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MINIMAX PROPERTIES OF M-, R-, AND L-ESTIMATORS OF LOCATION IN LEVY NEIGHBOURHOODS

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MINIMAX PROPERTIES OF M-, R-, AND L-ESTIMATORS OF LOCATION IN LÉVY NEIGHBOURHOODS[1]

(Abbreviated title: MINIMAX PROPERTIES OF ESTIMATORS)

by

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SUMMARY

We study the problem of robust estimation of a location parameter when the error distribution is assumed to lie in a Lévy neighbourhood of a symmetric distribution G: $\mathcal{P}_{\varepsilon,\delta}(G) = \{F | G(x-\delta)-\varepsilon \le F(x) \le G(x+\delta)+\varepsilon$ for all $x\}$. Under reasonably general conditions on G, Huber's (1964) theory is applied to obtain the distribution F_0 in $\mathcal{P}_{\varepsilon,\delta}(G)$ which minimizes Fisher information for location. Then it is shown that Huber's minimax property for M-estimators also holds for R-estimators in Lévy neighbourhoods. We also show that the minimax property for L-estimators fails to hold in Lévy neighbourhoods. The method of proof is to construct a sub-neighbourhood of distributions $\mathcal{F}_0$, with $F_0 \in \mathcal{F}_0 \subset \mathcal{P}_{\varepsilon,\delta}(G)$, such that the asymptotic variance of the L-estimator which is asymptotically efficient at F_0 is minimized over $\mathcal{F}_0$ at F_0.

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1. INTRODUCTION AND SUMMARY

Let $X_1, \ldots, X_n$ be a random sample from a distribution $F(x-\theta)$, where θ is an unknown location parameter. Huber (1964) presented a general theory of robust estimation of θ when F is assumed to be an unknown member of a specified convex, vaguely compact neighbourhood $\mathcal{F}$. In this paper we specialize Huber's theory to cases where $\mathcal{F} = \mathcal{P}_{\varepsilon,\delta}(G)$, a Lévy neighbourhood of a distribution G:

$$\mathcal{P}_{\varepsilon,\delta}(G) = \{F \mid G(x-\delta)-\varepsilon \le F(x) \le G(x+\delta)+\varepsilon \quad \text{for all} \quad x\} . \tag{1.1}$$

Here ε and δ are assumed to be fixed, with $0 \le \varepsilon < \frac{1}{2}$ and $\delta \ge 0$; G is a fixed distribution symmetric about 0.

If $\{T_n\}$ is a sequence of M-, R- or L-estimators of θ, then under mild regularity conditions, $n^{1/2}(T_n-\theta)$ converges in distribution to the normal law with mean 0 and variance $V(T,F)$, so that Huber's minimax variance theory applies.

The Lévy model, discussed in Chapter 2 of Huber (1981), is an important neighbourhood structure in robust estimation theory. It is based on the "Lévy distance", which metrizes the weak topology (Theorem 3.3 of Huber (1981)). From the point of view of practical application, the two-parameter family $\mathcal{P}_{\varepsilon,\delta}(G)$ allows wide flexibility in modelling the possible departures from G against which one wishes to protect. The choice $\delta = 0$ yields the important Kolmogorov neighbourhood model as a special case. The choice $\varepsilon = 0$ yields a Lévy band about G whose width at x decreases to 0 as x approaches $\pm\infty$; this may be a more realistic model than the fixed-width Kolmogorov band.

In Section 2 the distribution F_0 is found which minimizes Fisher information for location $I(F)$ ($= \int[(f')^2/f]dx$ if F has an absolutely continuous density f, $= \infty$ otherwise) over all F in $P_{\varepsilon,\delta}(G)$. This is carried out, for all choices of ε and δ, under regularity conditions on G which are only slightly stronger than strong unimodality, and which include the normal distribution $\Phi(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ and the logistic distribution as special cases. The minimum information F_0 is also found under some less restrictive conditions on G. The Cauchy and t-distributions are then included as special cases, although the solutions then require restrictions on the choice of ε and δ. The minimum information distributions obtained are, not surprisingly, qualitatively similar to solutions previously obtained in the special case of Kolmogorov neighbourhoods ($\delta = 0$) by Huber (1964) and Sacks and Ylvisaker(1972) when $G = \Phi$ and by Wiens (1986) for more general G.

Huber's (1964) theory yields the following minimax property for M-estimators: if T_0 denotes the M-estimator which is asymptotically efficient (i.e., $V(T_0,F_0) = 1/I(F_0)$) at the minimum information F_0 in $P_{\varepsilon,\delta}(G)$, then the minimum value of $\sup\{V(T,F): F \in P_{\varepsilon,\delta}(G)\}$ is $1/I(F_0)$, attained at T_0. In Section 3, we investigate whether the minimax property also holds for the R- and L-estimators that are asymptotically efficient at the minimum information F_0 in $P_{\varepsilon,\delta}(G)$. See Section 4.7 of Huber (1981) for further background for this problem. We show, under the conditions of Section 2, that the minimax property does hold for R-estimators, by generalizing Collins' (1983) proof for the special case $G = \Phi$ and $\delta = 0$.

We also show in Section 3, again under the conditions of Section 2, that the minimax property <u>fails</u> for L-estimators. This result was previously obtained in the special case $G = \Phi$, $\delta = 0$, $\varepsilon > .07$ by Sacks and Ylvisaker (1972). Their method was to show that there is an $F_1 \in P_{\varepsilon,0}(\Phi)$ for which $V(L_0,F_0) < V(L_0,F_1)$ where L_0 denote the L-estimator which is asymptotically efficient at F_0 . Unfortunately their method entails tedious numerical approximations that do not generalize easily. Our method is quite different and requires no approximations: we show that there is a subset $\mathcal{F}_0 \subset P_{\varepsilon,\delta}(G)$ over which $V(L_0,F)$ is non-constant and attains its minimum value at F_0 . The proof is based on a simple comparison of the influence curve of L_0 at F_0 , and at other $F \in \mathcal{F}_0$.

2. MINIMUM INFORMATION DISTRIBUTIONS IN $\mathcal{P}_{\varepsilon,\delta}$.

Throughout this paper, we shall assume:

A) The distribution function $G(x)$ is symmetric and fully stochastic, with an absolutely continuous density $g(x)$ and twice continuously differentiable (except possibly at zero) score function $\xi(x) = -g'(x)/g(x)$.

In Theorem 1 below, we shall as well assume:

B) The function $J(\xi)(x) = 2\xi'(x) - \xi^2(x)$ is strictly decreasing on $(0,\infty)$, and $\xi(0^+) \geq 0$.

In Theorem 2, we assume either B) or:

C) i) $\xi(x)$ is positive, and $x\xi(x)$ is strictly increasing, on $(0,\infty)$.

ii) $\xi(x)/x$ is non-increasing on $(0,\infty)$.

iii) $\xi(x)$ has no local minima in $(\bar{A},\infty)$, where $\bar{A}$ is defined by $\bar{A}\xi(\bar{A}) = 1$.

As at Lemma 1 of Wiens (1986), B) implies that ξ is positive and strictly increasing on $(0,\infty)$, so that g is strongly unimodal. This in turn implies C) i) and C) iii), and that

$$\rho(x) = \frac{g}{\xi}(x) - \bar{G}(x) , \tag{2.1}$$

where $\bar{G} = 1-G$, is strictly decreasing on $(0,\infty)$, with $\rho(\infty) = 0$.

Assumption C) i), together with the identity $(xg(x))' = g(x)(1-x\xi(x))$, implies that

$$\lim_{x\to\infty} xg(x) = 0 \quad , \quad \lim_{x\to 0} x\xi(x) < 1 \quad , \quad \lim_{x\to\infty} x\xi(x) > 1 ;$$

hence the existence of a unique point $\bar{A}$ as in C) iii). Now C) iii) implies that

$$\varepsilon_L(x) = \max(0, \rho(x)) \tag{2.2}$$

is non-increasing on $(\bar{A}, \infty)$. For if not, there must exist a point $x_0 > \bar{A}$ at which $\rho(x_0)$ is non-negative and increasing. By C) i), $\lim_{x\to\infty} \rho(x) = 0$, and so $\rho(x)$ must possess a local maximum. But since $\rho'(x) = -g(x)\xi'(x)/\xi^2(x)$, the local maxima of ρ are local minima of ξ , and this contradicts C) iii).

Examples of distributions satisfying B) are the logistic, normal, and more generally those with densities $g_k(x)$ proportional to $\exp(-|x|^k/k)$, $1 < k \le 2$. Some distributions satisfying C) but not B) are the "Student's" t , and those with densities $g_k(x)$, $k \le 1$.

The motivation behind Theorems 1 and 2 below is discussed in Wiens (1985, 1986), where they were proven for $\delta = 0$. Recall (Huber, 1981) that the necessary and sufficient condition for $F_0 \in \mathcal{P}_{\varepsilon,\delta}$ to minimize information there is

$$\int_{-\infty}^{\infty} J(\psi_0)(x)d(F-F_0)(x) \ge 0 \tag{2.3}$$

for all $F \in \mathcal{P}_{\varepsilon,\delta}$ with $I(F) < \infty$.

The proofs of Theorems 1 and 2 consist of verifying that the exhibited F_0 exist, belong to $\mathcal{P}_{\varepsilon,\delta}$, and satisfy (2.3). We make frequent use of the functions $\rho(x)$ and

$$\ell(\alpha) = \alpha\tan\alpha ,$$

$$k(\alpha) = \frac{\int_0^\alpha \cos^2 x dx}{\alpha\cos^2\alpha} = \frac{\ell'(\alpha)}{2\alpha} = \frac{\alpha^2 + \ell(\alpha) + \ell^2(\alpha)}{2\alpha^2} , \tag{2.4}$$

$$m(\alpha) = 2k(\alpha)\ell(\alpha) = (1+\ell(\alpha))\sec^2\alpha - 1 ,$$

$$h(\alpha) = \ell(\alpha)((1+m(\alpha)) = \ell(\alpha)\cos^2\alpha/(1+\ell(\alpha)) ,$$

$$\tau(x) = \frac{1}{2} - \bar{G}(x) - xg(x)k\circ\ell^{-1}(x\xi(x)/2) ;$$

for $\alpha \in [0,\pi/2]$, $x \in [0,\infty]$. Note that $\ell(\alpha)$ and $k(\alpha)$ are strictly increasing, to ∞ , with $\ell(0) = 0$, $k(0) = 1$, and

$$k'(\alpha) = \frac{1 - k(\alpha) + m(\alpha)}{\alpha} . \tag{2.5}$$

<u>THEOREM 1</u>. Under assumptions A) and B), there is $\varepsilon_*(G) > 0$ such that for $0 \le \varepsilon \le \varepsilon_*$ and $0 \le \delta \le \delta_*(\varepsilon)$, the minimum information $F_0 \in \mathcal{P}_{\varepsilon,\delta}(G)$ is described by

$$\psi_0(x) = -\psi_0(-x) = \{\lambda_1 \tan\frac{\lambda_1 x}{2} , \xi(x-\delta) , \lambda_2 = \xi(b-\delta)\}$$

$$f_0(x) = f_0(-x) = \left\{\frac{g(a-\delta)}{\cos^2\frac{\lambda_1 a}{2}}\cos^2\frac{\lambda_1 x}{2} , g(x-\delta), g(b-\delta)\exp(-\lambda_2(x-b))\right\}$$

on $[0,a]$, $[a,b]$, $[b,\infty)$ respectively. The constants $b \ge a \ge \delta$ and $\lambda_1 \ge 0$ are determined by

i) $F_0(a) = G(a-\delta) - \varepsilon$ ii) $F_0(\infty) = 1$ iii) $\psi_0(a^-) = \psi_0(a^+)$.

The curve $\delta_*(\varepsilon)$ is decreasing from ∞ at $\varepsilon = 0$ to 0 at $\varepsilon = \varepsilon_*$, and is defined by i) - iii) together with "b = a".

Proof. We first show that if solutions to i) - iii) are given, then $F_0 \in \mathcal{P}_{\varepsilon,\delta}$ and minimizes information there. Note that $f_0(x)$ and $\psi_0(x) = -f_0'(x)/f_0(x)$ are continuous, and that

$$J(\psi_0)(x) = \begin{cases} \lambda_1^2 & , x \in [0,a) \\ J(\xi)(x-\delta) & , x \in (a,b) \\ -\lambda_2^2 & , x \in (b,\infty) \end{cases} .$$

On $[b,\infty)$, $\psi_0(x) = \xi(b-\delta) \le \xi(x-\delta)$. Integrating this relationship over $[b,t]$, $t > b$, gives $f_0(t) \ge g(t-\delta) \ge g(t+\delta)$, which together with i) ensures that

$$G(x-\delta)-\varepsilon \le F_0(x) \le G(x+\delta)+\varepsilon \qquad (2.6)$$

for $x \ge a$. To establish (2.6) for $x \in [0,a]$, define

$$\eta_1(x) = \xi(x-\delta) - \psi_0(x) \quad , \quad \eta_2(x) = \xi(x+\delta) - \psi_0(x) \; ,$$

$$\gamma_1(x) = F_0(x) - [G(x-\delta)-\varepsilon] \quad , \quad \gamma_2(x) = G(x+\delta)+\varepsilon-F_0(x) \; ,$$

for $x \in [0,a]$. We require $\gamma_i(x) \ge 0$ on $[0,a]$; the behaviour of γ_i depends upon that of η_i .

By A) and iii), $\eta_1(0) \le 0$ and $\eta_1(a) = 0$. If $\eta_1(x) \le 0$ throughout $[0,a]$, then for $t \in [0,a]$, $1 \ge \exp(\int_t^a \eta_1(x)dx) = g(t-\delta)/f_0(t)$, contradicting (i). Thus η_1 has a zero $x_0 < a$, necessarily exceeding δ , at which it is increasing. We claim that $\eta_1(x) > 0$ on (x_0,a) , so that x_0 is the unique zero of η_1 . If not, then there is $x_1 \in (x_0,a)$ with $\eta_1(x_1) \le 0$, $\eta_1'(x_1) = 0, \eta''(x_1) \ge 0$. But, with $J'(\xi)(x) = d/dx\, J(\xi)(x)$,

$$2\eta_1'(x) = [J(\xi)(x-\delta) + \xi^2(x-\delta)] - [\lambda_1^2 + \psi_0^2(x)] \tag{2.7}$$

$$2\eta_1''(x) = J'(\xi)(x-\delta) + 2\xi(x-\delta)\eta_1'(x) + \eta_1(x)[\lambda_1^2 + \psi_0^2(x)] \tag{2.8}$$

and so

$$0 \le 2\eta_1''(x_1) = J'(\xi)(x_1-\delta) + \eta_1(x_1)[\lambda_1^2 + \psi_0^2(x)] \le J'(\xi)(x_1-\delta) ,$$

contradicting B). Thus x_0 is unique.

We have $\gamma_1(0) > 0$, $\gamma_1(a) = 0$. If γ_1 is ever negative, then there is a point t with $\gamma_1(t) < 0$, $\gamma_1'(t) = 0$, $\gamma_1''(t) \ge 0$, implying $f_0(t) = g(t-\delta)$ and $\eta_1(t) \ge 0$. Thus $t \ge x_0$ and $\eta_1(x) > 0$ on (t,a) , implying, by integration, that $f_0(t) < g(t-\delta)$, a contradiction.

Similarly, since $\gamma_2(x)$ is non-negative at 0 and at a , if it is ever negative then there is a point t with $\gamma_2(t) < 0$, $\gamma_2'(t) = 0$, $\gamma_2''(t) \ge 0$, implying $g(t+\delta) = f_0(t)$, $\eta_2(t) \le 0$. Since η_2 is non-negative at 0 and at a , there is a point v with $\eta_2(v) \le 0$, $\eta_2'(v) = 0$, $\eta_2''(v) \ge 0$. As before, this implies that $J'(\xi)(v+\delta) \ge 0$, contradicting B).

Thus $F_0 \in \mathcal{P}_{\varepsilon,\delta}$. We must check (2.3); as at Huber (1981, p. 89) it suffices to consider symmetric distributions F . For these, the left hand side of (2.3) is twice

$$\int_0^a \lambda_1^2 d(F-F_0) + \int_a^b J(\xi)(x-\delta)d(F-F_0) - \int_b^\infty \lambda_1^2 d(F-F_0) ,$$

which after an integration by parts becomes

$$(\lambda_1^2 - J(\xi)(a-\delta))(F-F_0)(a) + (\lambda_2^2 + J(\xi)(b-\delta))(F-F_0)(b) - \lambda_2^2(F-F_0)(\infty)$$
$$- \int_a^b (F-F_0)(x)J'(\xi)(x-\delta)dx \quad . \qquad (2.9)$$

Note that $F_0 \leq F$ on $[a,b]$, that $(F-F_0)(\infty) \leq 0$, and that by (iii),

$$\lambda_2^2 + J(\xi)(b-\delta) = 2\xi'(b-\delta) > 0 \ .$$

By (2.7), (2.8),

$$0 > 2\eta_1'(a) = J(\xi)(a-\delta) - \lambda_1^2 \ . \qquad (2.10)$$

Thus all terms in (2.9) are non-negative. It remains only to establish the existence of constants satisfying (i) - (iii). For this, set $\alpha = \lambda_1 a/2$, $A = a-\delta$, $B = b-\delta$. Then (i)-(iii) are equivalent to

I. $m(\alpha) = \xi(A)(\frac{1}{2} - \bar{G}(A) - \varepsilon)/g(A)$

II. $\delta = (2\ell(\alpha)/\xi(A)) - A$

III. $\varepsilon = \rho(B)$.

It is most convenient to solve for α, δ and B in terms of ε and A. Define $\varepsilon_0(A) = \min(\rho(A), \tau(A))$, $S = \{(A,\varepsilon) | 0 \leq A \leq \infty$, $0 \leq \varepsilon \leq \varepsilon_0(A)\}$, and let $(\varepsilon, A) \in S$. Since $\varepsilon \leq \tau(A) \leq \frac{1}{2} - \bar{G}(A)$, there is $\alpha \in [0, \frac{\pi}{2}]$ satisfying I. For this α, determine δ from II. Note that

$$\delta \geq 0 \iff \ell(\alpha) \geq \frac{A\xi(A)}{2} \iff m(\alpha) \geq m \circ \ell^{-1}(A\xi(A)/2) = A\xi(A) k \circ \ell^{-1}(A\xi(A)/2)$$
$$\iff \varepsilon \leq \tau(A)$$

and that $A \leq B$ iff $\varepsilon \leq \rho(A)$. Both inequalities then hold for

$(\varepsilon,A) \in S$. Put $B = \rho^{-1}(\varepsilon)$, to satisfy III.

We will show that $\frac{\partial\delta}{\partial A} > 0$ for all $\delta \geq 0$, and that $\tau'(A) > 0$. Then $\tau(A)$ increases from 0 at $A = 0$, $\rho(A)$ decreases to 0 at $A = \infty$, and so there is a unique point of intersection $(A_*,\varepsilon_*(A_*))$. The region S is then bounded above by $\tau(A)$ on $[0,A_*]$, and by $\rho(A)$ on $[A^*,\infty)$. For fixed $\varepsilon \in [0,\varepsilon_*]$, the solution is then valid for those $\delta = \delta(\varepsilon,A)$ between $\delta(\varepsilon,\tau^{-1}(\varepsilon)) = 0$ and $\delta(\varepsilon,\rho^{-1}(\varepsilon)) := \delta_*(\varepsilon)$ with $\delta_*(0) = \infty$, $\delta_*(\varepsilon_*) = 0$.

To show that $\frac{\partial\delta}{\partial A} > 0$ we first calculate, from II and I,

$$\xi^2(A)\delta'(A) = 2\ell'(\alpha)\alpha'(A)\xi(A) - 2\ell(\alpha)\xi'(A) - \xi^2(A) ,$$

$$m'(\alpha)\alpha'(A)\xi(A) = \xi^2(A)(1+m(\alpha)) + \xi'(A)m(\alpha) ,$$

whence

$$m'(\alpha)\xi^2(A)\delta'(A) = [2\ell'(\alpha)(1+m(\alpha))-m'(\alpha)]\xi^2(A)-4\ell^2(\alpha)k'(\alpha)\xi'(A) . \qquad (2.11)$$

In terms of the new variables, (2.10) is

$$2\xi'(A)\ell^2(\alpha) < \xi^2(A)(\alpha^2+\ell^2(\alpha)) ,$$

which in (2.11) gives

$$\begin{aligned} m'(\alpha)\delta'(A) &> 2\ell'(\alpha)[1+m(\alpha)-k(\alpha)] - 2k'(\alpha)[\alpha^2+\ell(\alpha)+\ell^2(\alpha)] \\ &= 4\alpha k(\alpha)[1+m(\alpha)-k(\alpha)-\alpha k'(\alpha)] \\ &= 0 , \end{aligned}$$

using (2.4), (2.5). Thus $\delta'(A) > 0$. Now put

$$\beta(A,\delta) = \frac{1}{2} - \bar{G}(A) - (\delta+A)g(A)k\circ\ell^{-1}((\delta+A)\xi(A)/2) .$$

By I and II, $\beta(A,\delta) = \varepsilon$, so that

$$\frac{\partial\delta}{\partial A} = -\frac{\partial\beta(A,\delta)}{\partial A}\Big/\frac{\partial\beta(A,\delta)}{\partial\delta} .$$

Since $\frac{\partial\beta(A,\delta)}{\partial\delta} < 0$, $\frac{\partial\delta}{\partial A} > 0$ implies that $\frac{\partial\beta(A,\delta)}{\partial A} > 0$ for all $\delta \geq 0$, and that in particular $\tau'(A) = \frac{\partial}{\partial A}\beta(A,0) > 0$. □

Remarks:

1. The curve $\delta_*(\varepsilon)$ is defined by placing $A = B$ in I - III. Equivalently,

$$2(1+m(\alpha)) = (\varepsilon+\bar{G}(A))^{-1} , \ \varepsilon = \rho(A) , \ \delta_* = (h(\alpha)/g(A))-A . \qquad (2.12)$$

Then (A_*,ε_*) are obtained by requiring as well $\delta_* = 0$. See Table I for values of $\delta_*(\varepsilon)$ in the case $G = \Phi$, the normal cumulative. Note that (A_*,ε_*) are the limiting values for the validity of the "small ε" solution in the Kolmogorov model $(\delta = 0)$, obtained by Huber (1964) if $G = \Phi$ (where $\varepsilon_* = .03033$, $A_* = 1.3496$) , and by Wiens (1986) under the conditions of Theorem 1.

2. For the neighbourhood $P_{\varepsilon,\varepsilon}(G)$, the relevant constants are obtained by placing $\delta = \varepsilon$ in I - III. The solution is valid for $0 \leq \varepsilon \leq \varepsilon_0$, where ε_0 is obtained by placing $\delta_* = \varepsilon$ in (2.12). See Table II for numerical values, if $G = \Phi$.

3. In the case $\varepsilon = 0$, the solution is valid for all $\delta \geq 0$, with $B = \infty$ and (α, A) determined from

$$m(\alpha) = \xi(A)(\tfrac{1}{2} - \overline{G}(A))/g(A) \quad , \quad \delta = (2\ell(\alpha)/\xi(A)) - A \; .$$

See Table III for numerical values, if $G = \Phi$.

4. If F_0 is as in Theorem 1, then

$$I(F_0) = 2 \int_A^B \xi'(x)dG(x) + 4\alpha\xi(A)g(A)/\sin 2\alpha \; .$$

<u>THEOREM 2</u>. Under assumptions A) and either B) or C), there exists $\varepsilon_*(G)$ such that for all $\varepsilon \in [\varepsilon_*, \frac{1}{2}]$ and all $\delta \in [0,\infty)$, the minimum information $F_0 \in \mathcal{P}_{\varepsilon,\delta}$ is described by

$$\psi_0(x) = -\psi_0(-x) = \left\{ \lambda_1 \tan \frac{\lambda_1 x}{2} \; , \; \lambda = \lambda_1 \tan \frac{\lambda_1 a}{2} \right\}$$

$$f_0(x) = f_0(-x) = \left\{ \frac{g(a-\delta)}{\cos^2 \frac{\lambda_1 a}{2}} \cos^2 \frac{\lambda_1 x}{2} \; , \; g(a-\delta)\exp(-\lambda(x-a)) \right.$$

on $[0,a]$, $[a,\infty)$ respectively. The constants are determined by i) $F_0(a) = G(a-\delta)-\varepsilon$, ii) $F_0(\infty) = 1$, and satisfy as well iii) $\psi_0(a) \leq \xi(a-\delta)$.

If B) holds, then this ε_* coincides with that of Theorem 1. The solution is then also valid for $0 \leq \varepsilon \leq \varepsilon_*$, $\delta_*(\varepsilon) \leq \delta < \infty$, where $\delta_*(\varepsilon)$ is as at (2.12).

Proof: We first establish the existence of constants satisfying i) - iii). With $\alpha = \lambda_1 a/2$, $A = a-\delta$, i) - iii) are equivalent to

1. $2(1+m(\alpha)) = (\varepsilon+\overline{G}(A))^{-1}$
2. $\delta = (h(\alpha)/g(A))-A$
3. $\varepsilon \geq \rho(A)$.

Let α, δ be defined, by 1. and 2., as functions of ε and A , for

$$\varepsilon_L(A) = \max(0,\rho(A)) \leq \varepsilon \leq \frac{1}{2} - \overline{G}(A) = \varepsilon_u(A) ,$$

so that 3. holds, with $\alpha \geq 0$. We require as well $\delta \geq 0$.

In Theorem 3 of Wiens (1985,1986), the present Theorem was proven in the special case $\delta = 0$. The existence of constants satisfying 1.-3., with $\delta = 0$, $A \geq \overline{A}$, was established for all sufficiently large ε under assumptions A) and C) i). Under assumption B), it was shown that such constants exist for all $\varepsilon \geq \varepsilon_*$, with ε_* coinciding with that of Theorem 1. Now note that the limiting solutions defined by (2.12) also satisfy 1.-3.

In order to show the existence of the constants for all $\delta \geq 0$ and sufficiently large ε , or $\delta \geq \delta_*(\varepsilon)$ of Theorem 1 (if $\varepsilon \leq \varepsilon_*(G)$ and B) holds), it then suffices to verify the following statement. If ε_0 , A_0 , $\delta_0 = \delta(\varepsilon_0, A_0)$, $\alpha_0 = \alpha(\varepsilon_0, A_0)$ is any solution to 1. and 2., with $\varepsilon_0 \in [\varepsilon_L(A_0), \varepsilon_u(A_0)]$ and $\varepsilon_L(A)$ non-increasing on (A_0, ∞) , then for every $A \geq A_0$, $\delta(\varepsilon_0, A)$ defined by 1. and 2. remains non-negative and tends to ∞ as $A \to \infty$.

If C) holds but not B), we take $A_0 \geq \overline{A}$. If B) holds, we may take any A_0 . In either case, $\varepsilon_L(A)$ is non-increasing and ε_0 remains in

$[\varepsilon_L(A),\varepsilon_u(A)]$ for $A \geq A_0$. Differentiating 1. and 2. gives

$$\frac{\partial\delta(\varepsilon,A)}{\partial A} = \frac{2\alpha^2\sec^2\alpha k'(\alpha)}{m'(\alpha)} + \frac{2\ell(\alpha)\xi(A)}{g(A)}[\varepsilon-\rho(A)] .$$

Since $\varepsilon_0 \geq \varepsilon_L(A) \geq \rho(A)$ for $A \geq A_0$, $\delta(\varepsilon_0,A)$ is an increasing, non-negative function of $A \geq A_0$. Furthermore,

$$\lim_{A\to\infty}\delta(\varepsilon_0,A) = \lim_{A\to\infty}\frac{h\circ m^{-1}\left(\frac{1}{2(\varepsilon_0+\bar{G}(A))}-1\right)-Ag(A)}{g(A)} = \infty .$$

This establishes the existence of F_0 . For symmetric $F \in \mathcal{P}_{\varepsilon,\delta}$, the left-hand side of (2.3) is $2(\lambda_1^2+\lambda^2)(F(a)-F_0(a))+2\lambda^2(1-F(\infty)) \geq 0$. It thus remains only to verify that $F_0 \in \mathcal{P}_{\varepsilon,\delta}$.

That (2.6) holds for $x \geq a$ follows from C) iii) in a manner very similar to that used for the case $\delta = 0$ in Wiens (1985). As there, (2.6) may hold on $[a,\infty)$ only for sufficiently large ε - the existence of solutions to 1. - 3. is not, in general, sufficient to guarantee (2.6). If B) holds however, (2.6) follows, for all x and all ε for which solutions to 1. - 3. exist, exactly as in Theorem 1.

To extend (2.6) to [0,a] if B) fails, note that C) ii) ensures that in (δ,a) , $\xi(x-\delta)$ remains above the line segment joining $(\delta,\xi(0))$ to $(a,\xi(a-\delta))$. This, and the convexity of ψ_0 , implies that $\xi(x-\delta)-\psi_0(x)$ has a unique root $x_0 \in (\delta,a)$. The first inequality in (2.6) now follows exactly as in Theorem 1, where B) was invoked to establish the uniqueness of x_0 . Similarly, C) ii) implies that $\xi(x+\delta)-\psi_0(x)$ has at most one root in $(0,a)$; the second inequality in (2.6) now follows as in Theorem 1.

□

COROLLARY 1. Under assumptions A) and B), the minimum information $F_0 \in \mathcal{P}_{\varepsilon,\delta}$ is as described by Theorem 1, for $0 \le \varepsilon \le \varepsilon_*$, $0 \le \delta \le \delta_*(\varepsilon)$; and by Theorem 2, for all remaining $\varepsilon \le \frac{1}{2}$, $\delta < \infty$.

Remarks:

5. Corollary 1 applies to the logistic and normal distributions, and more generally to those with densities $g_k(x)$, $1 < k \le 2$. For the Laplace distribution $(k = 1)$ it applies as well, with $\varepsilon_* = 0$. See Table II for numerical values for $\mathcal{P}_{\varepsilon,\varepsilon}(\Phi)$.

6. If F_0 is as in Theorem 2, then

$$I(F_0) = \frac{8\alpha g^2(A)(1+m(\alpha))}{\sin 2\alpha} .$$

3. MINIMAX PROPERTIES OF M-, R- AND L-ESTIMATORS

Consider the M-, R- and L-estimators of θ which are asymptotically efficient at the minimum information F_0 in $\mathcal{P}_{\varepsilon,\delta}(G)$. Using the definitions and notation of Chapter 3 of Huber (1981), the efficient M-, R- and L-estimators have score functions $\psi_0(x) = -f_0'(x)/f_0(x)$, $J_0(u) = \psi_0(F_0^{-1}(u))$ and $m_0(u) = \psi_0'[F_0^{-1}(u)]/I(F_0)$ respectively. It follows from general theory [see Huber (1964) or Section 4.6 of Huber (1981)] that the minimum possible value (among all M-estimators of θ) of the supremum of the asymptotic variance as F ranges over $\mathcal{P}_{\varepsilon,\delta}$ is $1/I(F_0)$, attained by ψ_0 at F_0. We now check whether this minimax property also holds for the R- and L-estimators which are asymptotically efficient at F_0. Throughout this section we shall use the usual formulas for the asymptotic variances of R- and L-estimators without discussion of the regularity conditions under which asymptotic normality holds. For such regularity conditions, see Huber (1981) or Serfling (1980).

Consider first the R-estimator with score function $J_0(u) = \psi_0(F_0^{-1}(u))$, $0 < u < 1$. Its asymptotic variance, under those distributions F in $\mathcal{P}_{\varepsilon,\delta}(G)$ with absolutely continuous density f, is given by

$$V(J_0,F) = \frac{\int J_0^2[F(x)]f(x)dx}{[-\int J_0[F(x)]f'(x)dx]^2} \tag{3.1}$$

<u>THEOREM 3</u>. Suppose that F_0 is the minimum information distribution in $\mathcal{P}_{\varepsilon,\delta}(G)$ which is either: (i) given by Theorem 1 under assumptions A) and B); or (ii) given by Theorem 2 under assumptions A) and either B) or C). Then, with J_0 defined by $J_0(u) = \psi_0[F_0^{-1}(u)]$, $V(J_0,F)$ is

maximized over $P_{\varepsilon,\delta}(G)$ at F_0, so that the minimax property holds.

Proof. First assume that A) and B) hold and consider case (i). Without loss of generality, it suffices to show that $V(J_0,F) \le V(J_0,F_0)$ for all F in $P_{\varepsilon,\delta}(G)$ that are strictly increasing with absolutely continuous density F. Since $\int J_0^2[F(x)]dF(x) \le \int J_0^2(t)dt = \int J_0^2[F_0(x)]dF_0(x)$, it suffices to show that

$$-\int_{-\infty}^{\infty} J_0^2[F(x)]f'(x)dx \ge -\int_{-\infty}^{\infty} J_0^2[F_0(x)]f_0'(x)dx \qquad (3.2)$$

for all such F. Proceeding as in the proof of Case A of the theorem of Collins (1983), the inequality (3.2) is equivalent to

$$\int_{-\infty}^{\infty} [\psi_0'f_0](q(x))p'(x)dx + \int_{-\infty}^{\infty} [\psi_0'f_0](q(x))[p'(x)]^2dx \ge 0 , \qquad (3.3)$$

where $q(x)$ and $p(x)$ are defined by

$$q(x) = x + p(x) = F_0^{-1}\circ F(x) .$$

Note that $p(x) \ge 0$ for all $x \in [a,b]$, since $F(x) \ge F_0(x) = G(x-\delta)-\varepsilon$ for all $x \in [a,b]$. Similarly note that $p(x) \le 0$ for all $x \in [-b,-a]$. Also note that

$$(\psi_0'f_0)(x) = \{0,(\xi'g)(x+\delta),C_0,(\xi'g)(x-\delta),0\}$$

on $(-\infty,b]$, $[-b,-a]$, $[-a,a]$, $[a,b]$, $[b,\infty)$ respectively, where $C_0 = \frac{1}{2}\lambda_1^2 g(a-\delta)/\cos^2(\lambda_1 a/2)$.

Since assumption B) implies that $\xi' > 0$, we have that $\int_{-\infty}^{\infty}(\psi_0' f_0)(q(x))[p'(x)]^2 dx \geq 0$. It remains to show that $\int_{-\infty}^{\infty}(\psi_0' f_0)(q(x))p'(x)dx \geq 0$. Set $x_a = q^{-1}(a)$, $x_b = q^{-1}(b)$, $x_{-a} = q^{-1}(-a)$ and $x_{-b} = q^{-1}(-b)$, and integrate by parts to obtain

$$\begin{aligned}\int_{-\infty}^{\infty}(\psi_0' f_0)(q(x))p'(x)dx &= (p(x_a)-p(x_{-a}))[C_0-(\xi' g)(a-\delta)] \\ &+ (p(x_b)-p(x_{-b}))(\xi' g)(b-\delta) \\ &- \int_{x_{-b}}^{x_{-a}} (\xi' g)'(q(x)+\delta)q'(x)p(x)dx \\ &- \int_{x_a}^{x_b} (\xi' g)'(q(x)-\delta)q'(x)p(x)dx \; . \end{aligned} \tag{3.4}$$

To show that (3.4) is non-negative, it suffices to show that

$$p(x) \geq 0 \quad \text{for all} \quad x \in [x_a, x_b] \tag{3.5}$$

$$p(x) \leq 0 \quad \text{for all} \quad x \in [x_{-b}, x_{-a}] \; , \tag{3.6}$$

$$C_0-(\xi' g)(a-\delta) \geq 0 \; , \tag{3.7}$$

$$-(\xi' g)'(y) \leq 0 \quad \text{for all} \quad y \in [-b+\delta, -a+\delta] \tag{3.8}$$

and $$-(\xi' g)'(y) \geq 0 \quad \text{for all} \quad y \in [a-\delta, b-\delta] \; . \tag{3.9}$$

But (3.5) and (3.6) follow exactly as in the proof in Collins (1983), and $C_0-(\xi' g)(a-\delta) = f_0(a)[\psi_0'(a-)-\xi'(a-\delta)] = \frac{1}{2} f_0(a)[\lambda_1^2-J(\xi)(a-\delta)] > 0$

by (2.10). Finally note that $(\xi'g)'(y) = -(\xi'g)'(-y) = \frac{1}{2}\, g(y)J'(\xi)(y) < 0$ for all $y > 0$ by assumption B) so that (3.8) and (3.9) hold, completing the proof in case (i).

The proof in case (ii) follows exactly as in the proof of case B) in Collins (1983). □

Now consider the L-estimator with score function $m_0(u) = \psi_0'[F_0^{-1}(u)]/I(F_0)$ for $u \in (0,1)$. The asymptotic variance of this estimator under $F \in \mathcal{P}_{\varepsilon,\delta}(G)$ is

$$V(m_0,F) = \int IC^2(x;F)dF ,$$

where the influence curve $IC(x;F)$ is given by

$$IC(x;F) = \int_{-\infty}^{x} m_0(F(y))dy - \int_{-\infty}^{\infty} [1-F(y)]m_0(F(y))dy .$$

Note that $V(m_0,F)$ can be written as $E_F IC^2(X;F) = \mathrm{var}_F IC(X;F)$, where X is a random variable with distribution F, since $E_F IC(X;F) = 0$ for all $F \in \mathcal{P}_{\varepsilon,\delta}(G)$. A useful alternative version is

$$IC(F^{-1}(u);F) = -\int_0^1 \{I[u \le t]-t\}m_0(t)dF^{-1}(t) .$$

If F is continuous, we then have

$$V(m_0,F) = \mathrm{Var}_U[IC(F^{-1}(U);F] ,$$

where U denotes a uniform random variable on [0,1]. Note also that $IC(F_0^{-1}(u);F_0) = \psi(F_0^{-1}(u))/I(F_0)$, with $V(m_0,F_0) = 1/I(F_0)$.

THEOREM 4. Suppose that F_0 is the minimum information distribution in $\mathcal{P}_{\varepsilon,\delta}(G)$ under the conditions of either Theorem 1 or Theorem 2. Then, with $m_0(u) = \psi_0'[F_0^{-1}(u)]/I(F_0)$, we have that

$$\sup\{V(m_0,F):\ F\in\mathcal{P}_{\varepsilon,\delta}(G)\} > V(m_0,F_0) ,$$

so that the minimax property fails for L-estimators.

Proof. Under the conditions of either Theorem 1 or Theorem 2, define a subset $\mathcal{F}_0$ of $\mathcal{P}_{\varepsilon,\delta}(G)$ as follows:

$$\mathcal{F}_0 = \{F \in \mathcal{P}_{\varepsilon,\delta}(G) \mid F \text{ is continuous and } F(x) = F_0(x) \text{ whenever } |x| \geq a\} .$$

We will show that $V(m_0,F)$ is non-constant on $\mathcal{F}_0$, and attains its *minimum* value there at F_0. The first part of the proof will be to show that, for all $F \in \mathcal{F}_0$,

$$\mathrm{Cov}[IC(F^{-1}(U);F), IC(F_0^{-1}(U);F_0)] = \mathrm{Var}[IC(F_0^{-1}(U);F_0)] . \tag{3.10}$$

Then (3.10) immediately implies that

$$V(m_0,F_0) = \rho_F^2 V(m_0,F) , \tag{3.11}$$

where ρ_F is the correlation between $IC(F_0^{-1}(U);F_0)$ and $IC(F^{-1}(U);F)$. The second part, completing the proof of the theorem, will be to show that $\rho_F^2 = 1$ for an $F \in \mathcal{F}_0$ if and only if $F \equiv F_0$.

To show that (3.10) holds for all F in $\mathcal{F}_0$, we first set

$$\eta(u,t) = -\{I[u \leq t] - t\}m_0(t) .$$

Then for $F \in \mathcal{F}_0$, we calculate that

$$[I(F_0)]^2\left\{\mathrm{Cov}[IC(F^{-1}(U);F),IC(F_0^{-1}(U);F_0)]-\mathrm{Var}[IC(F_0^{-1}(U);F_0)]\right\}$$

$$= I^2(F_0)\int_0^1 IC(F_0^{-1}(u);F_0)\left\{IC(F^{-1}(u);F)-IC(F_0^{-1}(u);F_0)\right\}du$$

$$= I^2(F_0)\int_0^1\int_0^1 \eta(u,s)dF_0^{-1}(s)\cdot\left\{\int_0^1 \eta(u,t)d(F^{-1}(t)-F_0^{-1}(t))\right\}du$$

$$= I^2(F_0)\int_0^1\left\{\int_0^1 \eta(u,t)\int_0^1 \eta(u,s)dF_0^{-1}(s)du\right\}d(F^{-1}(t)-F_0^{-1}(t))$$

$$= I(F_0)\int_0^1\left\{\int_0^1 \eta(u,t)\psi_0(F_0^{-1}(u))du\right\}d(F^{-1}(t)-F_0^{-1}(t)) . \tag{3.12}$$

The second-to-last step in (3.12) follows from Fubini's theorem. The change of variables $t = F_0(z)$ and $u = F(z)$ yields that (3.12) is equal to $\int_{-\infty}^{\infty} K(z)d(q_F^{-1}(z)-z)$, where $q_F^{-1}(z) = F^{-1}(F_0(z))$ and

$$K(z) = I(F_0)\int_{-\infty}^{\infty} \eta(F_0(x),F_0(z))\psi_0(x)dF_0(x)$$

$$= \int_{-\infty}^{\infty} f_0'(x)\psi_0'(z)[I(x\le z) - F_0(z)]dx$$

$$= \psi_0'(z)f_0(z) .$$

So to show that (3.10) holds for all $F \in \mathcal{F}_0$, it suffices to show that

$$\int_{-\infty}^{\infty} \psi_0'(z)f_0(z)d(q_F^{-1}(z)-z) = 0 \tag{3.13}$$

for all $F \in \mathcal{F}_0$. But (3.13) follows immediately from the fact that $\psi_0'(z)f_0(z) \equiv C_0$ for $|z| < a$ and that $q_F(z) \equiv z$ for $|z| \ge a$ by the definition of $\mathcal{F}_0$.

Now suppose that F is a member of $\mathcal{F}_0$ for which $\rho_F^2 = 1$. We need to show that this implies that $F \equiv F_0$. But $\rho_F^2 = 1$, together with $E[IC(F^{-1}(u);F)] = E[IC(F_0^{-1}(u); F_0] = 0$ implies that $IC(F^{-1}(u);F) = IC(F_0^{-1}(u);F_0)$ a.e. $u \in [0,1]$, or equivalently:

$$\int_0^1 tm_0(t)d(F^{-1}(t)-F_0^{-1}(t)) = \int_u^1 m_0(t)d(F^{-1}(t)-F_0^{-1}(t)) \quad \text{a.e.} \quad u \in [0,1] . \tag{3.14}$$

Letting $u \to 1$ shows that the right side of (3.14) is zero a.e. $u \in [0,1]$. The change of variable $t = F_0(z)$ then yields

$$\int_{F_0^{-1}(u)}^{\infty} \psi_0'(z)d(q_F^{-1}(z)-z) = 0 \quad \text{a.e.} \quad u \in [0,1] . \tag{3.15}$$

But since $q_F^{-1}(z) \equiv z$ for $|z| \geq a$ and $\psi_0'(z) > 0$ for $|z| < a$, (3.15) forces $q_F^{-1}(z) \equiv z$ for $|z| < a$. Thus $F(z) = F_0(z)$ for all z, and this completes the proof of the theorem. □

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TABLE I: VALUES OF $\delta_*(\varepsilon)$ FOR $P_{\varepsilon,\delta}(\Phi)$

ε	$\delta_*(\varepsilon)$	$A=B=\lambda_2$	α	$a=b$	λ_1	$1/I(F_0)$
0	∞	∞	$\pi/2$	∞	0	∞
.00002	5.2819	3.4577	1.4736	8.7396	.3372	9.3758
.0001	2.8295	3.0622	1.4152	5.8917	.4804	4.8134
.0005	1.3726	2.6332	1.3247	4.0058	.6614	2.7194
.001	.9587	2.4364	1.2723	3.3951	.7495	2.2104
.01	.1826	1.7241	1.0168	1.9067	1.0665	1.4139
.02	.0628	1.4921	.9071	1.5549	1.1668	1.3677
.03	.0016	1.3534	.8332	1.3550	1.2298	1.3823
.03033	.0000	1.3496	.8311	1.3496	1.2316	1.3832

TABLE II: VALUES OF THE CONSTANTS FOR $P_{\varepsilon,\varepsilon}(\Phi)$

$\varepsilon=\delta$	A	B	α	λ_1	$1/I(F_0)$
0	0	∞	0	$\sqrt{2}$	1
.0001	.4389	3.0622	.3055	1.3917	1.0028
.001	.7026	2.4364	.4776	1.3576	1.0205
.010	1.1428	1.7241	.7325	1.2707	1.1527
.020	1.3322	1.4921	.8277	1.2241	1.2918
$.02556=\varepsilon_0$	1.4084		.8634	1.2042	1.3725
.030	1.399		.8494	1.1890	1.4398
.050	1.386		.8005	1.1147	1.7800
.100	1.436		.7159	.9318	3.0015
.200	1.629		.5831	.6375	8.8573
.450	2.577		.2172	.1435	1062.1
.500	∞		0	0	∞

TABLE III: VALUES OF THE CONSTANTS FOR $P_{0,\delta}(\Phi)$

δ	A	α	λ_1	$1/I(F_0)$
0	0	0	$\sqrt{2}$	1
.0001	.3416	.2392	1.4004	1.0002
.001	.5465	.3774	1.3786	1.0015
.01	.8851	.5907	1.3199	1.0146
.05	1.2557	.7985	1.2231	1.0674
.10	1.4653	.9035	1.1544	1.1292
.25	1.8014	1.0529	1.0265	1.3053
.50	2.1065	1.1684	.8965	1.5926
1.00	2.4574	1.2785	.7396	2.1856
10.00	3.8737	1.5145	.2183	21.7091
∞	∞	$\pi/2$	0	∞

TABLE I: VALUES OF $\delta_*(\varepsilon)$ FOR $P_{\varepsilon,\delta}(\Phi)$

ε	$\delta_*(\varepsilon)$	$A=B=\lambda_2$	α	$a=b$	λ_1	$I(F_0)$	$1/I(F_0)$
0	∞	∞	$\pi/2$	∞	0	0	∞
.00002	5.2819	3.4577	1.4736	8.7396	.3372	.1067	9.3758
.00005	3.7370	3.2363	1.4436	6.9733	.4140	.1575	6.3507
.0001	2.8295	3.0622	1.4152	5.8917	.4804	.2078	4.8134
.0002	2.1042	2.8819	1.3809	4.9861	.5539	.2693	3.7130
.0005	1.3726	2.6332	1.3247	4.0058	.6614	.3677	2.7194
.001	.9587	2.4364	1.2723	3.3951	.7495	.4524	2.2104
.002	.6417	2.2317	1.2100	2.8735	.8422	.5406	1.8497
.005	.3414	1.9483	1.1093	2.2897	.9689	.6482	1.5427
.01	.1826	1.7241	1.0168	1.9067	1.0665	.7072	1.4139
.015	.1091	1.5892	.9550	1.6984	1.1246	.7264	1.3766
.02	.0628	1.4921	.9071	1.5549	1.1668	.7311	1.3677
.025	.0289	1.4160	.8675	1.4448	1.2008	.7291	1.3716
$.02556 = \varepsilon_0$	$=.02556$	1.4084	.8634	1.4339	1.2042	.7286	1.3725
.03	.0016	1.3534	.8332	1.3550	1.2298	.7234	1.3823
.03033	.0000	1.3496	.8311	1.3496	1.2316	.7230	1.3832

TABLE II: VALUES OF THE CONSTANTS FOR $P_{\varepsilon,\varepsilon}(\Phi)$

$\varepsilon=\delta$	A	B	α	a	b	λ_1	$I(F_0)$	$1/I(F_0)$
0	0	∞	0	0	∞	$\sqrt{2}$	1	1
.00001	.2759	3.6190	.1939	.2759	3.6190	1.4053	.9996	1.0004
.0001	.4389	3.0622	.3055	.4390	3.0623	1.3917	.9973	1.0028
.001	.7026	2.4364	.4776	.7036	2.4374	1.3576	.9799	1.0205
.01	1.1428	1.7241	.7325	1.1528	1.7341	1.2707	.8676	1.1527
.015	1.2494	1.5892	.7871	1.2644	1.6042	1.2451	.8186	1.2217
.020	1.3322	1.4921	.8277	1.3522	1.5121	1.2241	.7741	1.2918
.025	1.4013	1.4160	.8601	1.4263	1.4410	1.2061	.7330	1.3642
$.02556=\varepsilon_0$	1.4084		.8634	1.4339		1.2042	.7286	1.3725
.026	1.4073		.8619	1.4333		1.2027	.7251	1.3791
.030	1.399		.8494	1.4289		1.1890	.6946	1.4398
.040	1.388		.8228	1.4279		1.1525	.6243	1.6017
.050	1.386		.8005	1.4362		1.1147	.5618	1.7800
.075	1.403		.7546	1.4785		1.0208	.4325	2.3122
.100	1.436		.7159	1.5365		.9318	.3332	3.0015
.150	1.524		.6471	1.6742		.7731	.1963	5.0947
.200	1.629		.5831	1.8292		.6375	.1129	8.8573
.300	1.887		.4556	2.1866		.4167	.0317	31.550
.450	2.577		.2172	3.0273		.1435	.0009	1062.1
.500	∞		0	∞		0	0	∞

TABLE III: VALUES OF THE CONSTANTS FOR $P_{0,\delta}(\Phi)$

δ	A	α	a	λ_1	$I(F_0)$	$1/I(F_0)$
0	0	0	0	$\sqrt{2}$	1	1
.00001	.2147	.1512	.2147	1.4088	.99998	1.00002
.0001	.3416	.2392	.3417	1.4004	.9998	1.0002
.001	.5465	.3774	.5475	1.3786	.9985	1.0015
.01	.8851	.5907	.8951	1.3199	.9857	1.0146
.05	1.2557	.7985	1.3057	1.2231	.9369	1.0674
.10	1.4653	.9035	1.5653	1.1544	.8856	1.1292
.25	1.8014	1.0529	2.0514	1.0265	.7661	1.3053
.50	2.1065	1.1684	2.6065	.8965	.6279	1.5926
1.00	2.4574	1.2785	3.4574	.7396	.4575	2.1856
5.00	3.4222	1.4692	8.4223	.3489	.1145	8.7375
10.00	3.8737	1.5145	13.8737	.2183	.0461	21.7091
25.00	4.4679	1.5473	29.4679	.1050	.0011	91.9864
50.00	4.9009	1.5592	54.9009	.0568	.0032	312.1607
∞	∞	$\pi/2$	∞	0	0	∞