Theorem 1 Consider the continuous-time LQR cost function for a stable LTI continuous-time system $G$

$$J_c = \int_0^\infty (x'Q_c x + u'R_c u)dt$$

If $Q_d$, $R_d$ and $R_d$ are the equivalent weights in the discrete-time cost function $J_d$ for the single-rate system at the slow-rate, and $Q_L$, $R_L$, $N_L$ are the equivalent weightings in the discrete-time cost function $J_L$ for the lifted slow-sampled fast-control multirate system with sampling ratios of the outputs and inputs as $n$, then the following relations hold good :

$$Q_d = Q_L$$

$$R_d = \sum_{i=1}^{n} \sum_{j=1}^{n} R_{Lij}$$

$$N_{di} = \sum_{j=1}^{n} N_{Lij} \quad i = 1, \ldots, n_s$$

where $R_L = [R_{Lij}]$ and $N_L = [N_{Lij}]$ and $n_s$ is the order of the system $G$. $N_{di}$ is the weighting between the $i^{th}$ state and the input, $N_d = [N_{d1}, N_{d2}, \ldots, N_{dn_s}]^T$.

Proof: Denote the stable plant $G$ and the fast rate discretized plant $G_f$ as

$$\hat{g}(s) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} ; \quad \hat{g}(\lambda) = \begin{bmatrix} A_f & B_f \\ C & D \end{bmatrix}$$

respectively. Also, $A_f = e^{Ah_f}$ ; $B_f = \int_0^{h_f} e^{At}dtB$

If

$$\tilde{A} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}$$

then

$$\begin{pmatrix} A_f & B_f \\ 0 & I \end{pmatrix} = \exp \left\{ h_f \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \right\}$$

Consider the derivation of the equivalent discrete-system at the slow rate $h_b$, given $J_c$. Representing the integral $M$ at the slow rate as $M_s$, and $M_f$ at the fast rate $h_f = h_b/n$, we can write

$$M_s = \int_0^{h_b} e^{t\tilde{A}} \begin{pmatrix} Q_c & 0 \\ 0 & 0 \end{pmatrix} e^{t\tilde{A}}dt \quad ; \quad M_f = \int_0^{h_f} e^{t\tilde{A}} \begin{pmatrix} Q_c & 0 \\ 0 & 0 \end{pmatrix} e^{t\tilde{A}}dt$$
We know from earlier discussion that

\[
M_s = \begin{pmatrix} Q_d & N_d \\ N'_d & R_d \end{pmatrix}; \quad M_f = \begin{pmatrix} Q_f & N_f \\ N'_f & R_f \end{pmatrix}
\]

Using Cholesky factorization, we can write

\[
\begin{pmatrix} Q_c & 0 \\ R_c & 0 \end{pmatrix} = W^T W
\]

The integral denoted by \( M_s \) can be split into \( n \) integrals since \( h_b = nh_f \),

\[
M_s = \int_0^{h_f} e^{tA} W^T W e^{tA} dt + \int_{h_f}^{2h_f} e^{tA} W^T W e^{tA} dt + \cdots + \int_{(n-1)h_f}^{nh_f} e^{tA} W^T W e^{tA} dt
\]

Using equation (4) in equation (7), the \( k^{th} \) \((k = 2, \cdots, n)\) term in (7) can be written as

\[
\begin{pmatrix} A_f^{T^{k-1}} \\ B_f^T + B_f^T A_f^T + \cdots + B_f^T A_f^{T^{k-2}} \end{pmatrix} M_f \begin{pmatrix} A_f^{k-1} & B_f + B_f A_f + \cdots + B_f A_f^{k-2} \\ 0 & I \end{pmatrix}
\]

Substituting equation (5) in equation (8) and equating the individual terms on both the sides, gives

\[
Q_d = Q_f + A_f^T Q_f A_f + A_f^{T^2} Q_f A_f^2 + \cdots + A_f^{T^n-1} Q_f A_f^{n-1}
\]

\[
R_d = R_f + (B_f^T Q_f B_f + B_f^T N_f + N_f^T B_f + R_f) + \cdots + (B_f^T Q_f A_f^{T^2} B_f + \cdots + B_f^T Q_f A_f^2 B_f + B_f^T N_f + N_f^T A_f^{T^2} B_f + \cdots + N_f^T B_f + R_f)
\]

\[
N_d = N_f + (A_f^T Q_f B_f + A_f^T N_f) + (A_f^{T^2} Q_f A_f B_f + A_f^{T^2} Q_f A_f^2 B_f + A_f^{T^2} N_f) + \cdots + A_f^{T^{n-1}} Q_f A_f^{n-2} B_f + \cdots + A_f^{T^{n-1}} Q_f A_f^{n-1}
\]

Using Cholesky factorization, we can write

\[
\begin{pmatrix} Q_f & N_f \\ N'_f & R_f \end{pmatrix} = [ \tilde{C}_f \tilde{D}_f ]^T [ \tilde{C}_f \tilde{D}_f ]
\]

Note that

\[
z_k = \tilde{C}_f x_k + \tilde{D}_f u_k
\]

is the fictitious signal whose \( \| \cdot \|_2 \) we seek to minimize.
Observing from our results that for the lifted system, \( \tilde{G}_L = L_n G_f L^{-1} \) representing the model between \( z_k \) and \( u_k \), we can write,

\[
\tilde{g}_L(\lambda) = \begin{bmatrix}
A_f^n & A_f^{n-1}B_f & A_f^{n-2}B_f & \cdots & B_f \\
\tilde{C}_f & \tilde{D}_f & 0 & \cdots & 0 \\
\tilde{C}_f A_f & \tilde{C}_f B_f & \tilde{D}_f & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\tilde{C}_f A_f^{n-1} & \tilde{C}_f A_f^{n-2}B_f & \tilde{C}_f A_f^{n-2}B_f & \cdots & \tilde{D}_f
\end{bmatrix}
\]

\[
= \begin{bmatrix}
A_L & B_L \\
\tilde{C}_L & \tilde{D}_L
\end{bmatrix}
\]

Therefore, the equivalent weights in the cost function in the lifted system can be obtained as

\[
\begin{pmatrix}
Q_L & N_L \\
N_L^T & R_L
\end{pmatrix} = \begin{bmatrix}
\tilde{C}_L & \tilde{D}_L \end{bmatrix} \begin{bmatrix}
\tilde{C}_L & \tilde{D}_L
\end{bmatrix}^T
\]

where we make use of equations (13) and (14).

Thus, \( Q_L, N_L \) and \( R_L \) would be \( n_s x n_s \), \( n_s x n \) and \( n x n \) matrices respectively. From equations (9), (12) and (15), it is easy to see that

\[
Q_d = Q_L
\]

Obtaining \( R_L \) from equation (15) and adding up all the elements, it can be seen that

\[
R_d = \sum_{i=1}^{n} \sum_{j=1}^{n} R_{L_{i,j}}
\]

and similarly, adding up all the columns for each of the rows in \( N_L \), we get

\[
N_{d_i} = \sum_{j=1}^{n} N_{L_{i,j}} \quad i = 1, \cdots, n_s
\]