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Supplementary material for On the minimax robustness against correlation and heteroscedasticity of ordinary least squares among generalized least squares estimates of regression

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1. DETAILS RELATED TO REMARK 1

We take $\Phi(\Sigma) = \det(\Sigma)$ and consider the ratio $r(X, C_0)$ of the maximum loss of the GLS estimate to that of the OLS estimate. This is

$$r(X, C_0) = \frac{\max_{\mathcal{C}_M} \mathcal{L}(C \mid P_0)}{\max_{\mathcal{C}_M} \mathcal{L}(C \mid I_n)} = \frac{\mathcal{L}\left(\eta^2 I_n \mid P_0\right)}{\mathcal{L}\left(\eta^2 I_n \mid I_n\right)} = \frac{|X^{\mathrm{T}} C_0^{-2} X||X^{\mathrm{T}} X|}{|X^{\mathrm{T}} C_0^{-1} X|^2} = \frac{|Q^{\mathrm{T}} C_0^{-2} Q|}{|Q^{\mathrm{T}} C_0^{-1} Q|^2},$$

where the final term employs the QR-decomposition of X. Of course $r(X, C_0) \geq 1$; a direct proof follows from the observation that

$$Q^{\mathrm{T}} C_0^{-2} Q - \left(Q^{\mathrm{T}} C_0^{-1} Q\right)^2 = Q^{\mathrm{T}} C_0^{-1} \left\{I_n - QQ^{\mathrm{T}}\right\} C_0^{-1} Q \succeq 0.$$

For the equicorrelation model with $\rho > 0$ and $C_0 = (1-\rho)\left(I_n + \alpha 1_n 1_n^{\mathrm{T}}\right)$ for $\alpha = \rho/(1-\rho) \in (0, \infty)$, we calculate that

$$C_0^{-1} = (1-\rho)^{-1}\left(I_n - \beta 1_n 1_n^{\mathrm{T}}\right) \text{ for } \beta = \alpha/(1+n\alpha) \in \left(0, n^{-1}\right),$$
$$C_0^{-2} = (1-\rho)^{-2}\left(I_n - \gamma 1_n 1_n^{T}\right) \text{ for } \gamma = 2\beta - n\beta^2 \in \left(0, n^{-1}\right),$$

and then $r(X, C_0) = 1 + \left(S\beta^2(n-S) \,/\, (1-S\beta)^2\right)$, for $S = 1_n^{\mathrm{T}} QQ^{\mathrm{T}} 1_n \in [0, n]$.

For $0 < \varepsilon < n^{-1}$ suppose that ρ is sufficiently large that $\beta = n^{-1} - \varepsilon$, and that $S = n - \varepsilon$. Then

$$r(X, C_0) = 1 + \frac{\phi_n(\varepsilon)}{\varepsilon} \text{ for } \phi_n(\varepsilon) = \frac{(n-\varepsilon)\left(n^{-1}-\varepsilon\right)^2}{\left(n+n^{-1}-\varepsilon\right)^2} > 0,$$

and $r(X, C_0) \to \infty$ as $\varepsilon \to 0$ and $\phi_n(\varepsilon) \to 1/(1+n^2)$.

A simple example in which $S = n - \varepsilon$ is attained has $p = 1$, $X = x_{n\times 1}$. Then $Q = x/\|x\|$ and, with $\sigma_X^2 = \left(\sum x^2 - n\overline{x}^2\right)/n$, we have $S = n\overline{x}^2/(\sigma_X^2 + \overline{x}^2)$. Then $S = n - \varepsilon$ if the x_i are sufficiently concentrated that $\sigma_X^2 = \varepsilon\overline{x}^2/(n-\varepsilon)$. This example extends easily to arbitrary p.

2. THEORETICAL COMPLEMENTS FOR §4

PROOF OF THEOREM 1. Recall the constraints O and B:

$$\sum_{x\in\chi} f(x)\,\psi(x) = 0_{p\times 1}, \tag{S.1a}$$

$$\sum_{x\in\chi} \psi^2(x) \le \tau^2. \tag{S.1b}$$

Using (S.1a), the IMSPE $\sum_{x\in\chi} E[f^{\mathrm{T}}(x)\,\hat{\theta} - E\{Y(x)\}]^2$ decomposes as

$$\mathcal{I}(\xi, P \mid \psi, C) = \sum_{x\in\chi} f^{\mathrm{T}}(x)\,\mathrm{cov}\left(\hat{\theta} \mid C, P\right) f(x) + \sum_{x\in\chi} f^{\mathrm{T}}(x)\, b_{\psi,P} b_{\psi,P}^{\mathrm{T}} f(x) + \sum_{x\in\chi} \psi^2(x).$$

Here $b_{\psi,P} = E(\hat{\theta}) - \theta_0$ is the bias. Denote by ψ_X the $n\times 1$ vector consisting of the values of ψ corresponding to the rows of X, so that $b_{\psi,P} = \left(X^{\mathrm{T}}PX\right)^{-1} X^{\mathrm{T}}P\psi_X$. Recall that $X = JF$; note as well that $\psi_X = J\bar{\psi}$ for $\bar{\psi} = \left(\psi(x_1), ..., \psi(x_N)\right)^{\mathrm{T}}$. Then the above becomes

$$\begin{aligned} \mathcal{I}(\xi, P \mid \psi, C) &= \mathrm{tr}\left\{F\mathrm{cov}\left(\hat{\theta} \mid C, P\right) F^{\mathrm{T}}\right\} \\ &+ \bar{\psi}^{\mathrm{T}} J^{\mathrm{T}} PJF \left(F^{\mathrm{T}}J^{\mathrm{T}}PJF\right)^{-1} F^{\mathrm{T}}F \left(F^{\mathrm{T}}J^{\mathrm{T}}PJF\right)^{-1} F^{\mathrm{T}}J^{\mathrm{T}}PJ\bar{\psi} + \bar{\psi}^{\mathrm{T}}\bar{\psi}. \end{aligned} \tag{S.2}$$

As in §3, and taking $K = F^{\mathrm{T}}F$ in (iv) of that section, for $C \in \mathcal{C}_M$ the trace in (S.2) is maximized by $C = \eta^2 I_n$, with

$$\mathrm{tr}\left\{F\mathrm{cov}\left(\hat{\theta} \mid \eta^2 I_n, P\right) F^{\mathrm{T}}\right\} = \eta^2 \mathrm{tr}\left\{F\left(F^{\mathrm{T}}J^{\mathrm{T}}PJF\right)^{-1}\left(F^{\mathrm{T}}J^{\mathrm{T}}P^2JF\right)\left(F^{\mathrm{T}}J^{\mathrm{T}}PJF\right)^{-1}F^{\mathrm{T}}\right\}. \tag{S.3}$$

Extend the orthogonal basis for $\mathrm{col}(F)$ – formed by the columns of Q – by appending to Q the matrix $Q_* : N \times (N-p)$, whose columns form an orthogonal basis for the orthogonal complement $\mathrm{col}(F)^{\perp}$. Then $(Q\vdots Q_*) : N\times N$ is an orthogonal matrix and we have that $F = QR$ for a non-singular R. If the construction is carried out by the Gram-Schmidt method, then R is upper triangular.

Constraint (S.1a) dictates that $\bar{\psi}$ lie in $\mathrm{col}(Q_*)$. A maximizing ψ will satisfy the bound in constraint (S.1b) with equality, hence $\bar{\psi} = \tau Q_*\beta$ for some $\beta_{(N-p)\times 1}$ with unit norm. Combining these observations along with (S.2) and (S.3) yields that $\max_{\psi,C}\mathcal{I}(\xi, P \mid \psi, C)$ is given by

$$\begin{aligned} &\eta^2 \mathrm{tr}\left\{Q\left(Q^{\mathrm{T}}UQ\right)^{-1}\left(Q^{\mathrm{T}}VQ\right)\left(Q^{\mathrm{T}}UQ\right)^{-1}Q^{\mathrm{T}}\right\} \\ &+\tau^2 \max_{\|\beta\|=1}\left\{\beta^{\mathrm{T}}Q_*^{\mathrm{T}}UQ\left(Q^{\mathrm{T}}UQ\right)^{-1}Q^{\mathrm{T}}Q\left(Q^{\mathrm{T}}UQ\right)^{-1}Q^{\mathrm{T}}UQ_*\beta + 1\right\}. \end{aligned} \tag{S.4}$$

Here and below we use that $\mathrm{tr}AB = \mathrm{tr}BA$, and that such products have the same non-zero eigenvalues. Then (S.4) becomes $\left(\tau^2 + \eta^2\right)$ times $\mathcal{I}_\nu(\xi, P)$, given by

$$\begin{aligned} \mathcal{I}_\nu(\xi, P) &= (1-\nu)\,\mathrm{tr}\left\{\left(Q^{\mathrm{T}}UQ\right)^{-1}\left(Q^{\mathrm{T}}VQ\right)\left(Q^{\mathrm{T}}UQ\right)^{-1}\right\} \\ &+\nu\left\{\mathrm{ch}_{\max} Q_*^{\mathrm{T}}UQ\left(Q^{\mathrm{T}}UQ\right)^{-1}\cdot\left(Q^{\mathrm{T}}UQ\right)^{-1}Q^{\mathrm{T}}UQ_* + 1\right\}. \end{aligned} \tag{S.5}$$

The maximum eigenvalue (denoted above by $\mathrm{ch}_{\max}$) is also that of

$$\begin{aligned} \left(Q^{\mathrm{T}}UQ\right)^{-1}Q^{\mathrm{T}}UQ_*\cdot Q_*^{\mathrm{T}}UQ\left(Q^{\mathrm{T}}UQ\right)^{-1} &= \left(Q^{\mathrm{T}}UQ\right)^{-1}Q^{\mathrm{T}}U\left(I_N - QQ^{\mathrm{T}}\right)UQ\left(Q^{\mathrm{T}}UQ\right)^{-1} \\ &= \left(Q^{\mathrm{T}}UQ\right)^{-1}Q^{\mathrm{T}}U^2Q\left(Q^{\mathrm{T}}UQ\right)^{-1} - I_p; \end{aligned}$$

this in (S.5) gives (4) of Theorem 1. □

Theorem 2 below is used show that the experimenter can often design in such a way that $P = I_n$ is a minimax precision matrix, so that OLS is a minimax procedure. In particular, this holds if the design is uniform on its support. This is Remark 3 of the main article.

Recall the class $\mathcal{J}$ of indicator matrices. For $J \in \mathcal{J}$ let $J_+ : n \times q$ be the result of retaining only the non-zero columns of J, so that $JJ^{\mathrm{T}} = J_+ J_+^{\mathrm{T}}$, and $D_+ = J_+^{\mathrm{T}} J_+$ is the diagonal matrix containing the positive n_i. If the columns removed have labels $j_1, ..., j_{N-q}$ then let $Q_+ : q \times p$ be the result of removing these rows from Q, so that $JQ = J_+ Q_+$ and $Q^{\mathrm{T}} D Q = Q_+^{\mathrm{T}} D_+ Q_+$. Now define $\alpha = n/\mathrm{tr}\left(D_+^{-1}\right)$ and

$$P_0 = \alpha J_+ D_+^{-2} J_+^{\mathrm{T}}, \tag{S.6}$$

with $\mathrm{tr} P_0 = n$. Note that

$$\mathrm{rk}\left(P_0\right) = \mathrm{rk}\left(J_+ D_+^{-1}\right) = \mathrm{rk}\left(D_+^{-1} J_+^{\mathrm{T}} J_+ D_+^{-1}\right) = \mathrm{rk}\left(D_+^{-1}\right) = q,$$

so that P_0 is positive definite iff $q = n$. This is relevant in part (ii) of Theorem 2, where we deal with the possible rank deficiency of P_0 by introducing

$$P_\varepsilon = \left(P_0 + \varepsilon I_n\right) / \left(1 + \varepsilon\right); \tag{S.7}$$

for $\varepsilon > 0$, P_ε is positive definite with $\mathrm{tr}\left(P_\varepsilon\right) = n$.

THEOREM 2. *(i) Suppose that $q \leq N$ and the design is uniform on q points of χ, with $k \geq 1$ observations at each x_i. Then $n = kq$, $D_+ = kI_q$, and $P = I_n$ is a minimax precision matrix:*

$$\mathcal{I}_\nu\left(\xi, I_n\right) = \min_{P \succ 0} \mathcal{I}_\nu\left(\xi, P\right); \tag{S.8}$$

thus OLS *is minimax within the class of* GLS *methods. In particular this holds if $P_0 = I_n$, where P_0 is defined at (S.6).*
(ii) Suppose that a design ξ places mass on $q \leq N$ points of χ, that $P_0 \neq I_n$, and that neither of the following holds:

$$\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\left(Q_+^{\mathrm{T}} D_+^{-1} Q_+\right)\left(Q_+^{\mathrm{T}} Q_+\right)^{-1} = \left(Q_+^{\mathrm{T}} D_+ Q_+\right)^{-1}, \tag{S.9a}$$

$$ch_{\max}\left\{\left(Q_+^{\mathrm{T}} D_+ Q_+\right)^{-1}\left(Q_+^{\mathrm{T}} D_+^2 Q_+\right)\left(Q_+^{\mathrm{T}} D_+ Q_+\right)^{-1}\right\} = ch_{\max}\left\{\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\right\}. \tag{S.9b}$$

Then in particular D_+ is not a multiple of I_q and so the design is non-uniform. With P_ε as defined at (S.7), there is $\nu_0 \in (0, 1)$ for which, for each $\nu \in (\nu_0, 1]$, $\mathcal{I}_\nu\left(\xi, P_\varepsilon\right) < \mathcal{I}_\nu\left(\xi, I_n\right)$. Thus OLS *is not minimax for such (ξ, ν).*

Remark 4. The requirement of Theorem 2(ii) that (S.9a) and (S.9b) fail excludes more designs than those which are uniform on their supports, and is a condition on Q as well as on the design. For instance if $Q_+\left(Q_+^{\mathrm{T}} Q_+\right)^{-1/2} = A_{q \times p}$ is block-diagonal: $A = \oplus_{i=1}^m A_i$, where $A_i : q_i \times p_i$ ($\sum q_i = q, \sum p_i = p$) satisfies $A_i^{\mathrm{T}} A_i = I_{p_i}$, and if $D_+ = \oplus_{i=1}^m k_i I_{q_i}$, then

$$A^{\mathrm{T}} D_+^{-1} A = \left(A^{\mathrm{T}} D_+ A\right)^{-1}, \tag{S.10}$$

$$\left(A^{\mathrm{T}} D_+ A\right)^{-1} A^{\mathrm{T}} D_+^2 A \left(A^{\mathrm{T}} D_+ A\right)^{-1} = I_p. \tag{S.11}$$

Equation (S.10) gives (S.9a), and (S.11) asserts the equality of the two matrices in (S.9b), hence of their maximum eigenvalues. These equations are satisfied even though the design is non-uniform if the k_i are not all equal.

The proof of Theorem 2 follows that of the following preliminary result.

LEMMA 2. *(i) For a fixed design ξ and any $P \succ 0$, $\mathcal{I}_0(\xi, P) \geq \mathcal{I}_0(\xi, I_n)$ and $\mathcal{I}_1(\xi, P) \geq ch_{\max}\left\{\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\right\}$.*
If neither of the equations (S.9a), (S.9b) holds, then:
(ii) $\mathcal{I}_0(\xi, P_0) > \mathcal{I}_0(\xi, I_n)$ and $\mathcal{I}_1(\xi, I_n) > \mathcal{I}_1(\xi, P_0)$;
(iii) With P_ε as defined in Theorem 2(ii), and for sufficiently small $\varepsilon > 0$, $\Delta_0(\varepsilon) = \mathcal{I}_0(\xi, P_\varepsilon) - \mathcal{I}_0(\xi, I_n) > 0$ and $\Delta_1(\varepsilon) = \mathcal{I}_1(\xi, I_n) - \mathcal{I}_1(\xi, P_\varepsilon) > 0$.

PROOF OF LEMMA 2.

(i) From (4a), $\mathcal{I}_0(\xi, P) - \mathcal{I}_0(\xi, I_n)$ is the trace of

$$\begin{aligned} &\left(Q^{\mathrm{T}} J^{\mathrm{T}} P J Q\right)^{-1}\left(Q^{\mathrm{T}} J^{\mathrm{T}} P^2 J Q\right)\left(Q^{\mathrm{T}} J^{\mathrm{T}} P J Q\right)^{-1} - \left(Q^{\mathrm{T}} J^{\mathrm{T}} J Q\right)^{-1} \\ &= \left(Q^{\mathrm{T}} J^{\mathrm{T}} P J Q\right)^{-1} Q^{\mathrm{T}} J^{\mathrm{T}} P\left\{I_n - J Q\left(Q^{\mathrm{T}} J^{\mathrm{T}} J Q\right)^{-1} Q^{\mathrm{T}} J^{\mathrm{T}}\right\} P J Q\left(Q^{\mathrm{T}} J^{\mathrm{T}} P J Q\right)^{-1}, \end{aligned}$$

which is $\succeq 0$ with non-negative trace. For the second inequality first note that

$$\begin{aligned} &\left(Q^{\mathrm{T}} U Q\right)^{-1} Q^{\mathrm{T}} U^2 Q\left(Q^{\mathrm{T}} U Q\right)^{-1} - \left(Q_+^{\mathrm{T}} Q_+\right)^{-1} \\ &= \left(Q_+^{\mathrm{T}} J_+^{\mathrm{T}} P J_+ Q_+\right)^{-1} Q_+^{\mathrm{T}} J_+^{\mathrm{T}} P J_+ J_+^{\mathrm{T}} P J_+ Q_+\left(Q_+^{\mathrm{T}} J_+^{\mathrm{T}} P J_+ Q_+\right)^{-1} - \left(Q_+^{\mathrm{T}} Q_+\right)^{-1} \\ &= \left(Q_+^{\mathrm{T}} J_+^{\mathrm{T}} P J_+ Q_+\right)^{-1} Q_+^{\mathrm{T}} J_+^{\mathrm{T}} P J_+\left\{I_n - Q_+\left(Q_+^{\mathrm{T}} Q_+\right)^{-1} Q_+^{\mathrm{T}}\right\} J_+^{\mathrm{T}} P J_+ Q_+\left(Q_+^{\mathrm{T}} J_+^{\mathrm{T}} P J_+ Q_+\right)^{-1} \end{aligned}$$

is p.s.d., so that by Weyl's Monotonicity Theorem (p. 63 of Bhatia (1997)),

$$\mathcal{I}_1(\xi, P) = \mathrm{ch}_{\max}\left\{\left(Q^{\mathrm{T}} U Q\right)^{-1} Q^{\mathrm{T}} U^2 Q\left(Q^{\mathrm{T}} U Q\right)^{-1}\right\} \geq \mathrm{ch}_{\max}\left\{\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\right\}.$$

(ii) We use the following identities, which follow from (4a) and (4b):

$$\mathcal{I}_0(\xi, I_n) = \mathrm{tr}\left\{\left(Q_+^{\mathrm{T}} D_+ Q_+\right)^{-1}\right\} \tag{S.12a}$$

$$\mathcal{I}_1(\xi, I_n) = \mathrm{ch}_{\max}\left\{\left(Q_+^{\mathrm{T}} D_+ Q_+\right)^{-1}\left(Q_+^{\mathrm{T}} D_+^2 Q_+\right)\left(Q_+^{\mathrm{T}} D_+ Q_+\right)^{-1}\right\} \tag{S.12b}$$

$$\mathcal{I}_0(\xi, P_0) = \mathrm{tr}\left\{\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\left(Q_+^{\mathrm{T}} D_+^{-1} Q_+\right)\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\right\}, \tag{S.12c}$$

$$\mathcal{I}_1(\xi, P_0) = \mathrm{ch}_{\max}\left\{\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\right\}. \tag{S.12d}$$

To prove (ii) we show that if either inequality fails then one of (S.9a), (S.9b) holds – a contradiction. First note that

$$\begin{aligned} &\mathcal{I}_0(\xi, P_0) - \mathcal{I}_0(\xi, I_n) \\ &= \mathrm{tr}\left\{\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\left(Q_+^{\mathrm{T}} D_+^{-1} Q_+\right)\left(Q_+^{\mathrm{T}} Q_+\right)^{-1} - \left(Q_+^{\mathrm{T}} D_+ Q_+\right)^{-1}\right\} \qquad \text{(S.13)} \\ &= \mathrm{tr}\left\{\left(Q_+^{\mathrm{T}} Q_+\right)^{-1} Q_+^{\mathrm{T}} D_+^{-1/2}\left[I_q - D_+^{1/2} Q_+\left(Q_+^{\mathrm{T}} D_+ Q_+\right)^{-1} Q_+^{\mathrm{T}} D_+^{1/2}\right] D_+^{-1/2} Q_+\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\right\}, \end{aligned}$$

which is non-negative. If the first inequality fails, so that $\mathcal{I}_0(\xi, P_0) = \mathcal{I}_0(\xi, I_n)$, then the trace of the p.s.d. matrix at (S.13) is zero, hence all eigenvalues are zero and the matrix is the zero matrix. This is (S.9a).
That $\mathcal{I}_1(\xi, I_n) - \mathcal{I}_1(\xi, P_0) \geq 0$ is the first inequality in (i). If the second inequality of (ii) fails, then $\mathcal{I}_1(\xi, I_n) = \mathcal{I}_1(\xi, P_0)$ and their evaluations at (S.12b) and (S.12d) give (S.9b).
For (iii), that $\Delta_0(\varepsilon) > 0$ and $\Delta_1(\varepsilon) > 0$ for sufficiently small ε follow from the continuity of $\mathcal{I}_0(\xi, P_\varepsilon)$ and $\mathcal{I}_1(\xi, P_\varepsilon)$ as functions of ε: $\Delta_0(\varepsilon) \to \mathcal{I}_0(\xi, P_0) - \mathcal{I}_0(\xi, I_n) > 0$ and $\Delta_1(\varepsilon) = \mathcal{I}_1(\xi, I_n) - \mathcal{I}_1(\xi, P_0) > 0$ as $\varepsilon \to 0$. □

PROOF OF THEOREM 2.

(i) From the first inequality in Lemma 2 (i),

$$\mathcal{I}_0(\xi, I_n) = \min_{P \succ 0} \mathcal{I}_0(\xi, P).$$

If $P = I_n$ then $U = J^{\mathrm{T}} P J = J^{\mathrm{T}} J = D_+ = k I_q$, so that from (S.12b), and the second inequality in Lemma 2(i),

$$\mathcal{I}_1(\xi, I_n) = \mathrm{ch}_{\max}\left\{\left(Q_+^{\mathrm{T}} Q_+\right)^{-1}\right\} = \min_{P \succ 0} \mathcal{I}_1(\xi, P).$$

Now (S.8) is immediate. If $P_0 = I_n$ then $q = \mathrm{rk}(P_0) = n$, so that all n observations are made at distinct points, hence $D_+ = I_n$ and the design is uniform on its support.
(ii) By Lemma 2(iii) there is $\varepsilon_0 > 0$ for which $\Delta_0(\varepsilon) > 0$ and $\Delta_1(\varepsilon) > 0$ when $0 < \varepsilon \leq \varepsilon_0$. For ε in this range,

$$\mathcal{I}_\nu(\xi, I_n) - \mathcal{I}_\nu(\xi, P_\varepsilon) = \nu(\Delta_0(\varepsilon) + \Delta_1(\varepsilon)) - \Delta_0(\varepsilon) > 0,$$

for $\nu \in (\nu_0, 1]$ and $\nu_0 \equiv \Delta_0(\varepsilon) / (\Delta_0(\varepsilon) + \Delta_1(\varepsilon))$. □

3. TABLES OF SIMULATION RESULTS

Table 1. Minimax precision matrices; multinomial designs: means of performance measures $\pm$ 1 standard error.

Response	N	ν	$\%I_n$	$\mathcal{I}_\nu(\xi, I_n)$	$\mathcal{I}_\nu(\xi, P^\nu)$	T_1 (%)	T_2 (%)	T_3 (%)
	11	.5	1	$3.34 \pm .11$	$3.19 \pm .11$	$4.16 \pm .15$	$3.23 \pm .12$	$12.29 \pm .46$
linear	11	1	1	$3.72 \pm .16$	$3.27 \pm .14$	$11.81 \pm .35$	$9.18 \pm .31$	$14.40 \pm .52$
$n = 10$	51	.5	27	$11.19 \pm .21$	$11.05 \pm .21$	$1.24 \pm .07$	$.85 \pm .05$	$4.10 \pm .22$
	51	1	27	$9.80 \pm .23$	$9.35 \pm .22$	$4.34 \pm .22$	$2.96 \pm .16$	$4.82 \pm .26$
	11	.5	0	5.99 ± 1.25	5.57 ± 1.06	$4.62 \pm .15$	$3.15 \pm .11$	$14.16 \pm .50$
quadratic	11	1	0	7.30 ± 1.82	6.07 ± 1.27	$13.04 \pm .38$	$8.57 \pm .38$	$16.22 \pm .57$
$n = 15$	51	.5	4	$12.61 \pm .46$	$12.40 \pm .45$	$1.58 \pm .07$	$.99 \pm .05$	$5.75 \pm .27$
	51	1	4	$10.69 \pm .54$	$10.03 \pm .51$	$5.95 \pm .24$	$3.54 \pm .17$	$6.72 \pm .31$
	11	.5	0	9.71 ± 1.63	9.15 ± 1.52	$4.87 \pm .15$	$3.25 \pm .11$	$14.90 \pm .51$
cubic	11	1	0	12.98 ± 2.53	11.41 ± 2.22	$13.54 \pm .38$	$8.86 \pm .29$	$16.92 \pm .58$
$n = 20$	51	.5	0	21.67 ± 2.43	21.21 ± 2.38	$1.87 \pm .08$	$1.14 \pm .05$	$7.24 \pm .30$
	51	1	0	20.76 ± 2.9	19.29 ± 2.81	$7.39 \pm .26$	$3.75 \pm .16$	$8.45 \pm .35$

Table 2. Minimax precision matrices; symmetrized designs:
means of performance measures $\pm$ 1 standard error.

Response	N	ν	$\%I_n$	$\mathcal{I}_\nu(\xi, I_n)$	$\mathcal{I}_\nu(\xi, P^\nu)$	T_1 (%)	T_2 (%)	T_3 (%)
	11	.5	20	$2.10 \pm .03$	$2.05 \pm .03$	$2.45 \pm .11$	$1.64 \pm .07$	$8.46 \pm .41$
linear	11	1	20	$1.83 \pm .04$	$1.66 \pm .04$	$8.51 \pm .31$	$4.99 \pm .21$	$10.06 \pm .46$
$n = 10$	51	.5	80	$10.01 \pm .29$	$9.97 \pm .29$	$.40 \pm .05$	$.23 \pm .03$	$1.45 \pm .18$
	51	1	80	$7.77 \pm .29$	$7.67 \pm .29$	$1.48 \pm .16$	$.67 \pm .07$	$1.66 \pm .20$
	11	.5	0	$2.35 \pm .08$	$2.26 \pm .07$	$3.49 \pm .07$	$2.17 \pm .07$	$14.10 \pm .50$
quadratic	11	1	0	$2.01 \pm .09$	$1.74 \pm .08$	$14.40 \pm .37$	$8.57 \pm .22$	$18.05 \pm .58$
$n = 15$	51	.5	85	$10.58 \pm .70$	$10.53 \pm .68$	$.25 \pm .04$	$.15 \pm .02$	$1.02 \pm .16$
	51	1	85	$7.54 \pm .80$	$7.39 \pm .74$	$1.03 \pm .15$	$.49 \pm .07$	$1.19 \pm .19$
	11	.5	3	$2.64 \pm .19$	$2.55 \pm .19$	$3.56 \pm .12$	$1.99 \pm .07$	$15.18 \pm .56$
cubic	11	1	3	$2.39 \pm .27$	$2.11 \pm .26$	$15.19 \pm .44$	$9.09 \pm .28$	$19.69 \pm .70$
$n = 20$	51	.5	58	11.97 ± 1.91	11.80 ± 1.80	$.45 \pm .04$	$.24 \pm .02$	$2.11 \pm .19$
	51	1	58	8.44 ± 2.14	7.94 ± 1.80	$2.23 \pm .18$	$.86 \pm .07$	$2.47 \pm .21$

In Tables 1 and 2, uniform designs account for 100% and 95%, respectively, of the cases in which $P^\nu = I_n$ is optimal. Common exceptions in Table 2 are designs which are uniform apart from having points added or removed at $x = 0$ to maintain symmetry. Those designs for which I_n is not optimal all meet the conditions of Theorem 2 (ii). This was checked numerically: since (S.9a) implies that $\mathcal{I}_0(\xi, P_0) - \mathcal{I}_0(\xi, I_n) = 0$, and (S.9b) implies that $\mathcal{I}_1(\xi, I_n) - \mathcal{I}_1(\xi, P_0) = 0$, their failure is verified by checking that each of these differences is positive.

Table 3. Minimax designs and precision matrices:
performance measures ($T_1 = T_2 = T_3 = 0$ if $P^\nu = I_n$).

Response	N	ν	$\mathcal{I}_\nu(\xi, I_n)$	$\mathcal{I}_\nu(\xi, P^\nu)$	T_1(%)	T_2(%)	T_3(%)
	11	.5	1.60	1.60	0	0	0
linear	11	1	1.10	1.10	0	0	0
$n = 10$	51	.5	6.14	6.14	0	0	0
	51	1	5.10	5.10	0	0	0
	11	.5	1.61	1.53	4.67	2.81	16.06
quadratic	11	1	1.12	1.00	10.84	10.78	10.84
$n = 15$	51	.5	5.80	5.80	0	0	0
	51	1	3.40	3.40	0	0	0
	11	.5	1.55	1.53	1.46	1.13	6.04
cubic	11	1	1.12	1.00	11.03	4.84	11.03
$n = 20$	51	.5	5.56	5.56	0	0	0
	51	1	2.55	2.55	0	0	0

REFERENCES

BHATIA, R. (1997). *Matrix Analysis.* Berlin: Springer.