Superposition Principle

Superposition principle (for 1D waves): the resultant wave at each point in the region of overlap of two or more waves is the **algebraic sum** of the individual constituent waves at that location.

Note: Once the waves pass the intersecting region they will move away unaffected by the encounter.

• For example, if ψ_1 and ψ_2 are separate wavefunctions describing the constituent waves, then the resultant wavefunction is $(\psi_1 + \psi_2)$.

• Note that if ψ_1 and ψ_2 are solutions of the 1D wave equation, then $(\psi_1 + \psi_2)$ is also a solution of the 1D differential wave equation.

$$\frac{\partial^2 \psi_1}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi_1}{\partial t^2} \quad \text{and} \quad \frac{\partial^2 \psi_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi_2}{\partial t^2}$$

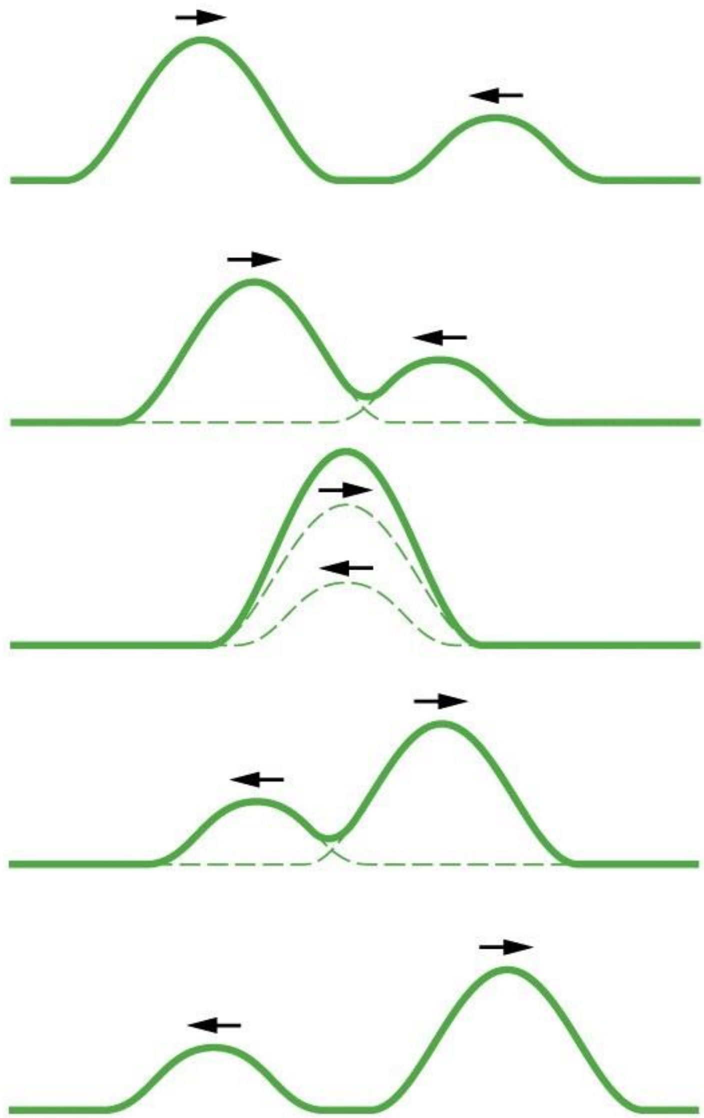
Adding these yields

$$\frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi_1}{\partial t^2} + \frac{1}{v^2}\frac{\partial^2 \psi_2}{\partial t^2}$$

and so
$$\frac{\partial^2}{\partial x^2}(\psi_1 + \psi_2) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}(\psi_1 + \psi_2)$$

which establishes that $(\psi_1 + \psi_2)$ is indeed a solution.

Superposition principle:

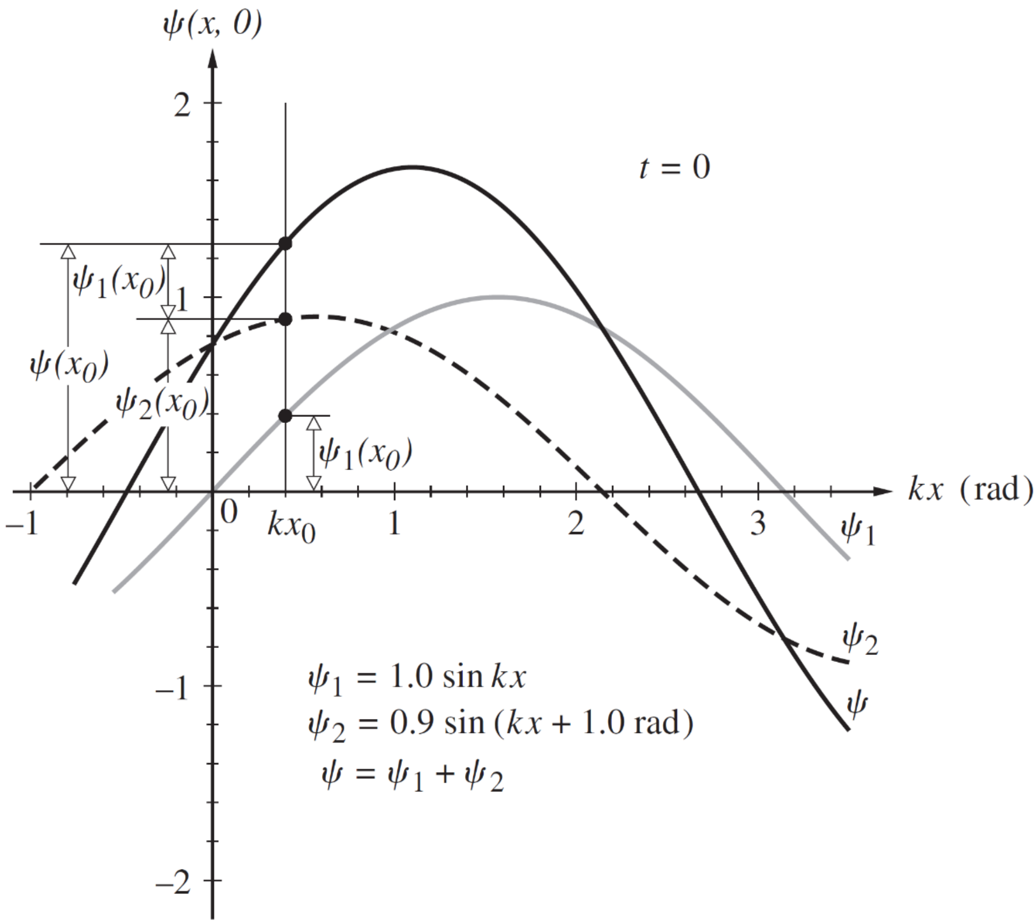


- Resultant wavefunction is just the algebraic sum of the two individual wavefunctions (whose shapes do not change).

- Overlapping waves do NOT in any way alter the travel of each other.

Superposition Principle (con't):

Example of two harmonic waves with the same k: (i.e., same λ)



- For any value of x, the resultant wave is the algebraic sum of the individual waves: $\psi(x) = \psi_1(x) + \psi_2(x)$.

- The resultant wave is still a harmonic wave (with the same k).

- The peak of the resultant wave ψ is between the peaks of the individual waves ψ_1 and ψ_2 (the phase of the resultant wave is "between" the phases of the individual waves).

Note: It can be shown that the resultant wave is $\psi = 1.668\,\sin(kx + 0.4713\text{ rad})$. [see end of handout for an example of this calculation]

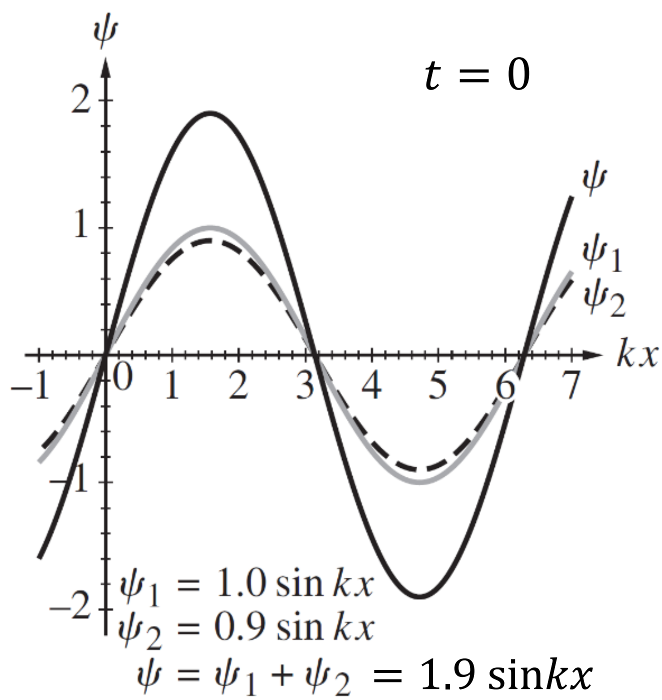
Superposition Principle: Constructive & Destructive Interference

- Example: Two waves are "***in-phase***",
  i.e., their constituent waves have the same phase.

$$\psi_1 = A_1 \sin(kx \mp \omega t)$$

$$\psi_2 = A_2 \sin(kx \mp \omega t)$$

$$\psi = (A_1 + A_2) \sin(kx \mp \omega t)$$



- Amplitude of the resultant wave is larger than the amplitudes of either constituent wave: ***constructive interference***

(this situation, where the phases are identical, is sometimes called "*fully constructive interference*")

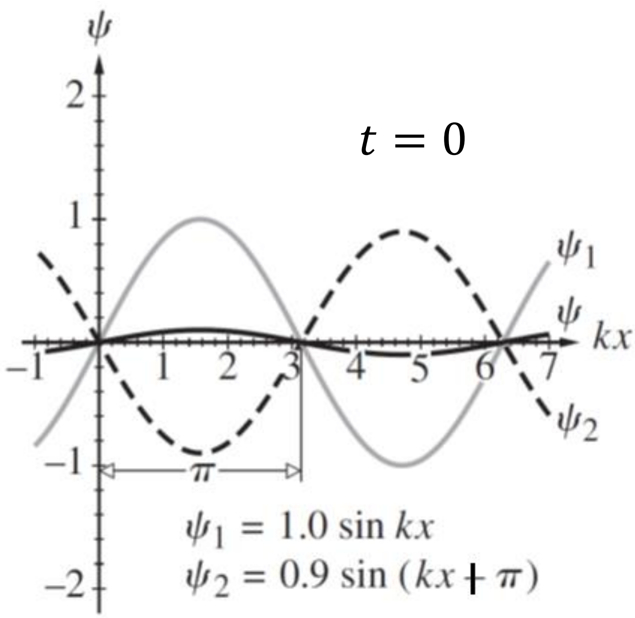
Constructive & Destructive Interference (con't)

- Example: Two waves are "***out-of-phase***",
  i.e., the phase of the constituent waves differ by $\Delta\varphi = \pi = 180^0$

$$\psi_1 = A_1 \sin(kx \mp \omega t)$$

$$\psi_2 = A_2 \sin(kx \mp \omega t + \pi)$$
$$= -A_2 \sin(kx \mp \omega t)$$

$$\psi = (A_1 - A_2) \sin(kx \mp \omega t)$$



$$\psi = \psi_1 + \psi_2 = 0.1\sin(kx)$$

- Amplitude of the resultant wave is smaller than at least one of the constituent waves: ***destructive interference***

  (this situation, where the phases differ by exactly π, is sometimes called "*fully destructive interference*")

Extra information: (Excerpts from document authored by H. Haber, UCSC)

One method for adding sine functions of different amplitude and phase

(below, "phase" means "initial phase")

In these notes, I will show you how to add two sinusoidal waves, each of different amplitude and phase, to get a third sinusoidal wave. That is, we wish to show that given

$$E_1 = E_{10} \sin \omega t \,, \tag{1}$$

$$E_2 = E_{20} \sin(\omega t + \delta) \,, \tag{2}$$

the sum $E_\theta \equiv E_1 + E_2$ can be written in the form:

$$\boxed{E_\theta = E_{10} \sin \omega t + E_{20} \sin(\omega t + \delta) = E_{\theta 0} \sin(\omega t + \phi)} \tag{3}$$

where the amplitude $E_{\theta 0}$ and phase ϕ are determined in terms of E_{10}, E_{20} and δ. In these notes, we shall derive that the amplitude $E_{\theta 0}$ is given by

$$\boxed{E_{\theta 0} = \sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20} \cos \delta}} \tag{4}$$

and the phase ϕ is determined modulo 2π by

$$\boxed{\sin \phi = \frac{E_{20} \sin \delta}{E_{\theta 0}}}, \qquad \boxed{\cos \phi = \frac{E_{10} + E_{20} \cos \delta}{E_{\theta 0}}}. \tag{5}$$

Derivation:

First, set $t = 0$ in eq. (3) to obtain $E_{20} \sin \delta = E_{\theta 0} \sin \phi$. Solving for $\sin \phi$ yields:

$$\boxed{\sin \phi = \frac{E_{20} \sin \delta}{E_{\theta 0}}}.$$

Next, set $\omega t = \pi/2$ in eq.(3). Noting that $\sin(\delta + \pi/2) = \cos \delta$, it follows that $E_{10} + E_{20} \cos \delta = E_{\theta 0} \cos \phi$. Solving for $\cos \phi$ yields:

$$\boxed{\cos \phi = \frac{E_{10} + E_{20} \cos \delta}{E_{\theta 0}}}.$$

Finally, using $\cos^2 \phi + \sin^2 \phi = 1$, and inserting the expressions for $\cos \phi$ and $\sin \phi$ just obtained, one finds:

$$\begin{aligned} E_{\theta 0}^2 &= (E_{10} + E_{20} \cos \delta)^2 + E_{20}^2 \sin^2 \delta \\ &= E_{10}^2 + 2E_{10}E_{20} \cos \delta + E_{20}^2 (\cos^2 \delta + \sin^2 \delta) \\ &= E_{10}^2 + E_{20}^2 + 2E_{10}E_{20} \cos \delta \,. \end{aligned}$$

By definition, $E_{\theta 0}$ is a non-negative number. Thus, we take the positive square root to obtain

$$\boxed{E_{\theta 0} = \sqrt{E_{10}^2 + E_{20}^2 + 2E_{10}E_{20} \cos \delta}}.$$

This completes our derivation.

For example,

If $\psi_1 = 1.0\sin(kx)$

$\psi_2 = 0.9\sin(kx + 1.0\text{ rad})$

Want $\psi_1 + \psi_2 = A\sin(kx + \phi)$

What is A and ϕ?

Answer:

In using the equations on the previous page, we are of course using different variable names, i.e., $\omega t \to kx$, $E_{\theta 0} \to A$, etc.

Use eq. (4) to find A:

$$A = \sqrt{1^2 + 0.9^2 + 2(1)(0.9)\cos(1.0\text{rad})} = 1.668$$

Use first of eqs. (5) to find ϕ:

$$\sin\phi = \frac{0.90\sin(1.0\text{rad})}{1.668} \quad \Rightarrow \quad \phi = 0.4713\text{ rad}$$

So $\psi_1 + \psi_2 = 1.668\sin(kx + 0.4713)$