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Robust Designs for Approximately Linear Regression: M-Estimated Parameters

by

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Summary

We obtain designs, to be used for investigations of response surfaces by regression techniques, when

i) the fitted, linear (in the parameters) response is incorrect; and

ii) the parameters are to be estimated robustly.

Loss is taken to be the integrated, asymptotic, squared bias or mean squared error of the fitted response. With respect to either loss function, minimax designs are determined for "small" departures from the fitted response. For mean squared error loss, we specialize to the case in which the experimenter fits a plane, when in fact the true response contains quadratic and interaction terms. In this case, minimax rotatable designs are derived, subject to a lower bound on the power of a robust test of model adequacy. The optimal designs place their mass at the centre of the design space, and on a sphere interior to the design space.

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(All integrals in this paper are over the design space S, unless it is explicitly indicated otherwise.) Condition (1.5) is imposed without loss of generality, since any function f which satisfies (1.6) may be modified, by subtracting its L_2- projection on $\mathbf{u}$, to satisfy (1.5) as well. In turn, (1.5) guarantees that $E[y(\mathbf{x})]$ is uniquely parameterized.

Under (1.4), the LSE $\widehat{\boldsymbol{\theta}}_{LS}$ may be biased, although the bias may be made small by an appropriate choice of design. If the $\mathbf{x}_i$ are governed by a design measure ξ, i.e. a probability measure on S given by

$$\xi(S') = \frac{\#\ of\ \mathbf{x}_i\ in\ S'}{n}$$

for $S' \subseteq S$, then the necessary and sufficient condition for unbiasedness of $\widehat{\boldsymbol{\theta}}_{LS}$ is

$$\int \mathbf{u}(\mathbf{x})f(\mathbf{x})d\xi(\mathbf{x}) = \mathbf{0}. \tag{1.7}$$

If this is to hold for all f satisfying (1.5) and (1.6), then it is required that ξ be the (continuous) uniform d.f. $\lambda(\mathbf{x})$ ($d\lambda(\mathbf{x}) = d\mathbf{x}$) on S.

Uniformity of ξ is of course attainable only asymptotically. Even if (1.7) holds only approximately, the corresponding design will likely be quite inefficient for model (1.1). There is a rich literature on the construction of designs for (1.1), concentrating on the minimization of some scalar-valued function of the covariance matrix of $\widehat{\boldsymbol{\theta}}_{LS}$. See, e.g., Fedorov (1972). These variance-optimal designs are typically supported on a small number of extreme points of S, and so differ markedly from any design which comes close to satisfying (1.7) for any broad class of functions f.

There is as well a body of work concerned with the choice of appropriate design measures for (1.4), or sub-models thereof, when $\widehat{\boldsymbol{\theta}}$ is the LSE and loss incorporates both bias and variance. Box and Draper (1959) studied designs for the sub-model in which $\boldsymbol{\theta}_0^T\mathbf{u}(\mathbf{x})$ and $f(\mathbf{x})$ are multinomials in $\mathbf{x}$, of degrees $d_1 < d_2$ respectively. Taking loss as the integrated mean squared error of $\widehat{y}(\mathbf{x})$:

$$\mathcal{L}(f,\xi) = \int E[\{\widehat{y}(\mathbf{x}) - (\boldsymbol{\theta}_0^T\mathbf{u}(\mathbf{x}) + f(\mathbf{x}))\}^2]d\mathbf{x} \tag{1.8}$$

they obtained designs to minimize $\mathcal{L}$, *averaged* over f. Huber (1975) obtained designs to minimize the *maximum* of $\mathcal{L}(f,\xi)$, for f ranging over the entire class defined by (1.5) and

2. Asymptotic theory

Let $y_1, \ldots, y_n$ be independent observations following model (1.4), and let $\xi_n(\mathbf{x})$ be the design measure of $\mathbf{x}_1, \ldots, \mathbf{x}_n$. Rather than (1.2) and (1.3) we work with

$$E_P[\psi(\frac{y - \boldsymbol{\theta}^T \mathbf{u}(\mathbf{x})}{\sigma})\mathbf{u}(\mathbf{x})] = \mathbf{0}, \tag{2.1}$$

$$E_P[\chi(\frac{y - \boldsymbol{\theta}^T \mathbf{u}(\mathbf{x})}{\sigma})] = 0 \tag{2.2}$$

When $P = P_n$, the empirical distribution function of $(\mathbf{x}_i, y_i)$, then (2.1) and (2.2) become (1.2) and (1.3). We assume that ξ_n has a weak limit ξ. Let G be the distribution function of ε. If P is the limiting, true distribution function of $(y, \mathbf{x})$, given by

$$P(y, \mathbf{x}) = \int \cdots \int_{\cap_{i=1}^{p}\{z_i \leq x_i\}} G(y - \boldsymbol{\theta}_0^T \mathbf{u}(\mathbf{x}) - f(\mathbf{x}))d\xi(\mathbf{x}), \tag{2.3}$$

then (2.1) and (2.2) define the asymptotic values of the estimates.

We make assumptions A1) - A7) of Maronna and Yohai (1981). The most important of these stipulate that

i) ψ is odd, bounded, continuous; $\psi(u) \geq 0$ for $u \geq 0, \psi(u)/u$ is non-increasing for $u > 0$.

ii) χ is continuous, bounded and even; $\chi(0) < 0, \chi(\infty) < \infty$, χ is strictly increasing in $(0, \chi^{-1}(\chi(\infty))$.

iii) $E_\xi[\|\mathbf{u}(\mathbf{x})\|] < \infty$.

We guarantee iii) for every ξ by further assuming

A8) The design space S is bounded.

Note that, although the results of Maronna and Yohai (1981) are derived under the implicit assumption of random carriers, they apply to the designed situation if the design measure ξ_n is substituted for the distribution function of the carriers.

Under the above conditions, Theorem 2.1 of Maronna and Yohai (1981) asserts that there exist solutions $(\boldsymbol{\theta}, \sigma)$ to (2.1) and (2.2). Asymptotic uniqueness may be infered under the

From (2.1) and (2.2), the asymptotic roots $(\boldsymbol{\theta}, \sigma)$ are then defined by

$$E_\xi[\ell(f(\mathbf{x}) - \boldsymbol{\theta}^T \mathbf{u}(\mathbf{x}), \sigma)\mathbf{u}(\mathbf{x})] = \mathbf{0}, \tag{2.9}$$

$$E_\xi[m(f(\mathbf{x}) - \boldsymbol{\theta}^T \mathbf{u}(\mathbf{x}), \sigma)] = 0. \tag{2.10}$$

Loss is taken to be the integrated, mean squared error of $\widehat{y}(\mathbf{x}) = \widehat{\boldsymbol{\theta}}^T \mathbf{u}(\mathbf{x})$, in estimating $E[y(\mathbf{x})] = f(\mathbf{x})$. We approximate $E[\widehat{y}(\mathbf{x})] = E[\widehat{\boldsymbol{\theta}}^T]\mathbf{u}(\mathbf{x})$ by its limit $\boldsymbol{\theta}^T\mathbf{u}(\mathbf{x})$, and $\mathrm{Var}[\widehat{y}(\mathbf{x})] = \mathbf{u}^T(\mathbf{x})\mathrm{Cov}[\widehat{\boldsymbol{\theta}}]\mathbf{u}(\mathbf{x})$ by

$$n^{-1}\mathbf{u}^T(\mathbf{x})(\lim \mathrm{Cov}[\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})])\mathbf{u}(\mathbf{x}) = n^{-1}\mathbf{u}^T(\mathbf{x})[D^{-1}CD^{-T}]_{11}\mathbf{u}(\mathbf{x}).$$

We define

$$A_1 = \int \mathbf{u}(\mathbf{x})\mathbf{u}^T(\mathbf{x})d\mathbf{x}.$$

Then loss is given by

$$\begin{aligned}
\int E[(\widehat{y}(\mathbf{x}) - f(\mathbf{x}))^2]d\mathbf{x} &= \int E[(\widehat{y}(\mathbf{x}) - E[\widehat{y}(\mathbf{x})])^2]d\mathbf{x} + \int (E[\widehat{y}(\mathbf{x})])^2 d\mathbf{x} + \int f^2(\mathbf{x})d\mathbf{x} \\
&= n^{-1}\int \mathbf{u}^T(\mathbf{x})[D^{-1}CD^{-T}]_{11}\mathbf{u}(\mathbf{x})d\mathbf{x} + \int (\boldsymbol{\theta}^T\mathbf{u}(\mathbf{x}))^2 d\mathbf{x} + \int f^2(\mathbf{x})d\mathbf{x} + o(n^{-1}) \\
&= n^{-1} tr A_1 [D^{-1}CD^{-T}]_{11} + \boldsymbol{\theta}^T A_1 \boldsymbol{\theta} + \int f^2(\mathbf{x})d\mathbf{x} + o(n^{-1}) \\
&:= \mathcal{V}(f,\xi) + \mathcal{B}(f,\xi) + \mathcal{Q}(f) + o(n^{-1}).
\end{aligned}$$

The mathematical formulation of the design problem is then to minimize, over ξ, the maximum, over f, of

$$\mathcal{L}(f,\xi) = \mathcal{V}(f,\xi) + \mathcal{B}(f,\xi) + \mathcal{Q}(f) \tag{2.11}$$

with $\boldsymbol{\theta}$ defined by (2.9), (2.10) and f constrained by (1.5), (1.6).

3. Optimality theory: $f = f_{\boldsymbol{\alpha}}$

Henceforth, we constrain $f(\mathbf{x})$ by (1.9) as well as by (1.5) and (1.6). Note that by (1.5),

$$\int \mathbf{u}\mathbf{v}^T d\mathbf{x} = O : r \times q. \tag{3.1}$$

Define constants

$$\begin{aligned} V_{\psi,G} &= \sigma^2(\mathbf{0})c(0,\sigma(\mathbf{0}))/d^2(0,\sigma(\mathbf{0})), \\ \tau_0 &= -\frac{m_{11}(0,\sigma(\mathbf{0}))}{\sigma(\mathbf{0})m_2(0,\sigma(\mathbf{0}))} \\ \tau_1 &= \frac{c_{11}(0,\sigma(\mathbf{0}))}{2c(0,\sigma(\mathbf{0}))} - \frac{d_{11}(0,\sigma(\mathbf{0}))}{d(0,\sigma(\mathbf{0}))} \\ \tau_2 &= \sigma(\mathbf{0})\frac{\tau_0}{\tau_1}\left(\frac{d_2(0,\sigma(\mathbf{0}))}{d(0,\sigma(\mathbf{0}))} - \frac{c_2(0,\sigma(\mathbf{0}))}{2c(0,\sigma(\mathbf{0}))} - \frac{1}{\sigma(\mathbf{0})}\right) \end{aligned}$$

and matrices

$$\Delta = B_{11}^{-1}B_{12}, \quad B_{22.1} = B_{22} - B_{21}B_{11}^{-1}B_{12},$$

$$P = E_\xi[(\mathbf{u}^T B_{11}^{-2}\mathbf{u})(\mathbf{v} - \Delta^T\mathbf{u})(\mathbf{v} - \Delta^T\mathbf{u})^T] - \tau_2(trB_{11}^{-1})B_{22.1}.$$

Note that

$$E_\xi[\mathbf{u}^T B_{11}^{-2}\mathbf{u}] = trB_{11}^{-1}, \quad E_\xi[(\mathbf{v} - \Delta^T\mathbf{u})(\mathbf{v} - \Delta^T\mathbf{u})^T)^T] = B_{22.1}.$$

A straightforward expansion of the terms in (2.11) gives the following result.

Theorem 3.1 *Make assumptions A1) - A12). Define* $\boldsymbol{\theta}(\boldsymbol{\alpha}), \sigma(\boldsymbol{\alpha})$ *by (2.9) and (2.10), where* $f = f_{\boldsymbol{\alpha}}$ *is given by (1.9), (3.1) and (3.2). Then:*

i) $\boldsymbol{\theta}(\boldsymbol{\alpha}) = \Delta\boldsymbol{\alpha} + O(\|\boldsymbol{\alpha}\|^3)$.

ii) $\sigma(\boldsymbol{\alpha}) = \sigma(\mathbf{0}) + \frac{1}{2}\tau_0\sigma(\mathbf{0})\boldsymbol{\alpha}^T B_{22.1}\boldsymbol{\alpha} + O(\|\boldsymbol{\alpha}\|^4)$.

iii) Ignoring terms which are $O(\|\boldsymbol{\alpha}\|^4)$,

$$[D^{-1}CD^{-T}]_{12} = 0.$$

Thus $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ *and* $\sqrt{n}(\widehat{\sigma} - \sigma)$ *are asymptotically uncorrelated, hence asymptotically independent.*

Remark 3.2: It is not obvious that the approximation to $[D^{-1}CD^{-T}]_{11}$, in Theorem 3.1. iii), yields a positive definite matrix when terms of $O(\|\boldsymbol{\alpha}\|^4)$ are neglected. In order that the approximation to $\mathbf{a}^T[D^{-1}CD^{-T}]_{11}\mathbf{a}$ be positive for all non-null vectors $\mathbf{a}$, we require

$$\begin{aligned}\tau_1\tau_2 &< \inf_{\mathbf{a}} \frac{\mathbf{a}^T\{B_{11}^{-1} + \tau_1 E[(\boldsymbol{\alpha}^T(\mathbf{v} - \Delta^T\boldsymbol{\alpha}))^2 B_{11}^{-1}\mathbf{u}\mathbf{u}^T B_{11}^{-1}]\}\mathbf{a}}{\mathbf{a}^T\{(\boldsymbol{\alpha}^T B_{22.1}\boldsymbol{\alpha})B_{11}^{-1}\}\mathbf{a}} \\ &= ch_{\min} \frac{\{I_p + \tau_1 E[(\boldsymbol{\alpha}^T(\mathbf{v} - \Delta^T\mathbf{u}))^2 B_{11}^{-1/2}\mathbf{u}\mathbf{u}^T B_{11}^{-1/2}]\}}{\boldsymbol{\alpha}^T B_{22.1}\boldsymbol{\alpha}}.\end{aligned}$$

(Here, $ch_{\min}$ denotes the minimum characteristic root). Hence it suffices for positive definiteness if $\tau_1 \geq 0$ and $\tau_1\tau_2 < \inf_{\|\boldsymbol{\alpha}\|\leq\eta}(\boldsymbol{\alpha}^T B_{22.1}\boldsymbol{\alpha})^{-1}$, i.e.

$$\tau_1 \geq 0, \quad \eta^2\tau_1\tau_2 ch_{\max} B_{22.1} < 1. \tag{3.6}$$

Remark 3.3: A possible approach to this problem is to consider arbitrary contamination within shrinking neighbourhoods. Replace η by $\eta/\sqrt{n}$ in (1.6) and (3.2). In Theorem 3.1 take $q = 1$. Then, ignoring terms which are $o(n^{-1})$, (3.5) gives

$$\begin{aligned}\max_{\alpha} \mathcal{L}(f_\alpha, \xi) &= n^{-1}[V_{\psi,G} tr B_{11}^{-1} + \eta^2(1 + \|B_{11}^{-1}\int \mathbf{u}(\mathbf{x})v(\mathbf{x})d\xi(\mathbf{x})\|^2)] \\ &:= \mathcal{L}_o(v, \xi).\end{aligned}$$

One can now consider the problem

$$P) : \min_{\xi} \max_{v} \mathcal{L}_o(v, \xi)$$

where the max is taken over all functions $v(\mathbf{x})$, of $L_2(S)$-norm 1, satisfying 3.1).

If least squares estimation is employed, then the ignored $o(n^{-1})$ term in fact vanishes identically, and $\mathcal{L}_o(v, \xi)$ is exact. For an arbitrary M-estimate, $\mathcal{L}_o(v, \xi)$ depends on (ψ, χ) only through $V_{\psi,G}$, which is constant with respect to v and ξ. Thus, the solutions to problem $P)$ for general M-estimates are identical to those in the least squares case, except that the error variance σ^2 is replaced by $V_{\psi,G}$. See Huber (1981) and Wiens (1990, 1991c) for details and special cases.

LSE, the remainder term in (3.8) vanishes, and the exact optimality of B-uniform designs is essentially the result in the Appendix of Box and Draper (1959).

The bias-optimality of B-uniform designs may be maintained even if $G(\varepsilon)$ is non-symmetric, if at least $\ell(0, \sigma(\mathbf{0})) = 0$ (Fisher consistency). In this case Theorem 3.1 i) holds, but with

$$\Delta = \{B_{11} - \tau_4 E_\xi[\mathbf{u}]E_\xi[\mathbf{u}^T]\}^{-1}\{B_{12} - \tau_4 E_\xi[\mathbf{u}]E_\xi[\mathbf{v}^T]\},$$
$$\tau_4 = \frac{\ell_2(0,\sigma(\mathbf{0}))m_1(0,\sigma(\mathbf{0}))}{\ell_1(0,\sigma(\mathbf{0}))m_2(0,\sigma(\mathbf{0}))}.$$

If $\mathbf{u}(\mathbf{x})$ contains a constant element, then $E_\xi[\mathbf{v}^T]$ is a row of B_{12}, so that $\Delta = 0$ if ξ is B-uniform. As at (3.8), bias-optimality follows.

Under conditions considerably more stringent than B-uniformity, we can obtain designs which are bias-optimal to higher orders in $\|\boldsymbol{\alpha}\|$. Again assuming symmetry of G, we may extend Theorem 3.1 i) to

$$\boldsymbol{\theta}(\boldsymbol{\alpha}) = \Delta\boldsymbol{\alpha} + \tau_5 E_\xi[(\boldsymbol{\alpha}^T(\mathbf{v} - \Delta^T\mathbf{u}))^3 B_{11}^{-1}\mathbf{u}]/6 + O(\|\boldsymbol{\alpha}\|^5),$$
$$\tau_5 = \ell_{111}(0,\sigma(\mathbf{0}))/\ell_1(0,\sigma(\mathbf{0})).$$

To this order, a design is bias optimal if it is B-uniform and if, as well, $E_\xi[(\boldsymbol{\alpha}^T(\mathbf{v} - \Delta^T\mathbf{u}))^3\mathbf{u}] = \mathbf{0}$ for all $\boldsymbol{\alpha}$, i.e.

$$E_\xi[v_i v_j v_k u_\ell] = 0; \quad i,j,k \le q, \ \ell \le p.$$

These conditions are attainable, but very restrictive. For instance, in the case of simple linear regression with quadratic contamination, i.e.

$$u(x) = (1, x\sqrt{12})^T, \ v(x) = \sqrt{180}(x^2 - \frac{1}{12}), \ -\frac{1}{2} \le x \le \frac{1}{2} \tag{3.9}$$

the requirements become

$$E_\xi[X] = 1/12, \quad E_\xi[(X^2 - \frac{1}{12})^3] = 0.$$

Such a design places an inordinate amount of mass near 0.

where

$$\Sigma_\lambda = \underset{\lambda}{cov}\ [\underset{s}{vec}\ \mathbf{XX}^T],\ \mathbf{j} = \underset{s}{vec}\ I_p,\ (\mu_1(\lambda)/p)\mathbf{j} = E_\lambda[\underset{s}{vec}\ \mathbf{XX}^T].$$

It turns out that the designs which minimize the maximum of $\mathcal{L}(f_{\boldsymbol{\alpha}}, \xi)$ have no power to detect interaction or curvature in the response - see subsection 4.3.2 below. We thus seek to minimax $\mathcal{L}(f_{\boldsymbol{\alpha}}, \xi)$, subject to a lower bound on the power of Wald's test of the hypothesis that $\boldsymbol{\alpha} = \mathbf{0}$. This test is carried out by fitting the full model $E[y] = \boldsymbol{\theta}^T \mathbf{u}(\mathbf{x}) + \boldsymbol{\alpha}^T \mathbf{v}(\mathbf{x})$, and then computing the test statistic

$$T = (n\widehat{\boldsymbol{\alpha}}^T B_{22.1}\widehat{\boldsymbol{\alpha}})/(\widehat{V}_{\psi,G}).$$

where $\widehat{V}_{\psi,G}$ is a consistent estimate of $V_{\psi,G}$. When ψ depends on $\mathbf{x}$ only through the residual, Wald's test is identical to the τ-test of Hampel et al (1986). See Huber (1981, ch. 7), Street, Carroll and Ruppert (1988), Wiens (1991b) for computational details.

For the asymptotics of the testing theory we must work in shrinking sub-neighbourhoods of (1.6). Under alternatives $K : \boldsymbol{\alpha} = \boldsymbol{\beta}/\sqrt{n}$, the asymptotic distribution of T is $\chi^2_q(\lambda^2)$, where the non-centrality parameter is

$$\lambda^2 = (\boldsymbol{\beta}^T B_{22.1}\boldsymbol{\beta})/(V_{\psi,G}).$$

On the sphere $\|\boldsymbol{\beta}\|/\sqrt{n} = \eta$, we have

$$\min \lambda^2 = \left(\frac{n\eta^2}{V_{\psi,G}}\right) ch_{min} B_{22.1}.$$

Since

$$\max_{\|\boldsymbol{\alpha}\| \le \eta} \mathcal{L}(f_{\boldsymbol{\alpha}}, \xi) = n^{-1} V_{\psi,G} tr B_{11}^{-1} + \eta^2 (1 + ch_{max}(B_{21} B_{11}^{-2} B_{12} + n^{-1} V_{\psi,G} \tau_1 P)), \qquad (4.3)$$

our problem is to find ξ to minimize (4.3), subject to

$$ch_{min} B_{22.1} \ge \delta^2_{min}, \qquad (4.4)$$

for fixed δ^2_{min}.

B2) holds. If $p = 2$, B3) is satisfied if

$$\sum x_i^2 = 1, \ \sum x_i^4 = \frac{3}{p+2}.$$

We then find $x_1 = \frac{1}{2}(2+\sqrt{2})^{1/2} = .9239, x_2 = \frac{1}{2}(2-\sqrt{2})^{1/2} = .3827$. If $p \geq 3$ we require as well

$$\sum x_i^6 = \frac{5}{(p+2)(p+4)}.$$

For $p = 3$ we find

$$x_1 = .86625, \ x_2 = .26664, \ x_3 = .42252.$$

For $p > 3$, some x_i may be arbitrarily set equal to 0, or to other x_j, thus reducing the required number of points.

The optimal distributions of Z, derived below, lead to *continuous* designs, in that the product nt need not even be an integer, let along a multiple of the number of points, required to be placed on $\|\mathbf{x}\| = r\sqrt{s}$, for rotatability. Some approximations are therefore necessary. Perhaps the simplest, and most practical, way around the problem is to employ a *randomized* design. Note that if $\mathbf{X}$ is a p-vector of i.i.d. normals, then $r\sqrt{s}\mathbf{X}/\|\mathbf{X}\|$ has all mass on the sphere of radius $r\sqrt{s}$, and satisfies B2) and B3). This follows from the facts that $\mathbf{X}/\|\mathbf{X}\|$ is distributed independently of $\|\mathbf{X}\|$, and $\mathbf{X}$ satisfies B2) and B3). Thus, to construct a randomized design for given n, s and t:

i) Obtain a value n_1 of $N_1 \sim bin\ (n, t)$.

ii) Place $n - n_1$ of the design points at $\mathbf{0}$.

iii) Generate n_1 values $\mathbf{x}_i$ of $\mathbf{X} \sim N(\mathbf{0}, I_p)$, and place the remaining n_1 design points at $r\sqrt{s}\mathbf{x}_i/\|\mathbf{x}_i\|$, $1 \leq i \leq n_1$.

In this case, the *expected* maximum loss is minimized by the optimal distribution of Z.

For further information on rotatable designs, see Herzberg (1988), and references cited therein.

and

$$\overline{\nu}_0(\mu_1(\xi),\delta) = \frac{(\mu_1(\xi)-\mu_1(\lambda))^2}{\sigma_\lambda^2}$$
$$+ n^{-1}V_{\psi,G}\tau_1\frac{\sigma^2(\mu_1(\xi),\delta)}{\sigma_\lambda^2}\left\{(1-\tau_2)\left(1+\frac{p\mu_1(\lambda)}{\mu_1(\xi)}\right)+\frac{p\mu_1(\lambda)}{\mu_1(\xi)}\left(\frac{\sigma^2(\mu_1(\xi),\delta)}{\mu_1^2(\xi)}-1\right)\right\},$$
$$\overline{\nu}_1(\mu_1(\xi),\delta) = n^{-1}V_{\psi,G}\tau_1\frac{\mu_2(\mu_1(\xi),\delta)}{\mu_2(\lambda)}\left\{(1-\tau_2)\left(1+\frac{p\mu_1(\lambda)}{\mu_1(\xi)}\right)+\frac{p\mu_1(\lambda)}{\mu_1(\xi)}\left(\frac{\mu_2(\mu_1(\xi),\delta)}{\mu_1^2(\xi)}-1\right)\right\}.$$

Theorem 4.1 *Make assumptions A1) - A12), B1) - B3). For the model defined by (4.2), the maximum loss (4.3) is minimized, subject to (4.4), by the design ξ_* described as follows.*

i)

$$P_{\xi_*}(Z=0) = 1-t,\ P_{\xi_*}(Z=\frac{\mu_1(\xi_*)}{t}) = t \tag{4.11}$$

where

$$t = \mu_1^2(\xi_*)/\mu_2(\mu_1(\xi_*),\delta_*), \tag{4.12}$$

$$(\mu_1(\xi_*),\delta_*) = \underset{J}{argmin}\ \gamma(\mu_1(\xi),\delta). \tag{4.13}$$

ii) On the sphere $Z=\mu_1(\xi_*)/t$, the conditional distribution of $\mathbf{X}$ satisfies B2) and B3). Furthermore:

iii) When $\overline{\nu}(\mu_1(\xi_*),\delta_*) = \overline{\nu}_0(\mu_1(\xi_*),\delta_*)$, the least favourable $f_{\boldsymbol{\alpha}}$ is

$$f_{\boldsymbol{\alpha}}^0(\mathbf{x}) = \frac{\eta}{\sigma_\lambda}(\sum_{i=1}^{p}X_i^2-\mu_1(\lambda)). \tag{4.14}$$

When $\overline{\nu}(\mu_1(\xi_*),\delta_*) = \overline{\nu}(\mu_1(\xi_*),\delta_*)$, a least favourable $f_{\boldsymbol{\alpha}}$, orthogonal to $f_{\boldsymbol{\alpha}}^0(\mathbf{x})$, is

$$f_{\boldsymbol{\alpha}}^1(\mathbf{x}) = \eta\left(\frac{2(p+2)}{(p-1)\mu_2(\lambda)}\right)^{1/2}\sum\sum_{1\le i<j\le p}X_iX_j. \tag{4.15}$$

iv) For any design matrix satisfying the rotatability conditions B2) and B3), and for any $\mathbf{x}_0$, the asymptotic variance of $\sqrt{n}\widehat{y}(\mathbf{x}_0)$ is as follows. Label the elements of $\boldsymbol{\alpha}$ as

$$\boldsymbol{\alpha}^T = (\alpha_{11},\alpha_{12},\ldots,\alpha_{1p},\alpha_{22},\alpha_{23},\ldots,\alpha_{2p},\ldots,\alpha_{pp})$$

For $p = 2$ the maximum characteristic root $\overline{\nu}$ in (4.3) tends to equal $\overline{\nu}_0$ for large n or small η, and then purely quadratic contamination, as at (4.14), is least favourable. For small n or large η, we have $\overline{\nu} = \overline{\nu}_1$ and contamination as at (4.15) is least favourable.

The optimal designs are quite insensitive to changes in c, and to the error distribution. It is tempting then to take $c = \infty$ and use a design whch is optimal for the LSE. However, even though the M-estimator tends to $\widehat{\boldsymbol{\theta}}_{LS}$ as $c \to \infty$, the optimal designs do not converge in an analogous fashion. The reason is that, for all positive, *finite* values of c, $\boldsymbol{\theta}_0$ cannot be estimated in a scale equivariant manner without also estimating scale. The bias in $\sigma(\boldsymbol{\alpha})$ then contributes components to $\mathcal{L}(f_{\boldsymbol{\alpha}}, \xi)$, through the covariance matrix of $\widehat{\boldsymbol{\theta}}$, which do not vanish as $c \to \infty$. When least squares estimation is employed, this covariance matrix depends only on the error variance, and not on $\sigma(\boldsymbol{\alpha})$. In this sense, the LSE represents a discontinuity point in the class of M-estimators.

We now consider some special cases in more detail.

4.3.1. $n \to \infty$: As $n \to \infty$, the contribution of $\mathcal{V}(f_{\boldsymbol{\alpha}}, \xi)$ to $\mathcal{L}(f_{\boldsymbol{\alpha}}, \xi)$ vanishes, and the problem becomes one of minimaxing $\mathcal{B}_1(f_{\boldsymbol{\alpha}}, \xi)$, as at (3.7). Ignoring terms of order $O(\|\boldsymbol{\alpha}\|^4)$, we have

$$\max_{\|\boldsymbol{\alpha}\| \leq \eta} \mathcal{B}_1(f_{\boldsymbol{\alpha}}, \xi) = \eta^2 \left(1 + \frac{(\mu_1(\xi) - \mu_1(\lambda))^2}{\sigma_\lambda^2}\right).$$

Note that this is independent of δ^2. It is minimized, subject to (4.4), by the design given by (4.11), with $\mu_1(\xi_*) = \mu(\lambda)$ (yielding B-uniformity) and

$$t = \frac{\mu_1^2(\lambda)}{\mu_2(\mu_1(\lambda), \delta)} = \begin{cases} \frac{\mu_1^2(\lambda)}{\delta^2 \mu_2(\lambda)}, & \delta^2 \leq 1, \\ \frac{\mu_1^2(\lambda)}{\delta^2 \sigma_\lambda^2 + \mu_1^2(\lambda)}, & \delta^2 \geq 1, \end{cases}$$

for chosen $\delta^2 = ch_{\min} B_{22.1}$.

4.3.2. $\delta_{\min} \to 0$: We have been unable to prove, but believe it to be true, that $\gamma(\mu_1(\xi), \delta)$ is always an increasing function of δ, so that $\delta_* = \delta_{\min}$. For such cases, which include all those in Table 1, $t \to 1$ as $\delta_{\min} \to 0$ and the limiting, "no-power" design places all mass at $Z = \mu_1(\xi_*) = \arg\min_{[0,r^2]} \gamma(\mu_1(\xi), 0)$.

non-centrality parameter λ^2 are given in Table 1. Where the error distribution is Cauchy, we have adjusted $\chi(0)$ so that $\sigma(\mathbf{0}) = 1$. For reference, a level .05 test of $\boldsymbol{\alpha} = \mathbf{0}$, on 3 d.f., has power of .5 when $\lambda^2 = 6$, and power of .98 when $\lambda^2 = 13$.

To choose c and η one might compare the relative efficiencies of ξ_*, optimal for fixed $\eta > 0$, with the ξ_0 of 4.3.4 above. When it is in fact the case that $\eta = 0$, the relative efficiency of ξ_0 is

$$e_- := \frac{\mathcal{L}(f_{\boldsymbol{\alpha}}, \xi_0)}{\mathcal{L}(f_{\boldsymbol{\alpha}}, \xi_*)}|_{\boldsymbol{\alpha}=\mathbf{0}} = (1 + p\frac{\mu_1(\lambda)}{\mu_U(\delta_{\min})})/(1 + p\frac{\mu_1(\lambda)}{\mu_1(\xi_*)}) < 1.$$

This may be viewed as a premium to be paid for protection against positive η. When $\eta > 0$ is the true value, a measure of the protection received for this premium is the relative efficiency, computed assuming maximal contamination for both designs,

$$e_+ := \frac{\max_{\|\boldsymbol{\alpha}\| \le \eta} \mathcal{L}(f_{\boldsymbol{\alpha}}, \xi_0)}{\max_{\|\boldsymbol{\alpha}\| \le \eta} \mathcal{L}(f_{\boldsymbol{\alpha}}, \xi_*)} = \frac{\gamma(\mu_U(\delta_{\min}), \delta_{\min})}{\gamma(\mu_*, \delta_*)} > 1.$$

The second equality above follows from the fact brought out in the proof of Theorem 4.1, that for any 2-point mass of the form (4.11) for some $\mu_1(\xi)$ and δ, the maximum loss is $\gamma(\mu_1(\xi), \delta)$. Values of e_- and e_+ are given in Table 1.

4.4 Proofs. The proof of Theorem 4.1 employs several intermediary results. We must first obtain the characteristic roots in (4.3) and (4.4).

Lemma 4.2 *i) The characteristic roots of $B_{22.1}$ are*

$$\delta_0^2 = \sigma_\xi^2/\sigma_\lambda^2, \text{ with multiplicity 1, and}$$

$$\delta_1^2 = \mu_2(\xi)/\mu_2(\lambda), \text{ with multiplicity } q-1.$$

ii) The characteristic roots of $B_{21}B_{11}^{-2}B_{12} + n^{-1}V_{\psi,G}\tau_1 P$ are

$$\nu_0 = \{(\mu_1(\xi) - \mu_1(\lambda))^2 + n^{-1}V_{\psi,G}\tau_1((1-\tau_2)(1 + p\frac{\mu_1(\lambda)}{\mu_1(\xi)})\sigma_\xi^2 + p\frac{\mu_1(\lambda)}{\mu_1^2(\xi)}\mu_3'(\xi)\}/\sigma_\lambda^2,$$

The Moore-Penrose inverse $G^+ = (G^T G)^{-1} G^T$ satisfies

$$G^+ \text{ vec } M = \underset{s}{\text{vec}}\, M \text{ for all symmetric } M : \; p \times p.$$

We have the identity

$$G^T G := D = \text{ diag } (1, 2, \ldots, 2; 1, 2, \ldots, 2; \ldots; 1, 2; 1) : \; q \times q.$$

There is a symmetric, orthogonal, permutation matrix $K : \; p^2 \times p^2$ defined uniquely by its action

$$K \text{ vec } M = \text{ vec } M^T \text{ for all } M : p \times p,$$

with the properties

$$GG^+ = \frac{1}{2}(I + K),$$

$$G^+(I + K)G^{+^T} = (\frac{1}{2}D)^{-1}.$$

Put

$$\mathbf{e} = \text{vec } I_p \, .$$

The following identities hold:

$$E_\xi[\text{ vec } \mathbf{x}\mathbf{x}^T] = \frac{\mu_1(\xi)}{p}\mathbf{e} \tag{4.18}$$

$$E_\xi[(\text{ vec } \mathbf{x}\mathbf{x}^T)(\text{ vec } \mathbf{x}\mathbf{x}^T)^T] = \frac{\mu_2(\xi)}{p(p+2)}(I + K + \mathbf{e}\mathbf{e}^T) \tag{4.19}$$

$$E_\xi[(\mathbf{x}^T\mathbf{x}) \text{ vec } \mathbf{x}\mathbf{x}^T] = \frac{\mu_2(\xi)}{p}\mathbf{e} \tag{4.20}$$

$$E_\xi[(\mathbf{x}^T\mathbf{x})(\text{ vec } \mathbf{x}\mathbf{x}^T)(\text{ vec } \mathbf{x}\mathbf{x}^T)^T] = \frac{\mu_3(\xi)}{p(p+2)}(I + K + \mathbf{e}\mathbf{e}^T)\, . \tag{4.21}$$

The derivation of (4.18) is straightforward. For derivations of the expectations in (4.19)-(4.21), not assuming B3), see Wiens (1991). The expressions above have been simplified considerably by virtue of this assumption.

If $\rho_1 + \frac{p}{2}\rho_2 = 0$, then $\nu = \nu_0$. If $\rho_1 = 0$, then $\nu = \nu_1$. This proves ii) in the statement of the Lemma. For iii), we note that if $\nu = \nu_0$, then from (4.23),

$$(I_q - \frac{1}{p}\mathbf{j}\mathbf{j}^T)\Sigma_\lambda^{-1/2}\boldsymbol{\alpha} = \mathbf{0},$$

whence $\Sigma_\lambda^{-1/2}\boldsymbol{\alpha}$ is proportional to $\mathbf{j}$. If $\nu = \nu_1$, then (4.23) requires $\boldsymbol{\alpha}$ to be orthogonal to $\Sigma_\lambda^{-1/2}\mathbf{j}$. □

The problem is now to minimize

$$\max_{\|\boldsymbol{\alpha}\| \le \eta} \mathcal{L}(f_{\boldsymbol{\alpha}}, \xi) = n^{-1}V_{\psi,G}\left(1 + p\frac{\mu_1(\lambda)}{\mu_1(\xi)}\right) + \eta^2(1 + \max(\nu_0, \nu_1)), \tag{4.24}$$

subject to

$$\min(\delta_0^2, \delta_1^2) = \delta^2 \ge \delta_{\min}^2 \,.$$

We do this first for *fixed* δ^2 *and* $\mu_1(\xi)$. This fixes the first two moments of Z, under ξ. From (4.24), we are then to minimize $\max(\nu_0, \nu_1)$. But from Lemma 4.2 ii), both ν_0 and ν_1 are minimized, for fixed $\mu_1(\xi), \mu_2(\xi)$ and $\sigma^2(\xi)$, by that ξ which has the smallest third moment.

Define

$$W = \frac{Z - \mu_1(\xi)}{r^2 - \mu_1(\xi)},\ a = \frac{\mu_1(\xi)}{r^2 - \mu_1(\xi)}\,.$$

Then $-a \le W \le 1$, $E[W] = 0$, and we seek to minimize $E[W^3]$, subject to $E[W^2]$ being fixed.

Lemma 4.3. *Let W be a random variable with support in $[-a, 1]$, $a > 0$.*

i) If $E[W] = 0$, then $E[W^2] \le a$.

ii) If $E[W] = 0$, and $E[W^2] = \sigma^2 \le a$, then

$$E[W^3] \ge \frac{\sigma^2}{a}(\sigma^2 - a^2), \tag{4.25}$$

with equality iff

$$P(W = -a) = \frac{\sigma^2}{\sigma^2 + a^2} = 1 - P(W = \frac{\sigma^2}{a})\,. \tag{4.26}$$

requiring $(\mu_1(\xi), \delta) \in J$.

If $p \geq 2$, then by Lemma 4.2 i) and (4.27), (4.28) we have

$$\delta_1^2 \leq r^2 \frac{\mu_1(\xi)}{\mu_2(\lambda)} \tag{4.29}$$

and $\delta_0^2 \leq g(\mu_1(\xi))$, i.e.

$$\mu_L(\delta_0) \leq \mu_1(\xi) \leq \mu_U(\delta_0). \tag{4.30}$$

Note that "$\delta_0^2 \leq \delta_1^2$" is equivalent to each inequality "$\delta_0\mu_1(\lambda) \leq \mu_1(\xi)$" and "$\delta_1\mu_1(\lambda) \leq \mu_1(\xi)$", so that

$$\delta_0^2 \leq \delta_1^2 \text{ iff } \delta\mu_1(\lambda) \leq \mu_1(\xi). \tag{4.31}$$

Now i) follows from (4.30) and (4.31), and ii) from (4.29) and (4.31). By ii), if $\delta = \delta_1$ then

$$\delta \leq \frac{r^2\mu_1(\lambda)}{\mu_2(\lambda)} = \frac{p+4}{p+2}. \tag{4.32}$$

It remains to establish this inequality when $\delta = \delta_0$. For this, it suffices to show that, when $\delta = \delta_0$, we have

$$\mu_L(\delta) \leq \delta\mu_1(\lambda). \tag{4.33}$$

Then, together with i), we will have that $\delta\mu_1(\lambda) \in [\mu_L(\delta), \mu_U(\delta)]$, so that $g(\delta\mu_1(\lambda)) \geq \delta^2$. This is (4.32).

Suppose that (4.33) fails. Then

$$\begin{aligned} 0 < g'(\delta\mu_1(\lambda)) &= \frac{g(\delta\mu_1(\lambda))}{\delta\mu_1(\lambda)} - \frac{\delta\mu_1(\lambda)}{\sigma_\lambda^2} \\ &< \delta\left(\frac{1}{\mu_1(\lambda)} - \frac{\mu_1(\lambda)}{\sigma_\lambda^2}\right) \\ &= -\frac{\delta(p+2)}{pr^2}\left(\frac{p(p+4)}{4} - 1\right) \\ &< 0, \end{aligned}$$

which is the desired contradiction. □

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TABLE 1 CONSTANTS FOR OPTIMAL DESIGNS

$p = 2,\ c = 1,\ \mu_1(\lambda) = .1592$

Gaussian errors: $\tau_0 = 1,\ \tau_1 = 1.094,\ \tau_2 = .0859,\ \sigma^2(\mathbf{0}) = 1,\ V_{\psi,G} = 1.1073$.

Cauchy errors: $\tau_0 = 1.2281,\ \tau_1 = .7977,\ \tau_2 = -.2784,\ \sigma^2(\mathbf{0}) = 1.2965,\ V_{\psi,G} = 2.6578$.

($V_{\psi,G}$, when estimate is adjusted for consistency at the Cauchy, is 2.5465)

n = 10:	$\eta = 0.1$:				$\eta = 1$:			
	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
$\delta_{\min} = \delta_*$	0.5000	0.5000	1.2000	1.2000	0.5000	0.5000	1.2000	1.2000
non-zero z/r^2	1.0000	0.9976	1.0000	1.0000	0.4477	0.4285	0.8000	0.8000
$p_{\xi_*}(Z = z)$	0.9787	0.97861	0.8606	0.8606	0.8822	0.8695	0.7500	0.7500
$\mu_1(\xi_*)$	0.3115	0.3108	0.2739	0.2739	0.1257	0.1186	0.1910	0.1910
e_-	1.0000	0.9988	1.0000	1.0000	0.5724	0.5488	0.8108	0.8108
e_+	1.0000	1.00000	1.0000	1.0000	2.5801	2.1197	1.5286	1.4423
$\min \lambda^2$	0.0226	0.0098	0.1300	0.0566	2.2578	0.9817	13.0050	5.6549
n = 100:	$\eta = 0.1$:				$\eta = 1$:			
	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
$\delta_{\min} = \delta_*$	0.5000	0.5000	1.2000	1.2000	0.5000	0.5000	1.2000	1.2000
non-zero z/r^2	0.6536	0.7452	0.8139	0.8845	0.5043	0.4882	0.9035	0.8594
$p_{\xi_*}(Z = z)$	0.9486	0.9610	0.7624	0.8108	0.9100	0.9032	0.5881	0.6499
$\mu_1(\xi_*)$	0.1974	0.2279	0.1975	0.2283	0.1461	0.1404	0.1691	0.1778
e_-	0.7738	0.8437	0.8279	0.9030	0.6359	0.6187	0.7501	0.7748
e_+	1.4725	1.1503	1.2162	1.0464	3.5673	3.3450	2.3535	2.1593
$\min \lambda^2$	0.2258	0.0982	1.3005	0.5655	22.5781	9.8175	130.0499	56.5487

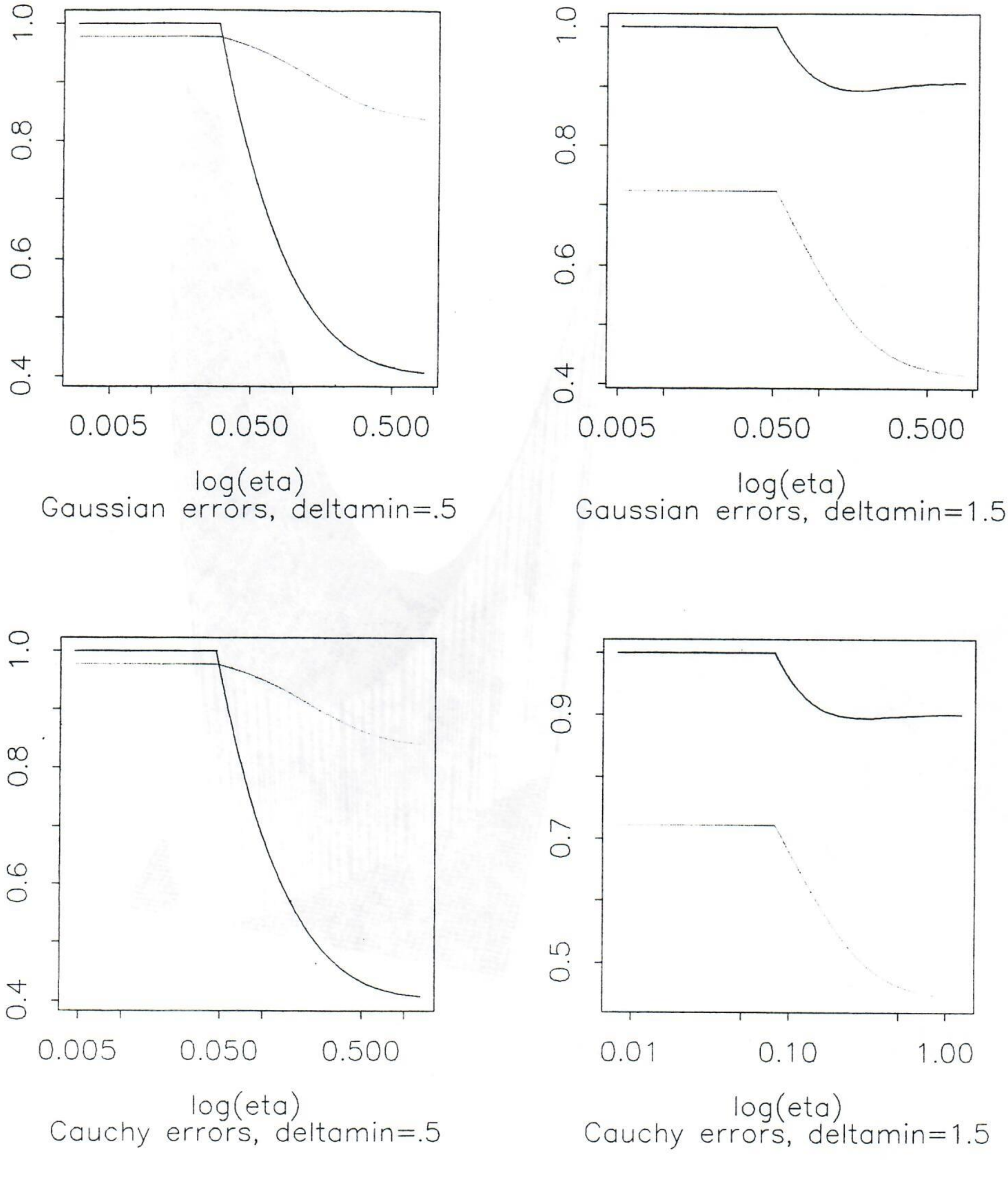

Fig. 1 Non-zero values of Z/rsqd (solid line) and corresponding mass (broken line) versus log(eta); p=1, c=1, n=30.

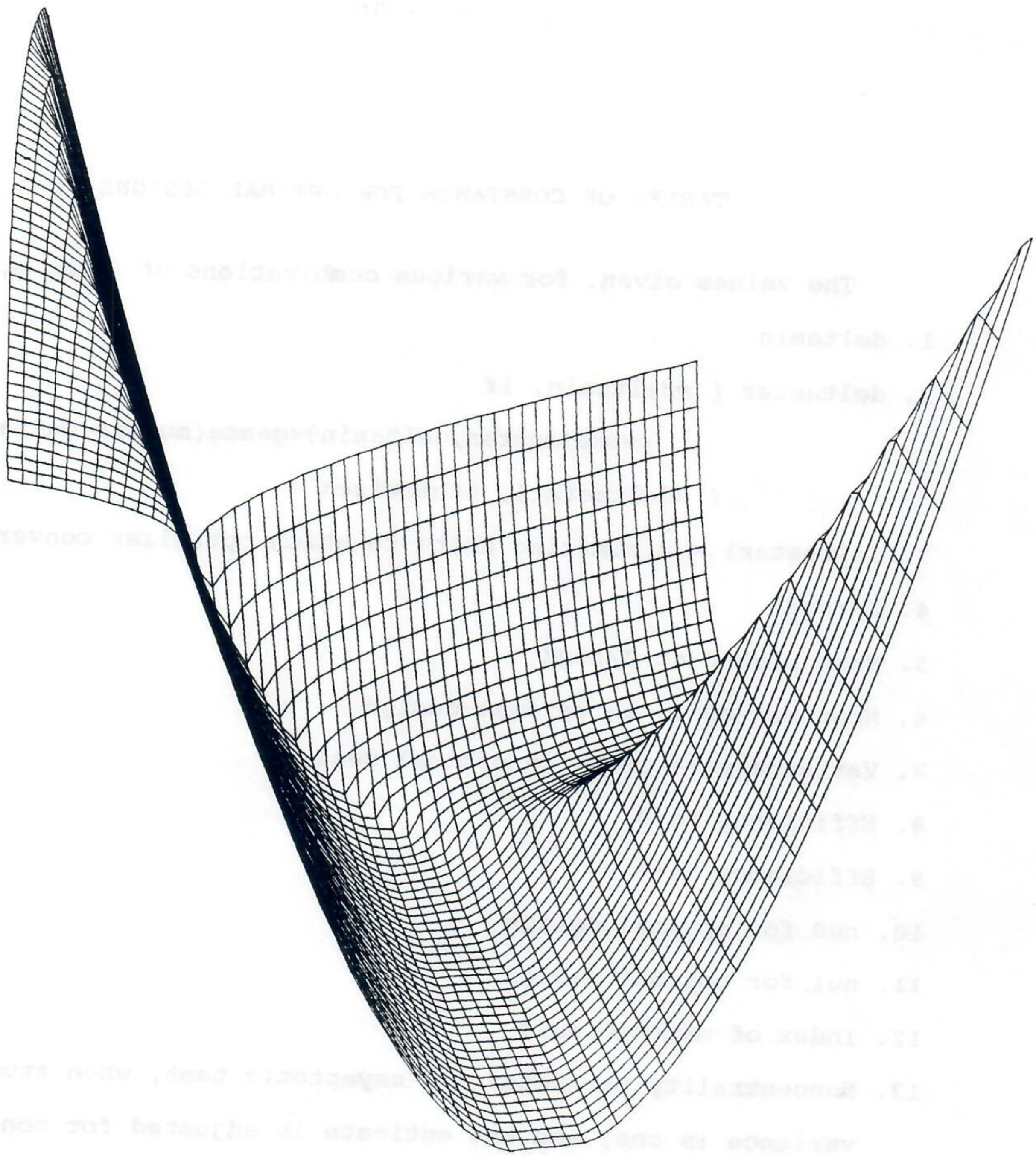

Fig.3 Smoothed version of Gamma(mu,delta) used by minimizer
Gaussian errors, p=2, c=1, n=100, eta=1, deltamin=.5

p= 2 mullam= 0.1592 c= 0.1

Gaussian errors:

tau0= 1 tau1= 1.0247 tau2= 0.0241
etabound= 3.6756 sigmasqdzero= 1 VpsiG= 1.4923
VpsiG, when estimate is adjusted for consistency at G= 1.4923

Cauchy errors:

tau0= 1.9881 tau1= 2.0108 tau2= 0.0153
etabound= 1.7184 sigmasqdzero= 1.0115 VpsiG= 2.3782
VpsiG, when estimate is adjusted for consistency at G= 2.3786

n= 10 :

eta= 0.01 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.5000000	0.5000000	1.200000	1.2000000
delta*	1.000e-01	1.000e-01	0.5000000	0.5000000	1.200000	1.2000000
delta*1	1.000e-01	1.010e-01	0.4999986	0.4758093	1.199996	1.1999965
Z/rsqd	1.000e+00	1.000e+00	1.0000000	1.0000000	1.000000	1.0000000
mass	9.992e-01	9.992e-01	0.9787136	0.9787136	0.860555	0.8605551
mu*	3.180e-01	3.180e-01	0.3115342	0.3115342	0.273923	0.2739232
var*	8.443e-05	8.443e-05	0.0021109	0.0021109	0.012159	0.0121585
e-	1.000e+00	1.000e+00	1.0000000	1.0000000	1.000000	1.0000000
e+	1.000e+00	1.000e+00	1.0000000	1.0000000	1.000000	1.0000000
nu0	2.991e+00	2.994e+00	2.7896215	2.8760335	1.891546	2.6099388
nu1	2.270e+00	7.123e+00	2.2229749	6.9769788	1.953265	6.1320106
numax	0.000e+00	1.000e+00	0.0000000	1.0000000	1.000000	1.0000000
ncp	6.701e-06	4.204e-06	0.0001675	0.0001051	0.000965	0.0006054
minloss	2.990e-01	4.767e-01	0.3020815	0.4816110	0.322933	0.5148906

eta= 0.1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	1.001e-01	9.994e-02	0.499984	0.499669	1.19974	1.19898
Z/rsqd	1.000e+00	1.000e+00	1.000000	1.000000	1.00000	1.00000
mass	9.992e-01	9.992e-01	0.978714	0.978714	0.86056	0.86056
mu*	3.180e-01	3.180e-01	0.311534	0.311534	0.27392	0.27392
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.01216	0.01216
e-	1.000e+00	1.000e+00	1.000000	1.000000	1.00000	1.00000
e+	1.000e+00	1.000e+00	1.000000	1.000000	1.00000	1.00000
nu0	2.991e+00	2.995e+00	2.789622	2.876034	1.89155	2.60994
nu1	2.270e+00	7.123e+00	2.222975	6.976979	1.95327	6.13201
numax	0.000e+00	1.000e+00	0.000000	1.000000	1.00000	1.00000
ncp	6.701e-04	4.204e-04	0.016753	0.010510	0.09650	0.06054
minloss	3.385e-01	5.571e-01	0.339599	0.560583	0.35217	0.58550

n= 100 :
eta= 0.01 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.200000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.200000
delta*1	1.000e-01	1.000e-01	0.500007	0.500020	1.20000	1.199999
Z/rsqd	1.000e+00	1.000e+00	0.999999	0.999998	1.00000	1.000000
mass	9.992e-01	9.992e-01	0.978714	0.978713	0.86056	0.860555
mu*	3.180e-01	3.180e-01	0.311534	0.311534	0.27392	0.273923
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.01216	0.012159
e-	1.000e+00	1.000e+00	1.000000	0.999999	1.00000	1.000000
e+	1.000e+00	1.000e+00	1.000000	0.999999	1.00000	1.000000
nu0	2.990e+00	2.990e+00	2.753955	2.762583	1.59315	1.664994
nu1	2.270e-01	7.123e-01	0.222297	0.697695	0.19533	0.613201
numax	0.000e+00	0.000e+00	0.000000	0.000000	0.00000	0.000000
ncp	6.701e-05	4.204e-05	0.001675	0.001051	0.00965	0.006054
minloss	3.026e-02	4.798e-02	0.030546	0.048458	0.03252	0.051684

eta= 0.1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	1.000e-01	1.000e-01	0.499998	0.499999	1.20000	1.20000
Z/rsqd	6.493e-01	7.026e-01	0.680216	0.731158	0.83371	0.87379
mass	9.980e-01	9.983e-01	0.952740	0.959379	0.77813	0.80468
mu*	2.063e-01	2.233e-01	0.206287	0.223281	0.20650	0.22381
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.01216	0.01216
e-	7.868e-01	8.249e-01	0.795011	0.833504	0.85071	0.89259
e+	1.379e+00	1.206e+00	1.337579	1.179156	1.14220	1.05941
nu0	2.632e-01	4.873e-01	0.267174	0.499953	0.30243	0.60846
nu1	6.826e-02	2.668e-01	0.074005	0.286037	0.10603	0.39295
numax	0.000e+00	0.000e+00	0.000000	0.000000	0.00000	0.00000
ncp	6.701e-03	4.204e-03	0.167528	0.105103	0.96496	0.60539
minloss	5.058e-02	7.256e-02	0.050621	0.072685	0.05095	0.07369

eta= 1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	9.996e-02	9.992e-02	0.499811	0.499394	1.20000	1.19979
Z/rsqd	4.554e-01	4.264e-01	0.499321	0.473616	0.89383	0.80000
mass	9.960e-01	9.954e-01	0.907970	0.896388	0.60080	0.75000
mu*	1.444e-01	1.351e-01	0.144312	0.135136	0.17094	0.19099
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.01941	0.01216
e-	6.243e-01	5.962e-01	0.630671	0.602523	0.75539	0.81077
e+	3.744e+00	3.515e+00	3.509798	3.279947	2.29882	2.01241
nu0	2.601e-02	6.893e-02	0.030324	0.082013	0.09936	0.24114
nu1	2.601e-02	6.893e-02	0.030324	0.082013	0.09923	0.28641
numax	1.000e+00	1.000e+00	1.000000	0.000000	0.00000	1.00000
ncp	6.701e-01	4.204e-01	16.752840	10.510290	96.49636	60.53927
minloss	1.074e+00	1.149e+00	1.078162	1.161813	1.14207	1.34983

eta= 0.1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
Z/rsqd	5.243e-01	5.362e-01	0.562609	0.573639	0.91231	0.88223
mass	9.970e-01	9.971e-01	0.929164	0.932075	0.57671	0.61670
mu*	1.664e-01	1.702e-01	0.166399	0.170192	0.16747	0.17318
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.02059	0.01864
e-	6.869e-01	6.971e-01	0.694059	0.704369	0.74537	0.76182
e+	2.976e+00	2.631e+00	2.811373	2.491841	1.98992	1.79069
nu0	6.227e-03	1.446e-02	0.006631	0.015757	0.01742	0.04769
nu1	3.806e-03	1.275e-02	0.004291	0.014303	0.01008	0.03089
numax	0.000e+00	0.000e+00	0.000000	0.000000	0.00000	0.00000
ncp	6.701e-02	4.204e-02	1.675284	1.051029	9.64964	6.05393
minloss	1.441e-02	1.697e-02	0.014413	0.016984	0.01450	0.01723

eta= 1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	5.000e-01	5.000e-01	1.20000	1.20000
delta*	1.000e-01	1.000e-01	5.000e-01	5.000e-01	1.20000	1.20000
delta*1	9.997e-02	9.999e-02	4.990e-01	4.993e-01	1.19988	1.19999
Z/rsqd	4.857e-01	4.742e-01	5.269e-01	5.164e-01	0.95178	0.93620
mass	9.965e-01	9.963e-01	9.183e-01	9.146e-01	0.52986	0.54765
mu*	1.541e-01	1.504e-01	1.540e-01	1.503e-01	0.16053	0.16320
var*	8.443e-05	8.443e-05	2.111e-03	2.111e-03	0.02286	0.02200
e-	6.525e-01	6.420e-01	6.593e-01	6.486e-01	0.72482	0.73279
e+	3.963e+00	3.930e+00	3.723e+00	3.690e+00	2.52630	2.47192
nu0	3.094e-03	9.157e-03	3.548e-03	1.056e-02	0.01148	0.03494
nu1	3.094e-03	9.157e-03	3.548e-03	1.056e-02	0.01043	0.03236
numax	1.000e+00	1.000e+00	1.000e+00	0.000e+00	0.00000	0.00000
ncp	6.701e+00	4.204e+00	1.675e+02	1.051e+02	964.96358	605.39271
minloss	1.008e+00	1.017e+00	1.008e+00	1.018e+00	1.01593	1.04196

p= 2 mullam= 0.1592 c= 1

Gaussian errors:

tau0= 1 tau1= 1.094 tau2= 0.0859
etabound= 1.8831 sigmasqdzero= 1 VpsiG= 1.1073
VpsiG, when estimate is adjusted for consistency at G= 1.1073

Cauchy errors:

tau0= 1.2281 tau1= 0.7977 tau2= -0.2784
etabound= 10000000000 sigmasqdzero= 1.2965 VpsiG= 2.6578
VpsiG, when estimate is adjusted for consistency at G= 2.5465

n= 10 :

eta= 0.01 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.5000000	5.000e-01	1.20000	1.2000000
delta*	1.000e-01	1.000e-01	0.5000000	5.000e-01	1.20000	1.2000000
delta*1	1.000e-01	1.010e-01	0.4999989	4.948e-01	1.20000	1.1999982
Z/rsqd	1.000e+00	1.000e+00	1.0000000	1.000e+00	1.00000	1.0000000
mass	9.992e-01	9.992e-01	0.9787136	9.787e-01	0.86056	0.8605551
mu*	3.180e-01	3.180e-01	0.3115342	3.115e-01	0.27392	0.2739232
var*	8.443e-05	8.443e-05	0.0021109	2.111e-03	0.01216	0.0121585
e-	1.000e+00	1.000e+00	1.0000000	1.000e+00	1.00000	1.0000000
e+	1.000e+00	1.000e+00	1.0000000	1.000e+00	1.00000	1.0000000
nu0	2.991e+00	2.993e+00	2.7776015	2.837e+00	1.79933	2.2193448
nu1	1.753e+00	3.531e+00	1.7165647	3.463e+00	1.50556	3.0660811
numax	0.000e+00	1.000e+00	0.0000000	1.000e+00	0.00000	1.0000000
ncp	9.031e-06	3.927e-06	0.0002258	9.817e-05	0.00130	0.0005655
minloss	2.219e-01	5.322e-01	0.2242395	5.378e-01	0.23968	0.5750267

eta= 0.1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	9.999e-02	1.076e-01	0.499931	0.526958	1.19990	1.19985
Z/rsqd	9.810e-01	9.999e-01	1.000000	0.997637	1.00000	1.00000
mass	9.991e-01	9.992e-01	0.978714	0.978610	0.86056	0.86056
mu*	3.120e-01	3.180e-01	0.311534	0.310765	0.27392	0.27392
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.01216	0.01216
e-	9.904e-01	9.999e-01	1.000000	0.998751	1.00000	1.00000
e+	1.000e+00	1.000e+00	1.000000	0.999226	1.00000	1.00000
nu0	2.768e+00	2.992e+00	2.777602	2.809714	1.79933	2.21934
nu1	1.660e+00	3.530e+00	1.716565	3.440625	1.50556	3.06608
numax	0.000e+00	1.000e+00	0.000000	1.000000	0.00000	1.00000
ncp	9.031e-04	3.927e-04	0.022578	0.009817	0.13005	0.05655
minloss	2.614e-01	5.771e-01	0.261638	0.582412	0.26739	0.61528

n= 100 :
eta= 0.01 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.5000000	1.20000	1.200000
delta*	1.000e-01	1.000e-01	0.500000	0.5000000	1.20000	1.200000
delta*1	1.000e-01	1.000e-01	0.500000	0.5000000	1.20000	1.200000
Z/rsqd	1.000e+00	1.000e+00	1.000000	1.0000059	1.20000	1.199999
mass	9.992e-01	9.992e-01	0.978714	0.9999995	1.00000	1.000000
mu*	3.180e-01	3.180e-01	0.311534	0.9787135	0.86056	0.860555
var*	8.443e-05	8.443e-05	0.002111	0.3115340	0.27392	0.273923
e-	1.000e+00	1.000e+00	1.000000	0.0021109	0.01216	0.012159
e+	1.000e+00	1.000e+00	1.000000	0.9999997	1.00000	1.000000
nu0	2.990e+00	2.990e+00	2.752760	0.9999997	1.00000	1.000000
nu1	1.753e-01	3.531e-01	0.171656	2.7587290	1.58393	1.625934
numax	0.000e+00	0.000e+00	0.000000	0.3462749	0.15056	0.306608
ncp	9.031e-05	3.927e-05	0.002258	0.0000000	0.00000	0.000000
minloss	2.255e-02	5.358e-02	0.022761	0.0009817	0.01300	0.005655
				0.0541093	0.02420	0.057725

eta= 0.1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	1.000e-01	1.000e-01	0.499999	0.499997	1.19999	1.19999
Z/rsqd	6.214e-01	7.172e-01	0.653616	0.745198	0.81393	0.88450
mass	9.978e-01	9.984e-01	0.948592	0.960960	0.76242	0.81083
mu*	1.974e-01	2.279e-01	0.197357	0.227943	0.19753	0.22829
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.01216	0.01216
e-	7.658e-01	8.349e-01	0.773767	0.843646	0.82790	0.90298
e+	1.526e+00	1.174e+00	1.472539	1.150272	1.21619	1.04639
nu0	1.729e-01	5.603e-01	0.175598	0.569871	0.20142	0.63733
nu1	4.565e-02	1.481e-01	0.049924	0.157755	0.07400	0.20998
numax	0.000e+00	0.000e+00	0.000000	0.000000	0.00000	0.00000
ncp	9.031e-03	3.927e-03	0.225781	0.098175	1.30050	0.56549
minloss	4.066e-02	7.930e-02	0.040687	0.079391	0.04093	0.08001

eta= 1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	9.998e-02	9.996e-02	0.499978	0.499689	1.20000	1.20000
Z/rsqd	4.613e-01	4.408e-01	0.504253	0.488240	0.90346	0.85941
mass	9.961e-01	9.957e-01	0.909959	0.903242	0.58806	0.64989
mu*	1.463e-01	1.397e-01	0.146056	0.140374	0.16911	0.17778
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.02003	0.01703
e-	6.299e-01	6.103e-01	0.635898	0.618730	0.75013	0.77480
e+	3.803e+00	3.571e+00	3.567264	3.345028	2.35351	2.15935
nu0	1.981e-02	4.523e-02	0.023071	0.052897	0.07616	0.16853
nu1	1.981e-02	4.523e-02	0.023071	0.052897	0.07616	0.15919
numax	0.000e+00	1.000e+00	1.000000	1.000000	1.00000	0.00000
ncp	9.031e-01	3.927e-01	22.578100	9.817477	130.04986	56.54867
minloss	1.055e+00	1.132e+00	1.058275	1.139742	1.10808	1.24269

eta= 0.1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.200000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.200000	1.20000
delta*1	1.000e-01	1.000e-01	0.499999	0.499998	1.200000	1.20000
Z/rsqd	5.189e-01	5.398e-01	0.557522	0.576957	0.923211	0.88336
mass	9.969e-01	9.971e-01	0.927756	0.932914	0.563170	0.61512
mu*	1.646e-01	1.713e-01	0.164644	0.171331	0.165497	0.17296
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.021245	0.01872
e-	6.821e-01	7.001e-01	0.689235	0.707433	0.739575	0.76119
e+	3.170e+00	2.544e+00	2.991130	2.411616	2.097687	1.75263
nu0	3.578e-03	1.758e-02	0.003844	0.018592	0.011963	0.03739
nu1	2.754e-03	7.308e-03	0.003121	0.008105	0.007748	0.01625
numax	0.000e+00	0.000e+00	0.000000	0.000000	0.000000	0.00000
ncp	9.031e-02	3.927e-02	2.257810	0.981748	13.004986	5.65487
minloss	1.328e-02	1.777e-02	0.013286	0.017781	0.013357	0.01792

eta= 1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	5.000e-01	0.500000	1.200e+00	1.20000
delta*	1.000e-01	1.000e-01	5.000e-01	0.500000	1.200e+00	1.20000
delta*1	1.000e-01	9.996e-02	4.991e-01	0.499882	1.200e+00	1.20000
Z/rsqd	4.878e-01	4.804e-01	5.288e-01	0.522635	9.537e-01	0.94629
mass	9.965e-01	9.964e-01	9.189e-01	0.916808	5.277e-01	0.53604
mu*	1.547e-01	1.523e-01	1.547e-01	0.152520	1.602e-01	0.16146
var*	8.443e-05	8.443e-05	2.111e-03	0.002111	2.297e-02	0.02256
e-	6.545e-01	6.476e-01	6.611e-01	0.654923	7.238e-01	0.72761
e+	3.970e+00	3.941e+00	3.730e+00	3.702575	2.535e+00	2.50026
nu0	2.317e-03	5.529e-03	2.663e-03	0.006291	8.517e-03	0.02093
nu1	2.317e-03	5.529e-03	2.663e-03	0.006291	7.951e-03	0.01713
numax	0.000e+00	1.000e+00	1.000e+00	1.000000	0.000e+00	0.00000
ncp	9.031e+00	3.927e+00	2.258e+02	98.174770	1.300e+03	565.48668
minloss	1.006e+00	1.014e+00	1.006e+00	1.014496	1.012e+00	1.02883

p= 2 mu1lam= 0.1592 c= 3

Gaussian errors:

tau0= 1 tau1= 1.0022 tau2= 0.0022
etabound= 12.192 sigmasqdzero= 1 VpsiG= 1.0004
VpsiG, when estimate is adjusted for consistency at G= 1.0004

Cauchy errors:

tau0= 0.1509 tau1= 0.0711 tau2= -1.0058
etabound= 10000000000 sigmasqdzero= 12.5967 VpsiG= 14.2444
VpsiG, when estimate is adjusted for consistency at G= 4.6785

n= 10 :

eta= 0.01 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.5000000	5.000e-01	1.200000	1.2000000
delta*	1.000e-01	1.000e-01	0.5000000	5.000e-01	1.200000	1.2000000
delta*1	1.000e-01	9.199e-02	0.4431790	4.679e-01	1.199999	1.1993067
Z/rsqd	1.000e+00	1.000e+00	1.0000000	1.000e+00	1.000000	1.0000000
mass	9.992e-01	9.992e-01	0.9787136	9.787e-01	0.860555	0.8605551
mu*	3.180e-01	3.180e-01	0.3115342	3.115e-01	0.273923	0.2739232
var*	8.443e-05	8.443e-05	0.0021109	2.111e-03	0.012159	0.0121585
e-	1.000e+00	1.000e+00	1.0000000	1.000e+00	1.000000	1.0000000
e+	1.000e+00	1.000e+00	1.0000000	1.000e+00	1.000000	1.0000000
nu0	2.991e+00	2.993e+00	2.7770870	2.829e+00	1.784215	2.1039631
nu1	1.501e+00	2.127e+00	1.4706227	2.090e+00	1.292996	1.8746252
numax	0.000e+00	0.000e+00	0.0000000	0.000e+00	0.000000	0.0000000
ncp	9.996e-06	2.137e-06	0.0002499	5.344e-05	0.001439	0.0003078
minloss	2.006e-01	2.850e+00	0.2026340	2.880e+00	0.216569	3.0800078

eta= 0.1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	9.999e-02	8.637e-02	0.499999	0.462710	1.19993	1.18364
Z/rsqd	9.569e-01	1.000e+00	0.977881	1.000000	1.00000	1.00000
mass	9.991e-01	9.992e-01	0.977717	0.978714	0.86056	0.86056
mu*	3.043e-01	3.180e-01	0.304333	0.311534	0.27392	0.27392
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.01216	0.01216
e-	9.779e-01	1.000e+00	0.988183	1.000000	1.00000	1.00000
e+	1.002e+00	1.000e+00	1.000495	1.000000	1.00000	1.00000
nu0	2.497e+00	2.993e+00	2.523367	2.828909	1.78422	2.10396
nu1	1.327e+00	2.127e+00	1.379958	2.090088	1.29300	1.87463
numax	0.000e+00	0.000e+00	0.000000	0.000000	0.00000	0.00000
ncp	9.996e-04	2.137e-04	0.024990	0.005344	0.14394	0.03078
minloss	2.396e-01	2.890e+00	0.239909	2.918150	0.24413	3.11074

n= 100 :
eta= 0.01 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.5000000	1.20000	1.200000
delta*	1.000e-01	1.000e-01	0.500000	0.5000000	1.20000	1.200000
delta*1	1.000e-01	1.000e-01	0.500000	0.4999999	1.20000	1.199999
Z/rsqd	1.000e+00	1.000e+00	1.000000	1.0000000	1.00000	1.000000
mass	9.992e-01	9.992e-01	0.978714	0.9787136	0.86056	0.860555
mu*	3.180e-01	3.180e-01	0.311534	0.3115342	0.27392	0.273923
var*	8.443e-05	8.443e-05	0.002111	0.0021109	0.01216	0.012159
e-	1.000e+00	1.000e+00	1.000000	1.0000000	1.00000	1.000000
e+	1.000e+00	1.000e+00	1.000000	1.0000000	1.00000	1.000000
nu0	2.990e+00	2.990e+00	2.752709	2.7578909	1.58242	1.614396
nu1	1.501e-01	2.127e-01	0.147062	0.2090088	0.12930	0.187463
numax	0.000e+00	0.000e+00	0.000000	0.0000000	0.00000	0.000000
ncp	9.996e-05	2.137e-05	0.002499	0.0005344	0.01439	0.003078
minloss	2.042e-02	2.854e-01	0.020601	0.2883619	0.02189	0.308231

eta= 0.1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	1.000e-01	1.000e-01	0.500000	0.500055	1.19999	1.19996
Z/rsqd	6.128e-01	1.000e+00	0.645566	0.999995	0.80811	1.00000
mass	9.978e-01	9.992e-01	0.947226	0.978713	0.75738	0.86056
mu*	1.946e-01	3.180e-01	0.194646	0.311533	0.19482	0.27392
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.01216	0.01216
e-	7.592e-01	1.000e+00	0.767171	0.999998	0.82086	1.00000
e+	1.583e+00	1.000e+00	1.524460	0.999999	1.24508	1.00000
nu0	1.492e-01	2.990e+00	0.152012	2.757835	0.17632	1.61440
nu1	3.884e-02	2.127e-01	0.042466	0.209006	0.06288	0.18746
numax	0.000e+00	0.000e+00	0.000000	0.000000	0.00000	0.00000
ncp	9.996e-03	2.137e-03	0.249899	0.053436	1.43942	0.30779
minloss	3.786e-02	3.249e-01	0.037884	0.325565	0.03811	0.33411

eta= 1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.20000	1.20000
delta*1	9.996e-02	9.998e-02	0.499868	0.499897	1.20000	1.19967
Z/rsqd	4.629e-01	5.719e-01	0.506202	0.604137	0.91248	0.85915
mass	9.961e-01	9.974e-01	0.910727	0.939226	0.57649	0.65029
mu*	1.468e-01	1.816e-01	0.146745	0.180616	0.16744	0.17784
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.02060	0.01701
e-	6.314e-01	7.268e-01	0.637949	0.731892	0.74527	0.77495
e+	3.819e+00	2.944e+00	3.583418	2.775025	2.36890	1.89661
nu0	1.826e-02	6.000e-02	0.021183	0.064479	0.07024	0.14343
nu1	1.826e-02	6.000e-02	0.021183	0.064479	0.06704	0.10556
numax	1.000e+00	1.000e+00	0.000000	1.000000	0.00000	0.00000
ncp	9.996e-01	2.137e-01	24.989946	5.343567	143.94209	30.77895
minloss	1.050e+00	1.452e+00	1.052887	1.457961	1.09926	1.54084

eta= 0.1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	0.500000	0.500000	1.200000	1.20000
delta*	1.000e-01	1.000e-01	0.500000	0.500000	1.200000	1.20000
delta*1	1.000e-01	1.000e-01	0.500000	0.499999	1.200000	1.20000
Z/rsqd	5.173e-01	6.447e-01	0.556078	0.675782	0.926503	0.83000
mass	9.969e-01	9.980e-01	0.927349	0.952085	0.559175	0.77534
mu*	1.641e-01	2.048e-01	0.164145	0.204801	0.164909	0.20484
var*	8.443e-05	8.443e-05	0.002111	0.002111	0.021439	0.01216
e-	6.807e-01	7.833e-01	0.687858	0.791528	0.737844	0.84656
e+	3.231e+00	1.400e+00	3.046788	1.357053	2.131187	1.15409
nu0	2.959e-03	2.467e-01	0.003239	0.247707	0.010610	0.25351
nu1	2.457e-03	7.742e-03	0.002775	0.008280	0.006785	0.01117
numax	0.000e+00	0.000e+00	0.000000	0.000000	0.000000	0.00000
ncp	9.996e-02	2.137e-02	2.498995	0.534357	14.394209	3.07789
minloss	1.297e-02	4.885e-02	0.012973	0.048861	0.013037	0.04891

eta= 1 :

	Normal	Cauchy	Normal	Cauchy	Normal	Cauchy
deltamin	1.000e-01	1.000e-01	5.000e-01	0.500000	1.200e+00	1.20000
delta*	1.000e-01	1.000e-01	5.000e-01	0.500000	1.200e+00	1.20000
delta*1	9.998e-02	1.000e-01	4.938e-01	0.501642	1.200e+00	1.20000
Z/rsqd	4.885e-01	5.219e-01	5.295e-01	0.559292	9.545e-01	0.94676
mass	9.965e-01	9.969e-01	9.192e-01	0.928251	5.269e-01	0.53551
mu*	1.549e-01	1.656e-01	1.549e-01	0.165255	1.601e-01	0.16138
var*	8.443e-05	8.443e-05	2.111e-03	0.002111	2.301e-02	0.02259
e-	6.551e-01	6.847e-01	6.619e-01	0.690919	7.235e-01	0.72737
e+	3.972e+00	3.840e+00	3.732e+00	3.609472	2.537e+00	2.45277
nu0	2.108e-03	4.968e-03	2.412e-03	0.005445	7.750e-03	0.01615
nu1	2.108e-03	4.968e-03	2.412e-03	0.005449	6.946e-03	0.01133
numax	1.000e+00	1.000e+00	0.000e+00	1.000000	0.000e+00	0.00000
ncp	9.996e+00	2.137e+00	2.499e+02	53.435672	1.439e+03	307.78947
minloss	1.005e+00	1.047e+00	1.005e+00	1.047130	1.011e+00	1.05849