.13), we can then write

\[
0 = m(w, \lambda) + m(w\sigma_s, \lambda) \\
-m(w\sigma_s, \lambda) = m(w, \lambda)
\tag{4.16}
\]

In conclusion, since the line (4.16) is independent of the chosen \( \mu \) to suit our arbitrarily fixed \( s \in \Pi^{\lambda+\rho} \) and since \( w \in W^{\lambda+\rho} \) is also arbitrary, this same equation (4.16) holds independently of the choices of pairs \( w \& s \).

Using the fact that \( W^{\lambda} \) is generated by the Weyl reflections \( \sigma_t \) for \( t \in \Pi^{\lambda+\rho} \), we are brought to concluding that

\[
m(w, \lambda) = \varepsilon(w)
\tag{4.17}
\]

It is then possible to rewrite the equation (4.7) as :

\[
\text{ch } L(\lambda) = \sum_{w \in W^{\lambda+\rho}} \varepsilon(w) \text{ ch } M(w, \lambda)
\]

... which is the Kac-Wakimoto character formula.

Q.E.D.

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This document is now complete ... "for its prupose" !!
References


